Finite generation of division subalgebras and of the group of eigenvalues for commuting derivations or automorphisms of division algebras

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Abstract

Let D be a division algebra such that $D \otimes D^o$ is a Noetherian algebra, then any division subalgebra of D is a *finitely generated* division algebra. Let Δ be a finite set of commuting derivations or automorphisms of the division algebra D, then the group $\mathrm{Ev}(\Delta)$ of common eigenvalues (i.e. *weights*) is a *finitely generated abelian* group. Typical examples of D are the quotient division algebra $\mathrm{Frac}(\mathcal{D}(X))$ of the ring of differential operators $\mathcal{D}(X)$ on a smooth irreducible affine variety X over a field K of characteristic zero, and the quotient division algebra $\mathrm{Frac}(U(\mathfrak{g}))$ of the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. It is proved that the algebra of differential operators $\mathcal{D}(X)$ is isomorphic to its opposite algebra $\mathcal{D}(X)^o$.

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1 Introduction

Throughout this paper, K is a field, $\otimes = \otimes_K$. Noetherian means left and right Noetherian. For a K-algebra A, A^o denotes the *opposite* algebra to A (recall that $A^o = A$ as abelian groups but the multiplication in A^o is given by the rule: $a * b = ba$), and $A^e := A \otimes A^o$ is called the *enveloping* algebra of A. The expressions ${}_AM$, M_A, and ${}_AM_A$ means that M is respectively a left, right A-module, and an A-bimodule. *Finitely generated division algebra* means a division algebra which is generated (as a division algebra) by a finite set of elements (i.e. $x_1, \ldots, x_n$ is a set of generators for a division K-algebra D if D is the only division K-subalgebra of D that contains $x_1, \ldots, x_n$).

For division algebras *finite dimensional* over K there is a well-developed theory where (commutative) subfields play a fundamental role. By contrast, if a division algebra is infinite dimensional little is known about its division subalgebras.

Question. Suppose that D is a finitely generated division K-algebra, is any division K-subalgebra of D finitely generated?

Certainly this is the case when D is a field. We will see that the answer is *affirmative* for many popular division algebras. For a similar question about *subfields* (= commutative division K-subalgebras), Resco, Small and Wadsworth give an affirmative answer in [5]: *Let D be a division algebra over a field K such that $D \otimes D^o$ is Noetherian, then every (commutative) subfield of D containing K is finitely generated.* One of the crucial steps in their proof is the following result of Vamos [7]: *Let L be a field extension of K. Then $L \otimes L$ is Noetherian iff L is a finitely generated over K.* M. Smith [6] showed that there is a division algebra D with centre K, containing two maximal subfields whose transcendence degrees are any two prescribed cardinal numbers.

Let $\Delta = \{\delta_1, \dots, \delta_t\}$ be a set of *commuting* K-derivations of a division K-algebra D. The set $\mathrm{Ev}(\Delta) := \{\lambda = (\lambda_1, \dots, \lambda_t) \in K^t \,|\, \delta_i(u) = \lambda_i u, i = 1, \dots, t \text{ for some } 0 \neq u \in D\}$ of common eigenvalues is an *additive* subgroup of K^t, and the Δ-*eigen-algebra* $D(\Delta) := \bigoplus_{\lambda \in \mathrm{Ev}(\Delta)} D_\lambda$ is a $\mathrm{Ev}(\Delta)$-graded algebra where $D_\lambda := \{u \in D \,|\, \delta_i(u) = \lambda_i u, i = 1, \dots, t\}$, $D_\lambda D_\mu \subseteq D_{\lambda+\mu}$ for all $\lambda, \mu \in \mathrm{Ev}(\Delta)$, and $0 \neq u \in D_\lambda$ implies $u^{-1} \in D_{-\lambda}$.

Let $\Delta = \{\delta_1, \dots, \delta_t\}$ be a set of *commuting* K-automorphisms of a division K-algebra D, and let $K^* := K \backslash \{0\}$ be the multiplicative group of the field K. The set $\mathrm{Ev}(\Delta) := \{\lambda = (\lambda_1, \dots, \lambda_t) \in K^{*t} \,|\, \delta_i(u) = \lambda_i u, i = 1, \dots, t \text{ for some } 0 \neq u \in D\}$ of common eigenvalues is an *multiplicative* subgroup of K^{*t}, and the Δ-*eigen-algebra* $D(\Delta) := \bigoplus_{\lambda \in \mathrm{Ev}(\Delta)} D_\lambda$ is a $\mathrm{Ev}(\Delta)$-graded algebra where $D_\lambda := \{u \in D \,|\, \delta_i(u) = \lambda_i u, i = 1, \dots, t\}$, $D_\lambda D_\mu \subseteq D_{\lambda\mu}$ for all $\lambda, \mu \in \mathrm{Ev}(\Delta)$, and $0 \neq u \in D_\lambda$ implies $u^{-1} \in D_{\lambda^{-1}}$.

The first statement of the next result is an extension of the mentioned above result of Resco-Small-Wadsworth to division subalgebras (with a short *different* proof given in Section 2).

