Covariant Conservation Laws and the Spin Hall Effect in Dirac-Rashba Systems

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We present a theoretical analysis of two-dimensional Dirac-Rashba systems in the presence of disorder and external perturbations. We unveil a set of exact symmetry relations (Ward identities) that impose strong constraints on the spin dynamics of Dirac fermions subject to proximity-induced interactions. This allows us to demonstrate that an arbitrary dilute concentration of scalar impurities results in the total suppression of nonequilibrium spin Hall currents when only Rashba spin-orbit coupling is present. Remarkably, a finite spin Hall conductivity is restored when the minimal Dirac-Rashba model is supplemented with a spin–valley interaction. The Ward identities provide a systematic way to predict the emergence of the spin Hall effect in a wider class of Dirac-Rashba systems of experimental relevance and represent an important benchmark for testing the validity of numerical methodologies.

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Systems exhibiting strong spin-orbit coupling (SOC) have received much attention because they host unique spin transport phenomena that can be harnessed for low-power spintronics [1,2]. The spin Hall effect (SHE) [3,4] is indubitably a landmark in this novel approach; combined with its reciprocal phenomenon (the inverse SHE), it allows all-electrical generation, detection and manipulation of nonequilibrium spin currents in nonmagnetic conductors [5–8]. The exploitation of the SHE has proved fruitful for manipulation of magnetic order via spin-orbit torque at interfaces [9–11] and has led to new discoveries, including the spin Hall magnetoresistance [12].

The interest in spin-orbit phenomena has been invigorated with the recent discovery of strong Rashba splitting of two-dimensional electron gases (2DEGs) at nonmagnetic metal surfaces and heterointerfaces [13–15]. Microscopically, the splitting can be understood as arising from a potential gradient normal to the surface, $\phi(z)$, which couples the electron spin $\mathbf{s}$ and in-plane momentum $\mathbf{p}$, i.e., in the simplest approximation, $H_{\mathrm{RB}} = \alpha\hat{z}\cdot(\mathbf{s}\times\mathbf{p})$, where $\alpha \propto \partial_z\phi$. The Rashba-Bychkov (RB) interaction H_{RB} (hereafter, Rashba interaction) mixes orbital states with opposite spins, leading to spin-split parabolic bands with counter-rotating spin textures [16]. The tangential spin winding of Rashba states enables efficient generation of nonequilibrium spin polarization by application of electric fields [17–23]. Strikingly, the very helical nature of these states enforces a vanishing SHE in the presence of (scalar) impurity scattering [24–28], so that, in practice, the current-induced spin polarization is not easily accompanied by the formation of spin Hall currents [29]. Given the universality of the Rashba effect (also observed in ultrathin metals [30,31], quantum wells [32,33], and surfaces of topological insulators [34–36]), it is of utmost importance to understand whether the absence of the SHE is a general property of nonmagnetic surfaces with broken inversion symmetry or, rather, a peculiarity of the 2DEG.

The interfacial enhancement of SOC in graphene has been recently demonstrated [37–42], making it a promising model system for exploring the above issue. The departure from the standard Rashba effect in a 2DEG can be readily appreciated for a minimal model of a graphene subject to $z \to -z$ asymmetric SOC. In the long-wavelength limit, the relevant spin-orbit interaction is obtained by replacing the momentum with the pseudospin operator $\mathbf{p} \to \boldsymbol{\sigma}$ in H_{RB} [43,44]. The Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{RB}}$ for the $\chi = \pm$ valley reads

$$\mathcal{H} = \psi_\chi^\dagger\{\chi[-\imath\hbar v\sigma^i\partial_i + \lambda(\boldsymbol{\sigma}\times\mathbf{s})_z] - \epsilon\}\psi_\chi, \qquad (1)$$

where v is the bare Fermi velocity of massless Dirac electrons, λ is the Rashba coupling, ϵ is the Fermi energy, and $\sigma_i (i = 1, 2)$ and $s_j (j = 1, 2, 3)$ are Pauli matrices acting on pseudospin and spin subspace, respectively. This model possesses two noteworthy features. First, the band splitting occurs along the energy axis [Fig. 1]. Second, the Dirac helical spin texture is momentum dependent, i.e., $|\langle\mathbf{s}\rangle|$ is not conserved [44]. Moreover, Eq. (1) admits a straightforward generalization by adding further interactions preserving the inherent SU(2) spin structure, such as a spin-valley coupling.

