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Gibbs states of continuum particle systems with unbounded spins: existence and uniqueness

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Abstract

We study an infinite system of particles chaotically distributed over a Euclidean space $\mathbb{R}^d$. Particles are characterized by their positions $x \in \mathbb{R}^d$ and an internal parameter (spin) $\sigma_x \in \mathbb{R}^m$, and interact via position-position and (position dependent) spin-spin pair potentials. Equilibrium states of such system are described by Gibbs measures on a marked configuration space. Due to the presence of unbounded spins, the model does not fit the classical (super-) stability theory of Ruelle. The main result of the paper is the derivation of sufficient conditions of the existence and uniqueness of the corresponding Gibbs measures.

1 Introduction

The aim of this paper is to study the equilibrium states of the following infinite particle system in continuum. We consider a countable collection γ of identical point particles chaotically distributed over a Euclidean space X $(= \mathbb{R}^d)$. Additionally, we assume that each particle $x \in \gamma$ possesses an internal structure described by a mark (spin) σ_x taking values in a single-spin space S $(= \mathbb{R}^m)$ and characterized by a single-spin measure χ on S. Each two particles $x, y \in \gamma$ interact via a pair potential given by the sum of two components:

(i) a purely positional (e.g. distance dependent, possibly singular or hard-core) potential

$$\Phi : X \times X \to \mathbb{R} \cup \{+\infty\}, \;\; \Phi(x,y) = \Phi(y,x), \;\; x, y \in X \tag{1.1}$$

(representing e.g. a molecular force);

(ii) a (position-dependent) spin-spin interaction of the form $W_{xy}(\sigma_x, \sigma_y)$, where

$$W_{xy} = W_{yx} : S \times S \to \mathbb{R}, \;\; W_{xy}(s,t) = W_{xy}(t,s), \;\; s, t \in S, \tag{1.2}$$

are symmetric functions of polynomial growth.

Our system can be seen as a combined type model, which carries features of both an infinite particle system in continuum (i.e., non-ideal classical gas) and an interacting system of unbounded spins on a discrete set (random graph) formed by positions of the particles. Therefore we have to take into account two possible catastrophic effects caused by dense particle configurations and by strong spin interactions, respectively. Notably, our model does not fit the setup of the previous papers on marked point processes, which have mostly been dealing with the case of compact spins. Thus its study requires development of new methods, involving an appropriate concept of thermodynamical stability. The corresponding physical systems are e.g. magnetic gases, ferrofluids, amorphous magnets, etc., see [16], [17], [36]. Such compound (with additional spin variables) models are of a special interest in mathematical physics because they provide some (of still very few) examples of continuum systems where the appearence of an (orientational odering) phase transition has been proved rigorously. This makes important an alternative question of the absence of phase transition, i.e. the uniqueness of thermal equilibrium states, expected e.g. in the low density regime. Such models are still poorly understood, to say nothing of the general case of non-compact (possibly multi-dimentional vector) marks and unbounded (not necessarily ferromagnetic or quadratic) spin interactions, which motivates our present study.

Once the interaction potentials have been specified, the whole system is governed by the heuristic Hamiltonian

$$H(\widehat{\gamma}) := \sum_{\{x,y\} \subset \gamma} \Phi(x,y) + \sum_{\{x,y\} \subset \gamma} W_{xy}(\sigma_x, \sigma_y)$$

on the phase space $\widehat{\Gamma}(X)$ consisting of marked configurations $\widehat{\gamma} = \{(x, \sigma_x)\}$, where the corresponding position configuration $\gamma = \{x\}$ belongs to the space

$$\Gamma(X) := \{\gamma \subset X : \; N(\gamma_\Lambda) < \infty \text{ for any } \Lambda \in \mathcal{B}_0(X)\}.$$

Here $\mathcal{B}_0(X)$ is the collection of all compact subsets of X and $N(\gamma_\Lambda)$ denotes the number of elements of $\gamma_\Lambda := \gamma \cap \Lambda$. In what follows, we will use the notation $\widehat{\gamma}_\Lambda := \{(x, \sigma_x), \; x \in \gamma_\Lambda\}$.

The equilibrium states of the system are described by certain probability measures on $\widehat{\Gamma}(X)$. In absence of the interaction (the so-called "free" case), the equilibrium state is unique and given by the marked Poisson measure

$$\widehat{\pi}(\mathrm{d}\widehat{\gamma}) = \bigotimes_{x \in \gamma} \chi(\mathrm{d}\sigma_x) \pi_z(\mathrm{d}\gamma),$$

where π_z is the Poisson measure on $\Gamma(X)$ with intensity (i.e., particle density) $z > 0$, see e.g. [12], [8]. If the interaction is present, the equilibrium states are given by marked Gibbs measures μ on $\widehat{\Gamma}(X)$, which are constructed as perturbations of $\widehat{\pi}$ by the (heuristic) density $\exp\{-H(\widehat{\gamma})\}$. Rigorously, any such μ is a probability measure on $\widehat{\Gamma}(X)$ with prescribed conditional distributions $\mu(\mathrm{d}\widehat{\gamma} \mid \widehat{\gamma} = \widehat{\eta}$ off $\Lambda)$, $\widehat{\eta} \in \widehat{\Gamma}(X)$, for an exhausting system of sets $\Lambda \in \mathcal{B}_0(X)$. These conditional distributions, or Gibbs specification kernels of our model, are explicitly given by formulae (2.23) and (2.24) below and will be denoted by $\Pi_\Lambda(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta})$. So, the study of Gibbs measures is reduced to the generic problem of reconstructing a Markov random field μ on $\widehat{\Gamma}(X)$ from its local specification $\Pi = \{\Pi_\Lambda\}_{\Lambda\in\mathcal{B}_0(X)}$. This constitutes the standard Dobrushin–Lanford–Ruelle (DLR) formalism described in details in Section 2.2.

We denote by $\mathcal{G}$ the set of all such measures (for fixed H and χ). The study of the structure of the set $\mathcal{G}$ is of a great importance. In particular, there are three fundamental questions arising here:

(E) *Existence*: is $\mathcal{G}$ not empty?

(U) *Uniqueness:* is $\mathcal{G}$ a singleton?

(M) *Multiplicity:* does $\mathcal{G}$ contain at least two (and hence infinitely many) elements?

In this paper, we derive sufficient conditions for (E) and (U). We introduce the set $\mathcal{G}^{\mathrm{t}} \subset \mathcal{G}$ of tempered Gibbs measures that are concentrated on the space $\widehat{\Gamma}_{\mathrm{t}}(X)$ of configurations with certain bounds on their density and spin growth, see (2.39), (2.40). Under reasonable assumptions on the interaction potentials Φ and W (responsible for the global stability of the system and listed under (A1)–(A6) below), we will prove that the set $\mathcal{G}^{\mathrm{t}}$ is not empty (Theorem 4) and, moreover, that $\mathcal{G}^{\mathrm{t}}$ is a singleton provided the particle density z is small enough (Theorem 5). To prove the existence, we use the extension of the analytic method developed in [24] for the case of interacting particle systems without spins. A crucial technical step here is to prove a uniform bound of certain exponential moments of the corresponding specification kernels $\Pi_\Lambda(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta})$ as $\Lambda \nearrow X$ for any boundary condition $\widehat{\eta} \in \widehat{\Gamma}_{\mathrm{t}}(X)$. This in turn allows to show the compactness (in the topology of local set convergence on $\widehat{\Gamma}(X)$) of the family $\{\Pi_\Lambda(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta}),\ \Lambda \in \mathcal{B}_0(X)\}$ and thus the existence of the limiting points, which can be identified with elements of $\mathcal{G}^{\mathrm{t}}$.

In order to study the uniqueness, we represent (via the natural embedding $\mathbb{Z}^d \subset X$) the configuration space $\widehat{\Gamma}(X)$ in the form $\widehat{\Gamma}(\mathrm{Q})^{\mathbb{Z}^d}$, where Q is an elementary cube in X, and construct a lattice model (with intricate non-linear spin space $\widehat{\Gamma}(\mathrm{Q})$) equivalent to the original continuum model. In this setting we can use the Dobrushin–Pechersky approach to the uniqueness problem for lattice-type systems, see [14], [6, Theorem 2.6] and also [32, Theorem 4] and [2, Theorem 3], where this method is applied to continuum systems (without spins) on $\Gamma(X)$. The uniform exponential moment bounds allow us to control the interaction growth and to check the conditions of the Dobrushin–Pechersky criterion for the lattice counterpart of the continuum model. As a by-product of our method we also prove a decay of correlations for the (unique) Gibbs measure (Corollary 7), which seems to be entirely new for such systems.

Let us note that a general theory of Gibbs measures with the Ruelle-type (super-) stable interactions on marked configuration spaces can be found e.g. in [1], [22], [26], [28] and [35]. However, it is essentially restricted to compact spins and hence does not apply to our model (see Remark 2.6). The case of unbounded vector spins interacting via potentials of superquadratic growth and position-position interaction with no hard core, including the existence and uniqueness problems for the associated Gibbs states, has not been treated so far in the literature.

The question of the existence of multipliple Gibbs states (phase transitions) has been discussed for ferromagnetic interactions in [39], [16], [5] (discrete spins), [17], [36] (hard core position-position interaction, continuous scalar spins) and in our complementary paper [9] (no hard core, continuous scalar spins). The appearance of Berezinskii-Kosterlitz-Thouless phase transition in a ferrofluid of hard-core particles with O(2)-invariant spins was shown in [18], see also references given there.

The structure of the paper is as follows. In Section 2 we give a rigorous description of our model (Subsections 2.1, 2.2, 2.4) and formulate the main results (Subsection 2.5). Section 3 is devoted to the derivation of moment bounds. In Section 4, we prove our main result on the existence problem – Theorem 4. Section 5 deals with the uniqueness problem. We start with the lattice representation of our model (Subsection 5.1) and prove Theorem 5 in Subsection 5.2. In Section 6, we present proofs of several technical lemmas.

2 The model and main results

2.1 Marked configuration spaces

As a location (phase) space X for our particle system, let us fix the d-dimensional ($d \geq 1$) Euclidean space $\mathbb{R}^d$. It is endowed with the Lebesgue measure $\mathrm{d}x$ on the Borel σ-algebra $\mathcal{B}(X)$. By $\mathcal{B}_0(X)$ we denote the ring of all bounded sets from $\mathcal{B}(X)$. The configuration space $\Gamma(X)$ consists of all locally finite subsets of X, that is,

$$\Gamma(X) = \{\gamma \subset X : \ N(\gamma_\Lambda) < \infty \text{ for any } \Lambda \in \mathcal{B}_0(X)\}, \tag{2.1}$$

where $N(\gamma_\Lambda)$ stands for the cardinality of the restriction $\gamma_\Lambda := \gamma \cap \Lambda$. Let $C_0(X)$ be the set of all continuous functions $f : X \to \mathbb{R}$ with compact support. The space $\Gamma(X)$ is equipped in the standard way with the *vague topology*, which is the weakest one that makes continuous all maps

$$\Gamma(X) \ni \gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_0(X).$$

It is well known (see, e.g., [19, Section 15.7.7]) that $\Gamma(X)$ is a Polish (i.e., separable completely metrizable) space in this topology; an explicit construction of the appropriate metric can be found in [23]. By $\mathcal{P}(\Gamma(X))$ we denote the space of all probability measures on the corresponding Borel σ-algebra $\mathcal{B}(\Gamma(X))$.

Let now S be another Euclidean space $\mathbb{R}^m$ (with $m \neq d$ in general) and consider the Cartesian product $\widehat{X} := X \times S$. For any element $\widehat{x} := (x, s)$ of $\widehat{X}$ its S-component s may be seen as a mark (spin, charge etc.) attached to a particle placed at position $x \in X$. Given a set $\Lambda \subset X$, we will often write for short $\widehat{\Lambda} := \Lambda \times S$. The canonical projection $p_X : X \times S \to X$ can be naturally extended to the configuration space $\Gamma(\widehat{X}) := \Gamma(X \times S)$. Observe that for a configuration $\widehat{\gamma} \in \Gamma(\widehat{X})$ its image $p_X(\widehat{\gamma})$ is a subset of X that possibly admits accumulation and multiple points, and hence does not in general belong to $\Gamma(X)$. The *marked* configuration space $\widehat{\Gamma}(X)$ is then defined in the following way (see e.g. [8], [12], [20]):

$$\widehat{\Gamma} := \widehat{\Gamma}(X) := \left\{ \widehat{\gamma} \in \Gamma(\widehat{X}) : \ p_X(\widehat{\gamma}) \in \Gamma(X) \right\}. \tag{2.2}$$

We will systematically use the notation

$$\gamma_\Lambda := \gamma \cap \Lambda \text{ and } \widehat{\gamma}_\Lambda := \widehat{\gamma} \cap \widehat{\Lambda}$$

for $\gamma \in \Gamma(X)$, $\widehat{\gamma} \in \widehat{\Gamma}(X)$, $\Lambda \subset X$ and cylinder sets $\widehat{\Lambda} := \Lambda \times S$.

We equip $\widehat{\Gamma}(X)$ with the so-called τ-topology defined as the weakest one that makes continuous the map

$$\widehat{\Gamma}(X) \ni \widehat{\gamma} \mapsto \langle g, \widehat{\gamma} \rangle := \sum_{(x,s) \in \widehat{\gamma}} g(x, s) \tag{2.3}$$

for any bounded continuous function $g : X \times S \to \mathbb{R}$ with $\operatorname{supp} g \subset \Lambda \times S$ for some $\Lambda \in \mathcal{B}_0(X)$, i.e. with spatially compact support. This topology has been employed in different frameworks in e.g. [1], [11] and [26]; for a short account of its properties see also [10]. An advantage of the τ-topology is that it makes $\widehat{\Gamma}(X)$ a Polish space, in contrast to the vague topology inherited from $\Gamma(\widehat{X})$ (which is generated by the maps (2.3) with $g \in C_0(\widehat{X})$). For an example of the τ-consistent metric on $\widehat{\Gamma}(X)$ see Section 2 of [7]. We then endow $\widehat{\Gamma}(X)$ with the associated Borel σ-algebra $\mathcal{B}(\widehat{\Gamma})$, also coinciding with the trace σ-algebra $\mathcal{B}(\Gamma(\widehat{X})) \cap \widehat{\Gamma}(X)$. This is the smallest σ-algebra for which the counting variable

$$\widehat{\gamma} \mapsto N(\widehat{\gamma} \cap \Delta) \tag{2.4}$$

is measurable for any $\Delta \in \mathcal{B}(X \times S)$ with $p_X(\Delta) \in \mathcal{B}_0(X)$.