Theorem 1.1 *Let D be a division K-algebra such that $D \otimes D$ is a Noetherian D-bimodule, and let $\Delta = \{\delta_1, \dots, \delta_t\}$ be either a set of commuting K-derivations or commuting K-automorphisms of the division K-algebra D. Then*

1. *D satisfies the ascending chain condition on division K-subalgebras, or equivalently, every division K-subalgebra of D is a finitely generated division K-algebra.*

2. *The group of eigenvalues* $\mathrm{Ev}(\Delta)$ *is a finitely generated abelian group, and so* $\mathrm{Ev}(\Delta) = \mathcal{T} \oplus \mathbb{Z}^r$ *where r is the rank of the group* $\mathrm{Ev}(\Delta)$ *and $\mathcal{T}$ is a finite abelian group.*

3. *The eigen-algebra $D(\Delta)$ is a Noetherian domain which isomorphic to an iterated skew Laurent extension. In more detail, $D_\mathcal{T} := \oplus_{\lambda \in \mathcal{T}} D_\lambda$ is a division algebra of right and left dimension $|\mathcal{T}|$ over the division algebra D_0, $D(\Delta)$ is isomorphic to the iterated skew Laurent extension $D_\mathcal{T}[x_1, x_1^{-1}; \sigma_1] \cdots [x_r, x_r^{-1}; \sigma_r]$ with coefficients from the division algebra $D_\mathcal{T}$.*

4. *For each subgroup F of* $\mathrm{Ev}(\Delta)$, *$\mathcal{F}(F) := \oplus_{\lambda \in F} D_\lambda$ is a Noetherian domain the quotient division algebra* $\mathrm{Frac}(\mathcal{F}(F))$ *of which is Δ-invariant and* $\mathrm{Ev}(\mathrm{Frac}(\mathcal{F}(F))) = F$, *any Δ-eigenvector $v \in$* $\mathrm{Frac}(\mathcal{F}(F))_\lambda$ *has the form $u^{-1}w$ for some $0 \neq u \in D_\mu$, $w \in D_{\lambda+\mu}$, and $\lambda, \mu \in F$.*

Remark 1. $D \otimes D$ is a *Noetherian* D-bimodule iff the algebra $D \otimes D^o$ is *Noetherian* iff the algebra $D \otimes D^o$ is *left* Noetherian iff the algebra $D \otimes D^o$ is *right* Noetherian as it follows from

$$_D D \otimes D_D \simeq {_D D} \otimes ({_{D^o}} D^o) \simeq {_{D\otimes D^o}} D \otimes D^o, \;\; {_D D} \otimes D_D \simeq D^o_{D^o} \otimes D_D \simeq D_D \otimes D^o_{D^o} \simeq (D \otimes D^o)_{D \otimes D^o}. \quad (1)$$

Remark 2. 'Finite generation' is built in in the structure of the eigen-algebra $D(\Delta)$ in the sense that it is a finitely generated algebra over a finitely generated division algebra.

In Section 3, it is proved that many division algebras that appear naturally in applications satisfy the conditions of Theorem 1.1 (Proposition 3.5, Lemma 3.1), *eg* $\mathrm{Frac}(\mathcal{D}(X))$ (Corollary 3.2) and $\mathrm{Frac}(U(\mathfrak{g}))$ (Corollary 3.4).

2 Proof of Theorem 1.1

Recall that any torsion free finitely generated abelian group is a free abelian group of finite rank, and vice versa. Any finitely generated abelian group G is isomorphic to $\mathcal{T} \oplus \mathbb{Z}^r$ where $r := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G)$ is the *rank* of the group G, $\mathcal{T}$ is the *torsion* subgroup of G, that is the subgroup of G that contains all the elements of finite order, it is a finite group.

Proof of Theorem 1.1. 1. Suppose that inside D one can pick a strictly ascending chain of division K-subalgebras $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_n \subset \cdots$, we seek a contradiction; this gives a strictly ascending chain of D-sub-bimodules, $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$, where $K_n = \ker(\phi_n)$ where $\phi_n : D \otimes D \to D \otimes_{\Gamma_n} D$, $x \otimes y \to x \otimes_{\Gamma_n} y$, (use the fact that D is a free left and right Γ_n-module and tensor product commutes with direct sum), a contradiction. Hence D satisfies the *acc* on division K-subalgebras.

2, 3, and 4. The proof of two cases are very similar, so we will treat them simultaneously by making some adjustments to our notation. So, let $\Delta = \{\delta_1, \ldots, \delta_t\}$ be either a set of commuting K-derivations or a set of commuting K-automorphisms of the division algebra D. In the first case, $\mathrm{Ev}(\Delta)$ is an *additive* subgroup of K^t, in the second case, $\mathrm{Ev}(\Delta)$ is a *multiplicative* subgroup of K^{*t}. In the second case, we still will write the group operation *additively*, i.e. $\lambda + \mu$ means $\lambda\mu$, $-\lambda$ means λ^{-1}, 0 means 1. Let D_0 be the set of Δ-*constants*: $D_0 := \cap_{i=1}^t \ker_D(\delta_i)$, in the case of derivations; and $D_0 := \{d \in D \mid \delta_i(d) = d, i = 1, \ldots, t\}$, in the case of automorphisms. In both cases, D_0 is a division subalgebra of D.

