**An ‘i' for an i, a truth for a truth[[1]](#footnote-1)**

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**Abstract**

Stewart Shapiro’s *ante rem* structuralism recognizes the structural or ‘algebraic’ aspects of mathematical practice while still offering a face value semantics. Fictionalism, as a purely ‘algebraic’ approach, is held to be at a disadvantage as compared with Shapiro’s structuralism, in not interpreting mathematics at face value. However, the face value reading of mathematical singular terms has difficulty explaining how we can use such terms to pick out a unique referent in cases where the relevant mathematical structures admit non-trivial automorphisms. Shapiro offers a solution to this difficulty, but his solution, I argue, evens the score between Shapiro’s structuralism and fictionalism.

1. **Introduction: *algebraic vs assertory views of mathematics***

What are mathematicians investigating when they do mathematics in the context of an axiomatic mathematical theory? On one view of axiomatic theories (articulated most clearly by Gottlob Frege in his correspondence with David Hilbert (Frege 1980)), mathematicians are involved in investigating some particular, independently grasped, subject matter (the natural numbers, in the case of Peano arithmetic), with the axioms viewed as assertions of some of the more fundamental truths about that subject matter. An alternative view, presented by Hilbert in his response to Frege, takes mathematicians not as investigating any one particular subject matter, but rather, as inquiring into what would have to be true of any system of objects satisfying the axioms in question (thus in the case of the Dedekind-Peano axioms, we inquire into what would have to be true in any ‘simply infinite system’, to use Richard Dedekind’s terminology).[[2]](#footnote-2) In this latter ‘algebraic’ approach to axiomatic theories, axioms should be thought of not as assertions of truths about a particular subject matter, but as defining mathematical structures, which may be exhibited in multiple different systems of objects.[[3]](#footnote-3)

Stewart Shapiro’s structuralism sees *both* algebraic and assertory readings as getting something right about our mathematical theories. Thus, Shapiro (2012, p. 388) tells us, his

…ante rem structuralism proposes to combine the algebraic and assertory approaches to at least some theories, including arithmetic, real analysis, complex analysis, various geometries, and set theory. I note different orientations toward mathematical theories. From one perspective, which I dubbed “places-are-offices,” they are algebraic, applying to whatever systems satisfy them. However, if the axioms of a branch of mathematics are satisfiable and categorical, then they characterize a single structure, and the axioms are true *of it*. I call this the “places-are-objects” perspective. The idea is that places in a structure are bona fide objects, and we can have quantifiers ranging over them. The structure itself is a chunk of reality, and the theory is assertory about that structure. So the same axioms are algebraic from one perspective, and assertory from another.

While some theories, such as group theory, *only* allow for an algebraic approach, both algebraic and assertory interpretations are available for those theories whose (2nd-order) axioms are satisfiable and categorical. In such cases, while these theories can be understood algebraically as characterizing ‘offices’ that could be filled by many different ‘officeholders’ in many different instantiating systems, according to Shapiro they also have an assertory interpretation, as truths about the offices themselves (places in the structure), considered as objects in their own right.

Mathematical fictionalism, by contrast, holding as it does that mathematical and scientific practice never requires us to understand our mathematical theories as assertions of truths about particular objects, can be viewed as offering a purely algebraic approach to our mathematical theories. As such, it has been challenged on the grounds that it offers a ‘non-face-value’ reading of our mathematical assertions, arguing that our apparently assertory mathematical claims either are or ought to be ‘cancelled’ by hidden prefaces. The purpose of this paper is to point out that Shapiro’s ante rem structuralism is in as much need of hidden reference-cancelling prefaces as is mathematical fictionalism, so in this regard, the additional ‘places-as-objects’ element of Shapiro’s fictionalism does not save it from one of the key challenges to purely algebraic accounts of mathematics. To see how the score between fictionalism and ante rem structuralism is evened out in this regard, it will be useful to understand how reference-cancelling prefaces feature in both accounts.