For a fixed $\Lambda \in \mathcal{B}_0(X)$, we consider the space

$$\widehat{\Gamma}_\Lambda := \widehat{\Gamma}_\Lambda(X) = \Big\{ \widehat{\gamma} \in \widehat{\Gamma}(X) : \; p_X(\widehat{\gamma}) \subset \Lambda \Big\} \tag{2.5}$$

of marked configurations located in the cylinder set $\widehat{\Lambda} := \Lambda \times S$. It will be equipped with the image topology $p_\Lambda \circ \tau$ induced from $\widehat{\Gamma}(X)$ under the natural projection

$$p_\Lambda : \; \widehat{\Gamma}(X) \ni \widehat{\gamma} \mapsto \widehat{\gamma}_\Lambda \in \widehat{\Gamma}_\Lambda(X) \tag{2.6}$$

and with the corresponding σ-algebra $\mathcal{B}(\widehat{\Gamma}_\Lambda) = \mathcal{B}(\widehat{\Gamma}) \cap \widehat{\Gamma}_\Lambda(X)$. Notably, $(\widehat{\Gamma}_\Lambda(X), \mathcal{B}(\widehat{\Gamma}_\Lambda))$ is a *standard Borel* space, which means that $\mathcal{B}(\widehat{\Gamma}_\Lambda)$ can be generated by some separable and complete metric on $\widehat{\Gamma}_\Lambda(X)$. We can now define the σ-algebra $\mathcal{B}_\Lambda(\widehat{\Gamma}) := p_\Lambda^{-1} \circ \mathcal{B}(\widehat{\Gamma}_\Lambda)$ on $\widehat{\Gamma}(X)$, which is constituted by the sets

$$\Big\{ \widehat{\gamma} \in \widehat{\Gamma}(X) : \widehat{\gamma}_\Lambda \in \Delta \Big\}, \quad \Delta \in \widehat{\Gamma}_\Lambda(X), \tag{2.7}$$

and hence is σ-isomorphic to $\mathcal{B}(\widehat{\Gamma}_\Lambda(X))$. In other words, $\mathcal{B}_\Lambda(\widehat{\Gamma}) \subset \mathcal{B}(\widehat{\Gamma})$ is the smallest σ-algebra generated by all variables (2.4) with $p_X(\Delta) \subset \Lambda$. Then $(\widehat{\Gamma}(X), \mathcal{B}(\widehat{\Gamma}))$ can be seen as a projective limit of the measurable spaces $(\widehat{\Gamma}_\Lambda(X), \mathcal{B}(\widehat{\Gamma}_\Lambda))$, $\Lambda \in \mathcal{B}_0(X)$, with respect to projection maps, cf. (2.6),

$$p_{\Lambda',\Lambda} : \; \widehat{\Gamma}_\Lambda(X) \ni \widehat{\gamma}_\Lambda \mapsto \widehat{\gamma}_{\Lambda'} \in \widehat{\Gamma}_{\Lambda'}(X), \quad \Lambda' \subset \Lambda. \tag{2.8}$$

In particular, this allows us to use a version of *Kolmogorov's theorem* (cf. [30, Theorem V.3.2]), according to which any probability measure $\mu \in \mathcal{P}(\widehat{\Gamma})$ is uniquely determined by its projections $\mu_\Lambda := p_\Lambda^* \mu \in \mathcal{P}(\widehat{\Gamma}_\Lambda)$, $\Lambda \in \mathcal{B}_0(X)$. Here and in what follows, we denote by $\mathcal{P}(\widehat{\Gamma})$ and $\mathcal{P}(\widehat{\Gamma}_\Lambda)$ the spaces of probaility measures on $\mathcal{B}(\widehat{\Gamma})$ and $\mathcal{B}(\widehat{\Gamma}_\Lambda)$, respectively.

We will also need the subset of marked configurations *finite* in all of $\widehat{X}$

$$\widehat{\Gamma}_0 := \widehat{\Gamma}_0(X) := \bigcup_{\Lambda \in \mathcal{B}_0(X)} \widehat{\Gamma}_\Lambda(X) \tag{2.9}$$

and the subalgebra of *local* events in $\widehat{\Gamma}(X)$

$$\mathcal{B}_0(\widehat{\Gamma}) := \bigcup_{\Lambda \in \mathcal{B}_0(X)} \mathcal{B}_\Lambda(\widehat{\Gamma}). \tag{2.10}$$

Remark 1 *The space* $\widehat{\Gamma}(X)$ *has a fibre bundle-type structure over* $\Gamma(X)$*, where the fibres* $p_X^{-1}(\gamma)$ *can be identified with the product spaces*

$$S^\gamma = \prod_{x\in\gamma} S_x, \quad S_x := S.$$

Thus each $\widehat{\gamma} \in \widehat{\Gamma}(X)$ *can be represented by the pair*

$$\widehat{\gamma} = (\gamma, \sigma_\gamma), \quad \textit{where } \gamma = p_X(\widehat{\gamma}) \in \Gamma(X), \;\; \sigma_\gamma = (\sigma_x)_{x\in\gamma} \in S^\gamma.$$

It follows directly from the definition of the corresponding topologies that the map $p_X : \widehat{\Gamma}(X) \to \Gamma(X)$ *is continuous. Hence for any configuration* γ *the space* $S^\gamma = p_X^{-1}(\gamma)$ *can be considered as a Borel subset of* $\widehat{\Gamma}(X)$*.*

From now on we fix a *single-spin distribution* $\chi \in \mathcal{P}(S)$ (=: the space of probability measures on S) and constant $z > 0$ called the *intensity* or *activity parameter*. Observe that each measurable $f : \widehat{\Gamma}_0(X) \to \mathbb{R}$ can be identified with a family of symmetric Borel functions $f_n : (X \times S)^n \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$f(\widehat{\gamma}) = f_n((x_1, \sigma_1), \ldots, (x_n, \sigma_n)) \text{ for } \widehat{\gamma} = \{(x_1, \sigma_1), \ldots, (x_n, \sigma_n)\}.$$

The *marked Lebesgue-Poisson measure* $\widehat{\lambda}_z$ is defined on $(\widehat{\Gamma}_0(X), \mathcal{B}(\widehat{\Gamma}_0))$ by the relation

$$\int_{\widehat{\Gamma}_0} f(\widehat{\gamma})\, \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}) = f(\emptyset) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{(X\times S)^n} f_n((x_1, \sigma_1), \ldots, (x_n, \sigma_n))\, \chi(\mathrm{d}\sigma_1)\mathrm{d}x_1 \cdots \chi(\mathrm{d}\sigma_n)\mathrm{d}x_n, \tag{2.11}$$

which has to hold for all measurable $f : \widehat{\Gamma}_0(X) \to \mathbb{R}_+$. For each $\Lambda \in \mathcal{B}_0(X)$ it is a finite measure on $\widehat{\Gamma}_\Lambda$ with mass $\widehat{\lambda}_z(\widehat{\Gamma}_\Lambda) = \exp\left\{z \int_\Lambda dx\right\}$. Likewise, the Lebesgue-Poisson measure λ_z on $(\Gamma_0(X), \mathcal{B}(\Gamma_0))$ is defined by

$$\int_{\Gamma_0} f(\gamma)\lambda_z(\mathrm{d}\gamma) = f(\emptyset) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{X^n} f_n(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n, \tag{2.12}$$

holding for all measurable $f : \Gamma_0(X) \to \mathbb{R}_+$.

It is clear that λ_z is an image of $\widehat{\lambda}_z$ under the projection $p_X : \widehat{\Gamma}_0(X) \to \Gamma_0(X)$, whereby $\widehat{\lambda}_z$ allows the disintegration

$$\widehat{\lambda}_z\,(\mathrm{d}\widehat{\gamma}) := \bigotimes_{x\in\gamma} \chi(\mathrm{d}\sigma_x)\; \lambda_z(\mathrm{d}\gamma). \tag{2.13}$$

2.2 The model

Following the DLR approach (for its comprehensive exposition see [15]), in this section we will give the rigorous definition of (grand canonical) Gibbs measures associated with the interaction potentials (1.1), (1.2) and a single-spin measure χ.

We define the *Hamiltonian* (or *energy functional*) $H : \widehat{\Gamma}_0(X) \to \mathbb{R}$ by the formula

$$H(\widehat{\gamma}) := U(\gamma) + E(\sigma_\gamma), \qquad \widehat{\gamma} = (\gamma, \sigma_\gamma) \in \widehat{\Gamma}_0(X), \tag{2.14}$$

involving the positional and spin counterparts

$$U(\gamma) := \sum_{\{x,y\}\subset\gamma} \Phi(x,y) \ \text{ and } \ E(\sigma_\gamma) := \sum_{\{x,y\}\subset\gamma} W_{xy}(\sigma_x, \sigma_y), \tag{2.15}$$

where the sums run over all (unordered) pairs of distinct points $x, y \in \gamma$. By convention, we put $H(\{\emptyset\}) = 0$ and $H(\{(x,\sigma_x)\}) = 0$ for all $(x,\sigma_x) \in \widehat{X}$.

For any $\Lambda \in \mathcal{B}_0(X)$ and $\widehat{\eta} = (\eta, \xi_\eta) \in \widehat{\Gamma}(X)$, the relative local energy is given by

$$H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta}) = H(\widehat{\gamma}_\Lambda) + \Delta H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta}) \tag{2.16}$$

where

$$\Delta H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta}) := \sum_{x\in\gamma_\Lambda} \sum_{y\in\eta_{\Lambda^c}} \Phi(x,y) + \sum_{x\in\gamma_\Lambda} \sum_{y\in\eta_{\Lambda^c}} W_{xy}(\sigma_x, \xi_y). \tag{2.17}$$

Separating different types of interactions, we may rewrite (2.16) as

$$H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta}) = U_\Lambda(\gamma_\Lambda|\eta) + E_\Lambda(\sigma_\Lambda|\xi) \tag{2.18}$$

with

$$U_\Lambda(\gamma_\Lambda|\eta) = U(\gamma_\Lambda) + \sum_{x\in\gamma_\Lambda} \sum_{y\in\eta_{\Lambda^c}} \Phi(x,y), \tag{2.19}$$

$$E_\Lambda(\sigma_{\gamma_\Lambda}|\xi) = E_\Lambda(\sigma_{\gamma_\Lambda}) + \sum_{x\in\gamma_\Lambda} \sum_{y\in\eta_{\Lambda^c}} W_{xy}(\sigma_x, \xi_y). \tag{2.20}$$

The *local Gibbs state* $\mu_\Lambda^{\widehat{\eta}} \in \mathcal{P}(\widehat{\Gamma}_\Lambda)$ with boundary condition $\widehat{\eta} \in \widehat{\Gamma}(X)$ fixed outside volume $\Lambda \in \mathcal{B}_0(X)$ is defined by the formula

$$\mu_\Lambda^{\widehat{\eta}}(\mathrm{d}\widehat{\gamma}_\Lambda) := Z_\Lambda(\widehat{\eta})^{-1} \exp\,\{-H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta})\}\ \widehat{\lambda}_{z,\Lambda}(\mathrm{d}\widehat{\gamma}_\Lambda), \tag{2.21}$$

where $\widehat{\lambda}_{z,\Lambda}$ is the restriction of the Lebesgue-Poisson measure $\widehat{\lambda}_z$ to $\mathcal{B}(\widehat{\Gamma}_\Lambda)$. We will often omit the subscript Λ and just write $\widehat{\lambda}_z\,(\mathrm{d}\widehat{\gamma}_\Lambda)$ and $\lambda(d\gamma_\Lambda)$. Here

$$Z_\Lambda(\widehat{\eta}) := \int_{\widehat{\Gamma}_\Lambda} \exp\,\{-H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta})\}\ \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_\Lambda) \tag{2.22}$$

is the normalizing factor (called the *partition function*) making $\mu_\Lambda^{\widehat{\eta}}$ a probability measure on $\widehat{\Gamma}_\Lambda(X)$ (provided $Z_\Lambda(\widehat{\eta}) < \infty$, which will be the case under certain conditions on the interaction potentials, cf. Corollary 10). Next, we introduce stochastic kernels

$$\widehat{\Gamma}(X) \times \mathcal{B}(\widehat{\Gamma}) \ni (\widehat{\eta}, B) \mapsto \Pi_\Lambda\,(B|\widehat{\eta}) \in [0,1]$$

by the formula

$$\Pi_\Lambda\left(B|\widehat{\eta}\right) := \mu_\Lambda^{\hat{\eta}}\left(B_{\Lambda,\widehat{\eta}}\right), \quad B \in \mathcal{B}(\widehat{\Gamma}), \tag{2.23}$$

where $B_{\Lambda,\widehat{\eta}} := \{\widehat{\gamma}_\Lambda : \widehat{\gamma}_\Lambda \cup \widehat{\eta}_{\Lambda^c} \in B\} \in \mathcal{B}(\widehat{\Gamma}_\Lambda)$. By construction, the projection of $\Pi_\Lambda(\cdot|\widehat{\eta})$ on $\widehat{\Gamma}_{\Lambda^c}$ is just the δ-measure concentrated at $\widehat{\eta}_{\Lambda^c}$. So, the integral relation

$$\int_{\widehat{\Gamma}} F(\widehat{\gamma})\Pi_\Lambda\left(\mathrm{d}\widehat{\gamma}|\widehat{\eta}\right) = Z_\Lambda(\widehat{\eta})^{-1}\int_{\widehat{\Gamma}_\Lambda} F(\widehat{\gamma}_\Lambda \cup \widehat{\eta}_{\Lambda^c})\exp\left\{-H_\Delta(\widehat{\gamma}_\Lambda|\widehat{\eta})\right\}\widehat{\lambda}_z\left(\mathrm{d}\widehat{\gamma}_\Lambda\right), \tag{2.24}$$

holds for any measurable function $F : \widehat{\Gamma}(X) \to \mathbb{R}_+$. Furthermore, the map $\widehat{\Gamma}(X) \ni \hat{\eta} \mapsto \Pi_\Lambda\left(B|\hat{\eta}\right)$ is measurable for each fixed $B \in \mathcal{B}(\widehat{\Gamma})$.

The family $\Pi = \{\Pi_\Lambda\}_{\Lambda\in\mathcal{B}_0(X)}$ constitutes a *Gibbsian specification* on $\widehat{\Gamma}(X)$ (in the standard sense of [15], [33]). In particular, it obeys the *consistency* property

$$\int_{\widehat{\Gamma}} \Pi_\Lambda\left(B|\widehat{\gamma}\right)\, \Pi_{\Lambda'}\left(\mathrm{d}\widehat{\gamma}|\widehat{\eta}\right) = \Pi_{\Lambda'}\left(B|\widehat{\eta}\right), \tag{2.25}$$

which holds for any $B \in \mathcal{B}(\widehat{\Gamma})$, $\widehat{\eta} \in \widehat{\Gamma}(X)$ and $\Lambda, \Lambda' \in \mathcal{B}_0(X)$ such that $\Lambda \subset \Lambda'$ (and thus $\widehat{\Lambda} \subset \widehat{\Lambda'}$).

Let μ be a probability measure on $\widehat{\Gamma}(X)$. We say that μ is a *Gibbs state* associated with the specification Π if it satisfies the *Dobrushin–Lanford–Ruelle* (DLR) equation

$$\mu(B) = \int_{\widehat{\Gamma}} \Pi_\Lambda\left(B|\widehat{\gamma}\right)\, \mu(\mathrm{d}\widehat{\gamma}) \tag{2.26}$$

for all $B \in \mathcal{B}(\widehat{\Gamma})$ and $\Lambda \in \mathcal{B}_0(X)$. We denote by $\mathcal{G} := \mathcal{G}(\widehat{\Gamma})$ the set of all such measures.

In the "free" case when both Φ and W vanish, the corresponding unique Gibbs state $\mu \in \mathcal{G}$ is just the *marked Poisson measure* $\widehat{\pi}$. Equation (2.26) then simplifies to Kolmogorov's theorem, which says that $\widehat{\pi}$ is fully determined by its local projections $\widehat{\pi}_\Lambda = [\widehat{\lambda}_z(\widehat{\Gamma}_\Lambda)]^{-1}\widehat{\lambda}_{z,\Lambda} \in \mathcal{P}(\widehat{\Gamma}_\Lambda)$, $\Lambda \in \mathcal{B}_0(X)$.

