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Article:

Bavula, V.V. (2016) The classical left regular left quotient ring of a ring and its semisimplicity criteria. Journal of Algebra and Its Applications, 16 (1750103). ISSN: 0219-4988

https://doi.org/10.1142/S0219498817501031

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The classical left regular left quotient ring of a ring and its semisimplicity criteria

V. V. Bavula

Abstract

Let R be a ring, $\mathcal{C}_R$ and ${}'\mathcal{C}_R$ be the set of regular and left regular elements of R ($\mathcal{C}_R \subseteq {}'\mathcal{C}_R$). Goldie's Theorem is a semisimplicity criterion for the classical left quotient ring $Q_{l,cl}(R) := \mathcal{C}_R^{-1}R$. Semisimplicity criteria are given for the classical left regular left quotient ring ${}'Q_{l,cl}(R) := {}'\mathcal{C}_R^{-1}R$. As a corollary, two new semisimplicity criteria for $Q_{l,cl}(R)$ are obtained (in the spirit of Goldie).

Contents

Key Words: Goldie's Theorem, the (classical) left quotient ring, the (classical) left regular left quotient ring.

Mathematics subject classification 2010: 16P50, 16P60, 16P20, 16U20.

1 Introduction

In this paper, R is a ring with 1, R^* is its group of units, module means a left module.

Semisimplicity criteria for the ring ${}'Q_{l,cl}(R)$. For each element $r \in R$, let $r\cdot : R \to R$, $x \mapsto rx$ and $\cdot r : R \to R$, $x \mapsto xr$. The sets ${}'\mathcal{C}_R := \{r \in R \,|\, \ker(\cdot r) = 0\}$ and $\mathcal{C}'_R := \{r \in R \,|\, \ker(r\cdot) = 0\}$ are called the *sets of left and right regular elements* of R, respectively. Their intersection $\mathcal{C}_R = {}'\mathcal{C}_R \cap \mathcal{C}'_R$ is the *set of regular elements* of R. The rings $Q_{l,cl}(R) := \mathcal{C}_R^{-1}R$ and $Q_{r,cl}(R) := R\mathcal{C}_R^{-1}$ are called the *classical left and right quotient rings* of R, respectively. Goldie's Theorem states that the ring $Q_{l,cl}(R)$ is a semisimple Artinian ring iff the ring R is semiprime, $\mathrm{udim}({}_RR) < \infty$ and the ring R satisfies the a.c.c. on left annihilators (udim stands for the uniform dimension).

In this paper, we consider/introduce the rings ${}'Q_{l,cl}(R) := {}'\mathcal{C}_R^{-1}R$ (the *classical left regular left quotient ring* of R) and ${}'Q_{r,cl}(R) := R{\mathcal{C}'_R}^{-1}$ (the *classical right regular right quotient ring* of R) and give several semisimplicity criteria for them. In view of left-right symmetry, it suffices to consider, say 'left' case.

A subset S of a ring R is called a *multiplicative set* if $1 \in S$, $SS \subseteq S$ and $0 \notin S$. Suppose that S and T are multiplicative sets in R such that $S \subseteq T$. The multiplicative subset S of T is called *dense* (or *left dense*) in T if for each element $t \in T$ there exists an element $r \in R$ such that $rt \in S$. Main results of the paper are semisimplicity criteria for the ring ${}'Q_{l,cl}(R)$. For a left ideal I of R, let ${}'\mathcal{C}_I := \{i \in I \,|\, \cdot i : I \to I,\ x \mapsto xi, \text{ is an injection}\}$. For a nonempty subset S of a ring R, let $\mathrm{ass}_R(S) := \{r \in R \,|\, sr = 0 \text{ for some } s \in S\}$.

Theorem 1.1 *Let R be a ring, $'\mathcal{C} = {}'\mathcal{C}_R$ and $\mathfrak{a} := \mathrm{ass}_R('\mathcal{C})$. The following statements are equivalent.*

1. *$'Q := {}'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. (a) *$\mathfrak{a}$ is a semiprime ideal of R,*
   (b) *the set $'\overline{\mathcal{C}} := \pi('\mathcal{C})$ is a dense subset of $'\mathcal{C}_{\overline{R}}$ where $\pi : R \to \overline{R} := R/\mathfrak{a}$, $r \mapsto \overline{r} := r + \mathfrak{a}$,*
   (c) *$\mathrm{udim}({}_{\overline{R}}\overline{R}) < \infty$, and*
   (d) *$'\mathcal{C}_V \neq \emptyset$ for all uniform left ideals V of $\overline{R}$.*
3. *$\mathfrak{a}$ is a semiprime ideal of R, $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$ and $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring.*

If one of the equivalent conditions holds then $'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$, $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$ and $'Q \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R} \simeq Q_{l,cl}(\overline{R})$. Furthermore, the ring $'Q$ is a simple ring iff the ideal $\mathfrak{a}$ is a prime ideal.

Let $\mathfrak{n} = \mathfrak{n}_R$ be the prime radical of the ring R. The following theorem is an instrumental in proving of several results of the paper including Theorem 1.1. It gives sufficient conditions for the set $'\mathcal{C}_R$ to be a left denominator set of the ring R such that the ring $'\mathcal{C}_R^{-1}R$ is a semisimple Artinian ring.

- (**Theorem 2.3**) *Let R be a ring. Suppose that $\mathrm{udim}({}_RR) < \infty$ and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R. Then $'\mathcal{C}_R \in \mathrm{Den}_l(R)$, the ring $'Q_{l,cl}(R)$ is a semisimple Artinian ring, $'Q_{l,cl}(R) \simeq Q_{l,cl}(R/\mathfrak{a})$ where $\mathfrak{a} := \mathrm{ass}_R('\mathcal{C})$ and $\mathfrak{n}_R \subseteq \mathfrak{a}$.*

For an arbitrary ring R, the set $\mathrm{max.Den}_l(R)$ of maximal left denominator sets is a non-empty set, [2]. The next semisimplicity criterion for the ring $'Q_{l,cl}(R)$ is given via the set $'\mathcal{M}$ of maximal denominator sets of R that contain $'\mathcal{C}_R$.

- (**Theorem 3.1**) *Let R be a ring, $\mathfrak{a} = \mathrm{ass}_R('\mathcal{C}_R)$ and $'\mathcal{M} := \{S \in \mathrm{max.Den}_l(R) \,|\, '\mathcal{C}_R \subseteq S\}$. The following statements are equivalent.*
  1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
  2. *$'\mathcal{M}$ is a finite nonempty set, $\bigcap_{S \in '\mathcal{M}} \mathrm{ass}_R(S) = \mathfrak{a}$, for each $S \in {}'\mathcal{M}$, the ring $S^{-1}R$ is a simple Artinian ring and the set $'\overline{\mathcal{C}} := \{c + \mathfrak{a} \,|\, c \in {}'\mathcal{C}_R\}$ is a dense subset of $\mathcal{C}_{R/\mathfrak{a}}$ in $R/\mathfrak{a}$.*

Theorem 3.3 below is a semisimplicity criterion for the ring $'Q_{l,cl}(R)$ that is given via the set $\mathrm{Min}_R(\mathfrak{a})$ of minimal primes of the ideal $\mathfrak{a}$. Theorem 3.3 describes explicitly the set $'\mathcal{M}$ in Theorem 3.1, see the full version of Theorem 3.3 in Section 3.

- (**Theorem 3.3**) *Let R be a ring, $'\mathcal{C} = {}'\mathcal{C}_R$ and $\mathfrak{a} = \mathrm{ass}_R('\mathcal{C})$. The following statements are equivalent.*
  1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
  2. (a) *$\mathfrak{a}$ is semiprime ideal of R and the set $\mathrm{Min}_R(\mathfrak{a})$ is a finite set.*
     (b) *For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the set $S_\mathfrak{p} := \{c \in R \,|\, c + \mathfrak{p} \in \mathcal{C}_{R/\mathfrak{p}}\}$ is a left denominator set of the ring R with $\mathrm{ass}_R(S_\mathfrak{p}) = \mathfrak{p}$.*
     (c) *For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the ring $S_\mathfrak{p}^{-1}R$ is a simple Artinian ring.*
     (d) *The set $'\overline{\mathcal{C}} = \{c + \mathfrak{a} \,|\, c \in {}'\mathcal{C}\}$ is a dense subset of $\mathcal{C}_{R/\mathfrak{a}}$.*

A ring R is called *left Goldie* if it satisfies the a.c.c. on left annihilators and $\mathrm{udim}({}_RR) < \infty$. Theorem 3.4 below is a semisimplicity criterion for the ring $'Q_{l,cl}(R)$ in terms of left Goldie rings.

- (**Theorem 3.4**) *The following statements are equivalent.*

1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. *(a) $\mathfrak{a}$ is a semiprime ideal of R and the set $\mathrm{Min}_R(\mathfrak{a})$ is finite.*
   (b) For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the ring $R/\mathfrak{p}$ is a left Goldie ring.
   (c) The set $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$.

Theorem 3.5 is a useful semisimplicity criterion for the ring $'Q_{l,cl}(R)$ as often we have plenty of simple Artinian localizations of a ring.

- (**Theorem 3.5**) *The following statements are equivalent.*
  1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
  2. *There are left denominator sets $S_1, \ldots, S_n$ of the ring R such that*
     (a) the rings $S_i^{-1}R$ are simple Artinian rings,
     (b) $\mathfrak{a} = \bigcap_{i=1}^n \mathrm{ass}_R(S_i)$, and
     (c) $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$.

Remark. Let R be a ring. If $'\mathcal{C}_R$ is a *right* denominator set of the ring R then $'\mathcal{C}_R = \mathcal{C}_R$ and $Q'_{r,cl}(R) = Q_{r,cl}(R)$ is the classical right quotient ring of R. Similarly, if $\mathcal{C}'_R$ is a *left* denominator set of the ring R then $\mathcal{C}'_R = \mathcal{C}_R$ and $'Q_{l,cl}(R) = Q_{l,cl}(R)$ is the classical left quotient ring of R.

Semisimplicity criteria for the ring $Q_{l,cl}(R)$. The next theorem shows that the a.c.c. condition on *left* annihilators in Goldie's Theorem can be replaced by the a.c.c. condition on *right* annihilators (or even by a weaker condition) and adding some extra condition.

Theorem 1.2 *Let R be a ring, $\mathcal{C} = \mathcal{C}_R$ and $'\mathcal{C} = {}'\mathcal{C}_R$. The following statements are equivalent.*

1. *$Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. *R is a semiprime ring, $\mathrm{udim}({}_RR) < \infty$, the ring R satisfies the a.c.c. on right annihilators and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R.*
3. *The ring R is a semiprime ring, $\mathrm{udim}({}_RR) < \infty$, the set $\{\ker(c_R\cdot) \,|\, c \in {}'\mathcal{C}\}$ satisfies the a.c.c. and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R.*
4. *The ring R is a semiprime ring, $\mathrm{udim}({}_RR) < \infty$, the set $\{\ker(r_R\cdot) \,|\, r \in R\}$ satisfies the a.c.c. and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R.*

Below is another semisimplicity criterion for the ring $Q_{l,cl}(R)$ via $'\mathcal{C}_R$.

Theorem 1.3 *Let R be a ring. The following statements are equivalent.*

1. *$Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. *R is a semiprime ring, $\mathrm{udim}({}_RR) < \infty$, $'\mathcal{C}_R = \mathcal{C}_R$ and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R.*

Apart from Goldie's Theorem, Theorem 1.2 and Theorem 1.3, there are several semisimplicity criteria for $Q_{l,cl}(R)$, [4].

The left regular left quotient ring $'Q_l(R)$ of a ring R and its semisimplicity criteria. Let R be a ring. In general, the classical left quotient ring $Q_{l,cl}(R)$ does not exists, i.e. the set of regular elements $\mathcal{C}_R$ of R is not a left Ore set. The set $\mathcal{C}_R$ contains the *largest left Ore set* denoted by $S_l(R)$ and the ring $Q_l(R) := S_l(R)^{-1}R$ is called the *(largest) left quotient ring* of R, [2]. Clearly, if $\mathcal{C}_R$ is a left Ore set then $\mathcal{C}_R = S_l(R)$ and $Q_{l,cl}(R) = Q_l(R)$. Similarly, the set $'\mathcal{C}_R$ of left regular elements of the ring R is not a left denominator set, in general, and so in this case the classical left regular left quotient ring $'Q_{l,cl}(R)$ does not exist. The set $'\mathcal{C}_R$ contains the *largest* left denominator set $'S_l(R)$ (Lemma 4.1.(1)) and the ring $'Q_l(R) := {}'S_l(R)^{-1}R$ is called the *left regular left quotient ring* of R. If $'\mathcal{C}_R$ is a left denominator set then $'\mathcal{C}_R = {}'S_l(R)$ and $'Q_{l,cl}(R) = {}'Q_l(R)$. Theorem 4.3 is a semisimplicity criterion for the ring $'Q_l(R)$.

- (**Theorem 4.3**) *Let R be a ring. Then*

  1. *$'Q_l(R)$ is a left Artinian ring iff $'Q_{l,cl}(R)$ is a left Artinian ring. If one of the equivalent conditions holds then $'S_l(R) = '\mathcal{C}_R$ and $'Q_l(R) = 'Q_{l,cl}(R)$.*
  2. *$'Q_l(R)$ is a semisimple Artinian ring iff $'Q_{l,cl}(R)$ is a semisimple Artinian ring. If one of the equivalent conditions holds then $'S_l(R) = '\mathcal{C}_R$ and $'Q_l(R) = 'Q_{l,cl}(R)$.*

So, all the semisimplicity criteria for the ring $'Q_{l,cl}(R)$ are automatically semisimplicity criteria for the ring $'Q_l(R)$.

The rings $'Q_{l,cl}(\mathbb{I}_1)$ and $Q'_{r,cl}(\mathbb{I}_1)$. Let K be a field of zero characteristic, $A_n = K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\rangle$ be the *Weyl algebra* and $\mathbb{I}_n = K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \int_1, \ldots, \int_n\rangle$ be the *algebra of polynomial integro-differential operators*. The ring $Q(A_n) := Q_{l,cl}(A_n)$ is a division ring and $Q_{l,cl}(A_n) = Q_l(A_n) = 'Q_{l,cl}(A_n) = 'Q_l(A_n)$.

- (**Lemma 4.8**) *1. For all K-algebras A and $n \geq 1$, the rings $Q_{l,cl}(\mathbb{I}_n \otimes A)$ and $Q_{r,cl}(\mathbb{I}_n \otimes A)$ do not exist.*

  2. For all K-algebras A and $n \geq 1$, the rings $Q_l(\mathbb{I}_n \otimes A)$ are not left Noetherian and the rings $Q_r(\mathbb{I}_n \otimes A)$ are not right Noetherian.

As an application of some of the results of the paper the rings $'Q_{l,cl}(\mathbb{I}_1)$ and $Q'_{r,cl}(\mathbb{I}_1)$ are found.

- (**Theorem 6.5**) *$'Q_{l,cl}(\mathbb{I}_1) \simeq Q(A_1)$ and $Q'_{r,cl}(\mathbb{I}_1) \simeq Q(A_1)$ are division rings.*

Explicit descriptions of the sets $'\mathcal{C}_{\mathbb{I}_1}$ and $\mathcal{C}'_{\mathbb{I}_1}$ are given in Theorem 6.7. This and some other results demonstrate that on many occasions the ring $'Q_{l,cl}(R)$ has 'somewhat better properties' than $Q_{l,cl}(R)$ which for $R = \mathbb{I}_n$ even does not exist.

Conjecture. *$'Q_{l,cl}(\mathbb{I}_n) \simeq Q_{l,cl}(A_n)$ is a division ring.*

2 Preliminaries, proofs of Theorem 1.1 and Theorem 1.2

The following notation is fixed in the paper.