1. **Fictionalism as an Algebraic Approach**

According to fictionalists, while there is a ‘face value’ reading of mathematical theories such as number theory and set theory, which views these theories as assertions of truths about realms of independently existing mathematical objects, we have no reason to believe that our mathematical theories are true on this face value reading, since we have no reason to believe that any such objects exist. The fictionalist views mathematicians as working out what follows logically from their mathematical axioms, without regard to the question of whether those axioms are true of any particular interpretation. As such, the fictionalist understanding is in line with Hilbert’s ‘algebraic’ approach, in understanding the “assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation…for which the axioms are satisfied” (Bernays (1967), p. 497). Fictionalists can and will agree with the algebraic approach that *what matters* about our mathematical theorems is not that they tell us what *is* the case, but rather, that they tell us what *would have to be* the case in any system of objects satisfying our mathematical axioms (regardless of whether there are in fact any such objects).

Given that set theoretic and number theoretic axioms are generally presented as apparent assertions (as compared with the axioms for group theory, which are often presented explicitly as stating what would have to be true of any set endowed with a binary operation in order for it to count as a group), those offering a global algebraic understanding of our mathematical theories must claim that this assertoric presentation is misleading as to what either is, or ought to be, really going on when mathematicians do mathematics. Following David Lewis’s terminology (2005, 315), one way of viewing apparently assertoric mathematical utterances as less than attempted assertions of truths is to see them as put forward against the backdrop of a *disavowing preface*, which cancels the apparent commitment to a unique intended interpretation in what follows. Thus, to take number theory as an example, although mathematicians practicing number theory may *appear* to be using numerical singular terms to refer to specific objects, such apparent reference to a unique subject matter would plausibly be cancelled if we read into their practice a disavowing preface. The fictionalist’s proposal is that we imagine mathematicians who make apparent assertions about *the* natural numbers as prefacing their theorizing with ‘*In what follows, suppose that* ℕ(0, 1, +, ×) *is any system of objects satisfying the (2nd order) Peano axioms…’*[[4]](#footnote-4) (perhaps adding, in the Hollywood style, ‘*Any resemblance to any actual systems of objects, living or dead, or actual events is purely coincidental’*). These *Hollywood disclaimers* are either to be understood as implicit in the actual practice of mathematicians (this is the so-called *hermeneutic* fictionalist proposal), or alternatively their uptake is proposed as a recommendation for amending that practice so as to satisfy ontological scruples (as *revolutionary* fictionalists suggest). In each case, though, the claim is that we need not take a face value reading of mathematical claims as assertions of truth in order to make sense of the value of mathematical theories and the uses to which those theories are put.

When we offer this Hollywood reading of mathematical theorizing, the upshot is that we no longer think of mathematical singular terms (such as ‘0’ and ‘1’ in our example) as genuine singular terms referring to specific objects in a domain of interpretation, and mathematical operation symbols (such as ‘+’ and ‘×’ in our example) as referring to specific operations on that domain. The effect of the Hollywood disclaimer is to cancel the presupposition that these are uniquely referring terms. Instead, ‘0’, ‘1’, ‘+’, and ‘×’ become what Richard Pettigrew (2008), in an extremely illuminating discussion of the algebraic (or what he calls ‘Aristotelian’) approach to mathematical theories labels *dedicated free variables*. By a *free variable* in mathematics, Pettigrew has in mind what are sometimes called *parameters*, symbols introduced in the context of a piece of mathematics with associated stipulations.[[5]](#footnote-5) Thus, for example, if I wish to prove that there is no largest prime number I may start my *reductio* by saying “Suppose, for a contradiction, that there is a largest prime number. Let *p* be that prime.” Here, while ‘*p*’, at first glance, may look like it is functioning as an ordinary uniquely referring constant symbol, naming a unique object, in fact we are using it as a *parameter* in our proof: we assume of it only what would have to be true of any object in order for it to fit the description ‘largest prime number’. Indeed, in this case, we discover in the course of the proof that there can be nothing that fits that description, so the term never succeeded in picking out a referent. In Pettigrew’s terminology, a *dedicated* free variable (or, we may say, a dedicated parameter), is a free variable in mathematics where the same symbol has conventionally come to be used with the same (usually implicit) stipulations whenever it is introduced (such as, for example, ‘ℕ’, which, as Pettigrew tells us, is “dedicated to being introduced, along with ‘0’, ‘1’, ‘+’, and ‘×’, by the stipulation ‘ℕ, 0, 1, +, and × satisfy Peano’s axioms”). As with the non-referring parameter *p*, the fictionalist will note that we can reason perfectly well about what would have to be true of any ‘ℕ’, ‘0’, ‘1’, ‘+’, and ‘×’ insofar as they satisfy the axioms, regardless of whether there are any actual objects that fit the bill.