Given a division algebra Γ, a group G, a group homomorphism $\varphi : G \to \mathrm{Aut}_K(\Gamma)$, and a '2-cocycle' $G \times G \to \Gamma^* := \Gamma\backslash\{0\}$, $(g, h) \mapsto (g, h)$. A *generalized crossed product* is an algebra $\Gamma * G = \oplus_{g \in G} \Gamma g$ which is a free left Γ-module with multiplication given by the rule

$$ag \cdot bh = a\varphi(g)(b)(g, h)gh, \;\; a, b \in \Gamma, \;\; g, h \in G.$$

It follows from $ag = g\varphi(g)^{-1}(a)$ that $\Gamma g = g\Gamma \simeq \Gamma_\Gamma$, and so $\Gamma * G = \oplus_{g \in G} g\Gamma$ is a free right Γ-module. A '2-cocycle' means that the multiplication of the generalized crossed product is associative. When $G = \mathbb{Z}$ and $(i, j) = 1$ for all $i, j \in \mathbb{Z}$, we have, so-called, a *skew Laurent extension* with coefficients from Γ denoted $\Gamma[x, x^{-1}; \sigma]$ where x is the group generator 1 for $\mathbb{Z}$ and $\sigma = \varphi(1) \in \mathrm{Aut}_K(\Gamma)$. So, the skew Laurent extension generated by Γ and x, x^{-1} subject to the defining relations $x^{\pm 1}a = \sigma^{\pm 1}(a)x^{\pm 1}$ for all $a \in \Gamma$. An *iterated skew Laurent extension* $A_n := \Gamma[x_1, x_1; \sigma_1] \cdots [x_n, x_n^{-1}; \sigma_n]$ is defined inductively as $A_n = A_{n-1}[x_n, x_n^{-1}; \sigma_n]$. Since the division algebra Γ is a Noetherian algebra then the iterated skew Laurent extension A_n is a Noetherian algebra (1.17, [3]).

For each $\lambda \in \mathrm{Ev}(\Delta)$, fix $0 \neq u_\lambda \in D_\lambda$. Then it is easy to see that the Δ-*eigen-algebra* is a free (left and right) D_0-module:

$$\mathcal{D} = D(\Delta) := \bigoplus_{\lambda \in \mathrm{Ev}(\Delta)} D_0 u_\lambda = \bigoplus_{\lambda \in \mathrm{Ev}(\Delta)} u_\lambda D_0, \;\; u_\lambda u_\mu = (\lambda, \mu) u_{\lambda+\mu}, \; (\lambda, \mu) := u_\lambda u_\mu u_{\lambda+\mu}^{-1} \in D_0, \tag{2}$$

for $a, b \in D_0$: $au_\lambda bu_\mu = a(u_\lambda bu_\lambda^{-1})u_\lambda u_\mu = a(u_\lambda au_\lambda^{-1})(u_\lambda u_\mu u_{\lambda+\mu}^{-1})u_{\lambda+\mu}$ (In general, this is not a generalized crossed product but if $\mathrm{Ev}(\Delta) \simeq \mathbb{Z}^r$ one can choose generators in such a way that it is). Given any finitely generated subgroup $F = \mathcal{T} \oplus (\oplus_{i=1}^s \mathbb{Z}v_i)$ of $\mathrm{Ev}(\Delta)$ where $\mathcal{T}$ is the torsion part of F. The algebra $D_\mathcal{T} := \oplus_{\lambda \in \mathcal{T}} D_0 u_\lambda = \oplus_{\lambda \in \mathcal{T}} u_\lambda D_0$ has left and right dimension $|\mathcal{T}| < \infty$ over the division algebra D_0 where $|\mathcal{T}|$ is the order of the group $\mathcal{T}$. The map

$$l : D_\mathcal{T} \to \mathrm{End}_{D_\mathcal{T}}(D_\mathcal{T}), \; a \mapsto (l_a : x \mapsto ax),$$

is an algebra isomorphism where $\mathrm{End}_{D_\mathcal{T}}(D_\mathcal{T})$ is the endomorphism algebra of the *right* $D_\mathcal{T}$-module $D_\mathcal{T}$. For each nonzero element $a \in D_\mathcal{T}$, l_a is a monomorphism, hence it is an isomorphism since (the right dimension over D_0) $\mathrm{r.dim}_{D_0}(D_\mathcal{T}) = \mathrm{r.dim}_{D_0}(aD_\mathcal{T}) = |\mathcal{T}| < \infty$. Therefore, $\mathrm{End}_{D_\mathcal{T}}(D_\mathcal{T})$ is a division algebra, hence so is its isomorphic copy $D_\mathcal{T}$. Let $F' = \oplus_{i=1}^s \mathbb{Z}v_i$, and so $F = \mathcal{T} \oplus F'$. The subalgebra $\mathcal{F}' = \mathcal{F}(F') := \oplus_{\lambda \in F'} D_0 u_\lambda$ of $D(\Delta)$ is isomorphic to the iterated skew Laurent extension $\mathcal{L} := D_0[x_1, x_1^{-1}; \sigma_1] \cdots [x_s, x_s^{-1}; \sigma_s]$ where $\sigma_i(d) = u_{v_i} d u_{v_i}^{-1}$ ($d \in D_0$) and $\sigma_i(x_j) = \lambda_{ij} x_j$, $j < i$, where $\lambda_{ij} := u_{v_i} u_{v_j} u_{v_i}^{-1} u_{v_j}^{-1} \in D_0$ (via the K-algebra epimorphism $\mathcal{L} \to \mathcal{F}'$, $d \mapsto d$, $x_i \mapsto u_{v_i}$, where $d \in D_0$). This follows easily from a definition of an iterated skew Laurent extension and the facts that D_0 is a division algebra, $u_{v_1}^{n_1} \cdots u_{v_s}^{n_s} \in D_{n_1 v_1 + \cdots + n_s v_s}$, and $F' = \oplus_{i=1}^s \mathbb{Z}v_i$. Then, by a similar reasoning (since $D_\mathcal{T}$ is a division algebra), the algebra $\mathcal{F}$ is isomorphic to the iterated skew Laurent extension $D_\mathcal{T}[x_1, x_1^{-1}; \sigma_1] \cdots [x_s, x_s^{-1}; \sigma_s]$ where $\sigma_i(d) = u_{v_i} d u_{v_i}^{-1}$ ($d \in D_\mathcal{T}$) and $\sigma_i(x_j) = \lambda_{ij} x_j$, $j < i$, λ_{ij} are as above.

