



Deposited via The University of Sheffield.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/id/eprint/124656/>

Version: Accepted Version

Article:

Bavula, V., Bekkert, V. and Futorny, V. (2018) Indecomposable generalized weight modules over the algebra of polynomial integro-differential operators. Proceedings of the American Mathematical Society, 146. pp. 2373-2380. ISSN: 0002-9939

<https://doi.org/10.1090/proc/13985>

© 2018 American Mathematical Society. This is an author produced version of a paper subsequently published in Proceedings of the American Mathematical Society. Uploaded in accordance with the publisher's self-archiving policy.

Reuse

This article is distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs (CC BY-NC-ND) licence. This licence only allows you to download this work and share it with others as long as you credit the authors, but you can't change the article in any way or use it commercially. More information and the full terms of the licence here: <https://creativecommons.org/licenses/>

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

INDECOMPOSABLE GENERALIZED WEIGHT MODULES OVER THE ALGEBRA OF POLYNOMIAL INTEGRO-DIFFERENTIAL OPERATORS

V. V. BAVULA, V. BEKKERT AND V. FUTORNY

ABSTRACT. For the algebra $\mathbb{I}_1 = K\langle x, \frac{d}{dx}, \int \rangle$ of polynomial integro-differential operators over a field K of characteristic zero, a classification of indecomposable, generalized weight \mathbb{I}_1 -modules of finite length is given. Each such module is an infinite dimensional uniserial module. Ext-groups are found between indecomposable generalized weight modules, it is proven that they are finite dimensional vector spaces.

Key Words: the algebra of polynomial integro-differential operators, generalized weight module, indecomposable module, simple module.

Mathematics subject classification 2000: 16D60, 16D70, 16P50, 16U20.

1. INTRODUCTION

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; $\mathbb{N}_+ := \{1, 2, \dots\}$ and $\mathbb{Z}_{\leq 0} := -\mathbb{N}$; K is a field of characteristic zero and K^* is its group of units; $P_1 := K[x]$ is a polynomial algebra in one variable x over K ; $\partial := \frac{d}{dx}$; $\text{End}_K(P_1)$ is the algebra of all K -linear maps from P_1 to P_1 , and $\text{Aut}_K(P_1)$ is its group of units (i.e. the group of all the invertible linear maps from P_1 to P_1); the subalgebras $A_1 := K\langle x, \partial \rangle$ and $\mathbb{I}_1 := K\langle x, \partial, \int \rangle$ of $\text{End}_K(P_1)$ are called the (first) *Weyl algebra* and the *algebra of polynomial integro-differential operators* respectively where $\int : P_1 \rightarrow P_1$, $p \mapsto \int p dx$, is the *integration*, i.e. $\int : x^n \mapsto \frac{x^{n+1}}{n+1}$ for all $n \in \mathbb{N}$. The algebra \mathbb{I}_1 is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals, [2].

In Section 2, a classification of indecomposable, generalized weight \mathbb{I}_1 -modules of finite length is given (Theorem 2.5). A similar classification is given in [1] for the generalized Weyl algebras where a completely different approach was taken. Properties of the algebras $\mathbb{I}_n := \mathbb{I}_1^{\otimes n}$ of polynomial integro-differential operators in arbitrary many variables are studied in [2] and [5]. The groups $\text{Aut}_{K\text{-alg}}(\mathbb{I}_n)$ are found in [3]. The simple \mathbb{I}_1 -modules are classified in [4].

Acknowledgment The first author is grateful to the University of São Paulo for hospitality during his visit and to Fapesp for financial support (processo 2013/24392-5). The third first author is supported in part by the CNPq (301320/2013-6), by the Fapesp (2014/09310-5).

2. CLASSIFICATION OF INDECOMPOSABLE, GENERALIZED WEIGHT \mathbb{I}_1 -MODULES OF FINITE LENGTH

In this section, a classification of indecomposable, generalized weight \mathbb{I}_1 -modules of finite length is given (Theorem 2.5).