1. **Problems with Global Algebraism**

Assertory accounts of mathematics are held to have a number of advantages over algebraic approaches. In particular, it has been argued that an interpretation of (at least some) mathematical theories as *assertions* of genuine *truths* about mathematical objects is required to make sense of

1. the role of mathematics in empirical scientific theories;
2. the modal claims required to assess mathematical theories understood algebraically;
3. the utterances of mathematicians when doing mathematics.

In the case of objection (a), it is argued that an adequate account of empirical science requires us to take the mathematical claims of our empirical scientific theories to be true at face value (this is the so-called Quine-Putnam indispensability argument, articulated and defended by, for example, Colyvan 2001). Objection (b) is raised by Shapiro (2005) against global algebraism, arguing that the metamathematical claims defenders of the algebraic picture will wish to make concerning logical possibility and logical consequence require us to adopt an assertory account of model theory. Objection (c) from mathematical practice is forcefully put by John Burgess (2004). The objection is that, given that Hollywood disclaimers are not explicit in mathematical practice, reading such prefaces as implicit in what mathematicians intend when they put forward their mathematical theories (as the hermeneutic fictionalist does), or proposing to revise mathematical practice to include the insertion of Hollywood disclaimers (as the revolutionary fictionalist does), is either simply “implausible” (p.28) (in the former case), or “*comically immodest*” (p. 30) in the latter.

I have challenged objections (a) and (b) extensively in other work (Leng 2007, Leng 2010), so will not consider them here. Rather, my focus in this paper will be on objection (c) from mathematical practice, according to which the global algebraism offered by fictionalism puts it at a disadvantage over views that interpret at least some mathematical theories as genuinely assertoric, since assertion-cancelling disclaimers are not explicit in mathematical practice, so would need to be either ‘read in’ by global algebraists to existing mathematical practice or alternatively advocated as revisions to that practice. By contrast, it is standardly thought, ‘assertory’ views have no need for such disclaimers, so are at an advantage on being able to advocate what Burgess (2005, p. 26) calls the “*default* interpretation”, according to which mathematical singular terms refer to mathematical objects, and mathematical utterances assert of these objects that they have various properties.

In response to this claim, I wish to consider the question of how, according to assertory views, we can understand our use of apparently referring singular terms such as ‘*i*’, where it is unclear what in our practice could single out a unique reference. Stewart Shapiro (2012) offers a plausible account of the use of terms such as ‘*i*’ on behalf of the *ante rem* structuralist’s assertory view of mathematics, but the approach could equally be used by more traditional non-structuralist ‘assertory’ accounts of mathematical theories. While I agree that Shapiro’s account offers a plausible solution to the problem of apparent reference to indiscernibles, my claim is that the adoption of this solution introduces hidden ‘disclaimers’ into mathematical practice in a way that evens the score between fictionalism’s global algebraism and the assertory account of mathematics. In both cases there is a move away from the so-called *default* or face value interpretation of mathematical singular terms, the only significant difference being in the positioning of the assertion-cancelling disclaimers.

1. ***Assertory* Structuralism and the Problem of Nonrigid Structures**

Let us consider, then, Shapiro’s *ante rem* structuralism, so as to understand the trouble it faces from apparent reference to indiscernibles. The problem that arises for Shapiro’s assertory ‘places as objects’ perspective is a result, as Shapiro (2012, p. 388) tells us, of the existence of theories that are satisfiable and categorical, but which apply to nonrigid structures. Non-rigid structures allow for non-trivial automorphisms, that is, mappings that permute some of their places, while leaving the overall structural properties fixed. One example of such a structure is Euclidean space, for which rigid transformation provides such an automorphism. And famously, the complex numbers provide such a structure, with the automorphism in question being the mapping *f*(*x* + *iy*) = *x* – *iy*, which permutes *i* and –*i*.