2.3 Assumptions on the interaction

Let us specify conditions on the interaction potentials Φ, W and single-spin distribution χ to be used in the proof of our main results. For that, we define a partition $(\mathrm{Q}_k)_{k\in\mathbb{Z}^d}$ of X by "elementary" volumes. Here Q_k is the half-open cube in X with side length 1 centered at point $k = (k^{(1)}, ..., k^{(d)}) \in \mathbb{Z}^d \subset X$, that is,

$$\mathrm{Q}_k := \left\{x = (x^{(1)}, ..., x^{(d)}) \in X :\ x^{(i)} \in \left[k^{(i)} - 1/2,\, k^{(i)} + 1/2\right)\right\}. \tag{2.27}$$

For $k \in \mathbb{Z}^d$ and $\gamma \in \Gamma(X)$ resp. $\widehat{\gamma} \in \widehat{\Gamma}(X)$, we then write for short

$$\gamma_k := \gamma_{\mathrm{Q}_k} \in \Gamma(\mathrm{Q}_k) =: \Gamma_k \ \text{ resp. } \ \widehat{\gamma}_k := \widehat{\gamma}_{\mathrm{Q}_k} \in \widehat{\Gamma}_{\mathrm{Q}_k}(X) =: \widehat{\Gamma}_k.$$

In what follows we always assume that the following conditions hold.

(A1) *Finite range of interactions*, that is, $\exists\, R > 0$ such that $\Phi(x, y) = 0$ and $W_{xy} = 0$ if $|x - y| \geq R$.

(A2) *Lower boundedness* of Φ, that is, $\exists\, M \geq 0$ such that

$$\inf_{x,y \in X} \Phi(x, y) \geq -M. \tag{2.28}$$

(A3) *Local strong superstability* of U, that is, $\exists\, P > 2$ such that for some $A_\Phi > 0$ and $B_\Phi \geq 0$

$$U(\gamma_k) \geq A_\Phi N(\gamma_k)^P - B_\Phi N(\gamma_k) \tag{2.29}$$

for any $k \in \mathbb{Z}^d$ and $\gamma \in \Gamma(X)$.

(A4) *Uniform polynomial bound* on $W_{xy}^- := -\min\{W_{xy},\, 0\}$, that is, $\exists\, r > 0$ and $\mathcal{J}, C_W \geq 0$ such that

$$W_{xy}^-(s, t) \leq \mathcal{J}\left(|s|^r + |t|^r + C_W\right), \quad s, t \in S, \tag{2.30}$$

for all $\{x, y\} \subset X$.

(A5) *Exponential moment bound* on χ, that is, $\exists\, q > r$ such that

$$\int_S e^{A_\chi |s|^q} \chi(\mathrm{d}s) < \infty \tag{2.31}$$

for some $A_\chi > 0$.

In addition, we require the following condition, which guarantees a *spin-position superstability* type estimate (3.1) crucial for our method:

(A6) P, q and r satisfy the relation

$$(P - 2)\,(q/r - 1) > 1. \tag{2.32}$$

Let us point out that neither translation invariance nor continuity of Φ and W is assumed.

Remark 2 *(i) For every potential Φ obeying (A1) and (A2), the local strong superstability (A3) readily implies the global one. More precisely, for any $A'_\Phi \in (0, A_\Phi)$ there exists a $B'_\Phi \geq 0$ such that*

$$U(\gamma) \geq A'_\Phi \sum_{k \in \mathbb{Z}^d} N(\gamma_k)^P - B'_\Phi N(\gamma), \quad \gamma \in \Gamma_0(X). \tag{2.33}$$

This can be easily seen from the following chain of estimates

$$\begin{aligned} U(\gamma) &\geq \sum_{k \in \mathbb{Z}^d} \left[A_\Phi N(\gamma_k)^P - B_\Phi N(\gamma_k)\right] - M \sum_{k \in \mathbb{Z}^d} \sum_{j \in \partial k} N(\gamma_k) N(\gamma_j) \\ &\geq \sum_{k \in \mathbb{Z}^d} \left[A_\Phi N(\gamma_k)^P - M \mathcal{N}_0 N(\gamma_k)^2 - B_\Phi N(\gamma_k)\right] \\ &\geq (A_\Phi - \delta) \sum_{k \in \mathbb{Z}^d} N(\gamma_k)^P - \left[\left(M \mathcal{N}_0 \delta^{-1}\right)^{\frac{2}{P-2}} + B_\Phi\right] N(\gamma), \end{aligned} \tag{2.34}$$

where in the last line we used Young's inequality (6.14) and $\mathcal{N}_0 := N(\partial k)$ *is cardinality of the set* ∂k*. By choosing small values of* $\delta > 0$*, we can get* A'_Φ *arbitrarily close to* A_Φ*.*

(ii) The size of the elementary cubes in the partition $X = \coprod_{k\in\mathbb{Z}^d} \mathrm{Q}_k$ *is irrelevant. Fix any* $\epsilon > 0$*, then (A3) clearly holds for all* $\mathrm{Q}_k^\epsilon := \epsilon(\mathrm{Q}_0 + k)$*,* $k \in \mathbb{Z}^d$*, with proper constants* $A_{\Phi,\epsilon} > 0$ *and* $B_{\Phi,\epsilon} \geq 0$*.*

(iii) One of the best-understood examples of strong superstable interactions is given by the so-called Dobrushin–Fisher–Ruelle *(DFR) potentials behaving at the diagonal like* $\Phi\,(x, y) \geq c\,|x - y|^{-d(1+\theta)}$ *as* $|x - y| \to 0$*, in which case* $P = 2 + \theta$*. For a detailed study and historical comments see [34] and also [24, Remark 4.1].*

(iv) Assumption (A5) is aimed to compensate the polynomial growth of W^- *allowed by (A4). It is obvious that any measure satisfying condition (2.31) is finite. Thus without loss of generality we can choose* χ *to be a probability measure. Furthermore, it is typically assumed that* $\chi(\mathrm{d}s) := e^{-V(s)}\mathrm{d}s$ *for some self-interaction potential* $V : S \to \mathbb{R}$ *growing fast enough:*

$$\exists\, A_V > A_\chi \text{ and } B_V \geq 0 : \;\; V(s) \geq A_V\,|s|^q - B_V, \;\; s \in S. \tag{2.35}$$

(v) The case of bounded W_{xy}^- *is essentially easier to handle. It can be covered by a (simplified) version of our method, which will also work for* $q = 0,\ P = 2$ *(excluded from the general case by condition (2.32)). This requires however* A_Φ *to be large enough. On the other hand, this case fits into Ruelle's superstability approach extended in a straightforward manner to marked configuration spaces (see a related comment in Section 2.6)*

(vi) Except for the finite range, we impose no further restrictions on the positive part $W_{xy}^+ := \max\{W_{xy},\, 0\}$ *of the spin-spin interaction. Indeed, adding any* $W_{xy}^+ \geq 0$ *could only improve our basic estimates in Section 2.5. Of a special interest here are ferromagnetic interactions* W_{xy} *of the form* $J_{xy}|s - t|^2$ *or* $-J_{xy}\langle s, t\rangle$ *with* $J_{xy} \geq 0$ *(notably, these two cases are not equivalent for our model insofar they cannot be reduced to each other by changing the single-spin measure* χ*), see also Remark 6.*

(vii) Assumption (2.32) is crucial for our method. It excludes the possibilty of $\Phi \equiv 0$ *(that is,* $P = 0$*, cf. (2.29)), which case can however be treated by modified arguments provided the spin-spin interaction is purely repulsive, that is,* $W_{xy} \geq 0$ *(as pointed out in Remark 22).*

(viii) The case of multi-particle potentials $\Phi(x_1, ...x_n)$ *and* $W(s_1, ...s_n)$ *with* $n > 2$ *can be studied by similar methods provided the superstability estimate of Proposition 8 holds for the corresponding local Hamiltonians.*

(ix) All the results below remain true if we take any non-atomic Radon measure $\sigma(\mathrm{d}x)$ *on* $(X, B(X))$ *obeying the bound* $sup_{k\in\mathbb{Z}^d}\,\sigma(Q_k) < \infty$ *as intensity measure of the point process* λ_z *(instead of the Lebesgue mass* dx*).*

2.4 Notations

Throughout the paper, we will use following shorthand notations (related to $\Lambda \in \mathcal{B}_0(X)$ and $k \in \mathbb{Z}^d$):

$\Gamma := \Gamma(X)$; $\widehat{\Gamma} := \widehat{\Gamma}(X)$
$\Gamma_\Lambda := \Gamma_\Lambda(X)$; $\gamma_\Lambda := \gamma \cap \Lambda$
$\widehat{\Gamma}_\Lambda := \widehat{\Gamma}_\Lambda(X)$; $\widehat{\gamma}_\Lambda := \widehat{\gamma} \cap (\Lambda \times S)$
$\Gamma_k := \Gamma_{\mathrm{Q}_k}$; $\gamma_k := \gamma_{\mathrm{Q}_k}$
$\widehat{\Gamma}_k := \widehat{\Gamma}_{\mathrm{Q}_k}$; $\widehat{\gamma}_k := \widehat{\gamma}_{\mathrm{Q}_k}$
$\partial k := \{j \neq k : \mathrm{dist}\,(\mathrm{Q}_k, \mathrm{Q}_j) \leq R\}$, where 'dist' is the Euclidean distance between two sets in $\mathbb{R}^d$
$\mathcal{N}_0 := N(\partial k)$- cardinality of the set ∂k; obviously, it is independent of $k \in \mathbb{Z}^d$ and finite;
$\gamma_{\partial k} := \cup_{j \in \partial k} \gamma_j$; $\widehat{\gamma}_{\partial k} := \cup_{j \in \partial k} \widehat{\gamma}_j$
$\partial \Lambda := \Lambda_R \setminus \Lambda = \Lambda_R \cap \Lambda^{\mathrm{c}}$
$|\Lambda| := \int_\Lambda \mathrm{d}x$ – volume of Λ
$\mathrm{Q}_{\mathcal{K}} := \bigcup_{j \in \mathcal{K}} \mathrm{Q}_j$, $\mathcal{K} \subset \mathbb{Z}^d$
$H_k(\widehat{\gamma}_k \,|\widehat{\eta}) := H_{\mathrm{Q}_k}(\widehat{\gamma}_{\mathrm{Q}_k} \,|\widehat{\eta})$
$U_k(\gamma_k \,|\eta) := U_{\mathrm{Q}_k}(\gamma_{\mathrm{Q}_k} \,|\eta)$

Further notations will be introduced as needed.

Remark 3 *By assumption (A1), both $\Phi(x, y)$ and W_{xy} vanish for all $x \in \mathrm{Q}_k$ and $y \in \mathrm{Q}_j$ whenever $j \notin \partial k$. The total number $\mathcal{N}_0 = N(\partial k)$ of "neighbor" cubes Q_j, $j \in \partial k$, is independent of k and can be roughly estimated by*

$$\mathcal{N}_0 \leq v_d \left(R + \sqrt{d}\,/2 \right)^d, \quad v_d = \frac{\pi^{d/2}}{\varGamma\,(1 + d/2)}, \tag{2.36}$$

where v_d is the volume of a unit ball in $\mathbb{R}^d$ and $\varGamma$ is the classical gamma function.

2.5 Main results

Let us fix parameters $\kappa, \vartheta > 0$ and define *control functions* $F : \widehat{\Gamma}_0(X) \to \mathbb{R}_+$ and $F_\alpha : \widehat{\Gamma}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ by formulae

$$F(\widehat{\gamma}) = \kappa N(\gamma)^P + \vartheta \sum_{x \in \gamma} |\sigma_x|^q\,, \;\; \widehat{\gamma} = (\gamma, \sigma), \tag{2.37}$$

and

$$F_\alpha(\widehat{\gamma}) = \sup_{k \in \mathbb{Z}^d} \left\{ e^{-\alpha |k|} F(\widehat{\gamma}_k) \right\}, \;\; \alpha > 0, \tag{2.38}$$

respectively. Introduce the space of *tempered configurations*

$$\widehat{\Gamma}_{\mathrm{t}}(X) := \left\{ \widehat{\gamma} \in \widehat{\Gamma}(X) : \; F_\alpha(\widehat{\gamma}) < \infty \text{ for any } \alpha > 0 \right\} \tag{2.39}$$

and the corresponding set $\mathcal{G}^{\mathrm{t}}$ of *tempered Gibbs measures* that are supported by $\widehat{\Gamma}_{\mathrm{t}}(X)$, i.e.

$$\mathcal{G}^{\mathrm{t}} := \left\{ \mu \in \mathcal{G} : \mu(\widehat{\Gamma}_{\mathrm{t}}(X)) = 1 \right\}. \tag{2.40}$$

Obviously, the spaces $\widehat{\Gamma}_{\mathrm{t}}(X)$ and $\mathcal{G}^{\mathrm{t}}$ are independent of the choice of positive κ and ϑ. Furthermore, $\widehat{\Gamma}_{\mathrm{t}}(X)$ can be characterized in the following way:

$$\widehat{\Gamma}_{\mathrm{t}}(X) = \left\{ \widehat{\gamma} \in \widehat{\Gamma}(X) : \sum_{k\in\mathbb{Z}^d} e^{-\alpha|k|} F(\widehat{\gamma}_k) < \infty \text{ for any } \alpha > 0 \right\}. \tag{2.41}$$

The next two theorems summarize the main results of this paper.

Theorem 4 *(Existence and a priori estimate)*

(i) The set $\mathcal{G}^{\mathrm{t}}$ is not empty.

(ii) For any given values

$$\kappa \in (0, A_\Phi) \quad \textit{and} \quad \vartheta \in (0, A_\chi), \tag{2.42}$$

there exists a (explicitly computable) positive constant $\Psi := \Psi(\kappa, \vartheta)$ such that each $\mu \in \mathcal{G}^{\mathrm{t}}$ obeys the moment estimate

$$\sup_{k\in\mathbb{Z}^d} \int_{\widehat{\Gamma}} \exp\left\{F(\widehat{\gamma}_k)\right\} \, \mu(\mathrm{d}\widehat{\gamma}) \le \Psi. \tag{2.43}$$

The proof will be given in Section 4. It is based on the uniform bound of exponential moments for the corresponding specification kernels (similar to (2.43), see Theorem 15) and local equicontinuity of this specification (Theorem 20), which in turn implies that it possesses a cluster point $\mu \in \mathcal{G}^{\mathrm{t}}$.

Theorem 5 *(Uniqueness) For any given $\mathcal{J}_0 > 0$ there exists $z_0 = z_0(\mathcal{J}_0) > 0$ such that $\mathcal{G}^{\mathrm{t}}$ is a singleton for all $\mathcal{J} \le \mathcal{J}_0$ and $z \le z_0$.*

Remark 6 *The threshold activity value z_0 can be computed explicitly. Observe that $[\lambda_z(\Gamma_\Lambda)]^{-1} \int_{\Gamma_\Lambda} N(\gamma_\Lambda)\, \mathrm{d}\lambda_z(\gamma_\Lambda) = z$ for any $\Lambda \in \mathcal{B}_0(X)$, so that z can be interpreted as the point density of the underlying Poisson point process, cf. [8, p. 41]. Thus the uniqueness regime is achieved in the systems with low particle density. On the other hand, for large z (that is, high particle density) one expects the existence of multiple Gibbs states, see [9] for the case of ferromagnetic spin-spin interactions, where sufficient conditions of such multiplicity (i.e., appearence of a phase transition) in our model are given.*

Our proof of the uniqueness employs a lattice representation of our system and the Dobrushin–Pechersky criterion, see Section 5.2. Sufficient conditions of this criterion are checked using the moment bounds from Section 3.