As an abstract algebra, the algebra \mathbb{I}_1 is generated by the elements ∂ , $H := \partial x$ and \int (since $x = \int H$) that satisfy the defining relations, [2, Proposition 2.2] (where $[a, b] := ab - ba$):

$$\partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra \mathbb{I}_1 ,

$$e_{ij} := \int^i \partial^j - \int_1^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \tag{1}$$

satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00} \partial^j$. The matrices of the linear maps $e_{ij} \in \text{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e.

$$e_{ij} * x^{[s]} = \begin{cases} x^{[i]} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

Let $E_{ij} \in \text{End}_K(K[x])$ be the usual matrix units, i.e. $E_{ij} * x^s = \delta_{js}x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!} E_{ij}, \quad (2)$$

$Ke_{ij} = KE_{ij}$, and $F := \bigoplus_{i,j \geq 0} Ke_{ij} = \bigoplus_{i,j \geq 0} KE_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices.

\mathbb{Z} -grading on the algebra \mathbb{I}_1 and the canonical form of an integro-differential operator, [2]. The algebra $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$ is a \mathbb{Z} -graded algebra ($\mathbb{I}_{1,i}\mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|} D_1 = D_1 \partial^{|i|} & \text{if } i < 0, \end{cases} \quad (3)$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of \mathbb{I}_1 , $He_{ii} = e_{ii}H = (i+1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i \in \mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals of D_1); $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; $D_1(D_1 \partial^i) \simeq D_1$, $d \partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \rightarrow D_1 \int^i$, $d \mapsto d \int^i$, and $\partial^i \cdot : D_1 \rightarrow \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$.

Each element a of the algebra \mathbb{I}_1 is the unique finite sum

$$a = \sum_{i>0} a_{-i} \partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (4)$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [2].

$$\text{Let } v_i := \begin{cases} \int^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ \partial^{|i|} & \text{if } i < 0. \end{cases}$$

Then $\mathbb{I}_{1,i} = D_1 v_i = v_i D_1$ and an element $a \in \mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i \in \mathbb{Z}} b_i v_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \quad (5)$$

where $b_i \in K[H]$ and $\lambda_{ij} \in K$. So, the set $\{H^j \partial^i, H^j, \int^i H^j, e_{st} \mid i \geq 1; j, s, t \geq 0\}$ is a K -basis for the algebra \mathbb{I}_1 . The multiplication in the algebra \mathbb{I}_1 is given by the rule:

$$\begin{aligned} \int H &= (H-1) \int, & H \partial &= \partial(H-1), & \int e_{ij} &= e_{i+1,j}, \\ e_{ij} \int &= e_{i,j-1}, & \partial e_{ij} &= e_{i-1,j}, & e_{ij} \partial &= \partial e_{i,j+1}, \\ He_{ii} &= e_{ii} H = (i+1)e_{ii}, & & & i \in \mathbb{N}, \end{aligned}$$

where $e_{-1,j} := 0$ and $e_{i,-1} := 0$.

The algebra \mathbb{I}_1 has the only proper ideal $F = \bigoplus_{i,j \in \mathbb{N}} Ke_{ij} \simeq M_\infty(K)$ and $F^2 = F$. The factor algebra \mathbb{I}_1/F is canonically isomorphic to the skew Laurent polynomial algebra $B_1 := K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H+1$, via $\partial \mapsto \partial$, $\int \mapsto \partial^{-1}$, $H \mapsto H$ (where $\partial^{\pm 1} \alpha = \tau^{\pm 1}(\alpha) \partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1}H$).

An \mathbb{I}_1 -module M is called a *weight module* if $M = \bigoplus_{\lambda \in K} M_\lambda$ where $M_\lambda := \{m \in M \mid Hm = \lambda m\}$. An \mathbb{I}_1 -module M is called a *generalized weight module* if $M = \bigoplus_{\lambda \in K} M^\lambda$ where $M^\lambda := \{m \in M \mid (H - \lambda)^n m = 0 \text{ for some } n = n(m)\}$. The set $\text{Supp}(M) := \{\lambda \in K \mid M^\lambda \neq 0\}$ is called the *support* of the generalized weight module M . For all $\lambda \in K$ and $n \geq 1$,

$$\partial^n M^\lambda \subseteq M^{\lambda-n} \quad \text{and} \quad \int^n M^\lambda \subseteq M^{\lambda+n}.$$

Let $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ be a short exact sequence of \mathbb{I}_1 -modules. Then M is a generalized weight module iff so are the modules N and L , and in this case

$$\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(L).$$

For each \mathbb{I}_1 -module M , there is a short exact sequence of \mathbb{I}_1 -modules

$$0 \rightarrow FM \rightarrow M \rightarrow \overline{M} := M/FM \rightarrow 0 \quad (6)$$

where

- (i) $F \cdot FM = FM$, and
- (ii) $F \cdot \overline{M} = 0$,

and the properties (i) and (ii) determine the short exact sequence (6) uniquely, i.e. if $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is a short exact sequence of \mathbb{I}_1 -modules such that $FM_1 = M_1$ and $FM_2 = 0$ then $M_1 \cong FM$ and $M_2 \simeq \overline{M}$.