Since $D_\mathcal{T}$ is a Noetherian algebra so is the algebra $\mathcal{F}$. So, $\mathcal{F}$ is a Noetherian domain, let $\mathrm{Frac}(\mathcal{F})$ be its quotient division algebra, so any element of $\mathrm{Frac}(\mathcal{F})$ is a fraction $a^{-1}b$ for some $0 \neq a, b \in \mathcal{F}$. Note that the elements a and b are finite sums $\sum a_\lambda$ and $\sum b_\lambda$ of eigenvectors $a_\lambda, b_\lambda \in D_\lambda$, $\lambda \in F$. If $0 \neq c = a^{-1}b \in D_\mu$ for some $\mu \in \mathrm{Ev}(\Delta)$, then $ac = b \neq 0$ implies that $a_\lambda c = b_\nu$ for some $a_\lambda \neq 0$ and $b_\nu \neq 0$ such that $\lambda + \mu = \nu$, and so $c = a_\lambda^{-1} b_\nu$ and $\mu = \nu - \lambda$. This proves that any Δ-eigenvector of $\mathrm{Frac}(\mathcal{F})$ is a fraction of the eigenvectors of $\mathcal{F}$ and that

$$\mathrm{Ev}(\Delta, \mathrm{Frac}(\mathcal{F}(F))) = F. \tag{3}$$

It follows immediately from this fact and statement 1 that $\mathrm{Ev}(\Delta)$ is finitely generated: otherwise one can find in $\mathrm{Ev}(\Delta)$ a strictly ascending chain of subgroups: $F_1 \subset F_2 \subset \cdots$, which gives, by (3), the strictly ascending chain of division subalgebras: $\mathrm{Frac}(\mathcal{F}(F_1)) \subset \mathrm{Frac}(\mathcal{F}(F_2)) \cdots$, a contradiction. This finishes the proof of statement 2 and 4. Then statement 3 has, in fact, been proved above. This finishes the proof of Theorem 1.1. $\square$

For an abelian monoid E, the set $\mathrm{tor}(E)$ of all the elements $e \in E$ such that $ne = 0$ is a group, so-called, the *torsion subgroup* of E.

Corollary 2.1 *Let a K-algebra A be a Noetherian domain with $D := \mathrm{Frac}(A)$ such that $D \otimes D$ is a Noetherian D-bimodule, $\Delta = \{\delta_1, \ldots, \delta_t\}$ be either a set of commuting K-derivations or commuting K-automorphisms of the algebra A. Then the abelian monoid of eigenvalues $\mathrm{Ev}(\Delta, A)$ for Δ in A is a submonoid of a finitely generated abelian group, and so the rank of $\mathrm{Ev}(\Delta, A)$ is finite, and the torsion subgroup $\mathrm{tor}(\mathrm{Ev}(\Delta, A))$ is a finite group.*

Proof. Note that each derivation (resp. an automorphism) δ_i of A can be uniquely extended to a derivation (resp. an automorphism) of the division algebra D by the rule $\delta_i(s^{-1}a) = s^{-1}a - s^{-1}\delta_i(s)s^{-1}a$ (resp. $\delta_i(s^{-1}a) = \delta_i(s)^{-1}\delta_i(a)$). So, the zero derivation $[\delta_i, \delta_j] = 0$ of A has zero extension to D, and by uniqueness it must be zero on D. Similarly, the identity automorphism $[\delta_i, \delta_j] = \delta_i\delta_j\delta_i^{-1}\delta_j^{-1}$ of A has the obvious extension to D, and by uniqueness it must be the identity map on D. So, Δ is a set of commuting derivations (resp. automorphisms) of D. Clearly, $\mathrm{Ev}(\Delta, A) \subseteq \mathrm{Ev}(\Delta, D)$, and the result follows from Theorem 1.1.(2). $\square$

Corollary 2.2 *Let a K-algebra A be a commutative affine domain with $D := \mathrm{Frac}(A)$, $\Delta = \{\delta_1, \ldots, \delta_t\}$ be either a set of commuting K-derivations or commuting K-automorphisms of the algebra A. Then the abelian monoid of eigenvalues $\mathrm{Ev}(\Delta, A)$ for Δ in A is a submonoid of a finitely generated abelian group, and so the rank of $\mathrm{Ev}(\Delta, A)$ is finite, and the torsion subgroup $\mathrm{tor}(\mathrm{Ev}(\Delta, A))$ is a finite group.*

In general, the eigen-algebra $D(\Delta)$ is not a finitely generated algebra even in the case of a commutative affine domain A since, in general, the Δ-constants D_0 is not a finitely generated algebra (Hilbert 14'th problem, etc).

