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The Uplink of Distributed MIMO: Wireless Network Coding v.s. Coordinated Multipoint

Dong Fang, Alister Burr, *Member, IEEE,*

***Abstract*—In this paper, we propose an advanced transmission protocol for the next generation wireless access network by using the so-called wireless network coding (WNC). As an alternative to conventional cooperative multipoint (CoMP) in 3GPP standard, WNC enables access points cooperation, which combines the flows using network coding function, instead of sending multiple separate information flows to the hub base station. It is worth noting that the wireless network coding in principle allows cooperation with less backhaul load compared to the CoMP transmission, while enjoying a superior performance compared with CoMP in terms of both bandwidth and energy efficiency.**

***Index Terms*—Wireless Network Coding, Distributed MIMO, Coordinated Multipoint.**

I. Introduction

In recent years, the concept of network MIMO, also known as coordinated multipoint, (CoMP), has attracted intensive attention [3] [4]. A cluster of several base stations cooperate to serve multiple MTs, so that the combination operates as a large multiuser virtual MIMO system. The combination of a large number of antennas results in a much increased multiplexing gain. It also means that these MTs can operate using the same radio resources without causing mutual interference, and thus has potential to greatly increase capacity of cellular systems. However it also potentially inevitably increases the backhaul load: on the uplink, a compressed version of the signals received by each BS must be forwarded to the hub. This is especially a problem for wireless backhaul systems.

Recently, wireless network coding (a.k.a. physical-layer network coding [1], [2]) has been introduced for wireless networks containing multiple nodes and multiple data flows. The approach here is that an intermediate or relay node in a multihop wireless network, which receives a superimposed combination of source symbols, decodes and forwards some joint function of these symbols rather than either of the symbols alone. Typically, this joint function is some linear operation on the Galois field containing the modulation constellation, a generalization of the modulo-2 (XOR) function employed in the original physical layer network coding [5]. This can then be combined at the destination with other information which may have come by a different route through the network to regenerate the original data symbols. This paper considers the application of this principle to uplink transmission in a distributed MIMO system. We will show that it can eliminate interference between the RBSs as effectively as the network MIMO, without increasing the backhaul load.

Dong Fang and Alister Burr are with Department of Electronics, University of York, York, YO10 5DD, UK, (email: {dong.fang, alister.burr}@york.ac.uk).

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II. System Model and Transmission Protocols

A simple system model of the distributed MIMO is illustrated in Fig. 1, where the access points (AP) are densely deployed for the date exchange between the hub base station (HBS) and mobile terminals (MT). In the uplink, the messages from the MTs are relayed by APs to HBS. The links connecting MTs and APs are wireless, referred to as the access link, while that connecting APs and HBS are wireline, referred to as the backhaul link.

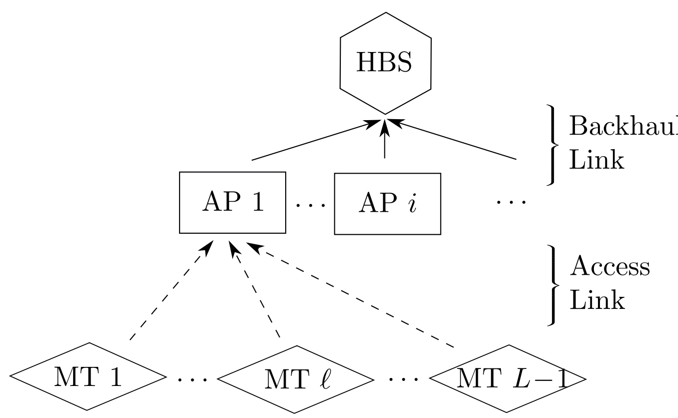

Fig. 1. The uplink of distributed MIMO.