Notation:

- R is a ring with 1, $\mathfrak{n} = \mathfrak{n}_R$ is its prime radical and $\mathrm{Min}(R)$ is the set of minimal primes of R;
- $\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of the ring R (i.e. $\mathcal{C}$ is the set of non-zero-divisors of the ring R);
- $Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the *classical left quotient ring* (the *classical left ring of fractions*) of the ring R (if it exists) and Q^* is the group of units of Q;
- $\mathrm{Ore}_l(R) := \{S \mid S \text{ is a left Ore set in } R\}$ and $\mathrm{ass}(S) := \{r \in R \mid sr = 0 \text{ for some } s = s(r) \in S\}$;
- $\mathrm{Den}_l(R) := \{S \mid S \text{ is a left denominator set in } R\}$;
- $\mathrm{Den}_l(R, I)$ is the set of left denominator sets S of R with $\mathrm{ass}(S) = I$ where I is an ideal of R;
- $\max.\mathrm{Den}_l(R)$ is the set of maximal left denominator sets of R (it is always a *non-empty* set, [2]).
- $'\mathcal{C} := '\mathcal{C}_R$ is the set of left regular elements of the ring R and $\mathfrak{a} := \mathrm{ass}_R('\mathcal{C})$,
- $'Q := 'Q_{l,cl}(R) := '\mathcal{C}_R^{-1}R$ is the *classical left regular left quotient ring* and $'Q^*$ is the group of units of Q;
- if $\mathfrak{a}$ is an ideal of R then $\overline{R} := R/\mathfrak{a}$, $\pi : R \to \overline{R}$, $r \mapsto \overline{r} := r + \mathfrak{a}$, and $'\overline{\mathcal{C}} := \pi('\mathcal{C})$;

- ${}'S_l(R)$ is the largest left denominator set in ${}'\mathcal{C}_R$ and ${}'Q_l(R) := {}'S_l(R)^{-1}R$ is the left regular left quotient ring of R;

- ${}'\mathfrak{a} := \mathrm{ass}_R({}'S_l(R))$ and ${}'\pi : R \to \overline{R}' := R/{}'\mathfrak{a}$, $r \mapsto \overline{r} := r + {}'\mathfrak{a}$.

Sufficient conditions for semisimplicity of the ring ${}'Q_{l,cl}(R)$. In this section, proofs are given of Theorem 1.1, Theorem 1.2 and Theorem 1.3. Let I be a nonzero left ideal of a ring R. Sufficient conditions are given for a right Noetherian ring to have a semisimple left quotient ring (Corollary 2.4). For each ideal $\mathfrak{a}$ of a ring R, the left singular ideal $\zeta_l(R, \mathfrak{a})$ of R over $\mathfrak{a}$ is introduced that, in the case when $\mathfrak{a} = 0$, coincides with the (classical) left singular ideal $\zeta_l(R)$ of R. It is proved that $\zeta_l(R, \mathfrak{a})$ is an ideal of R.

Let ${}'\mathcal{C}_I := \{i \in I \mid \cdot i : I \to I,\ x \mapsto xi \text{ is an injection}\}$. A nonzero module is called a *uniform module* if every two of its nonzero submodules have nonzero intersection.

Lemma 2.1 *Suppose that R is a ring, U is a left uniform ideal of R, $u \in {}'\mathcal{C}_U$ and $K = \ker(\cdot u_R)$. Then*

1. *$U \oplus K$ is an essential left ideal of R.*

2. *If I is a left ideal of R such that $U \subseteq I$ then $U \oplus (K \cap I)$ is an essential left R-submodule of I.*

Proof. 1. Clearly, $U \cap K = 0$ (since $\ker(\cdot u_U) = 0$ and $Ku = 0$). So, $U + K = U \oplus K$. Suppose that the left ideal $J := U \oplus K$ of R is not essential, we seek a contradiction. Then $J \cap V = 0$ for some nonzero left ideal V of R. The map $\cdot u_V : V \to U$ is an injection. So, $Vu \cap Uu \neq 0$, i.e. $vu = u'u$ for some nonzero elements $v \in V$ and $u' \in U$, and so $k := v - u' \in K$. This means that $0 \neq v = u' + k \in V \cap J$, a contradiction.

2. The left ideal J of R is essential (statement 1) and $I \neq 0$. Then the intersection $J \cap I = U \oplus I \cap K$ is an essential left R-submodule of I. $\square$

We say that $\mathrm{udim}({}_RR) < \infty$ if there are uniform left ideals $U_1, \ldots, U_n$ of R such that $\oplus_{i=1}^n U_i$ is an essential left ideal of R. Then $n = \mathrm{udim}({}_RR)$ does not depend on the choice of the uniform left ideals U_i and is called the *left uniform dimension* of R. Similarly, the *right uniform dimension* $\mathrm{udim}(R_R)$ of R is defined.

Let J be a nonzero ideal of a ring R. Let ${}'\mathcal{C}(R, J) := \{r \in R \mid \cdot r : J \to J,\ x \mapsto xr \text{ is an injection}\}$. We set ${}'\mathcal{C}_0(R, 0) := R$. For $r \in R$, let $\cdot r_R = \cdot r : R \to R$, $x \mapsto xr$, and $\cdot r_J : J \to J$, $y \mapsto yr$.

The *classical left quotient ring* $Q_{l,cl}(R) = \mathcal{C}_R^{-1}R$ often does not exists, i.e. the set $\mathcal{C}_R$ is not a left Ore set of R. The set $\mathcal{C}_R$ contains the *largest left Ore set* denoted $S_l(R)$ and the ring $Q_l(R) := S_l(R)^{-1}R$ is called the *(largest) left quotient ring* of R, [2]. If $\mathcal{C}_R \in \mathrm{Ore}_l(R)$ then $\mathcal{C}_R = S_l(R)$ and $Q_{l,cl}(R) = Q_l(R)$.

Theorem 2.2 *[2, Theorem 2.9] The ring $Q_l(R)$ is a semisimple ring iff the ring $Q_{l,cl}(R)$ is a semisimple ring. In this case, $S_l(R) = \mathcal{C}_R$ and $Q_l(R) = Q_{l,cl}(R)$.*

The next theorem gives sufficient conditions for the set ${}'\mathcal{C}_R$ to be a left denominator set of R such that the ring ${}'\mathcal{C}_R^{-1}R$ is a semisimple Artinian ring.

Theorem 2.3 *Let R be a ring and ${}'\mathcal{C} := {}'\mathcal{C}_R$. Suppose that $\mathrm{udim}({}_RR) < \infty$ and ${}'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R. Then*

1. *${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$ and ${}'Q := {}'\mathcal{C}^{-1}R$ is a semisimple Artinian ring (where $\mathfrak{a} := \mathrm{ass}_R({}'\mathcal{C})$).*

2. *${}'Q \simeq Q_{l,cl}(R/\mathfrak{a})$, an R-isomorphism.*

3. *Let $\pi = \pi_{\mathfrak{a}} : R \to R/\mathfrak{a}$, $r \mapsto \overline{r} = r + \mathfrak{a}$, and $\sigma : R \to {}'Q$, $r \mapsto \frac{r}{1}$. Then*

   (a) ${}'\mathcal{C} = \pi^{-1}({}'\mathcal{C}_{R/\mathfrak{a}}) \cap {}'\mathcal{C}(R, \mathfrak{a}) = \{c \in R \mid \cdot \overline{c}_{R/\mathfrak{a}} \text{ and } \cdot c_{\mathfrak{a}} \text{ are injections}\}$ and ${}'\mathcal{C} = \sigma^{-1}({}'Q^*) \cap {}'\mathcal{C}(R, \mathfrak{a})$.

(b) ${}'\mathcal{C}_{R/\mathfrak{a}} = \mathcal{C}_{R/\mathfrak{a}} = R/\mathfrak{a} \cap {}'Q^*$.

4. *For all essential left ideals I of R, $I \cap {}'\mathcal{C} \neq \emptyset$.*

5. *The prime radical $\mathfrak{n} = \mathfrak{n}_R$ of R is contained in the ideal $\mathfrak{a}$ (In general, $\mathfrak{n} \neq \mathfrak{a}$, eg $R = \mathbb{I}_1$, $\mathfrak{n}_{\mathbb{I}_1} = 0$ but $\mathfrak{a} = F \neq 0$ is the largest proper ideal of $\mathbb{I}_1$, Theorem 6.5.(1)).*

Proof. 4. We use the following fact repeatedly: *Given R-modules $K \subseteq L \subseteq M$ such that K is an essential submodule of L and L is an essential submodule of M then K is an essential submodule of M.* We also use repeatedly Lemma 2.1.

Let I be an essential left ideal of R. Fix a uniform left ideal, say U_1, of R such that $U_1 \subseteq I$ and fix an element $u_1 \in {}'\mathcal{C}_{U_1}$. Let $K_1 := \ker(\cdot(u_1)_R)$ and $K_1' = I \cap K_1$. By Lemma 2.1.(2), $U_1 \oplus K_1'$ is an essential left R-submodule of I, hence $U_1 \oplus K_1'$ is an essential left ideal of R such that $K_1' u_1 = 0$. Repeating the same argument for the left ideal K_1' we will find a uniform R-submodule U_2 of K_1' and an element $u_2 \in {}'\mathcal{C}_{U_2}$ such that $U_2 \oplus K_2'$ is an essential left R-submodule of K_1' where $K_2' = K_1' \cap K_2$ and $K_2 := \ker(\cdot(u_2)_R)$. So, $U_1 \oplus U_2 \oplus K_2'$ is an essential left ideal of R such that $(U_2 \oplus K_2')u_1 = K_2' u_2 = 0$. Repeating the same process several times and using the fact that $n := \mathrm{udim}({}_RR) < \infty$, we will find uniform submodules $U_1, \ldots, U_n$ of I and elements $u_1 \in {}'\mathcal{C}_{U_1}, \ldots, u_n \in {}'\mathcal{C}_{U_n}$ such that

(i) $J := \oplus_{i=1}^n U_i$ is an essential left ideal of R, and

(ii) $J_{i+1} u_i = 0$, $i = 1, \ldots, n-1$ where $J_s := \oplus_{t=s}^n U_t$, $s = 1, \ldots, n$.

Claim: $c = u_1 + \cdots + u_n \in {}'\mathcal{C} \cap I$.

Clearly, $c \in I$ since all $u_i \in U_i \subseteq I$. We aim to show that $\ker(\cdot c_R) = 0$, i.e. $c \in I \cap {}'\mathcal{C}$. Since J is an essential left ideal of R it suffices to show that $\ker(\cdot c_J) = 0$ (where $\cdot c_J : J \to J$, $x \mapsto xc$). The map $\cdot c_J$ respects the ascending filtration of left ideals

$$0 =: J_{n+1} \subset J_n \subset J_{n-1} \subset \cdots \subset J_1 = J,$$

i.e. $J_s c = J_s(u_s + \cdots + u_n) \subseteq J_s$ for all s (by (ii) and since $u_s + \cdots + u_n \in J_s$). Moreover, $J_s/J_{s+1} \simeq U_s$ and the map $\cdot c_{J_s/J_{s+1}} = \cdot(u_s)_{U_s}$ is an injection (since $u_s \in {}'\mathcal{C}_{U_s}$). Hence, the map $\cdot c_J$ is an injection. The proof of the Claim and of statement 4 is complete.

1. (i) ${}'\mathcal{C} \in \mathrm{Ore}_l(R)$: Given $r \in R$ and $c \in {}'\mathcal{C}$, we have to show that ${}'\mathcal{C}r \cap Rc \neq \emptyset$. Since $c \in {}'\mathcal{C}$, $\mathrm{udim}({}_RRc) = \mathrm{udim}({}_RR)$, i.e. Rc is an essential left ideal of R. Then the left ideal of R, $(Rc : r) := \{a \in R \,|\, ar \in Rc\}$, is an essential left ideal. By statement 4, we can fix an element $c' \in {}'\mathcal{C} \cap (Rc : r)$, and so $c'r = r'c$ for some $r' \in R$.

(ii) ${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$ *where* $\mathfrak{a} := \mathrm{ass}_R({}'\mathcal{C})$: This follows from the statement (i).

(iii) ${}'Q$ *is a semisimple Artinian ring*: Since $\mathrm{udim}({}_RR) < \infty$ we can fix a direct sum $J = \oplus_{i=1}^n U_i$ of uniform left ideals of R such that $I := J \oplus \mathfrak{a}$ is an essential left ideal of R. By statement 4, $I \cap {}'\mathcal{C} \neq \emptyset$. Hence, ${}'Q = {}'\mathcal{C}^{-1} I = \oplus_{i=1}^n {}'\mathcal{C}^{-1} U_i$ (since ${}'\mathcal{C}^{-1}\mathfrak{a} = 0$). It suffices to show that each ${}'Q$-module ${}'\mathcal{C}^{-1}U_i$ is a simple module. Suppose that, say ${}'\mathcal{C}^{-1}U_1$, is not a simple ${}'Q$-module, we seek a contradiction. Then it contains a proper submodule, say M. Since ${}_RU_1$ is essential in ${}_R{}'\mathcal{C}^{-1}U_1$, the intersection $U_1' = U_1 \cap M$ is a nonzero. The left ideal $I' = U_1' \oplus U_2 \oplus \cdots \oplus U_n \oplus \mathfrak{a}$ of R is an essential ideal such that ${}'\mathcal{C}^{-1}I'$ is a proper left ideal of ${}'Q$ (by the choice of M) but, by statement 2, $I' \cap {}'\mathcal{C} \neq \emptyset$, and so ${}'\mathcal{C}^{-1}I' = {}'Q$, a contradiction.

3(a) The second equality in the statement (a) is obvious. Let $\mathcal{R}$ be the RHS of the second equality in the statement (a). To prove that the first equality holds it remains to show that ${}'\mathcal{C} = \mathcal{R}$.

(i) ${}'\mathcal{C} \subseteq \mathcal{R}$: Let $c \in {}'\mathcal{C}$. Clearly, $c \in {}'\mathcal{C}(R, \mathfrak{a})$. If $\overline{rc} = 0$ for some elements $\overline{r} = r + \mathfrak{a} \in R/\mathfrak{a}$ then $\overline{r} = 0$ since $R/\mathfrak{a} \subseteq {}'Q$ and $\overline{c}$ is a unit of the ring ${}'Q$. So, $\cdot\overline{c}_{R/\mathfrak{a}}$ is an injection. Then $c \in \mathcal{R}$.

(ii) ${}'\mathcal{C} \supseteq \mathcal{R}$: If $c \in \mathcal{R}$ then $c \in {}'\mathcal{C}$ as the map $\cdot c_R$ respects the filtration $0 \subseteq \mathfrak{a} \subseteq R$.

The equality ${}'\mathcal{C} = \sigma^{-1}({}'Q^*) \cap {}'\mathcal{C}(R, \mathfrak{a})$ follows from the equality ${}'\mathcal{C} = \pi^{-1}({}'\mathcal{C}_{R/\mathfrak{a}}) \cap {}'\mathcal{C}(R, \mathfrak{a})$ and the statement 3(b): ${}'\mathcal{C} = \pi^{-1}({}'\mathcal{C}_{R/\mathfrak{a}}) \cap {}'\mathcal{C}(R, \mathfrak{a}) = \pi^{-1}(R/\mathfrak{a} \cap {}'Q^*) \cap {}'\mathcal{C}(R, \mathfrak{a}) = \sigma^{-1}({}'Q^*) \cap {}'\mathcal{C}(R, \mathfrak{a})$.