Such unique features make the Dirac-Rashba model an ideal test bed for reexamining the absence of the SHE in interfaces with spin-split states.

In this Letter, we investigate Dirac-Rashba models in the presence of disorder and external perturbations. The existence of a covariant conservation law for the spin current—stemming from SU(2) gauge invariance—allows us to obtain the analytic form of two-particle spin-current vertex functions directly from the self-energy of the Dirac fermions and show that the spin Hall conductivity in the minimal model [Eq. (1)] is zero for nonmagnetic disorder, irrespectively of the Fermi level position. Furthermore, we show that, when Eq. (1) is generalized to include additional interactions, the obtained Ward identity imposes strong constraints on the nonequilibrium spin responses. Remarkably, this allows us to predict what type of proximity spin-orbit interactions can lead to a robust SHE in Dirac-Rashba interfaces of experimental interest.

The suppression of the SHE in 2DEGs subject to uniform Rashba interactions occurs in the presence of an arbitrary small concentration of scalar impurities. Formally, the disorder corrections resulting from the resummation of ladder diagrams exactly cancel the "clean" spin Hall (SH) conductivity [24–28]. In Ref. [27], it was shown that this puzzling cancellation has its origin in the existence of a covariant conservation law for the spin current. For example, the spin-y component satisfies

$$\partial_t J_0^y(\mathbf{x},t) + \partial^i J_i^y(\mathbf{x},t) = -2\alpha m J_y^z(\mathbf{x},t), \qquad (2)$$

where J_0^a ($a = x, y, z$) is the spin density, J_i^a is the pure spin current flowing in the $i = x, y$ direction, m is the effective electron mass, and α is the Rashba parameter. The main difference with respect to the charge continuity equation originates from the non-Abelian nature of spin, which results in the additional contribution on the right hand side. Equation (2) suggests that in the steady state of a homogeneous system, J_y^z is zero irrespectively of the underlying relaxation mechanism. Below, we show that, albeit the drastically different nature of electronic states in the Dirac-Rashba model [Fig. 1], a similar covariant conservation law exists, and we discuss its consequences.

Conservation laws I.—A peculiarity of Dirac theories is the possible existence of quantum anomalies due to the joint effect of an infinite Dirac sea of filled electron states and an external field [45,46]. Let us consider the minimal coupling of Eq. (1) to a $U(1)$ gauge field $A_\mu \equiv (A_0, A_i)$ within a Minkowsky metric. To simplify notation, we take $\chi = +$ and omit this index hereafter. We also use natural units ($\hbar \equiv 1 \equiv e$) and the compact notation $\partial_\mu \equiv (\partial_t, \partial_i)$ with summation over dummy indices. The Dirac spin and charge currents are, respectively, $J_\mu^a(x) = \psi^\dagger(x) s^a v_\mu/2\psi(x)$ and $J_\mu(x) = \psi^\dagger(x) v_\mu \psi(x)$, where $v^\mu = (1, v\boldsymbol{\sigma})$ and $x \equiv (t, \mathbf{x})$. The Heisenberg equation of motion for the spin density reads

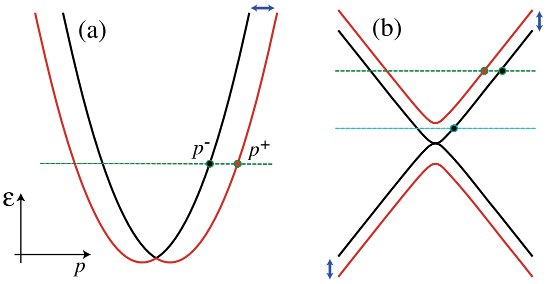

FIG. 1. Schematic of the splitting of electronic states due to the Rashba effect in a 2DEG (a) and graphene (b). The Fermi surface consists of two branches in a 2DEG. In graphene, for energies in the Rashba pseudogap $|\epsilon| < 2|\lambda|$, the Fermi surface is simply connected. Arrows indicate the type of splitting.