Let $\mathscr{A}_{2^m}$ denote the alphabet of 2^m-PSK/QAM modulation, where m is a positive integer. Let $\mathcal{M}_s : \mathbb{F}_2^m \to \mathscr{A}_{2^m}$ denote the mapping from bits to complex symbols used at each MT. Let $x_\ell = \mathcal{M}_s(\mathbf{s}_\ell) \in \mathscr{A}_{2^m}, \ell \in \{1, \ldots, L\}$ denote the transmitted complex symbol of the ℓ-th MT, where $\mathbf{s}_\ell \triangleq [s_{1,\ell}, \ldots, s_{m,\ell}] \in \mathbb{F}_2^m$ is m-length binary tuple of the ℓ-th MT.

A generalized model is described as follows: there are L MTs at the edge of M APs, where the i-th AP observes the superimposed signal as

$$y_i = \sum_{\ell=1}^{L} h_{i\ell} x_\ell + z_i, \qquad (1)$$

where z_i is the additive white Gaussian noise with zero mean and variance N_0 per dimension; and $h_{i\ell}$ denotes the channel coefficient between the ℓ-th MS and i-th AP.

A. Bandwidth-unlimited backhaul: ideal CoMP

In the ideal CoMP, it is assumed that the bandwidth of backhaul link is unlimited such that each AP can directly forwards the received noisy signal y_i to the HBS for implementing the joint multiuser ML detection:

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathscr{A}_{2^m}^L} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2, \quad (2)$$

where $\mathbf{x} \triangleq [x_1, \ldots, x_L]$ denotes the symbol vector of MTs and $\hat{\mathbf{x}}$ denotes its estimated version.

B. Bandwidth-limited backhaul: non-ideal CoMP

The non-ideal CoMP is a practical approach of network MIMO, where the backhaul link is bandwidth-limited.

In the non-ideal CoMP, each AP adopts the multiuser detection to extract the bit-wise log likelihood ratio (LLR) of each MT's symbol, given by

$$\Lambda_{j,\ell} = \frac{\sum\limits_{s_{j,\ell}=0} P(x_\ell)\, p(y_i|x_\ell)}{\sum\limits_{s_{j,\ell}=1} P(x_\ell)\, p(y_i|x_\ell)}, j \in \{1, \ldots, m\}, \quad (3)$$

where $\Lambda_{i,\ell}$ denotes the LLR corresponding to the j-th bit of the binary tuple $\mathbf{s}_\ell$; and the conditional probability $p(y_i|x_\ell)$ is calculated as (4), which is shown on the top of next page.

Then a scaler quantizer is applied for quantizing the analogue soft information into bits, given by

$$\tau_{j,\ell} = \mathcal{Q}(\Lambda_{j,\ell}). \quad (5)$$

The quantized index $\tau_{j,\ell}$ will be transmitted via the pipeline of backhaul link to HBS.

C. Proposed Design

Unlike the CoMP, the proposed PNC decodes the linear combination of each message from MTs rather than separately decoding them.

Each AP linearly maps the superimposed signal (SS) into the network coded symbol (NCS), given by

$$\mathcal{L}_i : \sum_{\ell=1}^{L} h_{i\ell} x_\ell \to \mathbf{s}_{\mathcal{L},i}. \quad (6)$$

where $s_{\mathcal{L},i}$ denotes the NCS generated from the i-th linear equation, taking the form:

$$\mathbf{s}_{\mathcal{L},i} = \bigoplus_{\ell=1}^{L} \mathbf{Q}_{i\ell} \otimes \mathbf{s}_\ell \quad (7)$$

where $\mathbf{Q}_{i\ell}$ is a binary matrix whose dimension is $n \times m$, where $n \geq m$. Here we note that $n > m$ results in extended mapping. We refer to $\mathbf{Q}_{i\ell}$ as the coefficient matrix of NCS.