The complex numbers have been thought by some (e.g., Keränen 2001) to present a counterexample to the structuralist view of mathematical objects as places in structures, whose only intrinsic properties are structural. Given the automorphism, *i* and –*i* are interchangeable from a structural perspective, so if these objects are taken to have *only* structural properties, the principle of identity of indiscernibles would lead, catastrophically, to the identification of *i* and –*i*. Shapiro (2008) responds to this challenge on behalf of *ante rem* structuralists by rejecting the principle of identity of indiscernibles, allowing that mathematical structures can contain multiple distinct places whose structural properties are nevertheless identical. But, Shapiro notes, the acceptance of indiscernible mathematical objects leaves a residual problem for *ante rem* structuralism.[[6]](#footnote-6) The problem, in short, is to answer the question, “How do we manage to talk about, and thus, in some sense, refer to indiscernible objects?” (Shapiro 2012, p. 381)

How can we use the term ‘*i*’ to pick out *the* positive square root of -1? Put this way, the question is jarring: we know that in the complex number structure there are two positions that could equally well do the job of being ‘the’ positive square root (and once we’ve picked one to do that job, the other one can happily do the job of the negative square root). But what could possibly determine that our choice of one as *positive* square root is the right one? Nothing at all, it would seem (and certainly nothing would hang on our getting that choice right). Suppose God had identified one of these objects as the positive square root, and suppose *we* somehow managed to pick out the *other* one as the referent for our term, ‘*i*’. The success of our mathematical theorizing would be in no way be affected by this mistake. So why should we worry about picking out *the* correct square root? When doing mathematics we certainly don’t worry about this, and Shapiro offers a diagnosis of what’s going on that explains why we’re right not to worry. The issue arises when we see ‘*i*’ as a referring term and ask what its referent could be. Shapiro’s solution is to challenge the idea that the function of the term ‘*i*’ is to refer to a specific *i*.

Shapiro’s diagnosis makes use of Craige Roberts’ (2003) work in linguistics on the semantics and pragmatics of definite noun phrases in natural languages (which can include names, definite descriptions, demonstratives, singular pronouns). Against the Russellian view of definite descriptions involving existence and uniqueness at the level of semantics, Roberts argues, with a wealth of illustrative examples, that the existence and, particularly, the uniqueness of a denotation for a definite noun phrases is registered instead at the level of pragmatics, and that in many cases, the pragmatic context is such that the uniqueness of an object denoted by a definite is not required for felicitous use of a definite noun phrase. An example (adapted by Shapiro (2012, p. 393) from Roberts (2003), p. 327) of a context where felicitous use doesn’t require uniqueness is as follows:

“Remember that chess set that came with an extra pawn? I could have used an extra king, but I never needed the extra pawn.”

Assuming that the set came with, let’s say, 9 white pawns and 8 black ones, there is no obvious unique referent singled out by the use of the definite description in ‘the extra pawn’ (from the context, we should not assume that any one of the nine whites was singled out as a spare – they may just have all been packed into the box together). Nevertheless the use of the definite description in ‘the extra pawn’ seems felicitous here, even if no particular pawn has been singled out as *the* extra one. Similarly, there are contexts where the felicitous use of a definite description does not require the *existence* of an object denoted, as in,

“If a strange man and a curious woman live here, the strange man will scare my cat, and the curious woman will make friends with it.” (Shapiro, 2012, p. 396, adapted from Roberts (2003), p. 311)

The existence assumptions that generally accompany the use of the definite descriptions in ‘the strange man’ and ‘the curious woman’ are cancelled by the hypothetical context (“*Suppose there is a strange man and a curious woman – though there might not be. Well if there is, then, …*”). According to Roberts’ account, the pragmatics of discourse are such as to set up *discourse referents*, which keep track of the apparently referring terms (‘the strange man’, ‘the extra pawn’, etc.), but the existence of these *discourse referents* does not presuppose the existence or uniqueness of objects denoted by the definite descriptions in question.