$$\partial^\mu J_\mu^a(x) = -\frac{2\lambda}{v}\epsilon_{bc}^a \epsilon^{bl} J_l^c(x) + \imath \int dy [J_0^a(x), J_\mu(y)] A^\mu(y), \qquad (3)$$

where ϵ^{bl} (ϵ_{bc}^a) is the Levi-Civita symbol of second (third) rank. The term on the left-hand side and the first term on the right-hand side result from the commutator of J_0^a, respectively, with the kinetic and the Rashba term and give a contribution identical to the one found in the 2DEG upon identification of $m \to 1/v$, c.f. Eq. (2). Both terms can be combined as the covariant derivative $D_\mu \mathcal{O}^a = \partial_\mu \mathcal{O}^a + 2\epsilon_{bc}^a \mathcal{A}_\mu^b \mathcal{O}^c$, where $\mathcal{A}_0^a = 0$, $\mathcal{A}_i^a = -\lambda/v\epsilon^{ai}$ is a SOC-induced, homogeneous gauge field. Hence, in the absence of an external field, Eq. (3) acquires the form of a covariant conservation law for the spin density $D^\mu J_\mu^a = 0$. The current commutator in the last term (Schwinger term) defines the anomaly. However, a careful analysis shows that, despite the Dirac nature of the theory, the commutator is identically zero–see Supplemental Material [47]; therefore, the argument of Ref. [27] implies a vanishing SHE in the Dirac-Rashba model. At first sight, this result contradicts the claims of Ref. [49], where the SH conductivity was evaluated using linear response theory $\sigma_{\rm SH} = \lim_{\omega\to 0}\lim_{q\to 0}\Theta_{yx}^z(\mathbf{q},\omega)/i\omega$, with the response function Θ_{yx}^z taken in the disorder-free approximation. Using the Matsubara propagator given in [47], we find

$$\sigma_{\rm SH} = -\frac{\epsilon}{16\pi\lambda}\left(\frac{2\lambda+\epsilon}{\epsilon+\lambda} + \theta(\epsilon - 2\lambda)\frac{2\lambda-\epsilon}{\epsilon-\lambda}\right), \qquad (4)$$

in agreement with Ref. [49]. Here, $\theta(.)$ is the Heaviside step function, and we assumed ϵ, $\lambda > 0$. The apparent contradiction is resolved by recalling that, without disorder, there is no true stationary state. In the following, we show that Eq. (4) misses on important physics related to scattering-induced relaxation that leads to $\sigma_{\rm SH} = 0$.

Conservation laws II: disorder effects.—Broadly speaking, the Fermi surface contribution to $\sigma_{\rm SH}$ is dominated by incoherent multiple scattering off impurities, which can be viewed as a series of skew scattering and side jump events

[50–52]. To determine how such effects change the above picture, we add to the bare Hamiltonian (1) a random scalar potential $V(\mathbf{x})$, which we will assume to be Gaussian distributed with zero mean: $\langle V(\mathbf{x})V(\mathbf{x}')\rangle = n_i\alpha_0^2\delta(\mathbf{x}-\mathbf{x}')$, where n_i is the impurity areal density and α_0 parametrizes the potential strength. This approximation is accurate in the limit of weak potential scattering provided cross sections are right-left symmetric (see below). We note that short-range impurities lead to scattering potentials that are off-diagonal in both sublattice and valley spaces. The intervalley scattering produced by such matrix disorder affects the charge conductivity σ_{xx} [53], but it does not change the covariant conservation law for the spin current.