3(b) Since ${}'Q$ is a semisimple Artinian ring, ${}'\mathcal{C}_{R/\mathfrak{a}} = \mathcal{C}_{R/\mathfrak{a}}$ and $\mathcal{C}_{R/\mathfrak{a}} = R/\mathfrak{a} \cap {}'Q^*$.

2. By statement 1, $\pi({}'\mathcal{C}) \in \mathrm{Den}_l(R/\mathfrak{a}, 0)$ and $\pi({}'\mathcal{C})^{-1}(R/\mathfrak{a}) \simeq {}'Q$ is a semisimple Artinian ring. Hence, $\pi({}'\mathcal{C}) \subseteq S_l(R)$ and $Q_l(R) \simeq {}'Q$ is a semisimple Artinian ring. Then $Q_l(R) \simeq Q_{l,cl}(R)$ is a semisimple Artinian ring (Theorem 2.2).

5. Since $'Q$ is a left Artinian ring, $'\mathcal{C}^{-1}I$ is an ideal of the ring $'Q$ for all ideals I of R. Let $\mathfrak{m}$ be a maximal ideal of $'Q$. Then $\sigma^{-1}(\mathfrak{m})$ is a prime ideal of the ring R: if $IJ \subseteq \sigma^{-1}(\mathfrak{m})$ for some ideals I and J of the ring R. We may assume that $\mathfrak{a} \subseteq I$ and $\mathfrak{a} \subseteq J$. Then $'\mathcal{C}^{-1}I'\mathcal{C}^{-1}J \subseteq {'\mathcal{C}^{-1}}\sigma^{-1}(\mathfrak{m}) = \mathfrak{m}$, and so one of the ideals $'\mathcal{C}^{-1}I$ or $'\mathcal{C}^{-1}J$ belongs to $\mathfrak{m}$. Say, $'\mathcal{C}^{-1}I \subseteq \mathfrak{m}$. Then $I \subseteq \sigma^{-1}('\mathcal{C}^{-1}I) \subseteq \sigma^{-1}(\mathfrak{m})$. Now, $\mathfrak{n} \subseteq \cap_{\mathfrak{m}\in \mathrm{Max}(R)}\sigma^{-1}(\mathfrak{m})$, and so $'\mathcal{C}^{-1}\mathfrak{n} \subseteq \cap_{\mathfrak{m}\in \mathrm{Max}(R)}{'\mathcal{C}^{-1}}\sigma^{-1}(\mathfrak{m}) = \cap_{\mathfrak{m}\in \mathrm{Max}(R)}\mathfrak{m} = 0$, i.e. $\mathfrak{n} \subseteq \mathfrak{a}$. $\square$

Let X be a non-empty subset of a ring R. The right ideal $\mathrm{r.ann}(X) := \{r \in R \,|\, Xr = 0\}$ is called the *right annihilator* of X. Right ideals of this kind are called *right annihilator ideals* of R.

Proof of Theorem 1.2. Let $Q = Q_{l,cl}(R)$. Then $R \subseteq Q$.

($1 \Rightarrow 2$) (i) $\mathrm{udim}({}_RR) < \infty$: Since ${}_RR$ is an essential submodule of ${}_RQ$, $\mathrm{udim}({}_RR) = \mathrm{udim}({}_RQ) = \mathrm{udim}({}_QQ) < \infty$.

(ii) *The ring R is a semiprime ring*: Let I be an ideal of R. Since Q is a left Artinian ring, $\mathcal{C}^{-1}I$ is an ideal of Q. If I is a nilpotent ideal then so is $\mathcal{C}^{-1}I$ (since $(\mathcal{C}^{-1}I)^t = \mathcal{C}^{-1}I^t$ for all $t \geq 1$). Then $\mathcal{C}^{-1}I = 0$, i.e. $I = 0$. Therefore, the ring R is a semiprime ring.

(iii) *$'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R*: The left ideal $M = QU = \mathcal{C}^{-1}U$ of Q is a simple left Q-submodule of Q and $Q = M \oplus N$ for some left Q-submodule N of Q. Then $1 = e_1 + e_2$ where $e_1 \in M$ and $e_2 \in N$ are idempotents of the ring Q. Notice that $M = Qe_1 = Me_1$ and $e_1M = e_1Qe_1 \simeq \mathrm{End}_Q(M)$ is a division ring. So, for each nonzero element $a \in e_1M$, the nonzero Q-module homomorphism $\cdot a : M \to M$, $x \mapsto xa$, of the simple Q-module M is an automorphism. Notice that $ue_1 = u$ for all elements $u \in U$. In particular, $Ue_1 = U$. The ring R is semiprime and $U \neq 0$. By (ii), $0 \neq U^2 = (Ue_1)^2 = Ue_1Ue_1$, and so $e_1Ue_1 \neq 0$. Fix a nonzero element $a = e_1ue_1 = e_1u$ in e_1Ue_1 where $u \in U$. The map $\cdot a_M : M \to M$, $m \mapsto ma$, is a bijection. Hence, the map $\cdot a_U : U \to U$, $u' \mapsto u'a = u'e_1ue_1 = u'u$, is an injection, and so $u \in {'\mathcal{C}_U}$.

(iv) *The ring R satisfies the a.c.c. on right annihilators*: Let X be a non-empty subset of R. Then $\mathrm{r.ann}_R(X) = R \cap \mathrm{r.ann}_Q(X)$. Since Q satisfies the a.c.c. on right annihilators, so does R.

($2 \Rightarrow 3$) This implication is obvious.

($3 \Rightarrow 1$) In view of Theorem 2.3.(2), it suffices to show that $\mathfrak{a} = 0$. By the assumption, the set $K := \{\ker(c_R\cdot) \,|\, c \in {'\mathcal{C}}\}$ satisfies the a.c.c.. Let $\mathfrak{b} := \ker(c_R\cdot)$ be the maximal element in K. Clearly, $\mathfrak{b} \subseteq \mathfrak{a}$. We claim that $\mathfrak{b} = \mathfrak{a}$. Otherwise, fix an element $a \in \mathfrak{a}\backslash\mathfrak{b}$ and an element $c' \in {'\mathcal{C}}$ such that $c'a = 0$. Since $'\mathcal{C} \in \mathrm{Ore}_l(R)$ (Theorem 2.3.(1)), $c''c' = rc$ for some elements $c'' \in {'\mathcal{C}}$ and $r \in R$. Then $c_1 := c''c' \in {'\mathcal{C}}$ and $\ker(c_1\cdot) \supseteq K + aR \supsetneqq K$, a contradiction. Therefore, $\mathfrak{b} = \mathfrak{a}$. The left ideal $\mathfrak{a}c$ is a nilpotent ideal ($\mathfrak{a}c \cdot \mathfrak{a}c = 0$). The ring R is a semiprime ring, so $\mathfrak{a}c = 0$. Then $\mathfrak{a} = 0$ since $c \in {'\mathcal{C}}$.

($2 \Rightarrow 4 \Rightarrow 3$) These implications are obvious. $\square$

The next corollary gives sufficient conditions for a *right* Noetherian ring to have a semisimple *left* quotient ring.

Corollary 2.4 *Let R be a semiprime, right Noetherian ring with $\mathrm{udim}({}_RR) < \infty$ and $'\mathcal{C}_U \neq \emptyset$ for all uniform left ideals U of R. Then $Q_{l,cl}(R)$ is a semisimple Artinian ring.*

Proof. The corollary follows from Theorem 1.2. $\square$

Lemma 2.5 *Suppose that $S, T \in \mathrm{Den}_l(R)$ and $S \subseteq T$. Then the map $\varphi : S^{-1}R \to S^{-1}T$, $s^{-1}r \mapsto s^{-1}r$, is a ring homomorphism (where $s \in S$ and $r \in R$).*

1. *φ is a monomorphism iff $\mathrm{ass}_R(S) = \mathrm{ass}_R(T)$.*
2. *φ is a epimorphism iff for each $t \in T$ there exists an element $r \in R$ such that $rt \in S + \mathrm{ass}_R(T)$.*
3. *φ is a isomorphism iff $\mathrm{ass}_R(S) = \mathrm{ass}_R(T)$ and for each element $t \in T$ there exists an element $r \in R$ such that $rt \in S$.*
4. *If, in addition, $T \subseteq {'\mathcal{C}_R}$, then φ is a isomorphism iff $\mathrm{ass}_R(S) = \mathrm{ass}_R(T)$ and for each element $t \in T$ there exists an element $r \in {'\mathcal{C}_R}$ such that $rt \in S$.*

Proof. 1. Statement 1 is obvious.

2. φ is a epimorphism iff for each element $t \in T$, $t^{-1} \in \mathrm{im}(\varphi)$ iff $t^{-1} = s^{-1}r$ for some elements $s \in S$ and $r \in R$ iff $rt - s \in \mathrm{ass}_R(T)$ iff $rt \in S + \mathrm{ass}_R(T)$.

3. Statement 3 follows from statements 1 and 2 and the fact that an element $a \in R$ belongs to $\mathrm{ass}_R(S)$ iff $sa = 0$ for some element $s \in S$.

4. Statement 4 follows from statement 3 and the inclusions $\ker(\cdot rt) \supseteq \ker(\cdot r)$ and $S \subseteq T \subseteq {}'\mathcal{C}_R$. $\square$

Proof of Theorem 1.1. $(3 \Rightarrow 1)$ (i) ${}'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$: Since ${}'\overline{\mathcal{C}} \subseteq \mathcal{C}_{\overline{R}}$, it suffices to show that ${}'\overline{\mathcal{C}} \in \mathrm{Ore}_l(\overline{R})$. Given elements $s \in {}'\overline{\mathcal{C}}$ and $a \in \overline{R}$. Then $as^{-1} = c^{-1}b \in Q_{l,cl}(\overline{R})$ for some elements $c \in \mathcal{C}_{\overline{R}}$ and $b \in \overline{R}$, and so $ca = bs$. The set ${}'\overline{\mathcal{C}}$ is dense in $\mathcal{C}_{\overline{R}}$. So, $s_1 c \in {}'\overline{\mathcal{C}}$ for some element $s_1 \in \overline{R}$. Now, $s_1 c \cdot a = s_1 bs$. This means that ${}'\overline{\mathcal{C}} \in \mathrm{Ore}_l(\overline{R})$.

(ii) ${}'\overline{\mathcal{C}}^{-1}\overline{R} = Q_{l,cl}(\overline{R})$: The equality follows from Lemma 2.5.(4) in view of (i) and the fact that ${}'\overline{\mathcal{C}}$ is dense in $\mathcal{C}_{\overline{R}}$.

(iii) ${}'\mathcal{C} \in \mathrm{Ore}_l(R)$: Given $s \in {}'\mathcal{C}$ and $r \in R$. By (i), $s_1 r \equiv r_1 s \mod \mathfrak{a}$ for some elements $s_1 \in {}'\mathcal{C}$ and $r_1 \in R$. Since $s_1 r - r_1 s \in \mathfrak{a}$, we can find an element $s_2 \in {}'\mathcal{C}$ such that $s_2(s_1 r - r_1 s) = 0$, and so $s_2 s_1 \cdot r = s_2 r_1 \cdot s$.

(iv) ${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$: By (iii), $\mathrm{ass}_R({}'\mathcal{C}) = \mathfrak{a}$. Since every element of ${}'\mathcal{C}$ is left regular, the statement (iv) follows.

(v) ${}'Q \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R}$ (obvious).

By (ii) and (v), ${}'Q \simeq Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring.

$(2 \Rightarrow 1)$ (i) ${}'\overline{\mathcal{C}} \cap I \neq \emptyset$ *for all essential left ideals* I *of* $\overline{R}$: By the assumptions (c), (d) and Theorem 2.3.(4), ${}'\mathcal{C}_{\overline{R}} \cap I \neq \emptyset$. Fix an element $c \in {}'\mathcal{C}_{\overline{R}} \cap I$. Since ${}'\overline{\mathcal{C}}$ is dense in ${}'\mathcal{C}_{\overline{R}}$, $sc \in {}'\overline{\mathcal{C}} \cap I$ for some element $s \in \overline{R}$.

(ii) ${}'\overline{\mathcal{C}} \in \mathrm{Ore}_l(\overline{R})$: Given elements $a \in \overline{R}$ and $c \in {}'\overline{\mathcal{C}}$, we have to show that ${}'\overline{\mathcal{C}}a \cap \overline{R}c \neq \emptyset$. Since $c \in {}'\overline{\mathcal{C}}$, $\mathrm{udim}({}_{\overline{R}}\overline{R}c) = \mathrm{udim}({}_{\overline{R}}\overline{R})$, the left ideal $\overline{R}c$ is an essential left ideal of $\overline{R}$. Then the left ideal of $\overline{R}$, $(\overline{R}c : a) := \{b \in \overline{R} \,|\, ba \in \overline{R}c\}$, is an essential left ideal. By the statement (i), we can find an element $c' \in {}'\overline{\mathcal{C}} \cap (\overline{R}c : a)$, and so $c'a = a'c$ for some element $a' \in \overline{R}$.

(iii) ${}'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$: In view of (ii), let us show that $\mathrm{ass}_{\overline{R}}({}'\overline{\mathcal{C}}) = 0$. If $\overline{cr} = 0$ for some elements $c \in {}'\mathcal{C}$ and $r \in R$ (where $\overline{c} = c + \mathfrak{a} \in {}'\overline{\mathcal{C}}$ and $\overline{r} = r + \mathfrak{a} \in \overline{R}$) then $cr \in \mathfrak{a}$. Hence, $c'cr = 0$ for some element $c' \in {}'\mathcal{C}$, and so $r \in \mathfrak{a}$ and $\overline{r} = 0$. Therefore, $\mathrm{ass}_{\overline{R}}({}'\overline{\mathcal{C}}) = 0$. It remains to show that if $\overline{rc} = 0$ for some elements $c \in {}'\mathcal{C}$ and $r \in R$ then $\overline{r} = 0$, i.e. $r \in \mathfrak{a}$. Clearly, $rc \in \mathfrak{a}$, hence $c_1 rc = 0$ for some element $c_1 \in {}'\mathcal{C}$. Since $c \in {}'\mathcal{C}$, $c_1 r = 0$, and so $r \in \mathfrak{a}$, as required.

(iv) ${}'\mathcal{C} \in \mathrm{Ore}_l(R)$: Given $s \in {}'\mathcal{C}$ and $r \in R$. By (ii), $s_1 r \equiv r_1 s \mod \mathfrak{a}$ for some elements $s_1 \in {}'\mathcal{C}$ and $r_1 \in R$. Since $s_1 r - r_1 s \in \mathfrak{a}$, we can find an element $s_2 \in {}'\mathcal{C}$ such that $s_2(s_1 r - r_1 s) = 0$, and so $s_2 s_1 \cdot r = s_2 r_1 \cdot s$ where $s_2 s_1 \in {}'\mathcal{C}$.

(v) ${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$: By (iii), $\mathrm{ass}_R({}'\mathcal{C}) = \mathfrak{a}$. Since every element of ${}'\mathcal{C}$ is left regular, the statement (v) follows.

(vi) ${}'Q \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R}$ (obvious).