Given the received signal y_i, the AP estimates the NCS $\mathbf{s}_{\mathcal{L},i}$ based on the maximal likelihood (ML) rule. Integrating the designed linear mapping in (7) into the ML detection, we have

$$\hat{\mathbf{s}}_{\mathcal{L},i} = \arg\max_{\mathbf{s}_{\mathcal{L},i}} p(y_i|\mathbf{s}_{\mathcal{L},i}), \quad (8)$$

where $\hat{\mathbf{s}}_{\mathcal{L},i}$ represents the estimated NCS; $p(y_i|\mathbf{s}_{\mathcal{L},i})$ is the likelihood function, given by

$$p(y_i|\mathbf{s}_{\mathcal{L},i}) \propto \sum_{(x_1,\cdots,x_\ell,\cdots,x_L):\mathbf{s}_{\mathcal{L},i}=\bigoplus\limits_{\ell=1}^{L}\mathbf{Q}_{i\ell}\otimes\mathbf{s}_\ell} p(y_R|x_{ss.}), \quad (9)$$

where $x_{ss.} \triangleq \sum\limits_{\ell=1}^{L} h_{i\ell}x_\ell$ represents the SS and the summation includes all transmitted symbols $(x_1, \cdots, x_\ell, \cdots, x_L)$ such that the SS is mapped into the NCS $\mathbf{s}_{\mathcal{L},i}$. The conditional probability density function (PDF) $p(y_i|x_{ss.})$ is given by

$$p(y_i|x_{ss.}) = \frac{1}{2\pi\sigma_w^2} \exp\left(-\frac{|y_i - x_{ss.}|^2}{2\sigma_w^2}\right). \quad (10)$$

A general mapping function is given by

$$\begin{bmatrix} \mathbf{s}_{\mathcal{L},1} \\ \vdots \\ \mathbf{s}_{\mathcal{L},i} \\ \vdots \\ \mathbf{s}_{\mathcal{L},L} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{Q}_{11} & \cdots & \mathbf{Q}_{1\ell} & \cdots & \mathbf{Q}_{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{Q}_{i\ell} & \cdots & \mathbf{Q}_{i\ell} & \cdots & \mathbf{Q}_{iL} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{Q}_{L1} & \cdots & \mathbf{Q}_{i\ell} & \cdots & \mathbf{Q}_{LL} \end{bmatrix}}_{\mathbf{G}} \begin{bmatrix} \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_\ell \\ \vdots \\ \mathbf{s}_L \end{bmatrix}, \quad (11)$$

where $\mathbf{G}$ is referred to as the generator matrix of the designed PNC, which also can be treated as the network transfer matrix.

Condition 1. *To successfully recover the MT's message, the linear mapping function $\mathcal{L}_i$ should be invertible. More specifically, if $\mathbf{Q}_{i\ell}$ is squared matrix, $\forall i \in \{1, ..., L\}$ and $\forall \ell \in \{1, ..., L\}$, i.e., $n = m$, then $\mathbf{G}$ should be full rank. If $\mathbf{Q}_{i\ell}$ is not a squared matrix, i.e., $n > m$, then $\mathbf{G}$ should be full column rank: that is, all columns are linearly independent.*

Proof: We note that the linear mapping in (11) in fact forms a set of linear Diophantine equations with 2^{mL} variables. These are soluble if and only if $\mathbf{G}$ are invertible, that is, have rank at least 2^{mL}. If $n = m$, $\forall i \in \{1, ..., L\}$ and $\forall \ell \in \{1, ..., L\}$, then $\mathbf{G}$ is a squared matrix. In this case, $\mathbf{G}$ should be full rank. For the case of $n > m$, since the row rank is equal to the column rank and there are 2^{mL} columns, $\mathbf{G}$ must have full column rank. Since there are only 2^{mL} columns, they are linearly independent if none of them is zero and they are different. Thus, this condition is proved. ■

In the proposed design, the criterion of selecting the optimal $\mathbf{G}$ should: 1) ensure the unambiguous decodability in 1. and 2) maximize the sum-rate:

$$\hat{\mathbf{G}} = \arg\max_{\mathbf{G}} R_{\text{Sum}}, \quad (12)$$

where R_{Sum} denotes the sum-rate, which is determined by the **Theorem 1**.

Theorem 1. *For the proposed WNC, the sum-rate of uplink in the distributed MIMO is given by*

$$R_{Sum} \leq \frac{m}{n} \sum_{i=1}^{L} I(Y_i; \mathbf{S}_{\mathcal{L},i}). \quad (13)$$

where $I(Y_i; \mathbf{S}_{\mathcal{L},i})$ denotes the mutual information between the received signal at AP and NCS.