Notice that

$$FM \simeq K[x]^I, \quad (7)$$

i.e. the \mathbb{I}_1 -module FM is isomorphic to the direct sum of I copies of the simple weight \mathbb{I}_1 -module $K[x]$. Clearly, \overline{M} is a B_1 -module.

The indecomposable \mathbb{I}_1 -modules $M(n, \lambda)$. For $\lambda \in K$ and a natural number $n \geq 1$, consider the B_1 -module

$$M(n, \lambda) := B_1 \otimes_{K[H]} K[H]/(H - \lambda)^n. \quad (8)$$

Clearly,

$$M(n, \lambda) \simeq B_1/B_1(H - \lambda)^n \simeq \mathbb{I}_1/(F + \mathbb{I}_1(H - \lambda)^n). \quad (9)$$

The \mathbb{I}_1 -module/ B_1 -module $M(n, \lambda)$ is a generalized weight module with $\text{Supp}M(n, \lambda) = \lambda + \mathbb{Z}$,

$$M(n, \lambda) = \bigoplus_{i \in \mathbb{Z}} M(n, \lambda)^{\lambda+i} \quad \text{and} \quad \dim M(n, \lambda)^{\lambda+i} = n \quad \text{for all } i \in \mathbb{Z}. \quad (10)$$

For an algebra A , we denote by $A - \text{Mod}$ its module category. The next proposition describes the set of indecomposable, generalized weight \mathbb{I}_1 -modules of finite length M with $FM = 0$.

Proposition 2.1. (1) $M(n, \lambda)$ is an indecomposable, generalized weight \mathbb{I}_1 -module of finite length n .

(2) $M(n, \lambda) \cong M(m, \mu)$ if and only if $n = m$ and $\lambda - \mu \in \mathbb{Z}$.

(3) Let M be a generalized weight B_1 -module of length n (i.e. let M be a generalized weight \mathbb{I}_1 -module such that $FM = 0$, by (6)). Then M is indecomposable if and only if $M \simeq M(n, \lambda)$ for some $\lambda \in K$.

Proof. 1. Since $(B_1)_{K[H]} = \bigoplus_{i \in \mathbb{Z}} \partial^i K[H]$ is a free right $K[H]$ -module, the functor

$$B_1 \otimes_{K[H]} - : K[H] - \text{Mod} \rightarrow B_1 - \text{Mod}, \quad N \mapsto B_1 \otimes_{K[H]} N,$$

is an exact functor. The $K[H]$ -module $K[H]/(H - \lambda)^n$ is an indecomposable, hence the B_1 -module $M(n, \lambda)$ is indecomposable and generalized weight of length n .

2. (\Rightarrow) Suppose that \mathbb{I}_1 -modules $M(n, \lambda)$ and $M(m, \mu)$ are isomorphic. Then $\text{Supp}(M(n, \lambda)) = \text{Supp}(M(m, \mu))$, i.e. $\lambda + \mathbb{Z} = \mu + \mathbb{Z}$, i.e. $\lambda - \mu \in \mathbb{Z}$. Then $n = m$, by (10).

(\Leftarrow) Suppose that $k := \lambda - \mu \in \mathbb{Z}$ and $n = m$. We may assume that $k \geq 1$. Using the equality $(H - \lambda)^n \partial^k = \partial^k (H - \lambda - k)^n = \partial^k (H - \mu)^n$, we see that the B_1 -homomorphism

$$M(n, \lambda) = B_1/B_1(H - \lambda)^n \rightarrow M(n, \mu) = B_1/B_1(H - \mu)^n, \quad 1 + B_1(H - \lambda)^n \mapsto \partial^k + B_1(H - \mu)^n,$$

is an isomorphism with the inverse given by the rule $1 + B_1(H - \mu)^n \mapsto \partial^{-k} + B_1(H - \lambda)^n$.

3. (\Leftarrow) This implication follows from statement 2.