Disorder enters the evaluation of response functions both in the propagator (as a self-energy) and the interaction vertex [54]. These two quantities are not independent of each other, but they are related by Ward identities (WIs); these relations are the key to establishing gauge invariance in quantum electrodynamics at a nonperturbative level [46]. Remarkably, we find that the non-Abelian WI associated to the spin current vertex completely determines the spin current J_i^z in the dc limit, and therefore, it can be used to directly evaluate the SH conductivity. To see this, consider the three-legged spin vertex function $\Lambda_\mu^y(x,x',x'') = \langle T_\tau J_\mu^y(x)\psi(x')\psi^\dagger(x'')\rangle$, where "$T_\tau$" stands for the imaginary time ordering operator. Moving to frequency-momentum space, we perform analytic continuation $\imath\omega_n \to \omega + \imath\,\mathrm{sign}(\omega)0^+$, where ω_n are fermionic Matsubara frequencies. Vertex corrections appear perturbatively as a series of impurity lines ladder diagrams, where only combinations of Green's functions, having poles on opposite sides of the real axis, contribute to the renormalization of the vertex [54]. In this way, by projecting the vertex function Λ_μ^y in the retarded (R)—advanced (A) sector, we find

$$q^\mu\Lambda_\mu^y = -\imath\frac{2\lambda}{v}\Lambda_y^z + \frac{1}{2}[S_y\mathcal{G}_{k+q}^R(\epsilon) - \mathcal{G}_k^A(\epsilon)S_y], \tag{5}$$

where k and q are three vectors. The disorder averaged Green's function ($a = A$, R) formally reads $\mathcal{G}_k^a(\epsilon) = [k_0 - H - \Sigma^a(\epsilon)]^{-1}$, where H is given by the first quantization form of Eq. (1) and $\Sigma^a(\epsilon)$ is the disorder induced self energy (see SM [47] for an explicit form). Owing to the non-Abelian nature of the WI, taking the dc ($q \to 0$) limit in Eq. (5) completely determines the effective vertex. The final step consists in recasting Λ_y^z in terms of the truncated vertex, $\Lambda_y^z = \mathcal{G}^A\tilde{j}_y^z\mathcal{G}^R$ [55], as appearing in the Kubo formula. After algebraic manipulations, we arrive at the important intermediate result

$$\tilde{j}_y^z = -\imath\frac{v}{4\lambda}\{[s_y,\tilde{H}]_- + \imath[s_y,\mathrm{Im}\Sigma^R(\epsilon)]_+\}. \tag{6}$$

where $\pm$ stands for the (anti-)commutator and $\tilde{H} = H + \mathrm{Re}\Sigma$ is the Hamiltonian renormalized by the real part of the self energy. This result provides an exact relation between the truncated spin current vertex and the self energy, and as such, it is independent of the particular approximation scheme used to evaluate disorder effects. Within the Gaussian approximation, we find $-\mathrm{Im}\Sigma^R(\epsilon) = 1/(2\tau)[1+\theta(2\lambda-\epsilon)\lambda/2\epsilon]\sigma_0 s_0 + \theta(2\lambda-\epsilon)[\lambda/(2\tau\epsilon)\sigma_3 s_3 - 1/(8\tau)(\boldsymbol{\sigma}\times\boldsymbol{s})_z]$, where $1/2\tau = n_i\epsilon\alpha_0^2/4v^2$ is the quasiparticle broadening. Using the expression of the self energy in Eq. (6), we arrive at

$$\tilde{j}_y^z = \frac{v}{2}\times\begin{cases}\sigma_y s_z - \frac{1}{2\lambda\tau}\sigma_0 s_y, & \epsilon > 2\lambda\\ \sigma_y s_z - \frac{1}{4\lambda\tau}(1+\frac{\lambda}{\epsilon})\sigma_0 s_y & \\ +\frac{1}{8\lambda\tau}\sigma_x s_0 + \frac{1}{4\pi\tau\lambda}\sigma_z s_x, & \epsilon \le 2\lambda.\end{cases} \tag{7}$$