The proof of **Theorem 1** is shown in the Appendix. The mutual information $I(Y_i; \mathbf{S}_{\mathcal{L},i})$ is calculated as (14), which is shown on the top of next page. We also note that the integration in (14) is typically unsolvable. However, we can use the Monte-Carlo integration instead, as shown in (15), on

$$\begin{aligned} p\left(y_i|x_\ell\right) &= \sum_{x_1}\cdots\sum_{x_k,\forall k\in\{1,\dots,L\}/\{\ell\}}\cdots\sum_{x_L}\Pr\left(x_1,\cdots,x_k,\cdots x_L\right)p\left(y_i|x_1,\cdots,x_\ell,\cdots x_L\right) \\ &\propto \frac{1}{2^{m(L-1)}}\sum_{x_1}\cdots\sum_{x_k,\forall k\in\{1,\dots,L\}/\{\ell\}}\cdots\sum_{x_L}\exp\left(-\frac{|z_i|^2}{2N_0}\right) \end{aligned} \tag{4}$$

$$\begin{aligned} I(Y_i;\mathbf{S}_{\mathcal{L},i}) &= H\left(\mathbf{S}_{\mathcal{L},i}\right)-H\left(\mathbf{S}_{\mathcal{L},i}|y_i\right)=\log_2(2^n)+\sum_{\mathbf{s}_{\mathcal{L},i}}\int_{y_i\in\mathbb{C}^2}p\left(y_i,\mathbf{s}_{\mathcal{L},i}\right)\log_2\left[p\left(\mathbf{s}_{\mathcal{L},i}|y_i\right)\right]dy_i \\ &= n+\sum_{\mathbf{s}_{\mathcal{L},i}}P\left(\mathbf{s}_{\mathcal{L},i}\right)\int_{y_i\in\mathbb{C}^2}p\left(y_i\left|\mathbf{s}_{\mathcal{L},i}\right.\right)\log_2\left[\frac{p\left(y_i\left|\mathbf{s}_{\mathcal{L},i}\right.\right)P\left(\mathbf{s}_{\mathcal{L},i}\right)}{\sum_{\mathbf{s'}_{\mathcal{L},i}}P\left(\mathbf{s'}_{\mathcal{L},i}\right)p\left(y_R\left|\mathbf{s'}_{\mathcal{L},i}\right.\right)}\right]dy_i \end{aligned} \tag{14}$$

$$I(Y_i;\mathbf{S}_{\mathcal{L},i})=n+\mathbb{E}\left\{\log_2\left[\frac{p\left(y_i\left|\mathbf{s}_{\mathcal{L},i}\right.\right)P\left(\mathbf{s}_{\mathcal{L},i}\right)}{\sum_{\mathbf{s'}_{\mathcal{L},i}}P\left(\mathbf{s'}_{\mathcal{L},i}\right)p\left(y_R\left|\mathbf{s'}_{\mathcal{L},i}\right.\right)}\right]\right\} \tag{15}$$

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D. Compute-and-forward

As a newly-emerged technique, CPF is proposed to harness the co-channel interference over wireless multiple access channels, which can be treated as a subset of WNC. In the following, we provide brief introduction of lattice construction for CPF, which is chosen to serve as our benchmark.