Remark 7 *A result that seems to be completely new for this type of systems is the decay of correlations of the Gibbs measures. Consider bounded functions $G_1, G_2 : \widehat{\Gamma}(X) \to \mathbb{R}$, such that G_1 is $\mathcal{B}_{\mathrm{Q}_{k_1}}(\widehat{\Gamma})$-measurable and G_2 is $\mathcal{B}_{\mathrm{Q}_{k_2}}(\widehat{\Gamma})$-measurable, for some $k_1, k_2 \in \mathbb{Z}^d$. Let $\|\cdot\|_\infty$ denote the usual* sup *norm. Set*

$$\mathrm{Cov}_\mu(G_1; G_2) := \mu(G_1 G_2) - \mu(G_1)\mu(G_2)$$

and assume that conditions of Theorem 5 are satisfied. Let μ be the corresponding unique tempered Gibbs measure. Then, there exist positive constants $\mathfrak{C}$ and $\mathfrak{a}$ such that

$$|\mathrm{Cov}_\mu(G_1; G_2)| \le \mathfrak{C} \|G_1\|_\infty \|G_2\|_\infty \exp\{-\mathfrak{a}|k_1 - k_2|\}. \tag{2.44}$$

This estimate is an immediate by-product of the (proof of) Theorem 5 and follows from [6, Theorem 2.7] adapted to our setting via the lattice representation of the initial continuum model, see Section 5.1. Such approach (even in the case of a system without marks) can be seen as a (simpler) alternative to the method of clusters expansions (the only method by which similar results on $\Gamma(X)$ have been obtained).

2.6 Comments

1. In [1, 22, 26, 28, 35], a theory of Gibbs measures (on marked configuration spaces) based on Ruelle's classical approach ([37, 38]) has been elaborated. To this end, one has to require either *stability* or, moreover, *superstability* of the energy functional, expressed by the inequalities

$$H(\widehat{\gamma}) \ge -C \cdot N(\gamma)$$

and

$$H(\widehat{\gamma}) \ge A \sum_{k \in Z^d} N(\gamma_k)^2 - B \cdot N(\gamma) \tag{2.45}$$

respectively, holding for any $\widehat{\gamma} \in \widehat{\Gamma}_0(X)$ with some $A, B, C > 0$. These bounds, which must be uniform in the variables $\sigma_x \in S$, obviously fail in the case of unbounded spin interactions like in (2.14)–(2.15).

It seems to be possible to establish an analogue of Ruelle's superstability estimates replacing the term $N(\gamma_k)^2$ in (2.45) by the control functional $F(\widehat{\gamma}_k)$ (defined by (2.37) and involving both particles' positions and their spins). This will allow us to construct the corresponding Gibbs states μ satisfying the regularity condition

$$\sup_{K \in \mathbb{N}} \left\{ K^{-d} \sum_{|k| \le K} F(\widehat{\gamma}_k) \right\} < \infty \text{ for } \mu\text{-a.a. } \widehat{\gamma} \in \widehat{\Gamma}(X).$$

As for the uniqueness problem for such Gibbs states, one has to develop a contraction theory of the Kirkwood–Salsburg equations for the corresponding marked correlation functions. So far, this was only done in [26] under condition (2.45) which, as already mentioned above, does not cover our model.

2. Gibbs measures $\mu \in \mathcal{G}$ represent so-called *annealed* thermodynamic states of our particle system; they describe the thermal equilibrium of this system as a whole. Alternatively, one can consider thermodynamic states of the spin system alone for a fixed typical configuration (sample) γ, which is distributed according to a Gibbs measure μ^Φ on $\Gamma(X)$ defined by the position-position interaction Φ. These are commonly referred to as *quenched* states, cf. [3, 4, 29]. The corresponding Gibbs measures μ_γ on the product spaces S^γ were constructed in [10]. The relationship between Gibbs measures of these two types can be expressed by the disintegration formula

$$\mu\,(\mathrm{d}\widehat{\gamma}) = \mu_\gamma(\mathrm{d}\sigma_\gamma)\,\mathcal{M}(\mathrm{d}\gamma), \tag{2.46}$$

where $\mathcal{M} := p_X^* \mu \in \mathcal{P}(\Gamma(X))$ is the projection of μ on $\Gamma(X)$, cf. Remark 1 and [11, formula (2.6)]. In general, the projected measure $\mathcal{M}$ does not coincide with the Gibbs measure μ^Φ and cannot be described in terms of position-position interactions alone. Thus it is not clear whether the existence result from [10] could be used in order to prove the existence of the annealed Gibbs measure μ. Furthermore, (2.46) indicates that one cannot directly compare (e.g., by means of various correlation inequalities known for measures on S^γ, see e.g. [15], [27]) any two annealed Gibbs states related to different spin-spin potentials W_{xy}.

Let us remark that the multiplicity (phase transition) problem for quenched Gibbs measures of ferromagnetic type has been studied in [11]. On the other hand, the question of uniqueness for quenched systems with unbounded spins remains so far open. The main source of difficulties here (making standard methods not applicable) is that the underlying discrete set $\gamma \subset \mathbb{R}^d$ is highly inhomogeneous, so that μ^Φ-a.s. it holds $\sup_{k\in\mathbb{Z}^d} N(\gamma_k) = +\infty$.

3. Analogously to the case of simple (i.e., unmarked) point processes, one can show that each $\mu \in \mathcal{G}^{\mathrm{t}}$ satisfies the so-called *Georgii–Nguen–Zessin* (GNZ) equation (see e.g. [26, 28]). It says that for any measurable function $G : \widehat{X} \times \widehat{\Gamma} \to \mathbb{R}_+$ the following identity holds:

$$\int_{\widehat{\Gamma}} \sum_{\widehat{x}\in\widehat{\gamma}} G(\widehat{x},\ \widehat{\gamma})\, \mu\,(\mathrm{d}\widehat{\gamma})$$

$$= \int_{\widehat{\Gamma}} \int_{\widehat{X}} G(\widehat{x},\ \widehat{\gamma} \cup \{\widehat{x}\}) \exp\left\{-\Delta H(\{\widehat{x}\}|\widehat{\gamma})\right\}\ \mu\,(\mathrm{d}\widehat{\gamma})\ \chi(\mathrm{d}\sigma_x)\mathrm{d}x.$$

Here, cf. (2.17),

$$\Delta H(\{\widehat{x}\}|\widehat{\gamma}) := \sum_{y\in\gamma} \left[\Phi(x,y) + W_{xy}(\sigma_x, \xi_y)\right], \quad \widehat{\gamma} = (\eta, \xi_\gamma).$$

3 Exponential moment estimate

3.1 One-point estimates

The following proposition is a starting point in the realization of our approach. It describes the *superstability* property of the system in terms of the control functional F. The proof involves simple but tedious calculations based on assumptions (A1)–(A6) and will be given in Section 6.

Proposition 8 *For any (arbitrarily small) $\delta > 0$ one finds a positive constant C_δ such that*

$$-H_k(\widehat{\gamma}_k\,|\widehat{\eta}\,) \le -\left(A_\Phi - \delta\right) N(\gamma_k)^P + \delta \sum_{x\in\gamma_k} |\sigma_x|^q + \delta \sum_{j\in\partial k} F(\widehat{\eta}_j) + C_\delta \tag{3.1}$$

for all $k \in \mathbb{Z}^d$ and $\widehat{\gamma}, \widehat{\eta} \in \widehat{\Gamma}(X)$. Here $C_\delta := C_\delta(\kappa, \vartheta; \mathcal{J})$ is a non-decreasing function of $\mathcal{J}$.

Remark 9 *Using the arguments from the proof of Proposition 8 (or, more precisely, Lemma 33) and the global superstability of $U(\gamma)$ (see Remark 2), we get the bound*

$$
\begin{aligned}
-H_\Lambda(\widehat{\gamma}_\Lambda|\widehat{\eta}) &\leq -(A_\Phi-\delta)\sum_{k\in\mathbb{Z}^d} N(\gamma_\Lambda\cap Q_k)^P + \delta\sum_{x\in\gamma_\Lambda}|\sigma_x|^q + C_{\Lambda,\delta}(\widehat{\eta}) \\
&\leq -(A_\Phi-\delta)P^{1-\mathcal{N}_\Lambda}N(\gamma_\Lambda)^P + \delta\sum_{x\in\gamma_\Lambda}|\sigma_x|^q + C_{\Lambda,\delta}(\widehat{\eta}), \quad (3.2)
\end{aligned}
$$

where $\mathcal{N}_\Lambda$ is the cardinality of the set $\{j\in\mathbb{Z}^d : \mathrm{Q}_j\cap\Lambda)\neq\emptyset\}$. Both inequalities in (3.2) hold for an arbitrary domain $\Lambda\in\mathcal{B}_0(X)$, any $\widehat{\eta}\in\widehat{\Gamma}(X)$ and $\delta\in(0,A_\Phi)$ with an appropriate constant $C_{\Lambda,\delta}(\widehat{\eta})\geq 0$ (the explicit value of which is irrelevant for our purposes).

Below we will frequently use the moment estimate

$$
\begin{aligned}
&\int_{\widehat{\Gamma}_\Lambda}\exp\left\{aN(\gamma_\Lambda)+b\sum_{x\in\gamma_\Lambda}|\sigma_x|^q\right\}\widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_\Lambda) \\
&= \sum_{n=0}^{\infty}\frac{z^n}{n!}|\Lambda|^n e^{an}\left(\int_S e^{b|s|^q}\,\chi(\mathrm{d}s)\right)^n = \exp\left\{z|\Lambda|e^a\int_S e^{b|s|^q}\,\chi(\mathrm{d}s)\right\}<\infty, \quad (3.3)
\end{aligned}
$$

which holds for any $\Lambda\in\mathcal{B}_0(X)$ and $a\in\mathbb{R}$, $b\leq A_\chi$ (cf. (2.31)) and follows from the definition of the Lebesgue-Poisson measure $\widehat{\lambda}_z$, assumption (A5) and disintegration formula (2.13).

Corollary 10 *The partition function $Z_\Lambda(\widehat{\eta})$ satisfies the estimate*

$$1\leq Z_\Lambda(\widehat{\eta})<\infty \quad (3.4)$$

for all $\Lambda\in\mathcal{B}_0(X)$ and $\widehat{\eta}\in\widehat{\Gamma}(X)$.

Proof. The lower bound can be immediately seen from the equalities $\widehat{\lambda}_{z,\Lambda}(\emptyset)=1$ and $U_\Lambda(\gamma_\Lambda\,|\eta)=E_{\gamma_\Lambda\cup\eta_{\Lambda^c}}(\sigma_{\gamma_\Lambda}\,|\xi)=0$ if $\gamma_\Lambda=\emptyset$. The upper bound follows from (3.2) and (3.3). □

Lemmas 11 and 14 below provide us with crucial estimates on the *"one-point"* kernels $\Pi_k(\mathrm{d}\widehat{\gamma}|\widehat{\eta}) := \Pi_{\mathrm{Q}_k}(\mathrm{d}\widehat{\gamma}|\widehat{\eta})$, $k\in\mathbb{Z}^d$, subject to varying boundary conditions $\widehat{\eta}\in\widehat{\Gamma}(X)$. To this end, let us fix some $\kappa\in(0,A_\Phi)$ and $\vartheta\in(0,A_\chi)$ in definition (2.37) of the functional F, cf. (2.42).

Lemma 11 *For any (arbitrarily small) $\delta>0$ there exists a constant $\Xi_\delta>0$ such that for all $k\in\mathbb{Z}^d$ and $\widehat{\eta}\in\widehat{\Gamma}(X)$*

$$\int_{\widehat{\Gamma}}\exp\{F(\widehat{\gamma}_k)\}\,\Pi_k\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta})\leq\exp\left\{\Xi_\delta+\delta\sum_{j\in\partial k}F(\widehat{\eta}_j)\right\}. \quad (3.5)$$

Proof. Without loss of generality we may assume that

$$\delta \leq \min\{A_\Phi - \kappa;\ A_\chi - \vartheta\}.$$

Taking into account that $Z_{\mathrm{Q}_k}(\widehat{\eta}) \geq 1$, cf. (3.4), and using estimate (3.1), we obtain

$$\int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\}\, \Pi_k\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta}) \\ \leq \exp\left\{C_\delta + \delta \sum_{j\in\partial k} F(\widehat{\eta}_j)\right\} \int_{\widehat{\Gamma}_k} \exp\left\{A_\chi \sum_{x\in\gamma_k} |\sigma_x|^q\right\} \widehat{\lambda}(\mathrm{d}\widehat{\gamma}_k). \quad (3.6)$$

The integral in the RHS of (3.6) is calculated explicitly in (3.3). Then we have

$$\int_{\Gamma_k} \int_{S^{\gamma_k}} \exp\left\{A_\chi \sum_{x\in\gamma_k} |\sigma_x|^q\right\} \bigotimes_{x\in\gamma_k} \chi(\mathrm{d}\sigma_x)\, \lambda(\mathrm{d}\gamma_k) = \exp\{z\mathcal{E}_\chi\}.$$

where $\mathcal{E}_\chi := \int_S \exp\{A_\chi\, |s|^q\}\ \chi(\mathrm{d}s)$ is finite because of (A5). Therefore (3.5) holds with

$$\Xi_\delta := C_\delta + z\mathcal{E}_\chi, \quad (3.7)$$

which depends on $\mathcal{J}$ through C_δ and hence is non-decreasing in $\mathcal{J}$ and z. □

A subsequent application of Jensen's inequality to both sides in (3.5) immediately implies the following estimate of Dobrushin's type (cf. [13]). It states a kind of *weak dependence* on boundary conditions, which could be achieved by choosing $\delta < \mathcal{N}_0^{-1}$.

Corollary 12 *Under assumptions of Lemma 11 we have the bound*

$$\int_{\widehat{\Gamma}} F(\widehat{\gamma}_k)\, \Pi_k\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta}) \leq \Xi_\delta + \delta \sum_{j\in\partial k} F(\widehat{\eta}_j). \quad (3.8)$$

Remark 13 *By virtue of (the first inequality of) (3.2) and (3.3) one can see that for any fixed $\kappa \in (0, A_\Phi)$ and $\vartheta \in (0, A_\chi)$*

$$\int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\}\ \Pi_\Lambda\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta}) \leq \mathcal{C}_k(\Lambda, \widehat{\eta}), \quad k \in \Lambda \in \mathcal{B}_0(X), \quad (3.9)$$

where $\mathcal{C}_k(\Lambda, \widehat{\eta}) < \infty$ is an increasing function of Λ. However, this estimate is too rough for our purposes and will be improved by more refined arguments employing the Markov property of the specification Π, see Section 3.2.

Here and in what follows, we denote by $\mathrm{d}_{\mathrm{var}}(\nu_1, \nu_2)$ the *total variation distance* between two measures ν_1 and ν_2 on a σ-algebra $\mathcal{F}$, that is,

$$\mathrm{d}_{\mathrm{var}}(\nu_1,\ \nu_2) := \sup\nolimits_{A\in\mathcal{F}}\ |\nu_1(A) - \nu_2(A)|.$$

Our second fundamental lemma evaluates this distance between local Gibbs states $\mu_k^{\widehat{\eta}}(\mathrm{d}\widehat{\gamma}_k) := \mu_{\mathrm{Q}_k}^{\widehat{\eta}}(\mathrm{d}\widehat{\gamma}_k)$ and $\mu_k^{\widehat{\varsigma}}(\mathrm{d}\widehat{\gamma}_k) := \mu_{\mathrm{Q}_k}^{\widehat{\varsigma}}(\mathrm{d}\widehat{\gamma}_k)$ on $\mathcal{B}(\widehat{\Gamma}_k)$ with boundary conditions $\widehat{\eta}$ and $\widehat{\varsigma}$ respectively, cf. (2.21).

Lemma 14 *There exists a non-decreasing function $\phi(z, \mathcal{J}, L)$ of $z, \mathcal{J}, L > 0$ such that*

$$\mathrm{d}_{\mathrm{var}}(\mu_k^{\widehat{\eta}},\ \mu_k^{\widehat{\varsigma}}) \leq z \cdot \phi(z, \mathcal{J}, L) \tag{3.10}$$

for all $k \in \mathbb{Z}^d$ and any pair of boundary conditions $\widehat{\eta}, \widehat{\varsigma} \in \Gamma(\widehat{X})$ obeying the constraint $\sup_{j\in\mathbb{Z}^d} \{F(\widehat{\eta}_j),\ F(\widehat{\varsigma}_j)\} \leq L$.