(\Rightarrow) Each indecomposable, generalized weight B_1 -module M is of the type $B_1 \otimes_{K[H]} N$ for an indecomposable $K[H]$ -module N of length n . Notice that $N \simeq K[H]/(H - \lambda)^n$ for some $\lambda \in K$. Therefore, $M \simeq M(n, \lambda)$. \square

Lemma 2.2. *Let M be an indecomposable, generalized weight \mathbb{I}_1 -module. Then $\text{Supp}(M) \subseteq \lambda + \mathbb{Z}$ for some $\lambda \in K$.*

Proof. Let $M = \bigoplus_{\mu \in \text{Supp}(M)} M^\mu$ be a generalized weight \mathbb{I}_1 -module. Then

$$M = \bigoplus_{\mu + \mathbb{Z} \in \text{Supp}(M)/\mathbb{Z}} M_{\mu + \mathbb{Z}}$$

is a direct sum of \mathbb{I}_1 -submodules $M_{\mu + \mathbb{Z}} := \bigoplus_{i \in \mathbb{Z}} M^{\mu + i}$ where $\text{Supp}(M)/\mathbb{Z}$ is the image of the support $\text{Supp}(M)$ under the abelian group epimorphism $K \rightarrow K/\mathbb{Z}$, $\gamma \mapsto \gamma + \mathbb{Z}$. The \mathbb{I}_1 -module M is indecomposable, hence $M = M_{\lambda + \mathbb{Z}}$ for some $\lambda \in K$, i.e. $\text{Supp}(M) \subseteq \lambda + \mathbb{Z}$. \square

The next lemma describes the set of indecomposable, generalized weight \mathbb{I}_1 -modules M with $FM = M$.

Lemma 2.3. *Let M be an indecomposable, generalized weight \mathbb{I}_1 -modules M . Then the following statements are equivalent.*

- (1) $FM = M$.
- (2) $M \simeq K[x]$.
- (3) $\text{Supp}(M) \subseteq \mathbb{N}$.

Proof. (1) \Rightarrow (2) : If $FM = M$ then $M \simeq K[x]^{(I)}$ for some set I necessarily with $|I| = 1$ since M is indecomposable, i.e. $M \simeq K[x]$.

(2) \Rightarrow (3) : $\text{Supp}(K[x]) = \{1, 2, \dots\} \subseteq \mathbb{N}$.

(3) \Rightarrow (1) : Suppose that $\text{Supp}(M) \subseteq \mathbb{N}$. Using the short exact sequence of \mathbb{I}_1 -modules $0 \rightarrow FM \rightarrow M \rightarrow \overline{M} := M/FM \rightarrow 0$ we see that $\text{Supp}(M) = \text{Supp}(FM) \cup \text{Supp}(\overline{M})$. Since $\text{Supp}(FM) = \text{Supp}(K[x]^{(I)}) = \{1, 2, \dots\}$ and $\text{Supp}(\overline{M})$ is an abelian group, we must have $\overline{M} = 0$ (since $\text{Supp}(M) \subseteq \mathbb{N}$), i.e. $M = FM$. \square

The following result is a key step in obtaining a classification of indecomposable, generalized weight \mathbb{I}_1 -modules of finite length.

Theorem 2.4. *Let M be a generalized weight \mathbb{I}_1 -module of finite length. Then the short exact sequence (6) splits.*

Proof. We can assume that $FM \neq 0$ and $\overline{M} \neq 0$. It is obvious that $FM \simeq K[x]^s$ for some $s \geq 1$ and the B_1 -module $\overline{M} \simeq \bigoplus_{i=1}^t M(n_i, \lambda_i)$ for some $n_i \geq 1$, $\lambda_i \in K$ and $t \geq 1$. It suffices to show that

$$\text{Ext}_{\mathbb{I}_1}^1(M(n, \lambda), K[x]) = 0 \tag{11}$$

for all $n \geq 1$ and $\lambda \in K$. If $\lambda \in \mathbb{Z}$ we can assume that $\lambda = 0$, by Proposition 2.1.(2).

(i) $F(H - \lambda)^n = F$: The equality follows from the equalities $e_{ij}(H - \lambda)^n = e_{ij}(j + 1 - \lambda)$ and the choice of λ .