3 Applications

Let Γ be a K-algebra, σ be a K-automorphism of Γ, and $\delta \in \mathrm{Der}_K(\Gamma)$ be a σ-*derivation* of Γ: $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for $a, b \in \Gamma$. The *Ore* extension $A = \Gamma[x; \sigma, \delta]$ is a K-algebra generated freely by Γ and an element x satisfying the defining relations: $xa = \sigma(a)x + \delta(a)$ for $a \in \Gamma$. Let Γ^o be the opposite algebra with multiplication given by the rule $a * b = ba$. Then $\sigma \in \mathrm{Aut}_K(\Gamma^o)$ as $\sigma(a * b) = \sigma(ba) = \sigma(b)\sigma(a) = \sigma(a) * \sigma(b)$, and so $\sigma^{-1} \in \mathrm{Aut}_K(\Gamma^o)$, and finally $\delta\sigma^{-1} \in \mathrm{Der}_K(\Gamma^o)$ is a σ^{-1}-derivation of the algebra Γ^o:

$$\begin{aligned}\delta\sigma^{-1}(a * b) &= \delta\sigma^{-1}(ba) = \delta(\sigma^{-1}(b)\sigma^{-1}(a)) = \delta\sigma^{-1}(b)\sigma^{-1}(a) + b\delta\sigma^{-1}(a) \\ &= \delta\sigma^{-1}(a) * b + \sigma^{-1}(a) * \delta\sigma^{-1}(b).\end{aligned}$$

$$A^o = \Gamma^o[x; \sigma^{-1}, -\delta\sigma^{-1}]. \quad (4)$$

Proof. The K-algebra A is generated by Γ and x that satisfy the defining relations: $x\sigma^{-1}(a) = ax + \delta\sigma^{-1}(a)$, $a \in \Gamma$, since σ an *automorphism* of Γ. Hence the K-algebra A^o is generated by the Γ^o and x that satisfy the defining relations: $x * a = \sigma^{-1}(a) * x - \delta\sigma^{-1}(a)$, $a \in \Gamma^o$, and we are done. $\square$

The *iterated Ore* $A = \Gamma[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is defined inductively as

$$(\Gamma[x_1; \sigma_1, \delta_1] \cdots [x_{n-1}; \sigma_{n-1}, \delta_{n-1}])[x_n; \sigma_n, \delta_n].$$

By (4) and induction on n,

$$(\Gamma[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n])^o \simeq \Gamma^o[x_1; \sigma_1^{-1}, -\delta_1\sigma_1^{-1}] \cdots [x_n; \sigma_n^{-1}, -\delta_n\sigma_n^{-1}]. \quad (5)$$

The tensor product of two iterated Ore extensions $A = \Gamma[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ and $B = \Delta[y_1; \tau_1, \partial_1] \cdots [y_m; \tau_m, \partial_m]$ is again an iterated Ore extension

$$A \otimes B = \Gamma \otimes \Delta[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n][y_1; \tau_1, \partial_1] \cdots [y_m; \tau_m, \partial_m]$$

where σ_i, δ_i and τ_j, ∂_j act *trivially* on the elements where they have not been defined. Recall that if Γ is a domain (resp. a Noetherian algebra) then so is the iterated Ore extension A. If $\Gamma^e = \Gamma \otimes \Gamma^o$ is a Noetherian algebra then so is the algebra Γ.

Lemma 3.1 *Suppose that $\Gamma^e = \Gamma \otimes \Gamma^o$ is a Noetherian domain (then so is Γ and an iterated Ore extension $A = \Gamma[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$). Let $D = \mathrm{Frac}(A)$. Then $D^e = D \otimes D^o$ is a Noetherian domain, and so the results of Theorem 1.1 hold.*

Proof. $A^e = A \otimes A^o$ is an iterated Ore extension with coefficients from the Noetherian domain Γ^e, hence A^e is a Noetherian domain (1.12, [3]), and so is its localization D^e. $\square$

Corollary 3.2 *Let X be a smooth irreducible affine variety over a field K of characteristic zero, $\mathcal{D}(X)$ be the ring of differential operators on X, and $D(X) = \mathrm{Frac}(\mathcal{D}(X))$ be its quotient division algebra. Then $D(X) \otimes D(X)^o$ is a Noetherian domain, and so the results of Theorem 1.1 hold.*

Proof. The coordinate algebra $\mathcal{O} = \mathcal{O}(X)$ of the variety X is a finitely generated domain with the field of fractions, say $\Gamma = \mathrm{Frac}(\mathcal{O})$. Let $S = \mathcal{O}\backslash\{0\}$. Then, by 15.2.6, [4], $S^{-1}\mathcal{D}(X) \simeq \Gamma[t_1; \frac{\partial}{\partial x_1}] \cdots [t_n; \frac{\partial}{\partial x_n}]$ is an iterated Ore extension (with trivial automorphisms: $\sigma_i = \mathrm{id}_\Gamma$) where $n = \dim(X)$ (the dimension of X), Γ contains a rational function field $Q_n := K(x_1, \ldots, x_n)$ where the $\frac{\partial}{\partial x_i}$ are partial derivations (extended uniquely from Q_n to Γ). Note that $\Gamma \otimes \Gamma^o = \Gamma \otimes \Gamma$ is a Noetherian domain as the localization of the domain $\mathcal{O}(X \times X) \simeq \mathcal{O}(X) \otimes \mathcal{O}(X)$, the variety $X \times X$ is a smooth irreducible affine variety. By Lemma 3.1, $D(X)^e$ is a Noetherian domain and every division K-subalgebra of $D(X)$ is a finitely generated division K-algebra. This proves the first two statements. Then statement 3 follows from Theorem 1.1.(2). $\square$