The proof is rather cumbersome and will be given in Section 6.

3.2 Volume estimates

The aim of this section is to prove a uniform estimate on exponential moments of the specification kernels, which in turn will be used in the proof of Theorem 4. For a finite subset $\mathcal{K} \subset \mathbb{Z}^d$, consider the union of elementary cubes $\mathrm{Q}_{\mathcal{K}} := \bigcup_{k\in\mathcal{K}} \mathrm{Q}_k$ (cf. (2.27)) and the corresponding cylinder set $\widehat{\mathrm{Q}}_{\mathcal{K}} = \mathrm{Q}_{\mathcal{K}} \times S$. Write for brevity $\Pi_{\mathcal{K}}(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma}) := \Pi_{\mathrm{Q}_{\mathcal{K}}}(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma})$. As usual, $\mathcal{K}\nearrow\mathbb{Z}^d$ means a limit taken along any ordered by inclusion and exhausting the whole $\mathbb{Z}^d$ sequence of such sets. Our strategy will be to start from the one-point estimate (3.5) and then by the consistency property (2.25) extend it to arbitrarily large cubic domains.

Theorem 15 *Under assumptions of Lemma 11 there exists a constant $\Psi := \Psi(\kappa, \vartheta) < \infty$ such that the estimate*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\}\ \Pi_{\mathcal{K}}(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma}) \leq \Psi \tag{3.11}$$

holds for all $k \in \mathbb{Z}^d$ and $\widehat{\varsigma} \in \widehat{\Gamma}_{\mathrm{t}}(X)$.

Proof. Introduce the notation

$$n_k(\mathcal{K}, \widehat{\varsigma}) := \ln \int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\}\ \Pi_{\mathcal{K}}\,(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma}) \geq 0,$$

whereby $n_k(\mathcal{K}, \widehat{\varsigma}) = F(\widehat{\varsigma}_k)$ if $k \notin \mathcal{K}$. An application of identity (2.25) and inequality (3.5) shows that for each $k \in \mathcal{K}$

$$\begin{aligned} n_k(\mathcal{K}, \widehat{\varsigma}) &= \ln \int_{\widehat{\Gamma}} \int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\}\, \Pi_k\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\eta})\ \Pi_{\mathcal{K}}\,(\mathrm{d}\widehat{\eta}|\widehat{\varsigma}) \\ &\leq \Xi_\delta + \ln \int_{\widehat{\Gamma}} \exp\left\{\delta \sum_{j\in\partial k} F(\widehat{\eta}_j)\right\}\ \Pi_{\mathcal{K}}\,(\mathrm{d}\widehat{\eta}|\widehat{\varsigma})\,. \end{aligned}$$

Assume without loss of generality that $\delta\mathcal{N}_0 < 1$. The multiple Hölder inequality then yields

$$\int_{\widehat{\Gamma}} \prod_{j\in\partial k} [\exp\{F(\widehat{\eta}_j)\}]^\delta\ \Pi_{\mathcal{K}}(\mathrm{d}\widehat{\eta}|\widehat{\varsigma}) \leq \prod_{j\in\partial k} \left[\int_{\widehat{\Gamma}} \exp\{F(\widehat{\eta}_j)\}\, \Pi_{\mathcal{K}}(\mathrm{d}\widehat{\eta}|\widehat{\varsigma})\right]^\delta.$$

Therefore

$$n_k(\mathcal{K},\widehat{\varsigma}) \le \Xi_\delta + \delta \sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K},\widehat{\varsigma}) + \delta \sum_{j\in\mathcal{K}^c\cap\partial k} F(\widehat{\varsigma}_j). \tag{3.12}$$

Fix arbitrary $k_0 \in \mathcal{K}$ and small enough $\alpha > 0$ so that $e^{\alpha\rho}\delta\mathcal{N}_0 < 1$, where

$$\rho = \sup_{k\in\mathbb{Z}^d} \max_{j\in\partial k} |j-k| \le R + \sqrt{d}.$$

Multiplying both sides of inequality (3.12) by $e^{-\alpha|k_0-k|}$ and taking into account that $|k_0-j| - |k_0-k| \le \rho$, we obtain the estimate

$$\begin{aligned} n_k(\mathcal{K},\widehat{\varsigma})e^{-\alpha|k_0-k|} &\le \Xi_\delta e^{-\alpha|k_0-k|} \\ &+ e^{\alpha\rho}\delta\left[\sum_{j\in\mathcal{K}\cap\partial k} n_j(\mathcal{K},\widehat{\varsigma})e^{-\alpha|k_0-j|} + \sum_{j\in\mathcal{K}^c\cap\partial k} F(\widehat{\varsigma}_j)e^{-\alpha|k_0-j|}\right]. \end{aligned} \tag{3.13}$$

Thus we can see that

$$\begin{aligned} &\sup_{k\in\mathcal{K}}\left\{n_k(\mathcal{K},\widehat{\varsigma})e^{-\alpha|k_0-k|}\right\} \\ &\quad\le \Xi_\delta + e^{\alpha\rho}\delta\left[\mathcal{N}_0 \sup_{k\in\mathcal{K}}\left\{n_k(\mathcal{K},\widehat{\varsigma})e^{-\alpha|k_0-k|}\right\} + \sum_{j\in\mathcal{K}^c} F(\widehat{\varsigma}_j)e^{-\alpha|k_0-j|}\right], \end{aligned}$$

so that

$$\begin{aligned} n_{k_0}(\mathcal{K},\widehat{\varsigma}) &\le \sup_{k\in\mathcal{K}}\left\{n_k(\mathcal{K},\widehat{\varsigma})e^{-\alpha|k_0-k|}\right\} \\ &\le \left(1 - e^{\alpha\rho}\delta\mathcal{N}_0\right)^{-1}\left[\Xi_\delta + e^{\alpha(\rho+|k_0|)}\delta\sum_{j\in\mathcal{K}^c} F(\widehat{\varsigma}_j)e^{-\alpha|j|}\right]. \end{aligned} \tag{3.14}$$

It follows from (2.41) that for any $\widehat{\varsigma} \in \widehat{\Gamma}_{\mathrm{t}}(X)$ we have

$$\sum_{j\in\mathcal{K}^c} F(\widehat{\varsigma}_j)e^{-\alpha|j|} \to 0 \ \text{ as } \ \mathcal{K} \nearrow \mathbb{Z}^d,$$

which in turn implies the bound

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K},\widehat{\varsigma}) \le \left(1 - e^{\alpha\rho}\delta\mathcal{N}_0\right)^{-1}\Xi_\delta.$$

Passage to the limit as $\alpha \to 0$ shows that

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} n_{k_0}(\mathcal{K},\widehat{\varsigma}) \le \left(1 - \delta\mathcal{N}_0\right)^{-1}\Xi_\delta =: \Psi_\delta,$$

which completes the proof. □

Corollary 16 *For any domain $\Lambda \in \mathcal{B}_0(X)$ and $N \ge 0$, there exists $\Psi_\Lambda(N) < \infty$ such that*

$$\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d} \int_{\widehat{\Gamma}} F(\widehat{\gamma}_\Lambda)^N\, \Pi_{Q_\mathcal{K}}(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma}) \le \Psi_\Lambda(N),$$

which holds uniformly for all $\widehat{\varsigma} \in \widehat{\Gamma}_{\mathrm{t}}(X)$.

4 Existence of Gibbs measures

In this section, we use the estimates obtained in Section 3 in order to prove that, for any $\widehat{\eta} \in \widehat{\Gamma}_{\mathrm{t}}(X)$, the family of Gibbsian specification kernels $\{\Pi_\Lambda(\cdot|\widehat{\eta}), \ \Lambda \in \mathcal{B}_0(X)\}$ contains a cluster point.

Definition 17 *(cf. [15, Def. 4.6]) We say that a sequence of probability measures* $\{\mu_m\}_{m\in\mathbb{N}}$ *on* $\widehat{\Gamma}(X)$ *is locally equicontinuous (LEC) if for any* $\Lambda \in \mathcal{B}_0(X)$ *and any* $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_\Lambda(\widehat{\Gamma})$ *with* $B_n \searrow \emptyset$ *as* $n \to \infty$*, we have*

$$\lim_{n\to\infty} \limsup_{m\in\mathbb{N}} \mu_m(B_n) = 0. \tag{4.1}$$

We equip the space $\mathcal{P}(\widehat{\Gamma})$ of probability measures on $\widehat{\Gamma}(X)$ with the topology of *local set convergence*, which is defined as the coarsest topology making the evaluation map $\mu \to \mu(B)$ continuous for each $B \in \mathcal{F}_0 := \mathcal{B}_0(\widehat{\Gamma})$. This topology (which is Hausdorff but *not* metrizable) is well suited to the study of local interactions (i.e., those having finite range as in assumption (A1)). In particular,

$$\mu_m \overset{\text{loc}}{\to} \mu \ \text{ iff } \ \mu_m(B) \to \mu(B) \ \text{ as } m \to \infty, \ \forall B \in \mathcal{F}_0. \tag{4.2}$$

The latter is equivalent to claiming that

$$\int_{\widehat{\Gamma}} f \, \mathrm{d}\mu_m \to \int_{\widehat{\Gamma}} f \, \mathrm{d}\mu \ \text{ as } m \to \infty, \tag{4.3}$$

for all bounded $\mathcal{F}_0$-measurable functions $f : \widehat{\Gamma}(X) \to \mathbb{R}$. Observe that the local set convergence is equivalent to convergence in the space $[0,1]^{\mathcal{F}_0}$.

Theorem 18 *(cf. [15, Prop. 4.9]) Any LEC sequence* $\{\mu_m\}_{m\in\mathbb{N}} \subset \mathcal{P}(\widehat{\Gamma})$ *has at least one cluster point, which is a probability measure on* $\widehat{\Gamma}(X)$*.*

Sketch of the proof. It is straightforward that the family $\{\mu_m\}_{m\in\mathbb{N}}$ contains a cluster point μ as an element of the compact space $[0,1]^{\mathcal{F}_0}$, and μ is an additive function on $\mathcal{F}_0$. The LEC property (4.1) implies that $\mu_\Lambda := p_\Lambda^* \mu$ is σ-additive on each $\mathcal{B}(\widehat{\Gamma}_\Lambda)$. Thus $\{\mu_\Lambda\}_{\Lambda\in\mathcal{B}_0(X)}$ forms a consistent (w.r.t. projective maps (2.8)) family of measures and by the corresponding version of the Kolmogorov theorem (see [30, Theorem V.3.2]) generates a probability measure on $\mathcal{B}(\widehat{\Gamma})$ (which obviously coincides with μ). $\square$

Remark 19 *It follows from [15, Prop. 4.15] that, although the topology of* $\mathcal{P}(\widehat{\Gamma})$ *is not metrizable, for each (topological) cluster point* μ *there exists a subsequence* $\{\mu_{m_j}\}_{j\in\mathbb{N}}$ *such that* $\mu_{m_j} \overset{\text{loc}}{\to} \mu$ *as* $j \to \infty$*.*

Let now $\{\mathcal{K}_m\}_{m\in\mathbb{N}}$ be any increasing sequence of finite subsets of $\mathbb{Z}^d$ such that $\mathcal{K}_m \nearrow \mathbb{Z}^d$ and hence $\mathrm{Q}_{\mathcal{K}_m} := \bigcup_{j\in\mathcal{K}_m} \mathrm{Q}_j \nearrow X$ as $m \to \infty$, and introduce notation $\Lambda_m := \Lambda_{\mathcal{K}_m}$ and $\Pi_m := \Pi_{\Lambda_{\mathcal{K}_m}}$.

Theorem 20 *For any* $\widehat{\varsigma} \in \widehat{\Gamma}_{\mathrm{t}}(X)$ *the family* $\{\Pi_m(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma})\}_{m\in\mathbb{N}}$ *is LEC.*

Proof. Fix $\Lambda \in \mathcal{B}_0(X)$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_\Lambda(\widehat{\Gamma})$ as in Definition 17. It is sufficient to prove that $\forall\varepsilon > 0$ there exist integers m_0 and n_0 such that $\Pi_m(B_n|\widehat{\varsigma}) \leq \varepsilon$ for any $m \geq m_0$ and $n \geq n_0$.

To this end, for $T > 0$ let us consider the set

$$\widehat{\Gamma}_T := \left\{ \widehat{\gamma} \in \widehat{\Gamma}(X) : \ F(\widehat{\gamma}_{\Lambda_R}) = \kappa N(\gamma_{\Lambda_R})^P + \vartheta \sum_{x\in\gamma_{\Lambda_R}} |\sigma_x|^q \leq T \right\}$$

(where Λ_R was defined in Section 2.4) and estimate the corresponding measures of $B_n \cap \widehat{\Gamma}_T$ and $B_n \cap [\widehat{\Gamma}_T]^{\mathrm{c}}$ separately. Observe (by analogy with (6.20) and (6.21)) that for any $1 \leq p \leq P$ and $1 \leq r \leq q$

$$\sup_{\widehat{\gamma}\in\widehat{\Gamma}_T} \left\{ N(\gamma_{\Lambda_R})^p; \ \sum_{x\in\gamma_{\Lambda_R}} |\sigma_x|^r \right\} \leq \frac{T}{\max\{\kappa; \vartheta\}}.$$

Using bound (3.2) we then see that there exists a constant $c_\Lambda(T)$ such that

$$\mathbf{1}_{\widehat{\Gamma}_T}(\widehat{\eta}_\Lambda \cup \widehat{\gamma}_{\Lambda^c}) \exp\{-H_\Lambda(\widehat{\eta}_\Lambda|\widehat{\gamma})\} \leq c_\Lambda(T). \tag{4.4}$$

uniformly for all $\widehat{\gamma}, \widehat{\eta} \in \widehat{\Gamma}(X)$.

Next, write

$$\Pi_m(B_n|\widehat{\varsigma}) = \Pi_m(B_n \cap [\widehat{\Gamma}_T]^{\mathrm{c}}|\widehat{\varsigma}) + \Pi_m(B_n \cap \widehat{\Gamma}_T|\widehat{\varsigma}).$$

According to Chebyshev's inequality applied to the measure $\Pi_m(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma})$ on $\widehat{\Gamma}(X)$ we have

$$\Pi_m(\{\widehat{\gamma} : f(\widehat{\gamma}) \geq T\}|\widehat{\varsigma}) \leq T^{-2} \int_{\widehat{\Gamma}} |f(\widehat{\gamma})|^2 \ \Pi_m(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma})$$

for any $T > 0$ and $f \in L^2(\widehat{\Gamma}, \ \Pi_m(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma}))$. Setting $f(\widehat{\gamma}) = F(\widehat{\gamma}_{\Lambda_R})$ we obtain, cf. Corollary 16,

$$\Pi_m(B_n \cap [\widehat{\Gamma}_T]^{\mathrm{c}}|\widehat{\varsigma}) \leq \Pi_m\left([\widehat{\Gamma}_T]^{\mathrm{c}}|\widehat{\varsigma}\right) \leq \varepsilon/2 \tag{4.5}$$

for any $\varepsilon > 0$ and T greater than some $T(\varepsilon)$.