Shapiro’s particular interest, given the problem of *i* and –*i*, is felicitous uses of definites where *uniqueness of denotation* is not presupposed. The model then is that of the chess pieces. Noting that we have two square roots of -1, we introduce the singular term ‘*i*’ to pick out (any) one of them. Then we go on to use ‘*i*’ as if it is genuine name with a particular referent, just like we used ‘the extra pawn’ *as if* we’d succeeded in picking out some particular pawn, even though our felicitous use of the definite doesn’t require that we have any particular one in mind. (Compare: ‘*Remember the chess set that came with an extra pawn? Let’s call the extra one ‘Jenny’. I could have used an extra king, but I never needed to use Jenny*’.) The use of a name in this case as a placeholder for any one of the chess pieces (or any one of the square roots) suggests that the names here are functioning more like the parameters (either dummy names or free variables) we find in natural deduction systems, than like genuinely referring terms. Thus, for example, in a natural deduction system, if ‘(*x*)(*x*)’ is a line in our proof we may allow the introduction of ‘(*b*)’, so long as systems are in place in our proof system to ensure that nothing is being assumed of *b* except that  holds of it.[[7]](#footnote-7) In relation to such examples, Shapiro (2012, p. 403) points out that,

In some ways, parameters function as constants; in others they function as variables. In the case of existential elimination, we have it that some object in the domain satisfies . The semantic role of the term *b* is to denote *one* such object. So in that sense, the parameter is like a constant. But it is crucial that we do not specify which such object, even if we could. The rules of engagement require the reasoner to avoid saying anything about *b* that does not hold of any object that satisfies . In that sense, *b* functions more like a variable, ranging over the ’s. Free variables, when they are allowed to occur in deductions, often have the same mixed role.

The point is that, although we use a symbol such as ‘*b*’ that *can* function in our language as a proper name, in these contexts ‘*b*’ is not functioning to name a particular object in our domain, but only as a placeholder for *any* object (assuming that there is one) that satisfies , the indication of its ‘placeholder’ status being given in the derivation by its being introduced via the existential instantiation rule, glossed in informal mathematical proof as the move from ‘There is at least one ’ to ‘Let *b* be a ’.

Shapiro’s proposal (2012, p. 405), then, in the peculiar case of *i* and –*i*, is to take the apparent referring term ‘*i*’as a parameter, analogous to the use of dummy names/free variables in natural deduction systems. In his “rational reconstruction” (Shapiro 2012, p. 399) of how the term ‘*i*’ might have come about, Shapiro imagines mathematicians either coming to realise, or perhaps simply assuming, that there is an algebraic closure of the reals. They then introduce the phrase ‘the complex numbers’ to pick out ‘the’ algebraic closure of the reals, though, as with the example of ‘the extra pawn’, there need be no assumption of *uniqueness* of referent. The mathematicians then “notice that in the complex numbers (or in any algebraic closure of the real numbers), there is square root of −1: ∃*x* (*x*2 = -1)” (*ibid.* p. 399) introducing a discourse referent “the square root of -1”, again along the lines of ‘the extra pawn’, not implying a unique referent. “At this point,” Shapiro conjectures,

the members of our community realize that they are going to continue to discuss these and other square roots of negative numbers, and so they introduce a singular term “*i*” for this purpose. (*ibid.*, p. 399)

But at no point have the mathematicians assumed that, in using the term ‘*i*’, they have succeeded in picking out any particular object of the several (in this case, two) that could do the job. Rather, on establishing the existence of at least one square root of -1, the mathematicians have introduced ‘*i*’ solely with the stipulation that ‘*i*’ is one such square root. What looked like a referring term is in fact a disguised parameter, and the question of how we manage to refer successfully to *the* positive square root of -1 is dodged: the answer is simply that *we don’t*. Despite appearances to the contrary, ‘*i*’ is not a referring term.