Assuming that there is a discrete subring of $\mathbb{C}$, denoted by R, forming a principal ideal domain (PID). The most widely-used PIDs for PNC are: 1) the ring of integer $\mathbb{Z}$; 2) the ring of Gaussian integer $\mathbb{Z}[i]$, where $\mathbb{Z}[i]=\{a+bi:a,b\in\mathbb{Z}\}$; and 3) the ring of Eisenstein integer $\mathbb{Z}[\omega]$, where $\mathbb{Z}[\omega]=\{a+b\omega:a,b\in\mathbb{Z}\}$ and $\omega=\frac{-1+\sqrt{3}i}{2}$. Let $\Lambda\subseteq\mathbb{C}^n$ and $\Lambda'\subseteq\Lambda$ be two full-rank R-lattices, which are referred to as the fine and coarse lattices, respectively. The quotient R-module Λ/Λ' forms a partition of Λ, which is the set of all of coset representatives of Λ' in Λ, i.e., $\Lambda/\Lambda'=\{\lambda+\Lambda':\lambda\in\Lambda\}$. Considering the complex-value fading, we pay particular attention to the complex Construction A such that the PID R would be either $\mathbb{Z}\left[i\right]$ or $\mathbb{Z}\left[\omega\right]$.

Let π be a prime in R of norm $|\pi|^2=q,q\in\mathbb{Z}^+$, such that $R/\pi R\cong\mathbb{F}_q$. This induces a ring isomorphism $\psi:R/\pi R\to\mathbb{F}_q$ and its inverse $\psi^{-1}:\mathbb{F}_q\to R/\pi R$. Note that a natural projection from R to $R/\pi R$ via $\mathrm{mod}\pi R$ operation also exists: $\mathrm{mod}\pi R: R\overset{\mathrm{mod}\,\pi R}{\to}R/\pi R$. Hence, a ring homomorphism is obtained:

$$\sigma: R\overset{\mathrm{mod}\,\pi R}{\to}R/\pi R\overset{\psi}{\to}\mathbb{F}_q, \tag{16}$$

such that $\sigma\left(\gamma\right)\in\mathbb{F}_q,\forall\gamma\in R$. This mapping function σ can also operate in a vector manner [10], given as $\sigma\left(\mathbf{a}\right)=\sigma\left(a_1,\dots,a_n\right)=\left[\begin{array}{ccc}\sigma\left(a_1\right) & \dots & \sigma\left(a_n\right)\end{array}\right]$, where $\mathbf{a}$ is a vector of length n. Based on these, the Construction A-based R-lattices [] can be represented by

$$\Lambda=\left\{\lambda\in\gamma R^n:\sigma\left(\gamma^{-1}\lambda\right)=\mathcal{C}\right\},\quad \Lambda'=\gamma(\pi R)^n, \tag{17}$$

where γ is the scaling factor to control the transmission power and $\mathcal{C}$ is a (n,k) linear code over $R/\pi R\cong\mathbb{F}_q$. Here, when the PID R is either $\mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$, Λ is either the Construction A based $\mathbb{Z}[i]$- or $\mathbb{Z}[\omega]$-lattice.

Current communication system normally requires the size of signal constellations to be a power of 2. According to [6,7], for $R/\pi R$, π would necessarily be the Gaussian prime and Eisenstein prime for $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, respectively. The details about Gaussian and Eisenstein prime can be found in [6,7]. We also note that those finite fields of characteristic 2 that can be represented by $R/\pi R$ for $R=\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ are only: $\mathbb{F}_2\cong\mathbb{Z}[i]/(1+i)\mathbb{Z}[i]$ and $\mathbb{F}_4\cong\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]$, respectively.

As far as we know, there is no lattice partition that is isomorphism to the finite field of size $2^m,m\geq 3$, which is for fair comparison with proposed WNC with 2^m-ary modulation. This is because that if 2^m can be factorized as $2^m=t^2$, t is neither a Gaussian nor Eisenstein prime and hence we cannot use it for the R-lattice partition. Similarly, the complex Construction A cannot form a R-lattice partition which is isomorphism to $\mathbb{F}_8$, $\mathbb{F}_{16}$ and $\mathbb{F}_{64}$ etc.. In [5], Yuan *et al.* adopt a (mn,mk) linear code over $\mathbb{F}_4$ to expand the message space from $\mathbb{F}_{4^m}$ to $\mathbb{F}_4^m$. However, the ring isomorphism is still $\mathbb{F}_4\cong\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]$ and hence the signal constellation (dimension-wise) does not change. Based on above discussion, we note that only $\mathbb{Z}[\omega]/2\mathbb{Z}[\omega]$ based CPF can be a feasible candidate for comparison with the proposed PNC (only for the case where the QPSK modulation is employed). The decoding algorithm of CPF follows the inter-forcing manner. Please refer to [4] for details.