On the other hand, there exists m_0 such that $\Lambda_m \supset \Lambda$ for $m \geq m_0$. For all such m, it follows from (2.24) and the consistency property (2.25) of the specification Π that

$$\Pi_m\left(B_n \cap \widehat{\Gamma}_T|\widehat{\varsigma}\right) = \int_{\widehat{\Gamma}} \left[\int_{\widehat{\Gamma}} \mathbf{1}_{B_n\cap\widehat{\Gamma}_T}(\widehat{\eta}_\Lambda \cup \widehat{\gamma}_{\Lambda^c}) \ \Pi_\Lambda(\mathrm{d}\widehat{\eta}|\widehat{\gamma}) \right] \Pi_m(\mathrm{d}\widehat{\gamma}|\widehat{\varsigma}). \tag{4.6}$$

Since $B_n \downarrow \emptyset$ as $n \to \infty$, by (3.4) and (4.4) we obtain

$$\int_{\widehat{\Gamma}} \mathbf{1}_{B_n\cap\widehat{\Gamma}_T}(\widehat{\eta}_\Lambda \cup \widehat{\gamma}_{\Lambda^c}) \ \Pi_\Lambda(\mathrm{d}\widehat{\eta}|\widehat{\gamma}) \leq c_\Lambda(T)\widehat{\lambda}_z(B_n) < \varepsilon/2$$

for n greater than some $n(\varepsilon, T)$. Hence, the right-hand side in (4.6) does not exceed $\varepsilon/2$ as well. Combining this with estimate (4.5) we can see that $\forall\varepsilon > 0$ and $m \geq m_0$, $n \geq n_0 = n(\varepsilon, T(\varepsilon))$ it holds

$$\Pi_m(B_n|\widehat{\varsigma}) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which completes the proof. $\square$

Now we are in a position to prove our first main result.

Proof of Theorem 4. *(i) Existence:* It follows from Theorems 18 and 20 that for any $\widehat{\varsigma} \in \Gamma^{\mathrm{t}}$ the family $\{\Pi_m\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\varsigma})\}_{m\in\mathbb{N}}$ has a cluster point $\mu = \mu(\widehat{\varsigma}) \in \mathcal{P}(\widehat{\Gamma})$. Therefore by Remark 19 there exists a subsequence $\Lambda_{m_j},\ j \in \mathbb{N}$, such that

$$\lim_{j\to\infty} \Pi_{m_j}\,(B\,|\widehat{\varsigma}) = \mu(B), \quad B \in \mathcal{B}_0(\widehat{\Gamma}). \tag{4.7}$$

Let us check that μ solves the DLR equation (2.26) for all $\Lambda \in \mathcal{B}_0(X)$ and $B \in \mathcal{B}_0(\widehat{\Gamma})$. As the interaction has finite range, the function $\widehat{\gamma} \mapsto \Pi_\Lambda\,(B\,|\widehat{\gamma})$ is $\mathcal{B}_0(\widehat{\Gamma})$-measurable. Using (4.3) and the consistency property (2.25) of the specification Π, we thus can pass to the limit

$$\begin{aligned}\int_{\widehat{\Gamma}} \Pi_\Lambda\,(B\,|\widehat{\gamma})\;\mu\,(\mathrm{d}\widehat{\gamma}) &= \lim_{j\to\infty}\int_{\widehat{\Gamma}} \Pi_\Lambda\,(B\,|\widehat{\gamma})\;\Pi_{m_j}\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\varsigma}) \\ &= \lim_{j\to\infty} \Pi_{m_j}\,(B\,|\widehat{\varsigma}) = \mu(B)\end{aligned}$$

and conclude that $\mu \in \mathcal{G}$. Finally, by (3.11) and Beppo Levi's monotone convergence theorem we see that

$$\begin{aligned}\int_{\widehat{\Gamma}} \sum_{k\in\mathbb{Z}^d} e^{-\alpha|k|} F(\widehat{\gamma}_k)\,\mu\,(\mathrm{d}\widehat{\gamma}) &= \lim_{K,L\to\infty}\lim_{j\to\infty} \sum_{|k|\le K} e^{-\alpha|k|}\int_{\widehat{\Gamma}} \{F(\widehat{\gamma}_k)\wedge L\}\,\Pi_{m_j}\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\varsigma}) \\ &\le \sum_{k\in\mathbb{Z}^d} e^{-\alpha|k|}\limsup_{j\to\infty}\int_{\widehat{\Gamma}} F(\widehat{\gamma}_k)\,\Pi_{m_j}\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\varsigma}) \le \Psi \sum_{k\in\mathbb{Z}^d} e^{-\alpha|k|} < \infty\end{aligned}$$

for all $\alpha > 0$, which by (2.41) implies that $\mu(\widehat{\Gamma}_{\mathrm{t}}(X)) = 1$ so that $\mu \in \mathcal{G}^{\mathrm{t}}$.

(ii) A priori estimate (2.43). Consider an arbitrary $\mu \in \mathcal{G}^{\mathrm{t}}$ (not necessarily given by the limit transition above). With the help of (2.26), Theorem 15 and Fatou's lemma we have

$$\begin{aligned}\int_{\widehat{\Gamma}_{\mathrm{t}}} \exp\{F(\widehat{\gamma}_k)\wedge L\}\,\mu\,(\mathrm{d}\widehat{\gamma}) &= \lim_{\mathcal{K}\nearrow\mathbb{Z}^d}\int_{\widehat{\Gamma}_{\mathrm{t}}}\int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\wedge L\}\,\Pi_{\mathcal{K}}\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\varsigma})\,\mu\,(\mathrm{d}\widehat{\varsigma}) \\ &\le \int_{\widehat{\Gamma}_{\mathrm{t}}}\left[\limsup_{\mathcal{K}\nearrow\mathbb{Z}^d}\int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\wedge L\}\,\Pi_{\mathcal{K}}\,(\mathrm{d}\widehat{\gamma}\,|\widehat{\varsigma})\right]\mu\,(\mathrm{d}\widehat{\varsigma}) \le \Psi\end{aligned}$$

for any $k \in \mathbb{Z}^d$ and $L > 0$, where $\Psi > 0$ is the same constant as in (3.11). By Levi's theorem this implies the bound

$$\int_{\widehat{\Gamma}} \exp\{F(\widehat{\gamma}_k)\}\,\mu\,(\mathrm{d}\widehat{\gamma}) = \lim_{L\to\infty}\int_{\widehat{\Gamma}_{\mathrm{t}}} \exp\{F(\widehat{\gamma}_k)\wedge L\}\,\mu\,(\mathrm{d}\widehat{\gamma}) \le \Psi,$$

and (2.43) is proved. $\square$

Remark 21 *A standard application of the Borel–Cantelli lemma to the moment bound (2.43) yields the following improved support property for any $\mu \in \mathcal{G}^{\mathrm{t}}$. Indeed, under the conditions of Theorem 4 all $\mu \in \mathcal{G}^{\mathrm{t}}$ are carried by the set*

$$\widehat{\Gamma}_{\mathrm{s}}(X) = \left\{ \widehat{\gamma} \in \widehat{\Gamma}(X) : \sup_{k \in \mathbb{Z}^d} \left[N(\gamma_k)^P + \sum_{x \in \gamma_k} |\sigma_x|^q \right] \cdot [\log(1 + |k|)]^{-1} < \infty \right\}, \tag{4.8}$$

which is smaller than $\widehat{\Gamma}_{\mathrm{t}}(X)$, cf. (2.39) and (2.41).

Remark 22 *Let us consider a special case when all the potentials are non-negative, i.e., $\Phi(x,y) \geq 0$ and $W_{xy}(s,t) \geq 0$. This would make superfluous the superstability assumptions (A3) and (A6). Indeed, in this case we can use the control functional*

$$\tilde{F}(\widehat{\gamma}) := \kappa N(\gamma) + \vartheta \sum_{x \in \gamma} |\sigma_x|^q, \ \ \widehat{\gamma} = (\gamma, \sigma),$$

instead of (2.37), with arbitary fixed $\kappa > 0$ and $\vartheta \in (0, A_\chi)$. Then we have the estimate

$$\int_{\widehat{\Gamma}_k} \exp\{\tilde{F}(\widehat{\gamma}_k)\}\, \mu_k^{\widehat{\eta}}(\mathrm{d}\widehat{\gamma}_k) \leq \int_{\widehat{\Gamma}_k} \exp\{\tilde{F}(\widehat{\gamma}_k)\}\, \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_k)$$
$$= \exp\left\{ z e^{\kappa} \int_S e^{\vartheta |s|^q}\, \chi(\mathrm{d}s) \right\} < \infty, \tag{4.9}$$

which holds uniformly for all $\widehat{\eta} \in \widehat{\Gamma}(X)$ and $k \in \mathbb{Z}^d$, cf. (3.3). This enables us to mimic the proof of Theorem 4 and construct in this way a Gibbs measure $\mu \in \mathcal{G}$ obeying the a priori *bound $\sup_k \int_{\widehat{\Gamma}} \exp\{\tilde{F}(\widehat{\gamma}_k)\}\, \mu(\mathrm{d}\widehat{\gamma}) < \infty$.*

5 Uniqueness of Gibbs measures

The aim of this section is to prove Theorem 5. First we will develop the lattice representation of our model, in order to use the abstract Dobrushin–Pechersky uniqueness criterion.

5.1 Lattice representation of the model

Let $\mathcal{Q} := \widehat{\Gamma}_{\mathrm{Q}_0}$, where Q_0 is the elementary cube centered at the origin, cf. (2.27). Recall that $(\mathcal{Q}, \mathcal{B}(\mathcal{Q}))$ is a standard Borel space and fix the Lebesgue-Poisson measure $\widehat{\lambda}_z$ thereon. Consider the product space $\mathcal{A} := \mathcal{Q}^{\mathbb{Z}^d} = \prod_{k \in \mathbb{Z}^d} \mathcal{Q}_k$, $\mathcal{Q}_k := \mathcal{Q}$, and endow it with the product topology and the corresponding Borel σ-algebra $\mathcal{B}(\mathcal{A})$. Elements of $\mathcal{A}$, to be called *lattice configurations*, are infinite sequences $\overline{\alpha} := (\alpha_k)_{k \in \mathbb{Z}^d}$ with $\alpha_k \in \mathcal{Q}$. By construction, $\mathcal{B}(\mathcal{A})$ is generated by cylinder sets

$$A^{k_1, \ldots, k_m}_{b_1, \ldots, b_m} := \{\overline{\alpha} \in \mathcal{A} : \ \alpha_{k_1} \in b_1, \ldots, \alpha_{k_m} \in b_m\} \tag{5.1}$$

with all possible choices of $k_i \in \mathbb{Z}^d$, $b_i \in \mathcal{B}(\mathcal{Q})$ and $1 \leq i \leq m \in \mathbb{N}$.

Remark 23 *Observe that in our notations $\mathcal{Q}_k$ is the k-th copy of $\mathcal{Q} = \widehat{\Gamma}_{\mathrm{Q}_0}$, so that $\mathcal{Q}_k \neq \widehat{\Gamma}_{\mathrm{Q}_k}$. These spaces are isomorphic via the translation by k.*

Define the map

$$\mathbb{T} : \widehat{\Gamma}(X) \ni \widehat{\gamma} \longmapsto \mathbb{T}(\widehat{\gamma}) = \overline{\alpha} \in \mathcal{A} \tag{5.2}$$

where $\overline{\alpha} := (\alpha_k)_{k\in\mathbb{Z}^d}$ with $\alpha_k = \widehat{\gamma}_k - k \in \widehat{\Gamma}_{\mathrm{Q}_0}$. Here we write

$$\widehat{\eta} - a := \{ \,...,\ (x - a, s),\ ...\ \}$$

for a marked configuration $\widehat{\eta} = \{ \,...,\ (x, s),\ ...\ \} \in \widehat{\Gamma}(X)$ and $a \in X$. Moreover, for any $B \in \widehat{\Gamma}(X)$ we define the shifted set $B - a$ constituted by all configurations $\widehat{\eta} - a$ with $\widehat{\eta} \in B$.

Lemma 24 $\mathbb{T}$: $\widehat{\Gamma}(X) \to \mathcal{A}$ *is a measurable bijection.*

Proof. The map $\mathbb{T}$ is clearly one-to-one by its construction. The inverse map $\mathbb{T}^{-1}$ acts as

$$\mathbb{T}^{-1} : \mathcal{A} \ni \overline{\alpha} \longmapsto \mathbb{T}^{-1}(\overline{\alpha}) = \widehat{\gamma} \in \widehat{\Gamma}(X) \tag{5.3}$$

where $\widehat{\gamma} := \bigcup_{k\in\mathbb{Z}^d}(\alpha_k + k)$. To establish the measurability of $\mathbb{T}$ it is sufficient to consider cylinder sets of the form (5.1). Then

$$\mathbb{T}^{-1}\left(A^{k_1,\ldots,k_m}_{b_1,\ldots,b_m}\right) = \bigcap_{1\le i\le m} B_{(k_i,b_i)} \in \mathcal{B}(\widehat{\Gamma}), \tag{5.4}$$

where $B_{(k,b)} \in \mathcal{B}_0(\widehat{\Gamma})$ is defined for each $k \in \mathbb{Z}^d$ and $b \in \mathcal{B}(\mathcal{Q})$ as follows:

$$B_{(k,b)} := \left\{\widehat{\gamma} \in \widehat{\Gamma}(X) :\ \widehat{\gamma}_k \in \tilde{b}_k\right\},\ \ \tilde{b}_k := b + k \in \mathcal{B}(\widehat{\Gamma}_{\mathrm{Q}_k}).$$

Furthermore, observe that such sets on the right-hand side in (5.4) generate the whole $\mathcal{B}(\widehat{\Gamma})$, which means the measurability of $\mathbb{T}^{-1}$ as well. □

Thus, for any $\mu \in \mathcal{P}(\widehat{\Gamma})$ we can define its push-forward image $\mathbb{T}_*\mu \in \mathcal{P}(\mathcal{A})$, where $\mathcal{P}(\mathcal{A})$ is the set of all probability measures on $\mathcal{A}$.

Lemma 25 *The map* $\mathbb{T}_* : \mathcal{P}(\widehat{\Gamma}) \to \mathcal{P}(\mathcal{A})$ *is injective.*

Proof. Let $\mu, \nu \in \mathcal{P}(\widehat{\Gamma})$ and $\mu \neq \nu$. Then there exists $B \in \mathcal{B}(\widehat{\Gamma})$ such that $\mu(B) \neq \nu(B)$. By Lemma 24, $A := \mathbb{T}(B) \in \mathcal{B}(\mathcal{A})$ and $\mathbb{T}^{-1}(A) = B$. Thus $\mathbb{T}_*\mu(A) = \mu(\mathbb{T}^{-1}(A)) \neq \nu(\mathbb{T}^{-1}(A)) = \mathbb{T}_*\nu(A)$, and the statement is proved. □

Define a family of one-point states $\mathfrak{M} = \left\{\mathfrak{m}^{\overline{\alpha}}_k :\ k \in \mathbb{Z}^d,\ \overline{\alpha} \in \mathcal{A}\right\}$ by the formula

$$\mathfrak{m}^{\overline{\alpha}}_k(b) := \mu^{\mathbb{T}^{-1}\overline{\alpha}}_k\left(b + k\right),\ \ b \in \mathcal{B}\left(\mathcal{Q}\right), \tag{5.5}$$

where $\mu_k := \mu_{\mathrm{Q}_k}$ is the local Gibbs state of the initial model given by (2.21). The corresponding one-point specification $\mathfrak{P} = \left\{\mathfrak{p}^{\overline{\alpha}}_k :\ k \in \mathbb{Z}^d,\ \overline{\alpha} \in \mathcal{A}\right\}$ is constituted by probability kernels

$$\mathcal{A} \times \mathcal{B}(\mathcal{A}) \ni (\overline{\alpha}, A) \mapsto \mathfrak{p}^{\overline{\alpha}}_k(A) := \Pi_k\left(\mathbb{T}^{-1}A \,\middle|\, \mathbb{T}^{-1}\overline{\alpha}\right),$$

cf. (2.23). It is clear that $\mathfrak{m}^{\overline{\alpha}}_k \in \mathcal{P}(\mathcal{Q}_k)$ coincides with the projection of $\mathfrak{p}^{\overline{\alpha}}_k \in \mathcal{P}(\mathcal{A})$ onto the k-th component of the product space $\mathcal{A}$.