(ii) $M(n, \lambda) = \mathbb{I}_1/\mathbb{I}_1(H - \lambda)^n$: By (i), $\mathbb{I}_1(H - \lambda)^n \supseteq F(H - \lambda)^n = F$. Hence,

$$M(n, \lambda) = \mathbb{I}_1/(F + \mathbb{I}_1(H - \lambda)^n) = \mathbb{I}_1/\mathbb{I}_1(H - \lambda)^n.$$

(iii) *The equality (11) holds:* Let $M = M(n, \lambda)$. By (ii), the short exact sequence of \mathbb{I}_1 -modules

$$0 \rightarrow \mathbb{I}_1(H - \lambda)^n \rightarrow \mathbb{I}_1 \rightarrow M \rightarrow 0 \tag{12}$$

is a projective resolution of the \mathbb{I}_1 -module M since the map

$$\cdot(H - \lambda)^n : \mathbb{I}_1 \rightarrow \mathbb{I}_1(H - \lambda)^n, \quad a \mapsto a(H - \lambda)^n,$$

is an isomorphism of \mathbb{I}_1 -modules, by the choice of λ . Then

$$\text{Ext}_{\mathbb{I}_1}^1(M, K[x]) \simeq Z^1/B^1$$

where $Z^1 = \text{Hom}_{\mathbb{I}_1}(\mathbb{I}_1(H - \lambda)^n, K[x]) \simeq K[x]$ and $B^1 \simeq (H - \lambda)^n K[x] = K[x]$, by the choice of λ . Hence, the equality (11) holds. The proof of the theorem is complete. \square

The next theorem is a classification of the set of indecomposable, generalized weight \mathbb{I}_1 -modules of finite length.

Theorem 2.5. *Each indecomposable, generalized weight \mathbb{I}_1 -module of finite length is isomorphic to one of the modules below:*

- (1) $K[x]$,
- (2) $M(n, \lambda)$ where $n \geq 1$ and $\lambda \in \Lambda$ where Λ is any fixed subset of K such that the map $\Lambda \rightarrow (K/\mathbb{Z}), \lambda \mapsto \lambda + \mathbb{Z}$, is a bijection.

The \mathbb{I}_1 -modules above are pairwise non-isomorphic, indecomposable, generalized weight and of finite length.

Proof. The theorem follows from Theorem 2.4, Proposition 2.1 and Lemma 2.3. \square

Corollary 2.6. *Every indecomposable, generalized weight \mathbb{I}_1 -module is an uniserial module.*

Proof. The statement follows from Theorem 2.5. \square

Homomorphisms and Ext-groups between indecomposables.

Proposition 2.7. (1) *Let M and N be generalized weight \mathbb{I}_1 -modules such that $\text{Supp}(M) \cap \text{Supp}(N) = \emptyset$. Then $\text{Hom}_{\mathbb{I}_1}(M, N) = 0$.*

- (2) $\text{Hom}_{\mathbb{I}_1}(M(n, \lambda), K[x]) = 0$.
- (3) $\text{Hom}_{\mathbb{I}_1}(K[x], M(n, \lambda)) = 0$.
- (4) $\text{Hom}_{\mathbb{I}_1}(M(n, \lambda), M(m, \lambda)) \simeq \text{Hom}_{K[H]}(K[H]/((H - \lambda)^n), (K[H]/((H - \lambda)^m))) \simeq K[H]/((H - \lambda)^{\min(n, m)})$.

Proof. 1. Statement 1 is obvious.

2. Statement 2 follows from the fact that $FM(n, \lambda) = 0$ and $Fp = K[x]$ for all nonzero elements $p \in K[x]$ (since $K[x]$ is a simple \mathbb{I}_1 -module, F is an ideal of the algebra \mathbb{I}_1 such that $FK[x] = K[x]$).

3. Statement 3 follows from the fact that $FK[x] = K[x]$ and $FM(n, \lambda) = 0$: $f(K[x]) = f(FK[x]) = Ff(K[x]) = 0$ for any $f \in \text{Hom}_{\mathbb{I}_1}(K[x], M(n, \lambda))$.

4. The first isomorphism is obvious. Then the second isomorphism follows. \square

Proposition 2.8. (1) $\text{Ext}_{\mathbb{I}_1}^1(K[x], K[x]) = 0$.

(2) $\text{Ext}_{\mathbb{I}_1}^1(M(n, \lambda), K[x]) = 0$.

(3) $\text{Ext}_{\mathbb{I}_1}^1(K[x], M(n, \lambda)) = 0$.