(vii) ${}'\overline{Q} := {}'\overline{\mathcal{C}}^{-1}\overline{R}$ *is a semisimple Artinian ring*: Since $\mathrm{udim}({}_{\overline{R}}\overline{R}) < \infty$, we can fix an essential direct sum $I = \oplus_{i=1}^n U_i$ of uniform left ideals U_i of the ring $\overline{R}$. By (i), $I \cap {}'\overline{\mathcal{C}} \neq \emptyset$. Hence, the $\overline{R}$-module R/I is ${}'\overline{\mathcal{C}}$-torsion. Therefore, ${}'\overline{Q} \simeq \oplus_{i=1}^n {}'\overline{\mathcal{C}}^{-1} U_i$, an isomorphism of ${}'\overline{Q}$-modules. It suffices to show that each ${}'\overline{Q}$-module $V_i = {}'\overline{\mathcal{C}}^{-1} U_i$ is simple. Suppose that, say V_1 is not, we seek a contradiction. Then it contains a proper ${}'\overline{Q}$-submodule, say M. By (iii), ${}_R U_1$ is an essential submodule of ${}_R V_1$, the intersection $U_1' = U_1 \cap M$ is nonzero and ${}'\overline{\mathcal{C}}^{-1} U_1' = M$. The left ideal $J = U_1' \oplus U_2 \oplus \cdots \oplus U_n$ of R is an essential left ideal such that ${}'\overline{\mathcal{C}}^{-1} J = M \oplus V_2 \oplus \cdots \oplus V_n$ is a proper left ideal of the ring ${}'\overline{Q}$. By (i), $I' \cap {}'\overline{\mathcal{C}} \neq \emptyset$, and so ${}'\overline{\mathcal{C}}^{-1} J = {}'\overline{Q}$, a contradiction.

$(2 \Rightarrow 3)$ We continue the proof of the implication $(2 \Rightarrow 1)$.

(viii) ${}'\overline{Q} = Q_{l,cl}(\overline{R})$ *is a semisimple Artinian ring*: This follows from (iii) and (vii).

(ix) ${}'\overline{\mathcal{C}}$ *is dense in* $\mathcal{C}_{\overline{R}}$: In view of the statements (iii) and (viii), this follows from Lemma 2.5.(3).

$(1 \Rightarrow 3)$ (α) ${}'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$ *and* ${}'Q \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R}$ (since ${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$).

(β) ${}'Q \simeq Q_{l,cl}(\overline{R})$ *is a semisimple Artinian ring* (by (α) and the simplicity of the ring ${}'Q \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R}$).

(γ) $\mathfrak{a}$ *is a semiprime ideal of* R (by (β)).

(δ) ${}'\overline{\mathcal{C}}$ *is dense in* $\mathcal{C}_{\overline{R}}$: By the statements ($\alpha$) and ($\beta$), ${}'\overline{\mathcal{C}}^{-1}\overline{R} \simeq Q_{l,cl}(\overline{R})$. Now, the statement ($\delta$) follows from Lemma 2.5.(3).

($3 \Rightarrow 2$) Recall that ($1 \Leftrightarrow 3$).

(a) $\mathfrak{a}$ is a semiprime ideal of R (this is given).

(b) ${}'\overline{\mathcal{C}}$ *is dense in* ${}'\mathcal{C}_{\overline{R}}$: Repeat the proof of the above statement (δ) bearing in mind that the statements (α) and (β) hold in view of the equivalence ($1 \Leftrightarrow 3$).

(c) $\mathrm{udim}({}_{\overline{R}}\overline{R}) < \infty$: This follows from the fact that $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring.

(d) ${}'\mathcal{C}_V \neq \emptyset$ *for all uniform left ideals* V *of* $\overline{R}$ (by Theorem 1.2.(2), since $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring). $\square$

Corollary 2.6 *We keep the notation of Theorem 1.1. If* ${}'Q$ *is a semisimple Artinian ring then* $\mathfrak{n} \subseteq \mathfrak{a}$ *(where* $\mathfrak{n}$ *is the prime radical of* R*).*

Proof. Repeat the proof of statement 5 of Theorem 2.3. $\square$

Proof of Theorem 1.3. ($1 \Rightarrow 2$) The first three conditions are obvious and the fourth holds by Theorem 1.2.

($2 \Rightarrow 1$) This implication follows from Theorem 1.1 and we keep its notation. Since ${}'\mathcal{C}_R = \mathcal{C}_R$, $\mathfrak{a} = 0$ and all the conditions (a) - (d) in Theorem 1.1.(2) hold, the ring ${}'Q = Q_{l,cl}(R)$ is a semisimple Artinian ring, by Theorem 1.1. $\square$

The left singular ideal of R **over** $\mathfrak{a}$. For a ring R, the set $\zeta_l(R) := \{r \in R \,|\, Ir = 0$ for some essential left ideal I of $R\}$ is called the *left singular ideal* of R. It is an ideal of R. Let $\mathfrak{a}$ be an ideal of R. The set $\zeta_l(R, \mathfrak{a}) := \{r \in R \,|\, Ir = 0$ for some essential left ideal I of R such that $\mathfrak{a} \subseteq I\}$ is called the *left singular ideal of* R *over* $\mathfrak{a}$. It is an ideal of R. Clearly, $\zeta_l(R, 0) = \zeta(R)$ and $\zeta_l(R, \mathfrak{a})$ is a right ideal of the ring R.

Lemma 2.7 *For all ideals* $\mathfrak{a}$ *of a ring* R*, the right ideal* $\zeta_l(R, \mathfrak{a})$ *is an ideal of the ring* R*.*

Proof. Let $r \in \zeta := \zeta_l(R, \mathfrak{a})$ and $Ir = 0$ for some essential left ideal I of R such that $\mathfrak{a} \subseteq I$. Let $r' \in R$. The map $f := \cdot r' : R \to R$, $x \mapsto xr'$, is an R-homomorphism. Then $f^{-1}(I) := \{a \in R \,|\, ar' \in I\}$ is an essential left ideal of R that contains the ideal $\mathfrak{a}$. Moreover, $f^{-1}(I) \cdot r'r \subseteq Ir = 0$, and so $r'r \in \zeta$. Therefore, ζ is an ideal of R. $\square$

Proposition 2.8 *Let* R *be a ring such that* ${}'Q_{l,cl}(R)$ *is a semisimple Artinian ring and* $\mathfrak{a} := \mathrm{ass}_R({}'\mathcal{C}_R)$*. Then* $\zeta_l(R, \mathfrak{a}) \subseteq \mathfrak{a}$*.*

Proof. We keep the notation of Theorem 1.1. Let $r \in \zeta := \zeta_l(R, \mathfrak{a})$. We have to show that $r \in \mathfrak{a}$. Fix an essential left ideal I of R such that $Ir = 0$ and $\mathfrak{a} \subseteq I$. Consider the set S of all left ideals J of R such that $J \subseteq I$ and $\mathfrak{a} \cap J = 0$. By Zorn's Lemma, let J be a maximal element in S. Then the left ideal $\mathfrak{a} + J = \mathfrak{a} \oplus J$ is an essential R-submodule of I, hence it is also an essential left ideal of R (since I is an essential left ideal of R).

Claim: $\overline{J} := \pi(J)$ *is an essential left ideal of* $\overline{R}$:

Suppose that this is not true. Then $\overline{J} \cap \overline{R}\overline{r}' = 0$ for some nonzero element $\overline{r}' = r' + \mathfrak{a} \in \overline{R}$ where $r' \in R$. The left ideal $\mathfrak{a} \oplus J$ of R is essential. So, $(\mathfrak{a} \oplus J) \cap Rr \neq 0$. Let $r''r' = a + j$ be a nonzero element in the intersection for some elements $r'' \in R$, $a \in \mathfrak{a}$ and $j \in J$. Then $ca = 0$ for some element $c \in {}'\mathcal{C}$, and so $cr''r' = cj \neq 0$ (otherwise, $cj = 0$, and so $j \in \mathfrak{a} \cap J = 0$, a contradiction). Now, $cj \in J \backslash \{0\}$, and so $0 \neq \overline{cr''r'} = \overline{cj} \in \overline{R}\overline{r}' \cap \overline{J} = 0$, a contradiction. So, $\overline{J}$ is an essential left ideal of the ring $\overline{R}$. This finishes the proof of the Claim.

Recall that $Ir = 0$ and $J \subseteq I$. In particular, $Jr = 0$, and so $\overline{J}\overline{r} = 0$ and $\overline{r} \in \zeta_l(\overline{R})$. By Theorem 1.1, ${}'Q := {}'Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring. Then $\zeta_l(\overline{R}) = 0$, by [9, Theorem 2.3.6]. Therefore, $\overline{r} = 0$. This means that $r \in \mathfrak{a}$, as required. $\square$

3 Semisimplicity criteria for the ring ${}'Q_{l,cl}(R)$

In this section, proofs are given of several semisimplicity criteria for the ring ${}'Q_{l,cl}(R)$ (Theorem 3.1, Theorem 3.3, Theorem 3.4 and Theorem 3.5). It is shown that the left localization radical $\mathfrak{l}_R$ of a ring R is contained in the ideal $\mathfrak{a} = \mathrm{ass}_R({}'\mathcal{C}_R)$ provided ${}'Q_{l,cl}(R)$ is a semisimple Artinian ring (Corollary 3.2). Theorem 3.6 gives sufficient conditions for semisimplicity of ${}'Q_{l,cl}(R)$ provided the ring $R/\mathfrak{n}$ is left Goldie (where $\mathfrak{n}$ is the prime radical of the ring R).

For a ring R, let ${}'\mathcal{C} = {}'\mathcal{C}_R$ and ${}'\mathcal{M} := \max.\mathrm{Den}_l(R, {}'\mathcal{C}) := \{S \in \max.\mathrm{Den}_l(R) \,|\, {}'\mathcal{C} \subseteq S\}$. The first semisimplicity criterion for ${}'Q_{l,cl}(R)$ is given via the set ${}'\mathcal{M}$ of maximal left denominator sets of R that contain the set ${}'\mathcal{C}_R$ of left regular elements of R.

Theorem 3.1 *Let R be a ring, ${}'\mathcal{C} = {}'\mathcal{C}_R$, $\mathfrak{a} = \mathrm{ass}_R({}'\mathcal{C})$ and ${}'\mathcal{M} = \{S \in \max.\mathrm{Den}_l(R) \,|\, {}'\mathcal{C} \subseteq S\}$. The following statements are equivalent.*

1. *${}'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. *${}'\mathcal{M}$ is a finite nonempty set, $\bigcap_{S \in {}'\mathcal{M}} \mathrm{ass}_R(S) = \mathfrak{a}$, for each $S \in {}'\mathcal{M}$, the ring $S^{-1}R$ is a simple Artinian ring and the set ${}'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{R/\mathfrak{a}}$ in $R/\mathfrak{a}$.*

Let $\overline{R} = R/\mathfrak{a}$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{a}$. If one of the equivalent conditions holds then

(a) the map ${}'\mathcal{M} \to \overline{\mathcal{M}} := \max.\mathrm{Den}_l(\overline{R})$, $S \mapsto \overline{S} := \pi(S)$, is a bijection with inverse $T \mapsto T' := \pi^{-1}(T)$.

(b) For all $S \in {}'\mathcal{M}$, $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$ and $\pi(\mathrm{ass}_R(S)) = \mathrm{ass}_{\overline{R}}(\overline{S})$. For all $T \in \overline{\mathcal{M}}$, $\pi^{-1}(\mathrm{ass}_{\overline{R}}(T)) = \mathrm{ass}_R(\pi^{-1}(T))$.

(c) For all $S \in {}'\mathcal{M}$, $S^{-1}R \simeq \overline{S}^{-1}\overline{R}$ is a simple Artinian ring.

(d) ${}'Q_{l,cl}(R) \simeq \prod_{S \in {}'\mathcal{M}} S^{-1}R \simeq \prod_{S \in {}'\mathcal{M}} \overline{S}^{-1}\overline{R} \simeq \prod_{T \in \overline{\mathcal{M}}} T^{-1}\overline{R} \simeq Q_{l,cl}(\overline{R})$ (semisimple Artinian rings).

Proof. $(1 \Rightarrow (a) - (d))$ To prove the implication, first, we prove statements (i)-(vi) below and from which then the statements (a)-(d) are deduced.

By Theorem 1.1, $\mathfrak{a}$ is a semiprime ideal of the ring R, ${}'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$ and $Q_{l,cl}(R) \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R} \simeq {}'Q$ is a semisimple Artinian ring.

(i) ${}'\overline{\mathcal{C}} \subseteq T$ *for all* $T \in \overline{\mathcal{M}}$: Recall that $S_l(\overline{R})$ is the largest left Ore set of $\overline{R}$ that consists of regular elements of the ring $\overline{R}$. Hence,

$$ {}'\overline{\mathcal{C}} \subseteq S_l(\overline{R}) \subseteq T, $$

for all $T \in \overline{\mathcal{M}}$, by [4, Proposition 2.10.(1)].

(ii) *For all* $T \in \overline{\mathcal{M}}$, $T' := \pi^{-1}(T) \in \mathrm{Den}_l(R)$, $\mathfrak{a} \subseteq \mathrm{ass}_R(T')$ *and* $T'^{-1}R \simeq T^{-1}\overline{R}$: Since ${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$ and ${}'\mathcal{C} \subseteq \pi^{-1}({}'\overline{\mathcal{C}}) \subseteq \pi^{-1}(T) = T'$ (by (i)), the result follows from [6, Lemma 2.11].

(iii) *Given distinct* $T_1, T_2 \in \overline{\mathcal{M}}$, *then* $T_1' \neq T_2'$: Suppose that $T_1' = T_2'$. Then, $T_1 = \pi(T_1') = \pi(T_2') = T_2$, a contradiction.

(iv) *For all* $T \in \overline{\mathcal{M}}$, $T' \in {}'\mathcal{M}$: By (ii), $T' \in \mathrm{Den}_l(R)$. Then $T' \subseteq S$ for some $S \in \max.\mathrm{Den}_l(R)$. Then $S \in {}'\mathcal{M}$, since

$$ {}'\mathcal{C} \subseteq \pi^{-1}({}'\overline{\mathcal{C}}) \overset{\text{(i)}}{\subseteq} \pi^{-1}(T) = T' \subseteq S. $$

Now, $T = \pi(T') \subseteq \pi(S) = \overline{S}$. Since $S \in {}'\mathcal{M}$, we have ${}'\mathcal{C} \subseteq S$ and so $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$. Therefore, $\overline{S} \in \mathrm{Den}_l(R, \mathrm{ass}_R(S)/\mathfrak{a})$, and so $\overline{S} \subseteq T_1$ for some $T_1 \in \overline{\mathcal{M}}$. Then $T \subseteq \overline{S} \subseteq T_1$, hence $T = T_1$ (since $T, T_1 \in \overline{\mathcal{M}}$) and $T = \overline{S}$. Now, $T' = \pi^{-1}(T) = \pi^{-1}(\overline{S}) \supseteq S \supseteq T'$. Therefore, $T' = S \in {}'\mathcal{M}$.

(v) *For all* $S \in {}'\mathcal{M}$, $S + \mathfrak{a} = S$ *and* $S = \pi^{-1}(\overline{S})$: For an arbitrary ring A and its maximal left denominator set S', $S' + \mathrm{ass}_A(S') = S'$, by [6, Corollary 2.12]. Since $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$, we have $S + \mathfrak{a} = S$, hence $S = \pi^{-1}(\overline{S})$.

(vi) *For all* $S \in {}'\mathcal{M}$, $\overline{S} \in \overline{\mathcal{M}}$: Since $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$, $\overline{S} \in \mathrm{Den}_l(\overline{R}, \mathrm{ass}_R(S)/\mathfrak{a})$. Therefore, $\overline{S} \subseteq T$ for some $T \in \overline{\mathcal{M}}$. Now,

$$S \overset{(v)}{=} \pi^{-1}(\overline{S}) \subseteq \pi^{-1}(T) = T' \in {}'\mathcal{M},$$

by (iv). Therefore, $S = T'$, and so $\overline{S} = \overline{T'} = T \in \overline{\mathcal{M}}$.

Now, we are ready to prove the statements (a) - (d).

(a) By (iv) and (vi), the maps ${}'\mathcal{M} \to \overline{\mathcal{M}}$, $S \mapsto \overline{S}$, and $\overline{\mathcal{M}} \to {}'\mathcal{M}$, $T \mapsto T' := \pi^{-1}(T)$, are well-defined. They are inverses of one another since $S \to \overline{S} \to \pi^{-1}(\overline{S}) = S$ (by (v)); and $T \to T' \to \pi(T') = T$ (since π is a surjection).