Lemma 26 *For any $k \in \mathbb{Z}^d$ and $\overline{\alpha},\ \overline{\alpha}' \in \mathcal{A}$ we have the following statements:*

(i) Measure $\mathfrak{m}_k^{\overline{\alpha}}$ has the form

$$\mathfrak{m}_k^{\overline{\alpha}}(\mathrm{d}\beta) = \mathcal{Z}^{-1} e^{-\mathcal{H}_k(\beta|\overline{\alpha})}\, \widehat{\lambda}(\mathrm{d}\beta),$$

where $\mathcal{H}_k(\beta\,|\overline{\alpha}) := H_{\mathrm{Q}_k}(\beta + k\,\big|\mathbb{T}^{-1}\overline{\alpha})$, $\beta \in \mathcal{Q} := \widehat{\Gamma}_{\mathrm{Q}_0}$ and $\mathcal{Z} := Z_{\mathrm{Q}_k}(\mathbb{T}^{-1}\overline{\alpha})$ is the normalizing factor (cf. (2.22)).

(ii) Assume that $\overline{\alpha}_{\partial k} = \overline{\alpha}'_{\partial k}$, where ∂k is defined in Sec. 2.4. Then $\mathfrak{m}_k^{\overline{\alpha}} = \mathfrak{m}_k^{\overline{\alpha}'}$ (Markovian property).

Proof. The statement immediately follows from the definition of measure $\mathfrak{m}_k^{\overline{\alpha}}$ and energy function H_k, cf. (2.16), and the translation invariance of the Lebesgue-Poisson measure $\widehat{\lambda}_z$. □

We denote by $\mathcal{M}(\mathfrak{P})$ the set of probability measures $\varpi \in \mathcal{P}(\mathcal{A})$ which are consistent with the singleton specification $\mathfrak{P}$, that is,

$$\int_{\mathcal{A}} \mathfrak{p}_k^{\bar{\alpha}}(A)\, \varpi(\mathrm{d}\overline{\alpha}) = \varpi(A), \;\; k \in \mathbb{Z}^d, A \in \mathcal{B}(\mathcal{A}). \tag{5.6}$$

For a measurable non-negative function $h : \mathcal{Q} \to \mathbb{R}$ define the subset $\mathcal{M}_h(\mathfrak{P})$ of those $\varpi \in \mathcal{M}(\mathfrak{P})$ that satisfy the bound

$$\sup_{k\in\mathbb{Z}^d} \int_{\mathcal{A}} h(\alpha_k)\, \varpi(\mathrm{d}\overline{\alpha}) < \infty. \tag{5.7}$$

Lemma 27 *Let $\mu \in \mathcal{G}^{\mathrm{t}}$. Then $\mathbb{T}_*\mu \in \mathcal{M}_{h_F}(\mathfrak{P})$ with $h_F = F_{\lceil \mathcal{Q}}$, where F is defined by formula (2.37).*

Proof. The consistency property (5.6) and bound (5.7) follow directly from the DLR equation (2.26) and estimate (2.43), respectively. □

The next statement is crucial for our approach.

Proposition 28 *We have $N\left(\mathcal{G}^{\mathrm{t}}\right) \le N\left(\mathcal{M}_{h_F}(\mathfrak{P})\right)$.*

Proof. Follows directly from Lemmas 25 and 27. □

Thus, in order to show that $\mathcal{G}^{\mathrm{t}}$ contains at most one element, it is sufficient to prove that $\mathcal{M}_{h_F}(\mathfrak{P})$ does so.

The uniqueness in question will be studied with the help of the Dobrushin–Pechesky criterion for lattice Gibbs states, extending Dobrushin's famous criterion [13] to the case of non-compact spins. This abstract result originally appeared in [14], see also [6, Theorem 2.6] for its further developments and [2, Theorem 3], [32, Theorem 4] resp. [31] for applications to some models of interacting particle systems (both in the continuum and on a lattice). More precisely, we will use the following adaptation of the Dobrushin–Pechesky criterion to our setting.

Theorem 29 (Uniqueness Criterion) *There exist a positive threshold value $\delta_* := \delta_*(d,R) < 1$ and a function $L^* : \mathbb{R}_+^3 \to (0,\infty)$ such that $N\left(\mathcal{M}_h(\mathfrak{P})\right) \le 1$ provided the family $\mathfrak{M}$ of one-point local Gibbs states satisfies the following two conditions:*

(DP-1) *There exist constants $\delta < \delta_*$ and $\Xi > 0$ such that*

$$\int_{\mathcal{Q}} h(\beta)\, \mathfrak{m}_k^{\bar{\alpha}}(\mathrm{d}\beta) \le \Xi + \delta \sum_{j \in \partial k} h(\alpha_j)$$

for any $k \in \mathbb{Z}^d$ and all boundary conditions $\bar{\alpha} \in \mathcal{A}$.

(DP-2) *There exists a constant $\ell < \mathcal{N}_0^{-1}$ such that*

$$\mathrm{d}_{\mathrm{var}}\left(\mathfrak{m}_k^{\overline{\alpha}},\ \mathfrak{m}_k^{\overline{\alpha}'}\right) < \ell$$

for any $k \in \mathbb{Z}^d$ and all boundary conditions $\overline{\alpha}, \overline{\alpha}' \in \mathcal{A}$ obeying the constraint

$$\sup_{j \in \mathbb{Z}^d} \{h(\overline{\alpha}_j);\ h(\overline{\alpha}'_j)\} \le L^*(\Xi, \delta, \ell). \tag{5.8}$$

Remark 30 *The original result is more refined in that precise threshold values δ_* and $L^*(\Xi, \delta, \ell)$ are given. We do not need this level of precision here and will show that (in our setting) the constants L^* and δ_* can be chosen arbitrarily large and small, respectively. Actually, $L^*(\Xi, \delta, \ell)$ tends to infinity as $\Xi \nearrow \infty$, $\delta \nearrow \delta_*$ or $\ell \nearrow \mathcal{N}_0^{-1}$. The values of δ_* and $L^*(\Xi, \delta, \ell)$ depend only on the geometry of the interaction (that is, the dimension d and interaction radius R only) and are the same for all control functions $h : \mathcal{Q} \to \mathbb{R}_+$.*

5.2 Proof of the uniqueness

In this section, we establish the uniqueness of tempered Gibbs measures due to small activity parameter $z > 0$ as stated in Theorem 5. For this, we will use the lattice representation of our model constructed in the previous section and verify for it both conditions (DP-1) and (DP-2) of Theorem 29.

Proof of Theorem 5. According to Proposition 28 it is sufficient to prove that $N(\mathcal{M}_{h_F}(\mathfrak{P})) \le 1$. To do so, we check conditions of Theorem 29 for $h := h_F$ defined in Lemma 27.

A simple change of variables shows that

$$\int_{\mathcal{Q}} h(\beta)\, \mathfrak{m}_k^{\overline{\alpha}}(\mathrm{d}\beta) = \int_{\widehat{\Gamma}_k} F(\widehat{\gamma}_k)\, \mu_k^{\mathbb{T}^{-1}\overline{\alpha}}(\mathrm{d}\widehat{\gamma}_k)$$

for any $\bar{\alpha} \in \mathcal{A}$. Set $\widehat{\eta} := \mathbb{T}^{-1}\overline{\alpha} \in \widehat{\Gamma}(X)$ and observe that $F(\widehat{\eta}_j) = h_F(\alpha_j)$. Corollary 12 implies that the inequality

$$\int_{\widehat{\Gamma}_k} F(\widehat{\gamma}_k)\, \mu_k^{\widehat{\eta}}(\mathrm{d}\widehat{\gamma}_k) \le \Xi + \delta \sum_{j \in \partial k} F(\widehat{\eta}_j) \tag{5.9}$$

holds for any $\delta > 0$ with a positive constant $\Xi := \Xi_\delta(\mathcal{J}, z)$, which is non-decreasing both in $\mathcal{J}$ and z. Thus (DP-1) is proved.

Let us now check (DP-2). Fix $L > 0$ and let $\overline{\alpha}, \overline{\alpha}' \in \mathcal{A}$ be boundary conditions satisfying

$$\sup_{j \in \mathbb{Z}^d} \{h(\overline{\alpha}_j);\ h(\overline{\alpha}'_j)\} \leq L. \tag{5.10}$$

By a change of variables it is easy to see that

$$\mathrm{d}_{\mathrm{var}}\left(\mathfrak{m}_k^{\overline{\alpha}},\ \mathfrak{m}_k^{\overline{\alpha}'}\right) = \mathrm{d}_{\mathrm{var}}\left(\mu_k^{\widehat{\eta}},\ \mu_k^{\widehat{\varsigma}}\right) \quad \text{for } \widehat{\eta} := \mathbb{T}^{-1}\overline{\alpha},\ \widehat{\varsigma} := \mathbb{T}^{-1}\overline{\alpha}'.$$

Condition (5.10) implies that $\sup_j \{F(\widehat{\eta}_j);\ F(\widehat{\varsigma}_j)\} = \sup_j\{h(\overline{\alpha}_j);\ h(\overline{\alpha}'_j)\} \leq L$. Thus, for given Ξ, δ as in (DP-1) and arbitrary ℓ and $\mathcal{J}_0$, by Lemma 14 we can find $z_0 > 0$ such that the bound

$$\mathrm{d}_{\mathrm{var}}(\mu_k^{\widehat{\eta}},\ \mu_k^{\widehat{\varsigma}}) \leq \ell$$

holds uniformly for any $z \leq z_0$, $\mathcal{J} \leq \mathcal{J}_0$ and all $\widehat{\eta}, \widehat{\varsigma}$ such that $F(\widehat{\eta}_j), F(\widehat{\varsigma}_j) \leq L$. This completes the proof. □

6 Proofs of auxiliary results

Our first aim is to prove Proposition 8**.** We start with some preparations.

Lemma 31 *For any $\gamma, \eta \in \Gamma(X)$ and $k \in \mathbb{Z}^d$ we have the estimate*

$$-U_k(\gamma_k|\eta) \leq -A_\Phi N(\gamma_k)^P + \frac{M\mathcal{N}_0}{2} N(\gamma_k)^2 + B_\Phi N(\gamma_k) + \frac{M}{2} \sum_{j \in \partial k} N(\eta_j)^2. \tag{6.1}$$

Proof. By definition (2.19) of the conditional energy $U_k(\gamma_k|\eta)$ and assumptions (A1)–(A3) on $\Phi(x, y)$, we immediately obtain

$$\begin{aligned} -U_k(\gamma_k|\eta) &= -U(\gamma_k) - \sum_{x \in \gamma_k} \sum_{y \in \eta_{\partial k}} \Phi(x, y) \qquad (6.2)\\ &\leq -\left[A_\Phi N(\gamma_k)^P - B_\Phi N(\gamma_k)\right] + M N(\gamma_k) \sum_{j \in \partial k} N(\eta_j) \\ &= -A_\Phi N(\gamma_k)^P + \frac{M\mathcal{N}_0}{2} N(\gamma_k)^2 + B_\Phi N(\gamma_k) + \frac{M}{2} \sum_{j \in \partial k} N(\eta_j)^2, \end{aligned}$$

and the proof is complete. □

Lemma 32 *For any $\varepsilon > 0$ the spin-spin energy $E_k(\sigma_k|\xi)$ satisfies the following estimate:*

$$\begin{aligned} -\mathcal{J}^{-1} E_k(\sigma_k|\xi) \leq & \left[(\mathcal{N}_0 + 1) \sum_{x \in \gamma_k} |\sigma_x|^{r(1+\varepsilon)} + \sum_{j \in \partial k} \sum_{y \in \eta_j} |\xi_y|^{r(1+\varepsilon)}\right] \\ & + \left(1 + \frac{1}{2} C_W\right) \left[(\mathcal{N}_0 + 1) N(\gamma_k)^{2+\varepsilon^{-1}} + \sum_{j \in \partial k} N(\eta_j)^{2+\varepsilon^{-1}}\right] \qquad (6.3) \end{aligned}$$

for all $k \in \mathbb{Z}^d$ and $\sigma_k \in S^{\gamma_k}, \xi \in S^\eta$.

Proof. By definition (2.20) of $E_k(\sigma_k|\xi)$ we have

$$-E_k(\sigma_k|\xi) \le \sum_{\{x,y\}\subset\gamma_k} W_{xy}^-(\sigma_x,\sigma_y) + \sum_{x\in\gamma_k}\sum_{y\in\eta_{\partial k}} W_{xy}^-(\sigma_x,\xi_y). \tag{6.4}$$

Let us estimate each sum in (6.4) by means of the classical Young inequality

$$ab \le \frac{a^{\mathrm{p}}}{\mathrm{p}} + \frac{b^{\mathrm{q}}}{\mathrm{q}}, \text{ for } a,b\ge 0 \text{ and } \mathrm{p},\mathrm{q}>1 \text{ s. t. } \mathrm{p}^{-1}+\mathrm{q}^{-1}=1. \tag{6.5}$$

To this end, observe that $\frac{1}{1+\varepsilon}+\frac{1}{1+\varepsilon^{-1}}=1$ for any $\varepsilon>0$. Using (A4) and then (6.5), we get

$$\begin{aligned}
\mathcal{J}^{-1}\sum_{\{x,y\}\subset\gamma_k} W_{xy}^-(\sigma_x,\sigma_y) &\le \sum_{\{x,y\}\subset\gamma_k}\left(|\sigma_x|^r+|\sigma_y|^r+C_W\right)\\
&\le [N(\gamma_k)-1]\sum_{x\in\gamma_k}|\sigma_x|^r + C_W\frac{N(\gamma_k)\,[N(\gamma_k)-1]}{2}\\
&\le \sum_{x\in\gamma_k}\left[\frac{|\sigma_x|^{r(1+\varepsilon)}}{1+\varepsilon}+\frac{N(\gamma_k)^{1+\varepsilon^{-1}}}{1+\varepsilon^{-1}}\right]+\frac{1}{2}C_W N(\gamma_k)^2\\
&\le \sum_{x\in\gamma_k}|\sigma_x|^{r(1+\varepsilon)}+\left(1+\frac{1}{2}C_W\right)N(\gamma_k)^{2+\varepsilon^{-1}}.
\end{aligned} \tag{6.6}$$

Similarly, for each $j\in\partial k$ we have

$$\begin{aligned}
\mathcal{J}^{-1}\sum_{x\in\gamma_k}\sum_{y\in\eta_j} W_{xy}^-(\sigma_x,\xi_y) &\le \sum_{x\in\gamma_k}\sum_{y\in\eta_j}\left(|\sigma_x|^r+|\xi_y|^r+C_W\right)\\
&\le N(\eta_j)\sum_{x\in\gamma_k}|\sigma_x|^r + N(\gamma_k)\sum_{y\in\eta_j}|\xi_y|^r + C_W N(\gamma_k)N(\eta_j)\\
&\le \sum_{x\in\gamma_k}\left[|\sigma_x|^{r(1+\varepsilon)}+N(\eta_j)^{1+\varepsilon^{-1}}\right]+\sum_{y\in\eta_j}\left[|\xi_y|^{r(1+\varepsilon)}+N(\gamma_k)^{1+\varepsilon^{-1}}\right]\\
&\quad+\frac{1}{2}C_W\left[N(\gamma_k)^2+N(\eta_j)^2\right].
\end{aligned} \tag{6.7}$$

Another application of Young's inequality yields the bound

$$N(\gamma_k)N(\eta_j)^{1+\varepsilon^{-1}} \le N(\gamma_k)^{2+\varepsilon^{-1}}\frac{1}{2+\varepsilon^{-1}} + N(\eta_j)^{2+\varepsilon^{-1}}\frac{1+\varepsilon^{-1}}{2+\varepsilon^{-1}},$$

by which we conclude that

$$\begin{aligned}
\mathrm{LHS}(6.7) &\le \sum_{x\in\gamma_k}|\sigma_x|^{r(1+\varepsilon)}+\sum_{y\in\eta_j}|\xi_y|^{r(1+\varepsilon)}\\
&\quad+\left(1+\frac{1}{2}C_W\right)\left[N(\gamma_k)^{2+\varepsilon^{-1}}+N(\eta_j)^{2+\varepsilon^{-1}}\right].
\end{aligned} \tag{6.8}$$

Combining (6.6)–(6.8), we obtain the desired estimate (6.3). □

Lemma 33 *For any $\epsilon > 0$ there exists a constant $D_\epsilon > 0$ such that the following superstability bound holds:*

$$
\begin{aligned}
&- H_k(\widehat{\gamma}_k|\widehat{\eta}) + A_\Phi N(\gamma_k)^P \\
&\qquad \le \epsilon \left[N(\gamma_k)^P + \sum_{x\in\gamma_k} |\sigma_x|^q + \sum_{j\in\partial k} \left(N(\eta_j)^P + \sum_{y\in\eta_j} |\xi_y|^q \right) \right] + D_\epsilon, \quad (6.9)
\end{aligned}
$$

for all $\widehat{\gamma}, \widehat{\eta} \in \widehat{\Gamma}(X)$ and $k \in \mathbb{Z}^d$. Furthermore, $D_\epsilon := D_\epsilon(\mathcal{J})$ can be chosen as a non-decreasing functions of $\mathcal{J}$.