1. **Evening the Score: Fictionalism vs *Ante Rem* Structuralism on the ‘face value’ reading of mathematical singular terms**

As an account of the use of the apparently uniquely referring term ‘*i*’, when there is a failure of uniqueness of referent, Shapiro’s disguised parameter solution seems quite plausible. But note that in order to provide such an account, Shapiro has had to move away from a ‘face value’ reading of some of our apparent referring terms in mathematics. There may be no explicit indicator, when mathematicians introduce the term ‘*i*’, that they don’t take it as naming a particular mathematical object, but only as a parameter that may equally pick out either one of two candidates to do the job. Nevertheless Shapiro proposes that we ‘read in’ as implicit in the practice a hidden *parameter-indicator* (‘Let ‘*i*’ be a square root of -1’…), which cancels the assumption that ‘*i*’ is functioning as an ordinary proper name.

It is this element of Shapiro’s picture that, I claim, evens the score between his assertory account of mathematical theorizing and the fictionalist’s global algebraism, at least as concerns Burgess’s challenge against global algebraism that it fails to offer a face value reading of mathematical claims. Indeed, the fact that Shapiro’s quite compelling story about our talk of ‘*i*’ ends up viewing an apparently uniquely referring term as actually functioning as a disguised parameter opens the way for considering the extent to which similar things might be going on *elsewhere* in mathematics when we have what appear to be uniquely referring terms. The fictionalist’s approach, as we have seen, follows Shapiro’s in reading hidden parameter-indicators into our mathematical practice,in the fictionalist’s case via disavowing prefaces (the Hollywood disclaimers) at the start of theorizing. The purpose of these disavowing prefaces is to conditionalize what follows on the supposition (which may or may not be true) that there are objects satisfying the relevant axioms, and the result of their introduction is that they cancel the assumption, which might otherwise be assumed to be in place, of the actual existence of objects referred to by the apparently referring terms of the discourse. Indeed, the hidden parameter-indicators, ‘Let ℕ, 0, 1, +, and × satisfy Peano’s axioms’, ‘Let ℂ be an algebraic closure of ℝ’…, when read in to the practice as disavowing prefaces at the beginning of mathematical theorizing, cancel existence assumptions in the discourse that follows by conditionalising them in much the same way that the antecedent ‘If a strange man and a curious woman live here’ cancels the assumption of existence behind the uses of definite descriptions that follow there.

The complaint against fictionalism as an algebraic view was that, by denying the assumption of *existence* of referents for the apparently referring terms of mathematics, and treating these instead as hidden parameters, the fictionalist offended against the ‘face value’ reading of mathematical theories, whereas assertory views of mathematics could take what mathematicians say as true at face value, without reading in any hidden parameters into the practice. But while the fictionalist proposes to cancel *existence* assumptions, Shapiro proposes to cancel *uniqueness* assumptions, at least in those cases where we have categorical theories picking out non-rigid structures, and again the strategy Shapiro uses to cancel these assumptions is to read hidden parameter-indicators into the practice. Uses of singular terms usually requires the existence *and* uniqueness of a referent. The fictionalist proposes hidden parameter-indicators at the start of mathematical theorizing to cancel the otherwise standard assumption of existence by showing the use of apparently referring terms to be conditional on the (undischarged, and unasserted) hypothesis that they have a referent. Shapiro’s *ante rem* structuralist, on the other hand, proposes hidden parameter-indicators dotted through mathematical discourse, where necessary, to cancel the otherwise standard assumption of uniqueness by showing the use of such terms to be functioning like parameters for existential instantiation.[[8]](#footnote-8) In both cases, though, the recommendation is that we *don’t* read what look like ordinary referring terms at face value.