III. Performance Evaluation

In this section, we evaluate the performance of the proposed WNC and the benchmarks. Without loss of generality, a 2MT-2AP-1HBS distributed MIMO system is set up, where each node is equipped with single antenna and each channel experiences the i.i.d. Rayleigh fading. For fair comparison, we choose lattice partition $\Lambda/\Lambda'=(\mathbb{Z}[\omega]/2\mathbb{Z}[\omega])^n$ based CPF as the benchmark for our proposed WNC using QPSK as both of their message space are $\mathbb{F}_4$. Similarly, both ideal and non-ideal CoMP are assumed to adopt the QPSK modulation at each MT.

The FER performance comparison is shown in Fig. 2. We can observe that the ideal CoMP clearly achieves the optimum FER performance. However, we also note that the limited number of quantized bits results in a severe performance degradation for non-ideal CoMP, compared with ideal CoMP. Even though the backhaul load is increased, e.g., quantization with 3bits, there is still a significant performance gap between the ideal and non-ideal CoMP. We can observe that the proposed WNC is superior to the non-ideal CoMP in terms of FER while achieving less backhaul load, i.e., only 4bits/symbol. In addition, the proposed WNC can also achieve the same 2nd order diversity as ideal and non-ideal CoMP. As another benchmark, we realize that the FER performance of CPF based on $\Lambda/\Lambda' = (\mathbb{Z}[\omega]/2\mathbb{Z}[\omega])^n$ is worse than that of the proposed WNC. The is because the signal constellation of this lattice partition is not optimized for the transmission power.

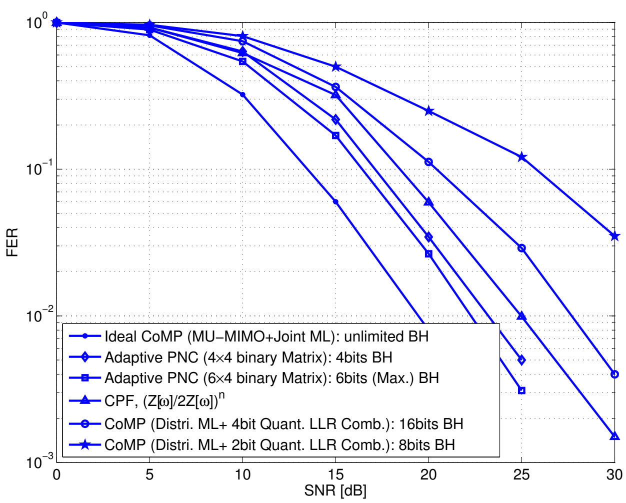

Fig. 2. FER performance.

IV. Conclusions

In this paper, we proposed an advanced transmission protocol for the next generation wireless access network by using the so-called wireless network coding (WNC). As an alternative to conventional cooperative multipoint (CoMP) in 3GPP standard, WNC enables access points cooperation, which combines the flows using network coding function, instead of sending multiple separate information flows to the hub base station. It is worth noting that the wireless network coding in principle allows cooperation with less backhaul load compared to the CoMP transmission, while enjoying a superior performance compared with CoMP in terms of both bandwidth and energy efficiency.

Appendix

Proof: Assume that the uplink is free from errors. Then the mapping procedure for $\mathbf{s}_{\mathcal{L},i}$ at the relay can be regarded as discrete memoryless source encoding in which the 2^{mL}-bit binary tuple $\mathbf{s} \triangleq (\mathbf{s}_1, ..., \mathbf{s}_\ell, , ...\mathbf{s}_L)$ is the input symbol; and $\mathbf{s}_{\mathcal{L},i}, \forall i \in \{1, ..., L\}$ is output symbol. Clearly, the mapping from $\mathbf{s}$ to $\mathbf{s}_{\mathcal{L},i}$ is a surjection.