Proof. It readily follows from (6.1) and (6.3) that

$$
\begin{aligned}
-H_k(\widehat{\gamma}_k|\widehat{\eta}) \le -A_\Phi N(\gamma_k)^P + B_{\Phi,\mathcal{J}} N(\gamma_k)^{2+\varepsilon^{-1}} + C_{\mathcal{J}} \sum_{j\in\partial k} N(\eta_j)^{2+\varepsilon^{-1}} \quad (6.10)\\
+ \mathcal{J}\Big[(\mathcal{N}_0 + 1) \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)} + \sum_{y\in\eta_{\partial k}} |\xi_y|^{r(1+\varepsilon)} \Big]
\end{aligned}
$$

for any $\widehat{\gamma}, \widehat{\eta} \in \widehat{\Gamma}(X)$, $k \in \mathbb{Z}^d$ and $\varepsilon > 0$. Here

$$
B_{\Phi,\mathcal{J}} := B_\Phi + C_{\mathcal{J}}(\mathcal{N}_0 + 1), \quad C_{\mathcal{J}} := \frac{M}{2} + \mathcal{J}\left(1 + \frac{1}{2} C_W\right), \quad (6.11)
$$

are both non-decreasing functions of $\mathcal{J}$. Now let us fix some $\varepsilon > 0$ such that

$$
t := r(1+\varepsilon) < q \ \text{ and } \ p := 2 + \varepsilon^{-1} < P, \quad (6.12)
$$

which is possible due to assumption (A6). Note that by (6.5) we have for any $\theta_1, \theta_2 > 0$

$$
\begin{aligned}
\sum_{x\in\gamma_k} |\sigma_x|^t &\le \theta_1 \sum_{x\in\gamma_k} |\sigma_x|^q + \theta_1^{\frac{t}{t-q}} N(\gamma_k), && (6.13)\\
N(\gamma_k)^p &\le \theta_2 N(\gamma_k)^P + \theta_2^{\frac{p}{p-P}}. && (6.14)
\end{aligned}
$$

Substituting both (6.13) and (6.14) into (6.10) and then taking θ_1, θ_2 small enough we get the required result. □

Proof of Proposition 8 . For any given δ the estimate (3.1) follows immediately from Lemma 33 with $\epsilon = \delta \max\{1, \kappa, \vartheta\}$ and $C_\delta(\kappa, \vartheta, \mathcal{J}) = D_\epsilon(\mathcal{J})$. □

Remark 34 *For $\widehat{\eta} = \emptyset$ we have the (slightly stronger than (6.10) and (6.11)) bound*

$$
-H(\widehat{\gamma}_k) \le -A_\Phi N(\gamma_k)^P + B^0_{\Phi,\mathcal{J}} N(\gamma_k)^{2+\varepsilon^{-1}} + \mathcal{J} \sum_{x\in\gamma_k} |\sigma_x|^{r(1+\varepsilon)} \quad (6.15)
$$

where the constant $B^0_{\Phi,\mathcal{J}} := B_\Phi + \mathcal{J}\left(1 + \frac{1}{2} C_W\right)$ is independent of $\varepsilon > 0$.

Proof of Lemma 14. To keep track of the dependence on the model parameters (z and $\mathcal{J}$ in particular), all constants in the estimates below will be written explicitly (although they need not be the best possible).

The general formula for the total variation distance between two probability measures states that

$$\begin{aligned}&\mathrm{d}_{\mathrm{var}}(\mu_k(\mathrm{d}\widehat{\gamma}_k|\widehat{\eta}),\ \mu_k(\mathrm{d}\widehat{\gamma}_k|\widehat{\varsigma}))\\&=\frac{1}{2}\int_{\widehat{\Gamma}_k}\left|Z_k^{-1}(\widehat{\eta})\exp\left\{-H_k(\widehat{\gamma}_k|\widehat{\eta})\right\}-Z_k^{-1}(\widehat{\varsigma})\exp\{-H_k(\widehat{\gamma}_k|\widehat{\varsigma})\}\right|\ \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_k).\end{aligned}\tag{6.16}$$

Multiplying the right-hand side by the expression $Z_k(\widehat{\eta})Z_k(\widehat{\varsigma})\geq 1$ and using (2.22), we see by an elementary calculation that

$$\begin{aligned}\mathrm{d}_{\mathrm{var}}(\mu_k(\mathrm{d}\widehat{\gamma}_k|\widehat{\eta}),\ \mu_k(\mathrm{d}\widehat{\gamma}_k|\widehat{\varsigma}))&\leq\min\left\{Z_k(\widehat{\eta}),\ Z_k(\widehat{\varsigma})\right\}\\&\times\int_{\widehat{\Gamma}_k}\left|\exp\{-H_k(\widehat{\gamma}_k|\widehat{\eta})\}-\exp\{-H_k(\widehat{\gamma}_k|\widehat{\varsigma})\}\right|\ \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_k).\end{aligned}\tag{6.17}$$

For simplicity, let us first set $\widehat{\varsigma}=\emptyset$ so that $H_k(\widehat{\gamma}_k|\emptyset)=H_k(\widehat{\gamma}_k)$. Observe that $H_k(\widehat{\gamma}_k|\widehat{\eta})=H_k(\widehat{\gamma}_k)=0$ for $\widehat{\gamma}_k=\emptyset$. Therefore

$$\begin{aligned}&\int_{\widehat{\Gamma}_k}\left|\exp\{-H_k(\widehat{\gamma}_k|\widehat{\eta})\}-\exp\{-H_k(\widehat{\gamma}_k)\}\right|\ \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_k)\\&\qquad=\int_{\widehat{\Gamma}_k\setminus\{\emptyset\}}\left|1-\exp\left\{-\Delta H_k(\widehat{\gamma}_k|\widehat{\eta})\right\}\right|\exp\left\{-H_k(\widehat{\gamma}_k)\right\}\ \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_k),\end{aligned}\tag{6.18}$$

where, cf. (2.17),

$$\Delta H_k(\widehat{\gamma}_k|\widehat{\eta}):=\sum_{x\in\gamma_k,y\in\eta_{\partial k}}\left[\Phi(x,y)+W_{xy}(\sigma_x,\xi_y)\right].$$

Obviously,

$$\begin{aligned}&\max\left\{\exp\left[-\Delta H_k(\widehat{\gamma}_k|\widehat{\eta})\right],\ \left|1-\exp\left[-\Delta H_k(\widehat{\gamma}_k|\widehat{\eta})\right]\right|\right\}\\&\leq\exp\left\{\left[\Delta H_k(\widehat{\gamma}_k|\widehat{\eta})\right]^-\right\}\leq\exp\left\{\sum_{x\in\gamma_k}\sum_{y\in\eta_{\partial k}}\left[\Phi^-(x,y)+W^-_{xy}(\sigma_x,\xi_y)\right]\right\},\end{aligned}\tag{6.19}$$

where superscript $^-$ denote the negative part of the corresponding function.

Recall that $\widehat{\eta}=(\eta,\xi)\in\widehat{\Gamma}(X)$ has to obey the bound $\sup_j F(\widehat{\eta}_j)\leq L$. Hence

$$\sup_{j\in\mathbb{Z}^d}\left\{N(\eta_j)^p,\ \sum_{y\in\eta_j}|\xi_y|^q\right\}\leq\frac{L}{\max\{\kappa,\vartheta\}}=:\mathcal{L}\tag{6.20}$$

for all $1\leq p\leq P$. Moreover, by (6.5) a similar estimate also holds for any $1\leq r\leq q$:

$$\sum_{y\in\eta_j}|\xi_y|^r\leq\frac{r}{q}\sum_{y\in\eta_j}|\xi_y|^q+\frac{q-r}{q}N(\eta_j)\leq\mathcal{L}.\tag{6.21}$$

Temporarily writing N for $N(\gamma_k)$ and taking into account that $\Phi^- \le M$, we immediately see by (6.2) and (6.20) that

$$\sum_{x\in\gamma_k}\sum_{y\in\eta_{\partial k}} \Phi^-(x,y) \le MN\mathcal{N}_0\mathcal{L}. \tag{6.22}$$

Next, we fix $\varepsilon > 0$, $t \in (r,q)$ and $p \in (2,P)$ as in (6.12). Then, by (6.7) and (6.21) we have

$$\begin{aligned}\mathcal{J}^{-1}\sum_{x\in\gamma_k}\sum_{y\in\eta_{\partial k}} W^-_{xy}(\sigma_x,\xi_y) &\le \mathcal{N}_0\mathcal{L}\sum_{x\in\gamma_k}|\sigma_x|^r + N\sum_{y\in\eta_{\partial k}}|\xi_y|^r + C_W N\mathcal{N}_0\mathcal{L}\\ &\le \mathcal{N}_0\mathcal{L}\left[\sum_{x\in\gamma_k}|\sigma_x|^r + (1+C_W)\,N\right]. \end{aligned}\tag{6.23}$$

Combining the above inequalities with the superstability bound (6.15) on $H_k(\widehat{\gamma}_k)$ and then setting

$$B'_{\Phi,\mathcal{J}} := B_\Phi + \mathcal{N}_0\mathcal{L}M + \mathcal{J}\,(1+C_W)\,(1+\mathcal{N}_0\mathcal{L}),$$

we obtain the estimate

$$\begin{aligned}&\max\left\{-H_k(\widehat{\gamma}_k|\widehat{\eta}),\ -H_k(\widehat{\gamma}_k) + \ln\left|1-\exp\left\{-\Delta H_k(\widehat{\gamma}_k|\widehat{\eta})\right\}\right|\right\}\\ &\qquad\le -A_\Phi N^P + B'_{\Phi,\mathcal{J}}N^p + \mathcal{J}\sum_{x\in\gamma_k}\left[|\sigma_x|^t + \mathcal{N}_0\mathcal{L}\,|\sigma_x|^r\right]. \end{aligned}\tag{6.24}$$

Notice that by Young's inequality the following uniform bound holds:

$$C_{\Phi,\mathcal{J}} := \max_{N\ge 0}\left\{-A_\Phi N^P + B'_{\Phi,\mathcal{J}}N^p\right\} \le (A_\Phi)^{-\frac{p}{P-p}}\left(B'_{\Phi,\mathcal{J}}\right)^{\frac{P}{P-p}}. \tag{6.25}$$

Thereafter, using the disintegration (2.11) we conclude (analogously to (3.3)) that

$$\begin{aligned}&\text{RHS (6.18)}\\ &\le e^{C_{\Phi,\mathcal{J}}}\int_{\Gamma_k\setminus\{\emptyset\}}\int_{S^{\gamma_k}}\exp\left\{\mathcal{J}\sum_{x\in\gamma_k}\left[|\sigma_x|^t + \mathcal{N}_0\mathcal{L}\,|\sigma_x|^r\right]\right\}\bigotimes_{x\in\gamma_k}\chi(\mathrm{d}\sigma_x)\lambda_z(\mathrm{d}\gamma_k)\\ &= e^{C_{\Phi,\mathcal{J}}}\int_{\Gamma_k\setminus\{\emptyset\}}\left[\mathcal{E}_{\mathcal{J}}\right]^{N(\gamma_k)}\lambda_z(\mathrm{d}\gamma_k) = e^{C_{\Phi,\mathcal{J}}}\sum_{n=1}^{\infty}\frac{(z\mathcal{E}_{\mathcal{J}})^n}{n!} = e^{C_{\Phi,\mathcal{J}}}\left[\exp\left\{z\mathcal{E}_{\mathcal{J}}\right\}-1\right], \end{aligned}\tag{6.26}$$

where

$$\mathcal{E}_{\mathcal{J}} := \int_S \exp\left\{\mathcal{J}\left(|s|^t + \mathcal{N}_0\mathcal{L}\,|s|^r\right)\right\}\,\chi(\mathrm{d}s) \tag{6.27}$$

is finite by assumption (A5).

We proceed in a similar way to obtain an upper bound on $Z_k(\widehat{\eta})$. Indeed, with the help of (6.24)–(6.27) one gets

$$Z_k(\widehat{\eta}) := \int_{\widehat{\Gamma}_k} \exp\{-H_k(\widehat{\gamma}_k|\widehat{\eta})\} \, \widehat{\lambda}_z(\mathrm{d}\widehat{\gamma}_k)$$

$$\le e^{C_{\Phi,\mathcal{J}}} \int_{\Gamma_k} \int_{S^{\gamma_k}} \exp\left\{ \mathcal{J} \sum_{x\in\gamma_k} \left(|\sigma_x|^t + \mathcal{N}_0 \mathcal{L} \, |\sigma_x|^r \right) \right\} \bigotimes_{x\in\gamma_k} \chi(\mathrm{d}\sigma_x) \lambda_z(\mathrm{d}\gamma_k)$$

$$= \exp\left\{ C_{\Phi,\mathcal{J}} + z\mathcal{E}_{\mathcal{J}} \right\}. \quad (6.28)$$

Putting (6.26) and (6.28) together and using the well-known inequality $e^a - 1 \le ae^a$ for all $a \ge 0$, we conclude that

$$\mathrm{d}_{\mathrm{var}}(\mu_k(\mathrm{d}\widehat{\gamma}_k|\widehat{\eta}),\ \mu_k(\mathrm{d}\widehat{\gamma}_k|\emptyset)) \le z\mathcal{E}_{\mathcal{J}} \exp\left\{ 2\left(C_{\Phi,\mathcal{J}} + z\mathcal{E}_{\mathcal{J}} \right) \right\}. \quad (6.29)$$

By the triangle inequality the above bound extends to general boundary conditions $\widehat{\varsigma} \neq \emptyset$. This yields the desired estimate (3.10) with

$$\phi(z, \mathcal{J}, L) := 2\mathcal{E}_{\mathcal{J}} \exp\left\{ 2\left(C_{\Phi,\mathcal{J}} + z\mathcal{E}_{\mathcal{J}} \right) \right\},$$

which is a non-decreasing function of $\mathcal{J}$, z, and L. □

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