Lemma 3.3 *Let A be a K-algebra and $A \to A^o$, $a \mapsto a^o$, be the canonical anti-isomorphism ($(\lambda a + \mu b)^o = \lambda a^o + \mu b^o$ and $(ab)^o = b^o * a^o$ for all $\lambda, \mu \in K$ and $a, b \in A$). Then*

1. *$(s^{-1})^o = (s^o)^{-1}$ for each unit $s \in A$.*

2. *If S is a left (resp. right) Ore subset of A then S^o is a right (resp. left) Ore subset of A^o and $(S^{-1}A)^o \simeq A^o(S^o)^{-1}$, $s^{-1}a \mapsto a^o * (s^o)^{-1}$ (resp. $(AS^{-1})^o \simeq (S^o)^{-1}A^o$, $as^{-1} \mapsto (s^o)^{-1} * a^o$) is the K-algebra isomorphism.*

3. *If A is a Noetherian domain then $\mathrm{Frac}(A^o) \simeq \mathrm{Frac}(A)^o$, $s^{-1}a \mapsto a^o * (s^o)^{-1}$, is the isomorphism of division K-algebras.*

4. *If A is a Noetherian domain such that $A \simeq A^o$ then $\mathrm{Frac}(A^o) \simeq \mathrm{Frac}(A)^o$.*

Proof. 1. $ss^{-1} = s^{-1}s = 1$ implies $s^o * (s^{-1})^o = (s^{-1})^o * s^o = 1$, and so $(s^{-1})^o = (s^o)^{-1}$.
2. Straightforward.
3. It is a particular case of statement 2.
4. By the universal property of localization, $A \simeq A^o$ implies $\mathrm{Frac}(A) \simeq \mathrm{Frac}(A^o)$, and by statement 3, $\mathrm{Frac}(A^o) \simeq \mathrm{Frac}(A)^o$. $\square$

Corollary 3.4 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field K, $U = U(\mathfrak{g})$ be its universal enveloping algebra, $D(\mathfrak{g}) = \mathrm{Frac}(U)$ be its quotient division algebra. Then $D(\mathfrak{g})^e \simeq D(\mathfrak{g}) \otimes D(\mathfrak{g})$ is a Noetherian domain, and so the results of Theorem 1.1 hold.*

Proof. $U \simeq U^o$, $g \mapsto -g$, $g \in \mathfrak{g}$. Hence, $\mathrm{Frac}(U) \simeq \mathrm{Frac}(U^o) \simeq \mathrm{Frac}(U)^o$ (Lemma 3.3) and $U^e = U \otimes U^o \simeq U \otimes U \simeq U(\mathfrak{g} \oplus \mathfrak{g})$, and so $D(\mathfrak{g})^e \simeq D(\mathfrak{g}) \otimes D(\mathfrak{g})$ is a Noetherian domain as a localization of $U(\mathfrak{g} \oplus \mathfrak{g})$, the rest follows from Theorem 1.1. $\square$

We have a great similarity in the proofs of the last two statements, one can repeat this pattern for other 'constructions' of algebras and their division algebras. To formalize the proofs in many similar situations let us introduce a concept of a *good construction* of algebras. We say that we have a *construction* of K-algebras, say $\mathcal{A}$, if, for a given K-algebra Γ, one attaches a set (class) of K-algebras $\mathcal{A}(\Gamma)$. Examples in mind are Ore extensions $\mathcal{A}(\Gamma) = \{\Gamma[x; \sigma, \delta]\}$, iterated Ore extensions, iterated skew Laurent polynomial algebras, etc. We say that the construction $\mathcal{A}$ is *good* if the following three properties hold:

($G1$) if Γ is a Noetherian domain then so is each algebra from the set $\mathcal{A}(\Gamma)$,

($G2$) $\mathcal{A}(\Gamma)^o \subseteq \mathcal{A}(\Gamma^o)$, and

($G3$) $\mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma') \subseteq \mathcal{A}(\Gamma \otimes \Gamma')$,

where $\mathcal{A}(\Gamma)^o := \{A^o \,|\, A \in \mathcal{A}(\Gamma)\}$ and similarly $\mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma') := \{A \otimes A' \,|\, A \in \mathcal{A}(\Gamma), A' \in \mathcal{A}(\Gamma')\}$.

For the definitions and properties of the algebras from the examples below the reader is refereed to [3] and [4].

Examples of good constructions. (1) Iterated Ore extensions.

(2) Iterated skew Laurent extensions: $\Gamma[x_1, x_1^{-1}; \sigma_1] \cdots [x_n, x_n^{-1}; \sigma_n]$ where σ_i are K-automorphisms (($G1$) - use the leading term and 1.17, [3]; ($G2$) - Exercise 1P, p. 17, [3]; ($G3$) - obvious).