Assuming that length of the sequence of 2^n-ary $(n \geq m)$ symbols $\mathbf{s}_{\mathcal{L},i}$ is N and $N \to \infty$. Let $\mathbf{b}_{\mathcal{L},i}$ denote the binary representation of the sequence of $\mathbf{s}_{\mathcal{L},i}$, where the average length of $\mathbf{b}_{\mathcal{L},i}$ is $\bar{K} = H(\mathbf{S}_{\mathcal{L},i}) \times N = nN$ (the length of $\mathbf{b}_{\mathcal{L},i}$ varies from frame to frame). For simplicity of notation, we omit the index of elements in $\mathbf{b}_{\mathcal{L},i}$ and similarly hereafter. The element of $\mathbf{b}_{\mathcal{L},i}$ is denoted as $b_{\mathcal{L},i}$.

Clearly, the length of the sequence of 2^{mL}-ary symbols $\mathbf{s}$ is also equal to N. We note that $\mathbf{s}$ is uniformly distributed with the probability of $\frac{1}{2^{mL}}$ as it is drawn from L 2^m-ary alphabets. Let $\mathbf{b}$ denote the binary representation of the sequence of $\mathbf{s}$, where the length of $\mathbf{b}$ is $T = \log_2(2^{mL}) \times N = mLN$. An element of $\mathbf{b}$ is denoted as b. Clearly, b is uniformly distributed since $\mathbf{s}$ is uniformly distributed. Hence, we have $H(B) = \log_2(2) = 1$.

We note that the entropy of $\mathbf{s}_{\mathcal{L},i}$ is constant during the whole mapping procedure. According to Shannon's variable-length source coding theorem, we have

$$\frac{H(B_{\mathcal{L},i})}{\log_2(2)} + \varepsilon \geq \frac{\bar{K}}{T} \geq \frac{H(B_{\mathcal{L},i})}{\log_2(2)}, \tag{18}$$

where $\varepsilon = \frac{1}{T} \to 0$ as $T = mLN \to \infty$. Hence, (18) can be further written as

$$\frac{\bar{K}}{T} = \frac{H(B_{\mathcal{L},i})}{\log_2(2)}. \tag{19}$$

Based on this, we know that when $N \to \infty$, the average code rate of the equivalent source encoder, denoted by $\bar{\mathscr{R}}$, is then given as $\bar{\mathscr{R}} = \frac{\sum_{i=1}^{L} H(\mathbf{S}_{\mathcal{L},i})}{mL} = \frac{n}{m}$.

Now we consider a distorted uplink. We note that the relay can receive up to $R_{\text{Sum,max}}$ bits per symbol for $\mathbf{s}$, where $R_{\text{Sum,max}}$ denotes the maximal achievable rate of $\mathbf{s}$. The linear mapper outputs the LNCC $\mathbf{s}_{\mathcal{L},i}$ with rate up to $\sum_{i=1}^{L} I(Y_R; \mathbf{S}_{\mathcal{L},i})$ bits per symbol. The compression efficiency, namely, the coding (mapping) rate which the linear mapping compresses $\mathbf{s}$ to $\mathbf{s}_{\mathcal{L},i}$, is independent of whether the MAC phase is distorted or not. The average code (mapping) rate $\bar{\mathscr{R}}$ is constant when $N \to \infty$ since each $\mathbf{s}$ corresponds an unique $\mathbf{s}_{\mathcal{L},i}$.

Based on this, we have

$$\bar{\mathscr{R}} = \frac{n}{m} = \frac{\sum_{i=1}^{L} I(Y_R; \mathbf{S}_{\mathcal{L},i})}{R_{\text{Sum,max}}}. \tag{20}$$

Based on (20), the input rate with respect to $\mathbf{s}$, namely, the sum-rate in the MAC phase, R_{Sum}, should be bounded by

$$R_{\text{Sum}} \leq \frac{m}{n} \sum_{i=1}^{L} I(Y_R; \mathbf{S}_{\mathcal{L},i}). \tag{21}$$

This completes the proof of **Theorem 2**. ■

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