(b) Let $S \in {}'\mathcal{M}$. Then ${}'\mathcal{C} \subseteq S$, and so $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$. Therefore, $\overline{S} \in \mathrm{Den}_l(\overline{R}, \mathrm{ass}_R(S)/\mathfrak{a})$, and so $\pi(\mathrm{ass}_R(S)) = \mathrm{ass}_{\overline{R}}(\overline{S})$.

If $T \in \overline{\mathcal{M}}$ then $T' \in {}'\mathcal{M}$ (by the statement (a)), and so $T = \pi(T') \in \mathrm{Den}_l(\overline{R}, \mathrm{ass}_R(T')/\mathfrak{a})$. It follows that $\pi^{-1}(\mathrm{ass}_{\overline{R}}(T)/\mathfrak{a}) = \mathrm{ass}_R(T')$ (since $\mathfrak{a} \subseteq \mathrm{ass}_R(T')$).

(c) By the statement (a), for all $S \in {}'\mathcal{M}$, $\overline{S} \in \overline{\mathcal{M}}$ and $S^{-1}R \simeq \overline{S}^{-1}\overline{R}$. Since $\overline{S} \in \overline{\mathcal{M}}$, the ring $\overline{S}^{-1}\overline{R}$ is a simple Artinian ring, by [4, Theorem 3.1] (since $Q_{l,cl}(\overline{R}) \simeq {}'Q$ is a semisimple Artinian ring).

(d)

$$\begin{aligned} {}'Q &\simeq Q_{l,cl}(\overline{R}) && \text{(Theorem 1.1)} \\ &\simeq \prod_{T \in \overline{\mathcal{M}}} T^{-1}\overline{R} && \text{([4, Theorem 3.1])} \\ &\simeq \prod_{S \in {}'\mathcal{M}} \overline{S}^{-1}\overline{R} && \text{(the statement (a))} \\ &\simeq \prod_{S \in {}'\mathcal{M}} S^{-1}R && \text{(the statement (c)).} \end{aligned}$$

($1 \Rightarrow 2$) Recall that ($1 \Rightarrow (a) - (d)$) and statement 2 follows from the statements (a)-(d). In more detail,

(i) $1 \leq |{}'\mathcal{M}| < \infty$: By the statement (a), $|{}'\mathcal{M}| = |\overline{\mathcal{M}}|$ and $\overline{\mathcal{M}}$ is a finite set, by [4, Theorem 3.1].

(ii) $\bigcap_{S \in {}'\mathcal{M}} \mathrm{ass}_R(S) = \mathfrak{a}$: By the statement (b), $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$ for all $S \in {}'\mathcal{M}$, and so $\mathfrak{a} \subseteq \mathfrak{a}' := \bigcap_{S \in {}'\mathcal{M}} \mathrm{ass}_R(S)$. We have to show that $\mathfrak{a} = \mathfrak{a}'$, that is $\mathfrak{a}'/\mathfrak{a} = 0$. Now,

$$\mathfrak{a}'/\mathfrak{a} = \bigcap_{S \in {}'\mathcal{M}} \mathrm{ass}_R(S)/\mathfrak{a} = \bigcap_{S \in {}'\mathcal{M}} \pi(\mathrm{ass}_R(S)) \overset{(b)}{=} \bigcap_{S \in {}'\mathcal{M}} \mathrm{ass}_{\overline{R}}(\overline{S}) \overset{(a)}{=} \bigcap_{\overline{S} \in \overline{\mathcal{M}}} \mathrm{ass}_{\overline{R}}(\overline{S}) = 0,$$

by [4, Theorem 3.1], since $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring (by the statement (d)).

(iii) For each $S \in {}'\mathcal{M}$, $S^{-1}R$ is a simple Artinian ring (by the statement (c)).

(iv) The set ${}'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$ (by Theorem 1.1).

($2 \Rightarrow 1$) It suffices to show that the conditions of Theorem 1.1.(3) hold. Since $\mathfrak{a} = \bigcap_{S \in {}'\mathcal{M}} \mathrm{ass}_R(S)$, the map

$$R/\mathfrak{a} \to \prod_{S \in {}'\mathcal{M}} S^{-1}R, \;\; r + \mathfrak{a} \mapsto (\frac{r}{1}, \ldots, \frac{r}{1}),$$

is an injection and the direct product is a semisimple Artinian ring. By [4, Theorem 6.2], $Q_{l,cl}(R/\mathfrak{a})$ is a semisimple Artinian ring. In particular, the ideal $\mathfrak{a}$ is a semiprime ideal of R. Now, by Theorem 1.1.(3), ${}'Q$ is a semisimple Artinian ring. $\square$

For a ring R, the ideal $\mathfrak{l}_R := \bigcap_{S \in \max.\mathrm{Den}_l(R)} \mathrm{ass}_R(S)$ is called the *left localization radical* of R, [2].

Corollary 3.2 *Let R be a ring such that ${}'Q_{l,cl}(R)$ is a semisimple Artinian ring. Then $\mathfrak{l}_R \subseteq \mathfrak{a}$ (where $\mathfrak{l}_R$ is the left localization radical of R and $\mathfrak{a} = \mathrm{ass}_R({}'\mathcal{C}_R)$).*

Proof. $\mathfrak{l}_R = \bigcap_{S\in \mathrm{max.Den}_l(R)} \mathrm{ass}_R(S) \subseteq \bigcap_{S\in '\mathcal{M}} \mathrm{ass}_R(S) = \mathfrak{a}$, by Theorem 3.1. $\square$

Let R be a ring and I be an ideal of R. We denote by $\mathrm{Min}_R(I)$ the set of minimal prime ideals over I. The map $\mathrm{Min}_R(I) \to \mathrm{Min}(R/I)$, $\mathfrak{p} \mapsto \mathfrak{p}/I$, is a bijection with the inverse $\mathfrak{q} \mapsto \pi_I^{-1}(\mathfrak{q})$ where $\pi_I : R \to R/I$, $r \mapsto r + I$.

The second semisimplicity criterion for $'Q_{l,cl}(R)$ is given via the minimal primes over $\mathfrak{a} = \mathrm{ass}_R('\mathcal{C}_R)$. It also gives an explicit description of the elements of the set $'\mathcal{M}$ (see Theorem 3.1 for a definition of $'\mathcal{M}$).

Theorem 3.3 *Let R be a ring, $'\mathcal{C} = '\mathcal{C}_R$ and $\mathfrak{a} = \mathrm{ass}_R('\mathcal{C})$. We keep the notation of Theorem 3.1.*

1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. *(a) $\mathfrak{a}$ is semiprime ideal of R and the set $\mathrm{Min}_R(\mathfrak{a})$ is a finite set.*
   (b) For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the set $S_\mathfrak{p} := \{c \in R \,|\, c + \mathfrak{p} \in \mathcal{C}_{R/\mathfrak{p}}\}$ is a left denominator set of the ring R with $\mathrm{ass}_R(S_\mathfrak{p}) = \mathfrak{p}$.
   (c) For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the ring $S_\mathfrak{p}^{-1}R$ is a simple Artinian ring.
   (d) The set $'\overline{\mathcal{C}} := \{c + \mathfrak{a} \,|\, c \in '\mathcal{C}\}$ is a dense subset of $\mathcal{C}_{R/\mathfrak{a}}$.

Let $\overline{R} = R/\mathfrak{a}$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{a}$. If one of the equivalent conditions holds then

(i) $'\mathcal{M} = \{S_\mathfrak{p} \,|\, \mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})\}$ and $\mathrm{ass}_R(S_\mathfrak{p}) = \mathfrak{p}$.

(ii) $\overline{\mathcal{M}} = \{S_{\overline{\mathfrak{p}}} \,|\, \overline{\mathfrak{p}} \in \mathrm{Min}(\overline{R})\}$ where $S_{\overline{\mathfrak{p}}} := \{\overline{r} \in \overline{R} \,|\, \overline{r} + \overline{\mathfrak{p}} \in \mathcal{C}_{\overline{R}/\overline{\mathfrak{p}}}\}$ and $\mathrm{ass}_{\overline{R}}(S_{\overline{\mathfrak{p}}}) = \overline{\mathfrak{p}}$.

(iii) For all $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, $S_\mathfrak{p}^{-1}R \simeq S_{\overline{\mathfrak{p}}}^{-1}\overline{R}$ is a simple Artinian ring.

(iv) $'Q_{l,cl}(R) \simeq \prod_{\mathfrak{p}\in\mathrm{Min}_R(\mathfrak{a})} S_\mathfrak{p}^{-1}R \simeq \prod_{\mathfrak{p}\in\mathrm{Min}(\overline{R})} S_{\overline{\mathfrak{p}}}^{-1}\overline{R} \simeq Q_{l,cl}(\overline{R})$ (semisimple Artinian rings).

Proof. $(1 \Rightarrow 2)$. By the assumption, $'Q$ is a semisimple Artinian ring. By Theorem 1.1, $\mathfrak{a}$ is semiprime ideal of R, the set $'\overline{\mathcal{C}}$ is dense in $\mathcal{C}_{\overline{R}}$ (this is the condition (d)) and the ring $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring.

By [4, Theorem 4.1], $\mathrm{Min}(\overline{R})$ is a finite set, $\overline{\mathcal{M}} = \{S_{\overline{\mathfrak{p}}} \,|\, \overline{\mathfrak{p}} \in \mathrm{Min}(\overline{R})\}$, $\mathrm{ass}_{\overline{R}}(S_{\overline{\mathfrak{p}}}) = \overline{\mathfrak{p}}$ and $S_{\overline{\mathfrak{p}}}^{-1}\overline{R}$ is a simple Artinian ring for all $\overline{\mathfrak{p}} \in \mathrm{Min}(\overline{R})$. By Theorem 3.1, $|\mathrm{Min}(\mathfrak{a})| = |\mathrm{Min}(\overline{R})| < \infty$, $'\mathcal{M} = \{S_\mathfrak{p} \,|\, \mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})\}$, $\mathrm{ass}_R(S_\mathfrak{p}) = \mathfrak{p}$ and $S_\mathfrak{p}^{-1}R \simeq S_{\overline{\mathfrak{p}}}^{-1}\overline{R}$ is a simple Artinian ring for all $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$. Therefore, the properties (a)-(d) hold.

$(2 \Rightarrow 1)$ It suffices to show that the conditions of statement 3 of Theorem 1.1 hold.

By the statement (a), the ideal $\mathfrak{a}$ is a semiprime ideal of R. By the statements (a) and (c), the map

$$R/\mathfrak{a} \to \prod_{\mathfrak{p}\in\mathrm{Min}(R,\mathfrak{a})} S_\mathfrak{p}^{-1}R, \;\; r + \mathfrak{a} \mapsto (\frac{r}{1}, \ldots, \frac{r}{1}),$$

is a ring monomorphism. The direct product above is a semisimple Artinian ring, by the statement (c). By [4, Theorem 6.2], the ring $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring. By Theorem 1.1, $'Q$ is a semisimple Artinian ring. So, the implication $(2 \Rightarrow 1)$ holds.

Now, the statements (i)-(iv) follow from Theorem 3.1. $\square$

A ring R is called *left Goldie* if it satisfies the a.c.c. on left annihilators and $\mathrm{udim}({}_RR) < \infty$. The third semisimplicity criterion for $'Q_{l,cl}(R)$ reveals its 'local nature' and is given via the rings $R/\mathfrak{p}$ where $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$.

Theorem 3.4 *We keep the notation of Theorem 3.3. The following statements are equivalent.*

1. *$'Q_{l,cl}(R)$ is a semisimple Artinian ring.*
2. *(a) $\mathfrak{a}$ is a semiprime ideal of R and the set $\mathrm{Min}_R(\mathfrak{a})$ is finite.*
   (b) For each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the ring $R/\mathfrak{p}$ is a left Goldie ring.

(c) *The set* $'\overline{\mathcal{C}}$ *is a dense subset of* $\mathcal{C}_{\overline{R}}$.

Proof. ($1 \Rightarrow 2$) Suppose that the ring $'Q = 'Q_{l,cl}(R)$ is a semisimple Artinian ring. By Theorem 3.3, the conditions (a) and (c) hold, and for each $\mathfrak{p} \in \mathrm{Min}_R(\mathfrak{a})$, the rings $S_\mathfrak{p}^{-1}R$ are simple Artinian rings (the statement (iii) of Theorem 3.3). Let $\pi_\mathfrak{p} : R \to R/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$. Then $\pi_\mathfrak{p}(S_\mathfrak{p}) \in \mathrm{Den}_l(R/\mathfrak{p}, 0)$ (since $\mathrm{ass}_R(S_\mathfrak{p}) = \mathfrak{p}$, the statement (i) of Theorem 3.3) and $\pi_\mathfrak{p}(S_\mathfrak{p})^{-1}(R/\mathfrak{p}) \simeq S_\mathfrak{p}^{-1}R$ is a simple Artinian ring. Then, $Q_{l,cl}(R/\mathfrak{p}) \simeq \pi_\mathfrak{p}(S_\mathfrak{p})^{-1}(R/\mathfrak{p})$ is a simple Artinian ring. So, the statement (b) holds.

($2 \Rightarrow 1$) Suppose that the conditions (a)-(c) hold. The conditions (a) and (b) means that the ring $\overline{R} = R/\mathfrak{a}$ is a semiprime ring with $|\mathrm{Min}(\overline{R})| = |\mathrm{Min}_R(\mathfrak{a})| < \infty$. By [4, Theorem 5.1], $Q_{l,cl}(R/\mathfrak{a})$ is a semisimple Artinian ring. By Theorem 1.1, $'Q$ is a semisimple Artinian ring. $\square$

The fourth semisimplicity criterion for $'Q_{l,cl}(R)$ is useful in applications as usually there are plenty of 'nice' left denominator sets.

Theorem 3.5 *We keep the notation of Theorem 3.4. The following statements are equivalent.*

1. $'Q_{l,cl}(R)$ *is a semisimple Artinian ring.*
2. *There are left denominator sets* $S_1, \ldots, S_n$ *of the ring* R *such that*
   (a) *the rings* $S_i^{-1}R$ *are simple Artinian rings,*
   (b) $\mathfrak{a} = \bigcap_{i=1}^n \mathrm{ass}_R(S_i)$, *and*
   (c) $'\overline{\mathcal{C}}$ *is a dense subset of* $\mathcal{C}_{\overline{R}}$.

Proof. ($1 \Rightarrow 2$) By Theorem 3.1, it suffices to take $'\mathcal{M} = \{S_1, \ldots, S_n\}$.

($2 \Rightarrow 1$) Suppose that the conditions (a)-(c) hold. By the conditions (a) and (b), the map

$$R/\mathfrak{a} \to \prod_{i=1}^n S_i^{-1}R, \;\; r + \mathfrak{a} \mapsto (\frac{r}{1}, \ldots, \frac{r}{1}),$$

is a ring monomorphism. The direct product above is a semisimple Artinian ring, by the statement (c). By [4, Theorem 6.2], the ring $Q_{l,cl}(\overline{R})$ is a semisimple Artinian ring. So, the conditions of Theorem 1.1.(3) hold, and so $'Q_{l,cl}(R)$ is a semisimple Artinian ring, by Theorem 1.1. $\square$

Sufficient conditions for semisimplicity of $'Q_{l,cl}(R)$ when $R/\mathfrak{n}$ is a left Goldie ring. Let R be a ring and I be its ideal. Let $\mathrm{Min}(R, I) := \{\mathfrak{p} \in \mathrm{Min}(R) \,|\, \mathfrak{p} \supseteq I\}$. An important case for applications is the one when the ring $R/\mathfrak{n}$ is a left Goldie ring, and therefore $Q_{l,cl}(R/\mathfrak{n})$ is a semisimple Artinian ring. In this case, the next theorem gives sufficient conditions for semisimplicity of the ring $'Q_{l,cl}(R)$.