Proposition 3.5 *Suppose that the enveloping algebra $\Gamma^e = \Gamma \otimes \Gamma^o$ of an algebra Γ is a Noetherian domain, $\mathcal{A}$ is a good construction, then each algebra $A \in \mathcal{A}(\Gamma)$ is a Noetherian domain and so is $D^e = D \otimes D^o$ where $D = \mathrm{Frac}(A)$. Hence, Theorem 1.1 is true for D.*

Proof. Γ^e is a Noetherian domain, then so is Γ, and then each algebra $A \in \mathcal{A}(\Gamma)$ is a Noetherian domain since the construction $\mathcal{A}$ is good. $A^e = A \otimes A^o \in \mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma)^o \subseteq \mathcal{A}(\Gamma) \otimes \mathcal{A}(\Gamma^o) \subseteq \mathcal{A}(\Gamma \otimes \Gamma^o)$, and so A^e is a Noetherian domain, as $\mathcal{A}$ is good. Hence, so is its localization D^e. The rest is obvious. $\square$

So, the algebras that satisfy conditions of Theorem 1.1 are fairly common.

Lemma 3.6 *Let D be a division algebra over a field K such that $D \otimes D$ is a Noetherian D-bimodule. Let Γ be a division K-subalgebra of D. Then*

1. *$\Gamma \otimes \Gamma$ is a Noetherian Γ-bimodule, and*
2. *$\mathrm{Kdim}({}_\Gamma(\Gamma \otimes \Gamma)_\Gamma) \leq \mathrm{Kdim}({}_D(D \otimes D)_D)$.*

Remark. $\mathrm{Kdim}({}_\Gamma M_\Gamma)$ means the Krull dimension of a Γ-bimodule M.

Proof. Suppose that $\Gamma \otimes \Gamma$ is not a Noetherian Γ-bimodule, we seek a contradiction. Then one can find a strictly ascending chain of Γ-sub-bimodules: $I_1 \subset I_2 \subset \cdots$. Note that $D \otimes D \simeq D \otimes_\Gamma (\Gamma \otimes \Gamma) \otimes_\Gamma D$, an isomorphism of D-bimodules. Since D_Γ is free, $D \otimes_\Gamma I_1 \subset D \otimes_\Gamma I_2 \subset \cdots$ is a strictly ascending chain of (D, Γ)-bimodules. Similarly, since ${}_\Gamma D$ is free, $D \otimes_\Gamma I_1 \otimes_\Gamma D \subset D \otimes_\Gamma I_2 \otimes_\Gamma D \subset \cdots$ is a strictly ascending chain of D-bimodules, a contradiction. We have proved that any strictly ascending chain of Γ-bimodules $\{I_i\}$ gives (by tensoring as above) the strictly ascending chain of D-bimodules $\{D \otimes_\Gamma I_i \otimes_\Gamma D\}$, hence $\mathrm{Kdim}({}_\Gamma(\Gamma \otimes \Gamma)_\Gamma) \leq \mathrm{Kdim}({}_D(D \otimes D)_D)$. $\square$

Given a K-algebra A, a K-linear map $\sigma : A \to A$ is called an *anti-isomorphism* iff $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$. Clearly, σ is an anti-isomorphism iff $\sigma : A \to A^o$, $a \mapsto \sigma(a)^o$, is a K-algebra isomorphism.

Lemma 3.7 *Let A_i, $i \in I$, be subalgebras of a K-algebra A, $B := \cap_{i \in I} A_i$, $\sigma : A \to A$ be an anti-isomorphism such that $\sigma(A_i) = A_i$ for all i. Then σ induces the anti-isomorphism of the algebra B.*

Proof. Clearly, σ^{-1} is an anti-isomorphism of the algebra A such that $\sigma^{-1}(A_i) = A_i$ for all $i \in I$. Then $\sigma(B) \subseteq (B)$ and $\sigma^{-1}(B) \subseteq B$ for all $i \in I$, hence $\sigma(B) = B$, and we are done. $\square$

Theorem 3.8 *Let X be a smooth irreducible affine variety over a field K of characteristic zero, $\mathcal{D}(X)$ be the ring of differential operators on X, and $D(X) = \mathrm{Frac}(\mathcal{D}(X))$ be its quotient division algebra. Then*

1. *$\mathcal{D}(X) \simeq \mathcal{D}(X)^o$, $d \mapsto d$, $\partial \mapsto -\partial$, where $d \in \mathcal{O}(X)$ and $\partial \in \mathrm{Der}_K(\mathcal{O}(X))$.*
2. *$D(X) \simeq D(X)^o$.*

3. *if X is a smooth affine variety (not necessarily irreducible) over K then still* $\mathcal{D}(X) \simeq \mathcal{D}(X)^o$.

Remark. 1. The algebra $\mathcal{D}(X)$ is generated by the coordinate algebra $\mathcal{O}(X)$ and the $\mathcal{O}(X)$-module $\mathrm{Der}_K(\mathcal{O}(X))$ of K-derivations of the algebra $\mathcal{O}(X)$ (5.6, [4]).

2. So, the ring of differential operators on a smooth irreducible algebraic variety is *symmetric* object indeed. If A is *not* smooth then, in general, the algebra $\mathcal{D}(A)$ need not be a finitely generated algebra nor a left or right Noetherian algebra, [2], the algebra $\mathcal{D}(A)$ can be finitely generated and right Noetherian yet not left Noetherian, [8].