1. **Conclusion**

Assertory views of mathematics, such as Shapiro’s *ante rem* structuralism, claim an advantage over global algebraic views such as mathematical fictionalism, this advantage being that they can offer a face value reading of mathematical theories, without ‘reading in’ hidden reference-cancelling prefaces into the discourse. The case of ‘*i*’ and ‘-*i*’ suggests that this advantage has been overstated. Plausibly, given the non-uniqueness of a suitable referent of ‘*i*’, the appropriate response to this issue is to take ‘*i*' to be functioning not as a genuinely referring term, but as a ‘dummy name’ or ‘dedicated free variable’, introduced by stipulation as *a* square root of -1. To make this move, the defender of the assertory view has to ‘read in’ a hidden reference-cancelling preface into the practice when ‘*i*’ is introduced. But if, it turns out, the apparently referring ‘*i*’ need not actually refer to a specific *i* in order for mathematical use of the apparent referring term to make sense, then why not think the same for apparently referring terms in mathematics globally? Shapiro’s case for a non-face-value reading of terms such as ‘*i*’ opens the way for considering whether other apparently referring terms in mathematics are genuine referring terms. If the apparently referring ‘*i*’ need not in fact refer to a specific *i* in order for us to make use of such a term in mathematical theorizing, then why think that apparent assertions of ‘truths’ in mathematics need to be viewed as genuine assertions of truth?

**References**

John Burgess (2004), ‘Mathematics and *Bleak House*’, *Philosophia Mathematica* 12: 18-36

Mark Colyvan (2001), *The Indispensability of Mathematics* (Oxford: OUP)

Gottlob Frege (1980), Philosophical and Mathematical Correspondence, G. Gabriel, et al. (eds.) (Oxford: Blackwell)

Geoffrey Hellman (1989), *Mathematics without Numbers: Towards a Modal-Structural Interpretation* (Oxford: OUP)

Geoffrey Hellman (2003), ‘Does Category Theory Provide a Framework for Mathematical Structuralism?’ *Philosophia Mathematica* 11: 129-157

Mark Eli Kalderon (2005) (ed.) , Fictionalism in Metaphysics (Oxford: Oxford University Press)

Jukka Keränen (2001), ‘The Identity Problem for Realist Structuralism’, *Philosophia Mathematica* 9: 308-330

Elaine Landry (2011), ‘How to be a Structuralist All the Way Down’, *Synthese* 179(3): 435-454

Mary Leng (2007), ‘What's there to know? A fictionalist account of mathematical knowledge’. In M. Leng, A. Paseau, & M. Potter (Eds.), Mathematical Knowledge (Oxford: Oxford University Press): pp. 84-108

Mary Leng (2010), *Mathematics and Reality* (Oxford: OUP)

David Lewis (2005), ‘Quasi-Realism is Fictionalism’, in Kalderon (2005): 314-321

Richard Pettigrew (2008), ‘Platonism and Aristotelianism in Mathematics’, *Philosophia Mathematica* 16: 310-332

Stewart Shapiro (2005), ‘Categories, Structures, and the Frege-Hilbert Controversy’, *Philosophia Mathematica* 13: 61-77

Stewart Shapiro (2006), ‘Structural Identity’, in MacBride (2006), 109-145.

Stewart Shapiro (2008), ‘Identity, Indiscernibility, and Ante Rem Structuralism: The Tale of *i* and –*i*’, *Philosophia Mathematica* 16: 285-509

Stewart Shapiro (2012), ‘An ‘i’ for an i: Singular Terms, Uniqueness, and Reference’, *Review of Symbolic Logic* 5 (3): 380-415

Neil Tennant (2000), ‘Deductive vs. Expressive Power: A Pre-Gödelian Predicament’, *Journal of Philosophy* 97(5): 257-277