Theorem 3.6 *Let* R *be a ring,* $'\mathcal{C} = '\mathcal{C}_R$ *and* $\mathfrak{a} = \mathrm{ass}_R('\mathcal{C})$. *Suppose that the ring* $Q_{l,cl}(R/\mathfrak{n})$ *is a semisimple Artinian ring such that* $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{p}_i$ *for some* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathrm{Min}(R)$. *If the set* $'\overline{\mathcal{C}} := \{c + \mathfrak{a} \,|\, c \in '\mathcal{C}_R\}$ *is dense in* $\mathcal{C}_{R/\mathfrak{a}}$ *then* $'Q_{l,cl}(R)$ *is a semisimple Artinian ring,* $\mathrm{Min}_R(\mathfrak{a}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ *and* $Q_{l,cl}(R/\mathfrak{n}) \simeq \prod_{i=1}^n S_{\mathfrak{p}_i}^{-1}R$ *where* $S_{\mathfrak{p}_i}^{-1}R$ *are simple Artinian rings and* $S_{\mathfrak{p}_i} := \{c \in R \,|\, c + \mathfrak{p}_i \in \mathcal{C}_{R/\mathfrak{p}_i}\} \in \mathrm{Den}_l(R, \mathfrak{p}_i)$.

Proof. It is obvious that $\mathrm{Min}_R(\mathfrak{a}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. The ring $Q_{l,cl}(R/\mathfrak{n})$ is a semisimple Artinian ring. By [4, Theorem 4.1], $\mathrm{Min}(R/\mathfrak{n}) = \{\overline{\mathfrak{p}}_1, \ldots, \overline{\mathfrak{p}}_m\}$ for some $m \geq n$ where $\overline{\mathfrak{p}}_i = \mathfrak{p}_i/\mathfrak{n}$ and $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$, $Q_{l,cl}(R/\mathfrak{n}) \simeq \prod_{i=1}^m S_{\overline{\mathfrak{p}}_i}^{-1}(R/\mathfrak{n})$ where $S_{\overline{\mathfrak{p}}_i}^{-1}(R/\mathfrak{n})$ are simple Artinian rings and $S_{\overline{\mathfrak{p}}_i} := \{c \in R/\mathfrak{n} \,|\, c + \mathfrak{p}_i/\mathfrak{n} \in \mathcal{C}_{R/\mathfrak{p}_i}\}$. The map

$$R/\mathfrak{a} \simeq (R/\mathfrak{n})/\bigcap_{i=1}^n \overline{\mathfrak{p}}_i \to \prod_{i=1}^n S_{\overline{\mathfrak{p}}_i}^{-1}(R/\mathfrak{n}), \;\; r + \mathfrak{a} \mapsto (\frac{r}{1}, \ldots, \frac{r}{1}),$$

is a monomorphism and the direct product is a semisimple Artinian ring. Since $\mathfrak{a}/\mathfrak{n} = \bigcap_{i=1}^n \overline{\mathfrak{p}}_i$, the conditions (a)-(c) of Theorem 3.5 hold (where $S_i = S_{\overline{\mathfrak{p}}_i}$), and so $'Q_{l,cl}(R)$ is semisimple Artinian ring (by Theorem 3.5). The rest follows from Theorem 3.3. $\square$

4 The left regular left quotient ring of a ring and its semisimplicity criteria

The aim of this section is to prove Theorem 4.3 and to establish some relations between the rings $Q_l(R)$ and $'Q_l(R)$ (Lemma 4.2, Proposition 4.4 and Corollary 4.5). In particular, to show that the rings $Q_l(R)$ and $'Q_l(R)$ are R-isomorphic iff $S_l(R) = {}'S_l(R)$ (Proposition 4.4.(4)). At the end of the section, some applications are given for the algebras of polynomial integro-differential operators.

The left regular left quotient ring $'Q_l(R)$ of a ring R. Let R be a ring. Its *opposite ring* R^{op} is a ring such that $R^{op} = R$ (as additive groups) but the multiplication in R^{op} is given by the rule $a \cdot b = ba$. Recall that $'\mathcal{C}_R$ and $\mathcal{C}'_R$ are the sets of left and right regular elements of the ring R, respectively, and $S_l(R)$ and $S_r(R)$ are the largest left and right Ore sets of R that consists of regular elements of R.

Lemma 4.1 *Let R be a ring.*

1. *In the set $'\mathcal{C}_R$ there exists the largest (w.r.t. inclusion) left denominator set of R, denoted by $'S_l(R)$. The set $S_r(R)$ is the largest (w.r.t. inclusion) right denominator set of R in $'\mathcal{C}_R$.*

2. *In the set $\mathcal{C}'_R$ there exists the largest (w.r.t. inclusion) right denominator set of R, denoted by $S'_r(R)$. The set $S_l(R)$ is the largest (w.r.t. inclusion) left denominator set of R in $\mathcal{C}'_R$.*

Proof. 1. If S and T are left denominator sets of the ring R such that $S, T \subseteq {}'\mathcal{C}_R$. The submonoid, denoted by ST, of $'\mathcal{C}_R$ that is generated by S and T does not contain 0. By [6, Lemma 2.4.(2)], ST is a left denominator set of R. Hence, the set $'S_l(R)$ exists and is the union of all left denominator sets of R in $'\mathcal{C}_R$.

If D is a right denominator set of R in $'\mathcal{C}_R$ then $\mathrm{ass}_R(D) = 0$, and so $D \subseteq \mathcal{C}_R$. Therefore, $S_r(R)$ is the largest right denominator set of R in $'\mathcal{C}_R$.

2. Statement 2 follows from statement 1 (by applying statement 1 to the opposite ring R^{op} of R). $\square$

Definition. The set $'S_l(R)$ is called the *largest left regular left denominator set* of R and the ring $'Q_l(R) := {}'S_l(R)^{-1}R$ is called the *left regular left quotient ring* of R. Similarly, the set $S'_r(R)$ is called the *largest right regular right denominator set* of R and the ring $Q'_r(R) := RS'_r(R)^{-1}$ is called the *right regular right quotient ring* of R.

If $'S_l(R) = {}'\mathcal{C}_R$ then $'Q_l(R) = {}'Q_{l,cl}(R)$. If $S'_r(R) = \mathcal{C}'_R$ then $Q'_r(R) = Q'_{r,cl}(R)$.

The next lemma shows that if the ring $Q_l(R)$ is a left Artinian ring (respectively, semisimple Artinian) ring then so is the ring $'Q_l(R)$. The reverse implication is usually wrong. For example, in the case of the algebra $\mathbb{I}_1 = K\langle x, \frac{d}{dx}, \int\rangle$ of the polynomial integro-differential operators over a field K of characteristic zero, the ring $Q_l(\mathbb{I}_1)$ is neither left nor right Noetherian ring and not a domain (see [2]) but the ring $'Q_l(\mathbb{I}_1)$ is a division ring and $'Q_l(\mathbb{I}_1) = {}'Q_{l,cl}(\mathbb{I}_1)$ (Theorem 6.5.(1)).

Lemma 4.2 *Let R be a ring.*

1. *If the ring $Q_l(R)$ is a left Artinian ring then $S_l(R) = \mathcal{C}_R = {}'\mathcal{C}_R = {}'S_l(R)$ and $Q_l(R) = Q_{l,cl}(R) = {}'Q_{l,cl}(R) = {}'Q_l(R)$ is a left Artinian ring.*

2. *If the ring $Q_l(R)$ is a semisimple Artinian ring then $S_l(R) = \mathcal{C}_R = {}'\mathcal{C}_R = {}'S_l(R)$ and $Q_l(R) = Q_{l,cl}(R) = {}'Q_{l,cl}(R) = {}'Q_l(R)$ is a semisimple Artinian ring.*

Proof. 1.If $Q_l(R)$ is a left Artinian ring then $'\mathcal{C}_R \subseteq Q_l(R)^*$. By [2, Theorem 2.8.(1)], $S_l(R) = R \cap Q_l(R)^*$. By intersecting with R the following inclusions of subsets of the ring $Q_l(R)$, $S_l(R) \subseteq \mathcal{C}_R \subseteq {}'\mathcal{C}_R \subseteq Q_l(R)^*$ and $S_l(R) \subseteq {}'S_l(R) \subseteq {}'\mathcal{C}_R \subseteq Q_l(R)^*$, we obtain the inclusions $S_l(R) \subseteq \mathcal{C}_R \subseteq {}'\mathcal{C}_R \subseteq S_l(R)$ and $S_l(R) \subseteq {}'S_l(R) \subseteq {}'\mathcal{C}_R \subseteq S_l(R)$. Therefore, $S_l(R) = \mathcal{C}_R = {}'\mathcal{C}_R = {}'S_l(R)$ and $Q_l(R) = Q_{l,cl}(R) = {}'Q_{l,cl}(R) = {}'Q_l(R)$ is a semisimple Artinian ring.

2. Statement 2 follows from statement 1. $\square$

Semisimplicity criteria for the ring $'Q_l(R)$. In general, the question of existence of the ring $'Q_{l,cl}(R)$ is a difficult one. In general, the ring $'Q_{l,cl}(R)$ does not exits but the ring $'Q_l(R)$ always does. If the ring $'Q_{l,cl}(R)$ exists then $'Q_{l,cl}(R) = {}'Q_l(R)$. The next theorem states that if the ring $'Q_l(R)$ is a left Artinian ring or a semisimple Artinian ring then so is the ring $'Q_{l,cl}(R)$, and vice versa.

Theorem 4.3 *Let R be a ring. Then*

1. *$'Q_l(R)$ is a left Artinian ring iff $'Q_{l,cl}(R)$ is a left Artinian ring. If one of the equivalent conditions holds then $'S_l(R) = {}'\mathcal{C}_R$ and $'Q_l(R) = {}'Q_{l,cl}(R)$.*

2. *$'Q_l(R)$ is a semisimple Artinian ring iff $'Q_{l,cl}(R)$ is a semisimple Artinian ring. If one of the equivalent conditions holds then $'S_l(R) = {}'\mathcal{C}_R$ and $'Q_l(R) = {}'Q_{l,cl}(R)$.*

Proof. 1. ($\Rightarrow$) Let $'S := {}'S_l(R)$, $'\mathfrak{a} := \mathrm{ass}_R('S)$ and $'\pi : R \to R/'\mathfrak{a}$, $r \mapsto \overline{r} := r + {}'\mathfrak{a}$.

(i) $'\pi('\mathcal{C}_R) \subseteq {}'\mathcal{C}_{R/'\mathfrak{a}}$: Suppose that $\overline{rc} = 0$ for some elements $r \in R$ and $c \in {}'\mathcal{C}_R$. We have to show that $r \in {}'\mathfrak{a}$. The element $a := rc$ belongs to the ideal $'\mathfrak{a}$. Then $0 = sa = src$ for some element $s \in {}'S$, and so $sr = 0$ (since $c \in {}'\mathcal{C}_R$). Therefore, $r \in {}'\mathfrak{a}$.

(ii) $'\mathcal{C}_{R/'\mathfrak{a}} = \mathcal{C}_{R/'\mathfrak{a}}$: Since $'S \in \mathrm{Den}_l(R, {}'\mathfrak{a})$, we have the inclusion $'\pi('S) \in \mathrm{Den}_l(R/'\mathfrak{a}, 0)$ and the ring $'Q_l(R) \simeq {}'\pi('S)^{-1}(R/'\mathfrak{a})$ is a left Artinian ring. Hence, $'\mathcal{C}_{R/'\mathfrak{a}} = \mathcal{C}_{R/'\mathfrak{a}}$.

(iii) $\mathfrak{a} := \mathrm{ass}_R('\mathcal{C}_R) = {}'\mathfrak{a}$: The inclusion $'S \subseteq {}'\mathcal{C}_R$ implies the inclusion $'\mathfrak{a} \subseteq \mathfrak{a}$. Then, by the statements (i) and (ii), $'\pi('\mathcal{C}_R) \subseteq \mathcal{C}_{R/'\mathfrak{a}}$, and so we must have $'\mathfrak{a} = \mathfrak{a}$.

(iv) $'\mathcal{C}_R \in \mathrm{Ore}_l(R)$: Let $c \in {}'\mathcal{C}_R$ and $r \in R$. By the statements (i) and (ii), $'\pi('\mathcal{C}_R) \subseteq \mathcal{C}_{R/'\mathfrak{a}}$. Hence, $\overline{c} \in \mathcal{C}_{R/'\mathfrak{a}}$. By the assumption, the ring $'Q_l(R) \simeq {}'\pi('S)^{-1}(R/'\mathfrak{a})$ is a left Artinian ring. Hence, $Q_{l,cl}(R/'\mathfrak{a}) \simeq {}'Q_l(R)$ is a left Artinian ring and the elements of the set $\mathcal{C}_{R/'\mathfrak{a}}$ are units in the ring $'Q_l(R)$. In particular, the element $\overline{c}$ is so. So, $\overline{rc}^{-1} = \overline{s}^{-1}\overline{a}$ for some elements $s \in {}'S$ and $a \in R$. Then $sr - ac \in {}'\mathfrak{a}$, and so $s'(sr - ac) = 0$ for some elements $s' \in {}'S$. So, $s's \cdot r = s'a \cdot c$ where $s's \in {}'S \subseteq {}'\mathcal{C}_R$, and the statement (iv) follows.

(v) $'\mathcal{C}_R \in \mathrm{Den}_l(R, {}'\mathfrak{a})$: This follows from the statements (iii) and (iv).

(vi) $'S = {}'\mathcal{C}_R$ (by the maximality of $'S$) and so $'Q_l(R) = {}'Q_{l,cl}(R)$ is a left Artinian ring.

($\Leftarrow$) This implication is obvious.

2. Statement 2 follows from statement 1. $\square$

In view of Theorem 4.3, *all the semisimplicity criteria for the ring $'Q_{l,cl}(R)$ are also semisimplicity criteria for the ring $'Q_l(R)$, and vice versa.*

The canonical homomorphism $\phi : Q_l(R) \to {}'Q_l(R)$. The next proposition shows that there is a canonical ring homomorphism $\phi : Q_l(R) \to {}'Q_l(R)$ and gives a criterion for ϕ to be an isomorphism.

Proposition 4.4 *Let R be a ring. Then*

1. *$S_l(R) \subseteq {}'S_l(R) \subseteq {}'\mathcal{C}_R$, and so $\mathrm{ass}_R(S_l(R)) \subseteq \mathrm{ass}_R('S_l(R)) \subseteq \mathrm{ass}_R('\mathcal{C}_R)$.*

2. *The map $\phi : Q_l(R) \to {}'Q_l(R)$, $s^{-1}r \mapsto s^{-1}r$, is a ring R-homomorphism with kernel $S_l(R)^{-1}{}'\mathfrak{a}$ where $'\mathfrak{a} = \mathrm{ass}_R('S_l(R))$.*

3. *ϕ is an isomorphism iff $'\mathfrak{a} = 0$ iff $S_l(R) = {}'S_l(R)$.*

4. *The rings $Q_l(R)$ and $'Q_l(R)$ are R-isomorphic iff one of the equivalent conditions of statement 3 holds.*

Proof. 1. Notice that $S_l(R) \subseteq \mathcal{C}_R \subseteq {}'\mathcal{C}_R$. By Lemma 4.1.(1), $S_l(R) \subseteq {}'S_l(R) \subseteq {}'\mathcal{C}_R$.