Proof. 1. We keep the notation of the proof of Corollary 3.2. In particular, the algebra $\mathcal{D}(X)$ is a subalgebra of its localization $A := S^{-1}\mathcal{D}(X) \simeq \Gamma[t_1; \frac{\partial}{\partial x_1}] \cdots [t_n; \frac{\partial}{\partial x_n}]$.

By 2.13 and 2.6, [4], there is a finite set of elements of the coordinate algebra $\mathcal{O}(X)$, say $c_1, \ldots, c_s$, such that the natural inclusion $\mathcal{D}(X) \rightarrow \prod_{i=1}^s \mathcal{D}(X)_{c_i}$ is a *faithfully flat* extension where $A_i := \mathcal{D}(X)_{c_i}$ is the localization of $\mathcal{D}(X)$ at the powers of the element c_i such that $\mathcal{D}(X)_{c_i} = \mathcal{O}(X)_{c_i}[t_1; \frac{\partial}{\partial x_1}] \cdots [t_n; \frac{\partial}{\partial x_n}]$. Note that $A_i \subseteq A$. Hence $\mathcal{D}(X) = \cap_{i=1}^s A_i$ (let $B := \cap_{i=1}^s A_i$ then $\mathcal{D}(X) \subseteq B \subseteq A_i$ for each i, then the localization of the chain of inclusions above at c_i gives $A_i := \mathcal{D}(X)_{c_i} \subseteq B_{c_i} \subseteq A_i$, and so, by the faithful flatness, we must have $\mathcal{D}(X) = B$). The map $\sigma : A \rightarrow A$ given by $d \rightarrow d$ ($d \in \Gamma$), $t_i \mapsto -t_i$, gives an anti-isomorphism of A such that $\sigma(A_i) = A_i$ for all i. By Lemma 3.7, σ gives the anti-isomorphism of the algebra $\mathcal{D}(X)$.

2. By Lemma 3.3, $D(X)^o = \mathrm{Frac}(\mathcal{D}(X))^o \simeq \mathrm{Frac}(\mathcal{D}(X)^o) \simeq \mathrm{Frac}(\mathcal{D}(X)) = D(X)$.

3. Then $X \simeq \prod_{i=1}^s X_i$ is a direct product of smooth irreducible affine varieties X_i over K. Then

$$\mathcal{D}(X) = \mathcal{D}(\prod_{i=1}^s X_i) \simeq \prod_{i=1}^s \mathcal{D}(X_i) \simeq \prod_{i=1}^s \mathcal{D}(X_i)^o \simeq (\prod_{i=1}^s \mathcal{D}(X_i))^o \simeq \mathcal{D}(X)^o. \quad \Box$$

It is well-known fact that if C is a commutative K-subalgebra of the ring of differential operators $\mathcal{D}(X)$ (see Theorem 3.8) then the Gelfand-Kirillov dimension $\mathrm{GK}\,(C) \leq n := \dim(X)$ (i.e. the transcendence degree $\mathrm{tr.deg}_K(\mathrm{Frac}(C)) \leq n$). It follows from Corollary 3.12, [1], that $\mathrm{Ev}(\Delta) \simeq \mathbb{Z}^r$ and $r \leq n$, where $\Delta = \{\delta_1, \ldots, \delta_t\}$ is a set of commuting *locally finite* K-derivations of the algebra $\mathcal{D}(X)$. A K-derivation δ of an algebra A is called *locally finite* if, for each $a \in A$, $\dim_K(\sum_{i\geq 0} K\delta^i(a)) < \infty$. In a view of Corollary 3.12, [1] and Theorem 3.8, the author propose the following conjecture.

Conjecture. If $\Delta = \{\delta_1, \ldots, \delta_t\}$ is a set of commuting K-derivations or K-automorphisms of the division algebra $D(X) = \mathrm{Frac}(\mathcal{D}(X))$ then the rank of the abelian group $\mathrm{Ev}(\Delta) \leq \dim(X)$.

Question 1. Given a division K-algebra D over a field K of characteristic zero such the algebra $D \otimes D^o$ is Noetherian and a set $\Delta = \{\delta_1, \ldots, \delta_t\}$ of commuting K-derivations or K-automorphisms of the division algebra D. Is it true that the rank of the abelian group $\mathrm{Ev}(\Delta) \leq \mathrm{Kdim}(D \otimes D^o)$?

Remark. If, in the question above, D is a (finitely generated) *field* then the result is obviously true (as it follows from (2) and $\mathrm{GK}\,(D(\Delta)) \leq \mathrm{GK}\,(D)$ that

$$\mathrm{GK}\,(D(\Delta)) = \mathrm{tr.deg}_K(D_0) + \mathrm{rank}(\mathrm{Ev}(\Delta)) \leq \mathrm{tr.deg}_K(D) = \mathrm{Kdim}(D \otimes D)$$

where D_0 is the subfield of Δ-constants in D).

Question 2. For a singular irreducible affine variety find a necessary and sufficient condition that $\mathcal{D}(X) \simeq \mathcal{D}(X)^o$.

Conjecture 2. Let X be a smooth irreducible affine curve over an algebraically closed field K of characteristic zero, and $\delta \in \mathrm{Der}_K(\mathcal{D}(X))$. Then the eigen-algebra $D(\delta)$ is a finitely generated Noetherian algebra.

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