The first term is just the bare spin current vertex $j_y^z = (v/2)\sigma_y s_z$, while, for $\epsilon > 2\lambda$, the second term, generated by the disorder, is the bare spin density vertex $\sigma_0 s_y/2$ apart from the factor $-v/2\lambda\tau$. This shows that the parameter $\lambda\tau$ plays a fundamental role in determining the importance of disorder. At first sight, one could be tempted to think that within the weak disorder limit ($\epsilon\tau \gg 1$) and for strong SOC ($\lambda\tau \gg 1$), all disorder corrections can be neglected. However, it turns out that the spin polarization response is of order $\lambda\tau$ (see below), whereas the bare spin current response, due to the first term in Eq. (7), is of order $(\lambda\tau)^0$. Hence, the two terms are of the same order irrespective of the disorder strength. Similar considerations also apply for $\epsilon < 2\lambda$.

SHE evaluation using the WI.—We start by computing the Fermi surface contribution

$$\begin{aligned}\sigma_{\mathrm{SH}}^{\mathrm{I}} &= \frac{1}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^2}\mathrm{tr}[\tilde{j}_y^z\mathcal{G}_{\mathbf{k}}^R(\epsilon)v_x\mathcal{G}_{\mathbf{k}}^A(\epsilon)]\\ &= \bar{\sigma}_{\mathrm{SH}} + \bar{\sigma}_{\mathrm{SG}} + \bar{\sigma}_{xx} + \bar{\sigma}_{zx},\end{aligned} \tag{8}$$

where $v_x = v\sigma_x s_0$ is the bare charge current vertex. Moreover, $\bar{\sigma}_{\mathrm{SH}}$, $\bar{\sigma}_{\mathrm{SG}}$, $\bar{\sigma}_{xx}$, and $\bar{\sigma}_{zx}$ are the conductivity "bubbles" corresponding to the various terms in Eq. (7), respectively, a spin Hall ($\sigma_y s_z$), spin galvanic (SG) ($\sigma_0 s_y$), longitudinal ($\sigma_x s_0$), and "staggered" ($\sigma_z s_x$) conductivities. Outside the pseudogap, where the Fermi surface splits into two branches [Fig. 1], we find $\bar{\sigma}_{xx} = \bar{\sigma}_{zx} = 0$ and $\bar{\sigma}_{\mathrm{SH}} = -\bar{\sigma}_{\mathrm{SG}}$, where

$$\bar{\sigma}_{\mathrm{SH}} = -\frac{1}{8\pi}\left(\frac{\epsilon^2}{\epsilon^2-\lambda^2} - \frac{1}{1+4\lambda^2\tau^2}\right), \tag{9}$$

and thus, the type I contribution to the SH conductivity is zero, $\sigma_{\mathrm{SH}}^{\mathrm{I}} = 0$. This result deserves a few comments: First, in the $\lambda\tau \gg 1$ limit, one recovers Eq. (4). Second, the "empty bubble" SH conductivity ($\bar{\sigma}_{\mathrm{SH}}$) is precisely counteracted by the corresponding empty bubble for the spin density-charge current response function ($\bar{\sigma}_{\mathrm{SG}}$) [56]. This means that the absence of the SHE is linked to the onset of a current-induced, in-plane spin polarization known as the inverse SG effect [17–19]. The remaining (type II) contribution

$$\sigma_{\mathrm{SH}}^{\mathrm{II}} = \frac{-1}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{0}dk_0\mathrm{Re}\,\mathrm{tr}[\mathcal{G}_k^R(\epsilon)j_y^z\overleftrightarrow{\partial}_{k_0}\mathcal{G}_k^R(\epsilon)v_x], \tag{10}$$