1. I presented versions of this paper at the Foundations of Mathematical Structuralism conference in Munich (2016), at the Department of Mathematics at the University of Athens, and at a meeting of the *Mind and Reason* research group at the University of York. I am grateful to the audiences on all three of these occasions for helpful discussions. I first attempted to write a paper along these lines in 2010, and I thank Stewart Shapiro for his patient comments on that much earlier, rather half-baked, incarnation of this article. For detailed and constructive comments on this later, hopefully more fully-baked, version, I am very grateful to Neil Tennant, acting as referee for this journal, and a further anonymous referee. [↑](#footnote-ref-1)
2. Paul Bernays describes Hilbert’s approach to axioms aptly as follows: “the axiomatic method is presented and practiced in the spirit of the abstract conception [*the algebraic structuralist conception*] of axiomatics that arose at the end of the nineteenth century and which has been adopted in modern mathematics. It consists in … understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation … for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure. (Bernays (1967), p. 497, quoted in Landry (2011), p. 438 – the insertion ‘*the algebraic structuralist conception*’ is due to Landry.) [↑](#footnote-ref-2)
3. The terminology, ‘algebraic’ and ‘assertory’, in relation to these two approaches to mathematical axioms, is due to Geoffrey Hellman (2003). A distinction somewhat along these lines is picked up on by Neil Tennant (2000), who uses ‘polymathematics’ and ‘monomathematics’ respectively to distinguish between mathematical theories understood as applying equally to multiple structures and those thought of as picking out a particular intended structure. However, Tennant’s distinction, if mapped directly on to Hellman’s, would suggest that, where a theory is intended to pick out only isomorphic structures, it should automatically be viewed as *assertory*. The global algebraic approach I have in mind, by contrast, plays down the distinction between categorical and non-categorical theories, at least as concerns the interpretation of axioms. In all cases, according to the algebraic approach, we should interpret our theories as applying equally well to all systems of objects instantiating the axioms, to the extent that there are any such systems, and not as picking out one particular system as its intended subject matter. In the categorical case, this will include multiple different systems, all of which are isomorphic (such as, in the case of the 2nd order Peano axioms, the Zermelo ordinals and the von Neumann ordinals), whereas in the noncategorical case this will include non-isomorphic systems. [↑](#footnote-ref-3)
4. NB the qualification ‘2nd order’ is important here, as the 1st order axioms are not categorical, and yet fictionalists do not think that when mathematicians talk about ‘the natural numbers’ they have in mind any old model of the 1st order axioms (including non-standard models). In what follows, unless otherwise stated I will take the ‘Peano axioms’ to mean the second order formulation of these axioms, using the full induction axiom rather than the first order axiom scheme of induction. [↑](#footnote-ref-4)
5. ‘Parameters’ are more often thought of as being introduced using constant symbols (‘a’, ‘b’,…), whereas ‘free variables’ suggests the use of variable symbols (‘x’, ‘y’,…). It is generally a matter of convention which symbols play the role, though constant symbols are more common in standard mathematical proofs. In formal natural deduction systems, either variables or constant symbols can play this role, depending on the system, though again the use of constant symbols is more common. [↑](#footnote-ref-5)
6. Indeed, as Shapiro (2012, p. 388) points out, this problem arises for other assertory views, even those that hold that *i* and –*i* can have non-structural properties that may serve to individuate them: “the issue arises for any philosophy that holds that there are assertory mathematical theories of nonrigid structures, especially if some such theory has a singular term that (apparently) denotes one of a batch of indiscernible objects. It does not matter, at least at first, what the metaphysical nature of mathematical objects may be. Our philosopher may be a traditional platonist, or she may hold that mathematical objects are somehow mental constructions, or social constructs, or perhaps she is a quietist about mathematical ontology, or whatever. All that matters, for now, is that the languages be understood literally, as assertory, and that some numerically distinct objects are indistinguishable.” [↑](#footnote-ref-6)
7. Typically, in natural deduction systems, ‘ϕ(*b*)’ will be introduced as an *assumption* in the proof, to be discharged further down in the proof by the existential elimination rule. Shapiro (2012, p. 403) suggests an alternative formal rule of ‘existential instantiation’ would better suit the informal practice of mathematicians, where the rule would be:

   “from a sentence in the form,

   (∃*x*)ϕ(*x*) ,

   one can infer

   ϕ(*b*),

   provided that the term *b* does not occur previously in the deduction. The

   inferred formula ϕ(*b*) rests on whatever premises and assumptions the

   existential formula (∃*x*)ϕ(*x*) rests on.” [↑](#footnote-ref-7)
8. Pettigrew (2008) makes a similar point about the parallel between a global algebraist’s reading of mathematical theories as involving hidden parameters on a global scale, as compared with Shapiro’s view of mathematics as involving only local hidden parameters. Pettigrew’s interest, though, is in algebraic vs non-algebraic versions of structuralism. The relevance of this parallel has not, to my knowledge, been appreciated in relation to fictionalism, whose status as an ‘algebraic’ approach to mathematics has not been widely appreciated. [↑](#footnote-ref-8)