2. Statement 2 follows from statement 1: By statement 1, the map ϕ is well-defined. If $\phi(s^{-1}r) = 0$ then $\frac{r}{1} = 0$ in $'Q_l(R)$, and so $r \in {}'\mathfrak{a}$, i.e. $\ker(\phi) = S_l(R)^{-1}{}'\mathfrak{a}$.

3. ϕ is an isomorphism iff $\ker(\phi) = 0$ and ϕ is a surjection iff $'\mathfrak{a} = 0$ (since $S_l(R) \subseteq \mathcal{C}_R$) and ϕ is a surjection iff $S_l(R) = {}'S_l(R)$ and ϕ is a surjection iff $S_l(R) = {}'S_l(R)$ iff $'\mathfrak{a} = 0$.

4. We have to show that ($3 \Leftrightarrow 4$). The implication ($3 \Rightarrow 4$) is obvious. Conversely, suppose that $\varphi : Q_l(R) \to {}'Q_l(R)$ is an R-isomorphism ($\varphi(rq) = r\varphi(q)$ for all $r \in R$ and $q \in Q_l(R)$). Then $R \subseteq Q_l(R)$ and $'\mathfrak{a} \subseteq \ker(\varphi) = 0$, i.e. $'\mathfrak{a} = 0$. So, statement 3 holds. $\square$

The next corollary shows that if $S_l(R) \neq {}'S_l(R)$ then the ring $Q_l(R)$ is not left Artinian.

Corollary 4.5 *1. Let R be a ring such that $S_l(R) \neq {}'S_l(R)$ or, equivalently, ${}'\mathfrak{a} := \mathrm{ass}_R({}'S_l(R)) \neq 0$ (Proposition 4.4.(3)). Then $Q_l(R)$ is not a left Artinian ring. In particular, $Q_l(R)$ is not a semisimple Artinian ring.*

2. If, in addition, the ring R is a K-algebra over a field K then, for all algebras A, $Q_l(R \otimes A)$ is not a left Artinian ring. In particular, $Q_l(R \otimes A)$ is not a semisimple Artinian ring.

Proof. 1. Suppose that the ring $Q_l(R)$ is a left Artinian ring. Then, by Lemma 4.2.(1), ${}'S_l(R) = \mathcal{C}_R$, and so ${}'\mathfrak{a} = 0$, a contradiction.

2. Clearly, ${}'\mathcal{C}_R \subseteq {}'\mathcal{C}_{R\otimes A}$. Hence, ${}'S_l(R) \subseteq {}'S_{R\otimes A}$ (by Lemma 4.1.(1)), and so $0 \neq {}'\mathfrak{a} = \mathrm{ass}_R({}'S_l) \subseteq \mathrm{ass}_{R\otimes A}({}'S_{R\otimes A})$. By statement 1, $Q_l(R \otimes A)$ is not a left Artinian ring. $\square$

Applications to the algebras of polynomial integro-differential operators. Let K be a field of characteristic zero, $K[x]$ be a polynomial algebra in a single variable x, $\partial := \frac{d}{dx}$ and $\int : K[x] \to K[x]$, $x^n \mapsto \frac{x^{n+1}}{n+1}$ for all $n \geq 0$ be the integration. The following subalgebras of $\mathrm{End}_K(K[x])$, $A_1 = K\langle x, \partial\rangle$ and $\mathbb{I}_1 = K\langle x, \partial, \int\rangle$, are called the *first Weyl algebra* and the *algebra of polynomial integro-differential operators*, respectively. By definition, $A_n := A_1^{\otimes n}$ is called the n'th *Weyl algebra* and $\mathbb{I}_n := \mathbb{I}_1^{\otimes n}$ is called the *algebra of polynomial integro-differential operators.* The Weyl algebra A_n is a Noetherian domain, and so $Q_{l,cl}(A_n)$ is a division ring. For the algebra $\mathbb{I}_n$, neither the ring $Q_{l,cl}(\mathbb{I}_n)$ nor the ring $Q_{r,cl}(\mathbb{I}_n)$ exists (Lemma 4.8.(1)). The rings $Q_l(\mathbb{I}_n)$ and $Q_r(\mathbb{I}_n)$ are neither left nor right Noetherian and not domains (Lemma 4.8.(2)).

As an easy application of Corollary 4.5 we have the next result. A more strong result of that kind is Lemma 4.8 where different arguments are used in its proof.

Corollary 4.6 *For all $n \geq 1$, the rings $Q_l(\mathbb{I}_n)$ (resp., $Q_r(\mathbb{I}_n)$) are not left (resp., right) Artinian. Moreover, for all algebras A, the rings $Q_l(\mathbb{I}_n \otimes A)$ (resp., $Q_r(\mathbb{I}_n \otimes A)$) are not left (resp., right) Artinian.*

Proof. The ring $\mathbb{I}_n$ is isomorphic to its opposite ring [2]. In view of this fact and Corollary 4.5, it suffices to show that $S_l(\mathbb{I}_1) \neq {}'S_l(\mathbb{I}_1)$. The set $S_\partial := \{\partial^i \mid i \geq 0\}$ is a left denominator set of the algebra $\mathbb{I}_1$ (see [2]) with $\mathrm{ass}_{\mathbb{I}_1}(S_\partial) \neq 0$ since $\partial \cdot (1 - \int \partial) = \partial - \partial \int \partial = \partial - 1 \cdot \partial = 0$. Clearly, $\partial \in {}'\mathcal{C}_{\mathbb{I}_n}$ since $\partial \int = 1$. Then, $\partial \in {}'S_l(\mathbb{I}_1) \backslash S_l(\mathbb{I}_1)$, as required. $\square$

By Proposition 4.4, the ring homomorphism ϕ is the composition of the following ring homomorphisms:

$$\phi : Q_l(R) \stackrel{\pi'}{\to} \overline{Q} := Q_l(R)/S_l(R)^{-1}{}'\mathfrak{a} \to {}'Q_l(R) \simeq T^{-1}\overline{Q} \tag{1}$$

where $\pi'(a) = a + S_l(R)^{-1}{}'\mathfrak{a}$ and $T \in \mathrm{Den}_l(\overline{Q})$ is the multiplicative subset of $(\overline{Q}, \cdot)$ generated by the group of units $\overline{Q}^*$ of the ring $\overline{Q}$ and the set $\pi'({}'S_l(R))$.

Corollary 4.7 *Let R be a ring. Suppose that $\mathcal{P}$ is a property of rings that is preserved by left localizations and passing to factor ring. If the ring $Q_l(R)$ satisfies the property $\mathcal{P}$ then so does the ring ${}'Q_l(R)$. In particular, if the ring $Q_l(R)$ is a semisimple (respectively, left Artinian; left Noetherian) then so is the ring ${}'Q_l(R)$.*

Proof. The corollary follows from (1). $\square$

The next lemma gives plenty of examples of algebras for which neither left nor right classical quotient ring exists. This is true for the algebras $\mathbb{I}_n$.

Lemma 4.8 *Let A be an algebra over K.*

1. *The set $\mathcal{C}_{\mathbb{I}_1 \otimes A}$ of regular elements of the algebra $\mathbb{I}_n \otimes A$ is neither a left nor right Ore set. Therefore, the rings $Q_{l,cl}(\mathbb{I}_n \otimes A)$ and $Q_{r,cl}(\mathbb{I}_n \otimes A)$ do not exist.*

2. *The algebras $Q_l(\mathbb{I}_n \otimes A)$ contain infinite direct sums of nonzero left ideals and so they are not left Noetherian algebras. Similarly, the algebras $Q_r(\mathbb{I}_n \otimes A)$ contain infinite direct sums of nonzero right ideals and so they are not right Noetherian algebras.*

Proof. 1. Clearly, $\mathbb{I}_n \otimes A = \mathbb{I}_1 \otimes \mathbb{I}_{n-1} \otimes A$ and $\mathcal{C}_{\mathbb{I}_1} \subseteq \mathcal{C}_{\mathbb{I}_n \otimes A}$. Let $e_{00} := 1 - \int \partial$. The element $a := \partial_1 + \int_1$ belongs to the set $\mathcal{C}_{\mathbb{I}_1}$, $\mathbb{I}_1 e_{00} \cap \mathbb{I}_1 a = 0$ and $e_{00}\mathbb{I}_1 \cap a\mathbb{I}_1 = 0$, see the proof of [1, Theorem 9.7]. Hence, $a \in \mathcal{C}_{\mathbb{I}_n \otimes A}$ and $(\mathbb{I}_n \otimes A)e_{00} \cap (\mathbb{I}_n \otimes A)a = (\mathbb{I}_1 \otimes \mathbb{I}_{n-1} \otimes A)e_{00} \cap (\mathbb{I}_1 \otimes \mathbb{I}_{n-1} \otimes A)a = (\mathbb{I}_1 e_{00} \cap \mathbb{I}_1 a) \otimes \mathbb{I}_{n-1} \otimes A = 0$, and similarly $e_{00}(\mathbb{I}_n \otimes A) \cap a(\mathbb{I}_n \otimes A) = 0$. This means that $\mathcal{C}_{\mathbb{I}_n \otimes A} \notin \mathrm{Ore}_l(\mathbb{I}_n \otimes A) \cup \mathrm{Ore}_r(\mathbb{I}_n \otimes A)$.

2. The algebra $\mathbb{I}_n$ contains infinite direct sums of nonzero left ideals [1], hence so do the algebras $\mathbb{I}_n \otimes A$, and statement 2 follows. $\square$

5 Properties of $'S_l(R)$ and $'Q_{l,cl}(R)$

In this section, some properties of $'S_l(R)$ and $'Q_{l,cl}(R)$ are established (Theorem 5.3). The main motive is to develop practical tools for finding the ring $'Q_{l,cl}(R)$. A key idea is that in order to find $'Q_{l,cl}(R)$ there is no need to know explicitly the set $'S_l(R)$. It suffices to replace $'S_l(R)$ with another left denominator set that yields the same result, see Theorem 5.3.(5). Further developing this idea sufficient conditions are found for the ring $'Q_{l,cl}(R)$ to be isomorphic to $Q_{l,cl}(R/\mathrm{ass}_R('S_l(R)))$ (Theorem 5.4).

Lemma 5.1 *Let R be a ring, S be a multiplicative subset of $'\mathcal{C}_R$ such that $\mathfrak{a}' := \mathrm{ass}_R(S)$ is an ideal, $\pi' : R \to \overline{R}' := R/\mathfrak{a}'$, $r \to r + \mathfrak{a}'$. If $\pi'(S) \in \mathrm{Ore}_l(\overline{R}')$ then $S \in \mathrm{Den}_l(R, \mathfrak{a}')$.*

Proof. Since $S \subseteq {}'\mathcal{C}_R$, it suffices to show that $S \in \mathrm{Ore}_l(R)$. Given $s \in S$ and $r \in R$. Then $\overline{s}_1\overline{r} = \overline{r}_1\overline{s}$ for some elements $s_1 \in S$ and $r_1 \in R$ (since $\pi'(S) \in \mathrm{Ore}_l(\overline{R}')$). Then $s_1 r - r_1 s \in \mathfrak{a}'$ and so $s_2(s_1 r - r_1 s) = 0$ for some elements $s_2 \in S$, and we are done (since $s_2 s_1 \cdot r = s_2 r \cdot s$). $\square$

Proposition 5.2 *Let R be a ring, S be a multiplicative subset of $'\mathcal{C}_R$ such that $\mathfrak{a}' := \mathrm{ass}_R(S)$ is an ideal of R, $\pi' : R \to \overline{R}' := R/\mathfrak{a}'$, $r \to r + \mathfrak{a}'$. We keep the notation of Theorem 3.1. Then*

1. *$S \subseteq \{c \in R \mid \cdot c_{\mathfrak{a}'}$ and $\cdot c_{R/\mathfrak{a}'}$ are injections$\}$. If, in addition, $S = {}'\mathcal{C}_R$ then $'\mathcal{C}_R = \{c \in R \mid \cdot c_{\mathfrak{a}}$ and $\cdot c_{R/\mathfrak{a}}$ are injections$\}$ (where $\mathfrak{a} = \mathrm{ass}_R('\mathcal{C}_R)$ is an ideal of R, by the assumption).*
2. *$\pi'(S) \subseteq \mathcal{C}_{\overline{R}'}$. In particular, $\pi('\mathcal{C}_R) \subseteq \mathcal{C}_{\overline{R}}$ provided $\mathfrak{a}$ is an ideal of R.*
3. *$S \subseteq \{c \in R \mid \pi'(c) \in \mathcal{C}_{\overline{R}'}, \cdot c_{\mathfrak{a}'}$ is an injection$\}$. If $\mathfrak{a}$ is an ideal of R then $'\mathcal{C}_R = \{c \in R \mid \pi(c) \in \mathcal{C}_{\overline{R}}, \cdot c_{\mathfrak{a}}$ is an injection$\}$.*
4. *$S \in \mathrm{Den}_l(R, \mathfrak{a}')$ iff $\pi'(S) \in \mathrm{Den}_l(\overline{R}', 0)$ iff $\pi'(S) \in \mathrm{Ore}_l(\overline{R}')$ iff $S \in \mathrm{Ore}_l(R)$.*
5. *If $\mathfrak{a}$ is an ideal of R then $'\mathcal{C}_R \in \mathrm{Den}_l(R, \mathfrak{a})$ iff $\pi('\mathcal{C}_R) \in \mathrm{Den}_l(\overline{R}, 0)$ iff $\pi('\mathcal{C}_R) \in \mathrm{Ore}_l(\overline{R})$ iff $'\mathcal{C}_R \in \mathrm{Ore}_l(R)$.*

Proof. 1. Clearly, $T := \{c \in R \mid \cdot c_{\mathfrak{a}'}$ and $\cdot c_{R/\mathfrak{a}'}$ are injections$\} \subseteq {}'\mathcal{C}_R$. Given $c \in S$. In order to show that $S \subseteq T$, we have to prove that $s \in T$. Since $c \in S \subseteq {}'\mathcal{C}_R$, the map $\cdot c_{\mathfrak{a}'}$ is an injection. It remains to show that $\cdot c_{R/\mathfrak{a}'}$ is also an injection. If $\overline{rc} = 0$ then $rc \in \mathfrak{a}'$, and so $src = 0$ for some $s \in S$. Then $sr = 0$ (since $c \in S \subseteq {}'\mathcal{C}_R$), and so $r \in \mathfrak{a}'$, i.e. $\overline{r} = 0$. Therefore, $S \subseteq T$.

If $S = {}'\mathcal{C}_R$ then $'\mathcal{C}_R = T$ (since $T \subseteq {}'\mathcal{C}_R$).

2. By statement 1, $\pi'(S) \subseteq {}'\mathcal{C}_{\overline{R}'}$. To prove that the inclusion $\pi'(S) \subseteq \mathcal{C}_{\overline{R}'}$ holds it remains to show that each element $\overline{s} := s + \mathfrak{a}'$ (where $s \in S$) is a right regular element of the ring $\overline{R}'$. Suppose that $\overline{sr} = 0$ for some element $r \in R$. Then $sr \in \mathfrak{a}'$, and so $tsr = 0$ for some element $t \in S$. This implies that $r \in \mathfrak{a}'$, i.e. $\overline{r} = 0$, and so $\overline{s}$ is a right regular element of R.

3. Statement 3 follows from statements 1 and 2.

4. Since $S \subseteq {}'\mathcal{C}_R$ and $\mathfrak{a}' = \mathrm{ass}_R(S)$, $S \in \mathrm{Den}_l(R, \mathfrak{a}')$ iff $S \in \mathrm{Ore}_l(R)$. By statement 2, $\pi'(S) \subseteq \mathcal{C}_{\overline{R}'}$. So, $\pi'(S) \in \mathrm{Den}_l(\overline{R}', 0)$ iff $\pi'(S) \in \mathrm{Ore}_l(\overline{R})$. It remains to show that the first 'iff' holds.