(4) $\text{Ext}_{\mathbb{I}_1}^1(M(n, \lambda), M(m, \mu)) = \begin{cases} K & \text{if } \lambda - \mu \in \mathbb{Z}, \\ 0 & \text{if } \lambda - \mu \notin \mathbb{Z}. \end{cases}$

Proof. 1. Let $0 \rightarrow K[x] \rightarrow N \rightarrow K[x] \rightarrow 0$ be a s.e.s. of \mathbb{I}_1 -modules. Then $FN = N$ (since $FK[x] = K[x]$), and so N is an epimorphic image of the semisimple \mathbb{I}_1 -module $F \oplus F$. Hence, $N \simeq K[x] \oplus K[x]$ (since ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$).

2. See (11).

3. Let $0 \rightarrow M = M(n, \lambda) \rightarrow L \rightarrow K[x] \rightarrow 0$ be a s.e.s. of \mathbb{I}_1 -modules. Since $FM = 0$, we have $FL = FK[x] \simeq K[x]$ is a submodule of L such that $FL \cap M = 0$ (since otherwise $FL \subseteq M$ by simplicity of the \mathbb{I}_1 -module $FL \simeq K[x]$, and so $0 \neq K[x] \simeq FL = F^2L \subseteq FM = 0$, a contradiction). Then $FL \oplus M \subseteq L$. Furthermore, $FL \oplus M = L$ since $l_{\mathbb{I}_1}(FL \oplus M) = l_{\mathbb{I}_1}(L)$. This means that the s.e.s. splits.

4. Let $0 \rightarrow M_1 \rightarrow M \rightarrow M_1 \rightarrow 0$ be a s.e.s. of generalized weight \mathbb{I}_1 -modules. If $\text{Supp}(M_1) \cap \text{Supp}(M_2) = \emptyset$, it splits. In particular, $\text{Ext}_{\mathbb{I}_1}^1(M(n, \lambda), M(m, \mu)) = 0$ if $\lambda - \mu \notin \mathbb{Z}$. If $\lambda - \mu \in \mathbb{Z}$ we can assume that $\lambda = \mu$ (since $M(m, \lambda) \simeq M(m, \mu)$). Using (12), where we assume that $\lambda = 0$ if $\lambda \in \mathbb{Z}$, we see that $\text{Ext}_{\mathbb{I}_1}^1(M(n, \lambda), M(m, \lambda)) \simeq M(m, \lambda)/(H - \lambda)M(m, \lambda) \simeq K$. \square

Since the left global dimension of the algebra \mathbb{I}_1 is 1, [6], Proposition 2.7 and Proposition 2.8 describe all the Ext-groups between indecomposable, generalized weight \mathbb{I}_1 -modules. This is also obvious from the proofs of the propositions.

REFERENCES

- [1] V. V. Bavula and V. Bekkert, Indecomposable representations of generalized Weyl algebras, *Comm. Algebra*, **28** (2000), no. 11, 5067-5100.
- [2] V. V. Bavula, The algebra of integro-differential operators on a polynomial algebra, *J. Lond. Math. Soc. (2)*, **83** (2011), no. 2, 517-543.
- [3] V. V. Bavula, The group of automorphisms of the algebra of polynomial integro-differential operators, *J. Algebra*, **348** (2011), 233-263.
- [4] V. V. Bavula, The algebra of integro-differential operators on an affine line and its modules, *J. Pure Appl. Algebra* **217** (2013), no. 3, 495-529.
- [5] V. V. Bavula, The algebra of polynomial integro-differential operators is a holonomic bimodule over the subalgebra of polynomial differential operators. *Algebr. Represent. Theory* **17** (2014), no. 1, 275-288.
- [6] V. V. Bavula, The global dimension of the algebra of integro-differential operators, submitted.

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SHEFFIELD, HICKS BUILDING, SHEFFIELD S3 7RH, UK
E-mail address: v.bavula@sheffield.ac.uk

DEPARTAMENTO DE MATEMÁTICA, ICEx, UNIVERSIDADE FEDERAL DE MINAS GERAIS, AV. ANTÔNIO CARLOS, 6627, CP 702, CEP 30123-970, BELO HORIZONTE-MG, BRASIL
E-mail address: bekkert@mat.ufmg.br

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, CAIXA POSTAL 66281, SÃO PAULO, CEP 05315-970, BRASIL
E-mail address: futorny@ime.usp.br