accounts for processes away from the Fermi surface [57]. Explicit evaluation shows that $\sigma_{\mathrm{SH}}^{\mathrm{II}}=0$, and thus, $\sigma_{\mathrm{SH}}=\sigma_{\mathrm{SH}}^{\mathrm{I}}+\sigma_{\mathrm{SH}}^{\mathrm{II}}$ is zero, in agreement with our earlier argument viz., Eqs. (2)–(3). Interestingly, in the 2DEG-Rashba model, the type II term is only zero in the formal limit $\epsilon\tau\to\infty$ and can attain large values for $\lambda\tau\approx 1$ [58]. The exact vanishing of the off-Fermi surface contribution is a unique feature of the Dirac theory. We now move gears to the regime $\epsilon<2\lambda$, where only one subband is occupied. We note that this regime has no analogue in the 2DEG model, for which the Fermi surface always consists of two disconnected rings [Fig. 1]. Thus, the mechanism leading to $\sigma_{\mathrm{SH}}=0$ is far from obvious. To investigate this issue, we evaluate the Fermi surface contribution making use of the WI [see Eq. (7)] and the type II contribution using Eq. (10). After a lengthy calculation, we find, for both contributions,

$$\sigma_{\mathrm{SH}}^{\mathrm{I}}=\frac{1}{16\pi}\frac{\epsilon}{\lambda},\qquad \sigma_{\mathrm{SH}}^{\mathrm{II}}=-\sigma_{\mathrm{SH}}^{\mathrm{I}}. \tag{11}$$

so that $\sigma_{\mathrm{SH}}=0$. Note that, since $\sigma_{\mathrm{SH}}^{\mathrm{I}}$ is of order τ^0, we can evaluate the type II contribution [Eq. (10)] directly in the absence of disorder. Therefore, the suppression of the SHE in the regime $0<\epsilon<2\lambda$ results from a compensation between scattering corrections to the clean SH conductivity and off-Fermi surface processes.

Diagrammatic evaluation.—Now, we show the consistency of our results with a standard diagrammatic evaluation. The renormalized charge current vertex satisfies the following Bethe-Salpeter coupled equations [see Fig. 2]:

$$\tilde{v}_{x,\mu a}=v\delta_{\mu 1}\delta_{a0}+T_{\mu a\rho d}{}^{\nu b\lambda c}I_{\nu b\lambda c}\tilde{v}_x^{\rho d}, \tag{12}$$

$$T_{\mu a\rho d}{}^{\nu b\lambda c}=\mathrm{tr}[\sigma_\mu s_a\sigma_\nu s_b\sigma_\rho s_d\sigma_\lambda s_c], \tag{13}$$

$$I_{\nu b\lambda c}=\frac{n\alpha_0^2}{4}\int\frac{d\mathbf{k}}{(2\pi)^2}\mathcal{G}^R_{\mathbf{k},\nu b}(\epsilon)\mathcal{G}^A_{\mathbf{k},\lambda c}(\epsilon). \tag{14}$$

In principle, I spans the entire Clifford Algebra. However, not all matrix elements contribute to the renormalization of the charge vertex. It is convenient to consider the effect of a single impurity density insertion, for which the vertex has the structure: $\bar{v}_x=\delta v_{10}\sigma_1 s_0+\delta v_{23}\sigma_2 s_3+\delta v_{02}\sigma_0 s_2+\delta v_{31}\sigma_3 s_1$, with δv_{ij} some nonzero matrix elements. This result suggests the form of the ansatz for $\tilde{v}_x$ to use in Eq. (12). Since no new matrix element is generated in this procedure, the ansatz

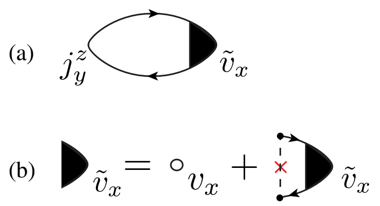

FIG. 2. Feynman diagrams for (a) dressed SH conductivity. (b) Charge vertex renormalization. The empty dot represents the bare charge vertex while the red x and the black dots represent, respectively, impurity density and scattering potential insertions.

closes the system. In addition to the renormalized charge vertex $\tilde{v}_x^{10}$, we find that disorder induces effective SH ($\tilde{v}_x^{23}$), SG ($\tilde{v}_x^{02}$), and staggered ($\tilde{v}_x^{31}$) interactions. Their explicit form reads (for $\epsilon>2\lambda$): $\tilde{v}_x^{10}=2v$, $\tilde{v}_x^{02}=-2v(\lambda/\epsilon)$, $\tilde{v}_x^{31}=0$, and $\tilde{v}_x^{23}=0$. In order to evaluate the SH conductivity, now, we use Eq. (8), with the ladder series now included in the charge vertex (i.e., $\tilde{j}_y^z\to j_y^z$ and $v_x\to\tilde{v}_x$). Using Eq. (12), it is now easy to relate the renormalized vertex directly to the SH and Drude conductivity

$$\sigma_{\mathrm{SH}}=\frac{1}{2\pi}\left(\frac{2v}{n_i\alpha_0^2}\right)\tilde{v}_x^{23}=0, \tag{15}$$

$$\sigma_{xx}=\frac{1}{2\pi}\left(\frac{4v}{n_i\alpha_0^2}\right)(\tilde{v}_x^{10}-v)=\frac{2\epsilon\tau}{\pi}. \tag{16}$$

Discussion.—We mentioned earlier that higher-order scattering contributions to the self energy (and ladder series) could generate important corrections. This happens when impurities in the system lead to skew scattering. In the 2DEG, it is well known that skew scattering is absent (unless other ingredients, such as spin-orbit active impurities are considered). The absence of skewness has, in fact, an intuitive explanation: the spin of Rashba eigenstates is locked in-plane, so that, in a given scattering event, quasiparticles cannot distinguish left and right. The same picture holds in the Dirac-Rashba model and, so, here, too, there should be no skewness. We verified this by means of the self-consistent diagrammatic approach introduced in Ref. [52] together with the WI [Eq. (6)].

The formalism developed in this Letter also allows us to predict the behavior of more complicated systems. For instance, it is easy to see that a nonzero SH conductivity emerges when adding suitable interactions to Eq. (1), altering the covariant conservation law expressed in Eq. (3) and, hence, the WI [Eq. (6)]. For example, let us consider a spin-valley interaction of the form $\mathcal{A}_0^a=\chi\lambda'\delta_{az}$ with λ' a constant. This interaction generates, in Eq. (3), a new term proportional to $\langle s_x^\chi\rangle$, where $\langle s_x^\chi\rangle$ is the nonequilibrium average of the $\hat{x}$-spin polarization at a given valley. Taking the steady state of a homogeneous system, we find an exact relation between the spin Hall current and the difference between the nonequilibrium spin density at the two inequivalent valleys, namely

$$\langle J_y^z\rangle=v\frac{\lambda'}{\lambda}(\langle s_x^{\chi=1}\rangle-\langle s_x^{\chi=-1}\rangle). \tag{17}$$

This suggests that SHE can emerge provided there is a mechanism to generate $\langle s_x^\chi\rangle\neq 0$ with opposite signs for $\chi=\pm 1$. A strong candidate is skew scattering. In principle, skewness is now allowed since the spin-valley interaction takes the spin of bare eigenstates out of the plane. We have computed both (nonvanishing) sides of Eq. (17) diagrammatically and verified that the identity holds at all orders in the scattering potential strength (not shown). This is a significant finding since the spin-valley coupling

λ' can attain sizable values in graphene with proximity SOC [59,60]. The possibility to have skew scattering exclusively driven by SOC in the band structure appears to be a unique feature of Dirac systems.

In this context, we note in passing that random spatial fluctuations in the Rashba coupling (e.g., due to corrugations) provide an alternative source of SHE [61]. The skew scattering contribution discussed above is dominant in clean samples due to its characteristic scaling (n_i^{-1} opposed to n_i^0 in the random mechanism) and the relatively small size of the fluctuations expected for atomically flat interfaces.

Our work constitutes a major step towards a unified theory of spin and charge dynamics for Dirac-Rashba models in generic nonstationary conditions. Real-space methodologies for numerical evaluation of transverse conductivities have recently been proposed [62,63], which can help tackling more complex scenarios. The exact symmetry relations presented here provide a stringent test for real-space numerical approaches, for which the achievable energy resolutions still represent a major limiting factor.

Data availability statement (EPSRC).—No new data were created during this study.

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