Suppose that $S \in \mathrm{Den}_l(R, \mathfrak{a}')$. Then $\pi'(S) \in \mathrm{Den}_l(\overline{R}, 0)$.

Suppose that $\pi'(S) \in \mathrm{Den}_l(\overline{R}, 0)$. By Lemma 5.1, $S \in \mathrm{Den}_l(R, \mathfrak{a}')$.

5. Statement 5 is a particular case of statement 4 where $S = {}'\mathcal{C}_R$. $\square$

For a ring R, let ${}'\mathrm{Ass}_l(R) := \{\mathrm{ass}_R(S) \,|\, S \in \mathrm{Den}_l(R), S \subseteq {}'\mathcal{C}_R\}$. The set $({}'\mathrm{Ass}_l(R), \subseteq)$ is a poset.

Theorem 5.3 *Let R be a ring and ${}'\mathfrak{a} := \mathrm{ass}_R({}'S_l(R))$. Then*

1. ${}'S_l(R) = {}'\mathcal{C}_R \cap ({}'S_l(R) + {}'\mathfrak{a})$.
2. ${}'\mathfrak{a}$ *is the largest element in* ${}'\mathrm{Ass}_l(R)$.
3. ${}'S_l(R)$ *is the largest element (w.r.t. inclusion) in the set* $\{S \in \mathrm{Den}_l(R) \,|\, S = {}'\mathcal{C}_R \cap (S + \mathrm{ass}_R(S))\}$.
4. ${}'S_l(R) + {}'\mathfrak{a} \in \mathrm{Den}_l(R, {}'\mathfrak{a})$.
5. ${}'Q_l(R) \simeq ({}'S_l(R) + {}'\mathfrak{a})^{-1}R$.

Proof. 1. Let ${}'S = {}'S_l(R)$ and $T := {}'\mathcal{C}_R \cap ({}'S_l(R) + {}'\mathfrak{a})$. Clearly, ${}'S \subseteq T$. In order to prove that ${}'S \supseteq T$, it suffices to show that $T \in \mathrm{Den}_l(R)$ (since $T \subseteq {}'\mathcal{C}_R$ and by maximality of ${}'S$ (Lemma 4.1.(1)), ${}'S \supseteq T$). By the very definition, T is a multiplicative set in R such that ${}'\pi(T) = {}'\pi({}'S)$ where ${}'\pi : R \to R/{}'\mathfrak{a}$, $r \mapsto \overline{r} := r + {}'\mathfrak{a}$. Since $T \subseteq {}'\mathcal{C}_R$, $T \in \mathrm{Den}_l(R)$ iff $T \in \mathrm{Ore}_l(R)$. Let us show that $T \in \mathrm{Ore}_l(R)$. Given elements $t \in T$ and $r \in R$. Then $\overline{t} \in {}'\pi(T) = {}'\pi({}'S)$. Since ${}'\pi({}'S) \in \mathrm{Den}_l(R/{}'\mathfrak{a}, 0)$, $\overline{sr} = \overline{r}_1\overline{t}$ for some elements $s \in {}'S$ and $r_1 \in R$. Then $sr - r_1t \in {}'\mathfrak{a}$, and so $s'(sr - r_1t) = 0$ for some element $s' \in {}'S$, i.e. $s's \cdot r = s'r \cdot t$ where $s's \in {}'S \subseteq T$. Therefore, $T \in \mathrm{Ore}_l(R)$.

2. Statement 2 follows from the maximality of ${}'S$ (Lemma 4.1.(1)).

3. Every element $S \in \mathrm{Den}_l(R)$ such that $S = {}'\mathcal{C}_R \cap (S + \mathrm{ass}_R(S))$ consists of left regular elements, hence $S \subseteq {}'S$. Now, statement 3 follows from statement 1.

4. Let $T' := {}'S_l(R) + {}'\mathfrak{a}$. Then ${}'\pi(T') = {}'\pi({}'S)$, and so the set T' is a multiplicative set of R.

(i) $T' \in \mathrm{Ore}_l(R)$: Given elements $t \in T'$ and $r \in R$. Then $\overline{t} \in {}'\pi(T') = {}'\pi({}'S)$. Since ${}'\pi({}'S) \in \mathrm{Den}_l(R/{}'\mathfrak{a}, 0)$, $\overline{sr} = \overline{r}_1\overline{t}$ for some elements $s \in {}'S$ and $r_1 \in R$. Then $sr - r_1t \in {}'\mathfrak{a}$, and so $s'(sr - r_1t) = 0$ for some element $s' \in {}'S$, i.e. $s's \cdot r = s'r \cdot t$ where $s's \in {}'S \subseteq T'$. Therefore, $T' \in \mathrm{Ore}_l(R)$.

(ii) $\mathrm{ass}_R(T') = {}'\mathfrak{a}$: The inclusion ${}'S \subseteq T'$ implies the inclusion ${}'\mathfrak{a} \subseteq \mathfrak{b} := \mathrm{ass}_R(T')$. If $r \in \mathfrak{b}$, i.e. $tr = 0$ for some element $t = s + a \in T'$ where $s \in {}'S$ and $a \in {}'\mathfrak{a}$. Fix an element $s' \in {}'S$ such that $s'a = 0$. Then $0 = s'tr = s'(s + a)r = s'sr$, and so $r \in {}'\mathfrak{a}$ since $s's \in {}'\mathfrak{a}$.

(iii) $T' \in \mathrm{Den}_l(R)$: We have to show that if $rt = 0$ for some elements $r \in R$ and $t \in T'$ then $r \in {}'\mathfrak{a}$. The element $t \in T'$ is a sum $s + a$ where $s \in {}'S$ and $a \in {}'\mathfrak{a}$. Then the equality $0 = rt = r(s + a)$ can be written as $rs = -ra \in {}'\mathfrak{a}$. Hence, $s'rs = 0$ for some element $s' \in {}'S$, and so $s'r = 0$ (since $s \in {}'S \subseteq {}'\mathcal{C}_R$). Therefore, $r \in {}'\mathfrak{a}$ (as $s' \in {}'S$).

5. By statement 4, ${}'Q_l(R) \simeq {}'\pi({}'S)^{-1}(R/{}'\mathfrak{a}) = {}'\pi({}'S + {}'\mathfrak{a})^{-1}(R/{}'\mathfrak{a}) \simeq ({}'S + {}'\mathfrak{a})^{-1}R$. $\square$

Sufficient conditions for ${}'Q_{l,cl}(R) \simeq Q_{l,cl}(R/\mathrm{ass}_R({}'\mathcal{C}_R))$. The next theorem gives sufficient conditions for the ring ${}'Q_{l,cl}(R)$ to be isomorphic to the ring $Q_{l,cl}(R/\mathrm{ass}_R({}'\mathcal{C}_R))$.

Theorem 5.4 *Let R be a ring, ${}'\mathcal{C} = {}'\mathcal{C}_R$ and $\mathfrak{a} = \mathrm{ass}_R({}'\mathcal{C})$. Suppose that $\mathfrak{a}$ is an ideal of the ring R, the set ${}'\overline{\mathcal{C}} := \pi({}'\mathcal{C})$ is a dense subset of $\mathcal{C}_{\overline{R}}$ in $\overline{R} := R/\mathfrak{a}$ where $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{a}$, and $\mathcal{C}_{\overline{R}} \in \mathrm{Ore}_l(R)$. Then*

1. ${}'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$ *and* ${}'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$.
2. $\pi^{-1}(\mathcal{C}_{\overline{R}}) \in \mathrm{Den}_l(R, \mathfrak{a})$ *and* ${}'\mathcal{C} \subseteq \pi^{-1}(\mathcal{C}_{\overline{R}})$.
3. ${}'Q_{l,cl}(R) \simeq {}'\overline{\mathcal{C}}^{-1}\overline{R} \simeq \pi^{-1}(\mathcal{C}_{\overline{R}})^{-1}R \simeq Q_{l,cl}(\overline{R})$.

Remark. If $\mathfrak{a}$ is an ideal of the ring R then ${}'\overline{\mathcal{C}} \subseteq \mathcal{C}_{\overline{R}}$, by Proposition 5.2.(2).

Proof. 1. (i) ${}'\mathcal{C} \in \mathrm{Ore}_l(R)$: Clearly, ${}'\mathcal{C}$ is a multiplicative set of R. Given elements $c \in {}'\mathcal{C}$ and $r \in R$. We have to find elements $c' \in {}'\mathcal{C}$ and $r' \in R$ such that $c'r = r'c$. By the assumption,

$\mathcal{C}_{\overline{R}} \in \mathrm{Ore}_l(\overline{R})$. So, let $\overline{Q} := Q_{l,cl}(\overline{R})$. Since $'\overline{\mathcal{C}} \subseteq \mathcal{C}_{\overline{R}}$, we have $\overline{r}\overline{c}^{-1} = \overline{s}^{-1}\overline{r}_1$ for some elements $\overline{s} = s + \mathfrak{a} \in \mathcal{C}_{\overline{R}}$ (where $s \in R$) and $r_1 \in R$. We can write $\overline{s}_1\overline{r} = \overline{r}_1\overline{c}$. By the assumption, the set $'\overline{\mathcal{C}}$ is a dense subset of $\mathcal{C}_{\overline{R}}$. Fix an element $\overline{r}_2 \in \overline{R}$ (where $r_2 \in R$) such that $\overline{c}_1 := \overline{r}_2\overline{s} \in {'\overline{\mathcal{C}}}$ for some element $c_1 \in {'\mathcal{C}}$. Then the equality $\overline{r}_2\overline{s}_1\overline{r} = \overline{r}_2\overline{r}_1\overline{c}$ can be written as $\overline{c}_1\overline{r} = \overline{r}_2\overline{r}_1\overline{c}$. Hence, $c_1r - r_2r_1c \in \mathfrak{a}$, and so there exists an element $c_2 \in {'\mathcal{C}}$ such that $c_2(c_1r - r_2r_1c) = 0$. Notice that $c' := c_2c_1 \in {'\mathcal{C}}$, $r' := c_2r_2r_1 \in R$ and $c'r = r'c$.

(ii) $'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$: The inclusion follows from Lemma 5.1 and (i).

By (ii), $'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$ and $'Q_{l,cl}(R) \simeq {'\overline{\mathcal{C}}}^{-1}\overline{R}$.

2. By statement 1, $'\mathcal{C} \in \mathrm{Den}_l(R, \mathfrak{a})$ and $'\mathcal{C} \subseteq \pi^{-1}({'\mathcal{C}_{\overline{R}}})$. By the assumption, $\mathcal{C}_{\overline{R}} = \pi(\pi^{-1}(\mathcal{C}_{\overline{R}})) \in \mathrm{Den}_l(\overline{R}, 0)$. Therefore, $\pi^{-1}(\mathcal{C}_{\overline{R}}) \in \mathrm{Den}_l(R, \mathfrak{a})$.

3. By statement 2, $\pi^{-1}(\mathcal{C}_{\overline{R}})^{-1}R \simeq \mathcal{C}_{\overline{R}}^{-1}\overline{R} = \overline{Q}$. By the assumption, the set $'\overline{\mathcal{C}}$ is dense in $\mathcal{C}_{\overline{R}}$. By statement 1, $'\overline{\mathcal{C}} \in \mathrm{Den}_l(\overline{R}, 0)$. Hence, $'Q \simeq {'\overline{\mathcal{C}}}^{-1}\overline{R} \simeq {'Q_{l,cl}(R)}$. $\square$

6 The classical left regular left quotient ring of the algebra of polynomial integro-differential operators $\mathbb{I}_1$

The aim of this section is to find for the algebra of polynomial integro-differential operators $\mathbb{I}_1$ its classical left regular left quotient ring $'Q := {'Q_{l,cl}(\mathbb{I}_1)}$ and classical right regular right quotient ring $Q' := {'Q_{r,cl}(\mathbb{I}_1)}$, and to show that both of them are canonically isomorphic to to the classical quotient ring of the Weyl algebra A_1 (Theorem 6.5). The sets $'\mathcal{C}_{\mathbb{I}_1}$ and $\mathcal{C}'_{\mathbb{I}_1}$ are described in Theorem 6.7. The algebra A_1 is a Noetherian domain so $Q(A_1) := Q_{l,cl}(A_1) \simeq Q_{r,cl}(A_1)$. The key idea in finding the rings $'Q$ and Q' is to use Theorem 1.1. The most difficult part is to verify that the set $'\overline{\mathcal{C}}$ is dense in $\mathcal{C}_{\mathbb{I}_n}$ (see, Corollary 6.4).

We start this section with collecting necessary facts about the algebra $\mathbb{I}_1$ that are used in the proofs (their proofs are given in [1]).

The algebra $\mathbb{I}_1$ of polynomial integro-differential operators. Let us recall some of the properties of the algebra $\mathbb{I}_1$. Let K be a field of characteristic zero, $P_1 = K[x]$ and $E_1 = \mathrm{End}_K(P_1)$ be the algebra of all K-linear maps from P_1 to P_1. Recall that the algebra $\mathbb{I}_1$ of polynomial integro-differential operators is the subalgebra of E_1 generated by the elements x, $\partial = \frac{d}{dx}$ and $\int$. The algebra $\mathbb{I}_1$ contains the Weyl algebra $A_1 = K\langle x, \partial\rangle$. The algebra A_1 is a Noetherian domain but the algebra $\mathbb{I}_1$ is neither left nor right Noetherian domain. Moreover, it contains infinite direct sums of nonzero left and right ideals. Neither left nor right classical quotient ring exits (Lemma 4.8.(1)). The largest left quotient ring $Q_l(\mathbb{I}_1)$ and the largest right quotient ring $Q_r(\mathbb{I}_1)$ are neither left nor right Noetherian rings (Lemma 4.8.(2)). The algebra $\mathbb{I}_1$ admits a single proper ideal $F = \oplus_{i,j\in\mathbb{N}}Ke_{ij}$ where $e_{ij} = \int^i\partial^j - \int^{i+1}\partial^{j+1}$, and ${}_{K[\int]}F_{K[\partial]} = K[\int]e_{00}K[\partial] \simeq K[\int]\otimes K[\partial]$. The factor algebra $\mathbb{I}_1/F$ is isomorphic to the algebra $A_{1,\partial}$ which is a the localization of the Weyl algebra A_1 at the powers of the element ∂. Each element $a \in \mathbb{I}_1$ is a unique sum

$$a = \sum_{i>0} a_{-i}\partial^i + a_0 + \sum_{i>0}\int^i a_i + \sum_{i,j}\lambda_{ij}e_{ij} \qquad (2)$$

where $a_k \in K[H]$, $H := \partial x$ and $\lambda_{ij} \in K$. Since $\partial\int = 1$, we have the equalities $\partial e_{ij} = e_{i-1,j}$, $e_{ij}\partial = e_{i,j+1}$, $\int e_{ij} = e_{i+1,j}$ and $e_{ij}\int = e_{i,j-1}$ (where $e_{-1,j} := 0$ and $e_{i,-1} := 0$). The algebra $\mathbb{I}_1$ is generated by the elements ∂, $H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations:

$$\partial\int = 1,\ \ [H, \int] = \int,\ \ [H, \partial] = -\partial,\ \ H(1 - \int\partial) = (1 - \int\partial)H = 1 - \int\partial.$$

Since the algebra $\mathbb{I}_1/F$ is a domain, $'\mathcal{C}_{\mathbb{I}_1} = \{a \in \mathbb{I}_1 \,|\, \ker(\cdot a_F) = 0\}$ where $\cdot a_F : F \to F$, $f \mapsto fa$. The right $\mathbb{I}_1$-module F is the direct sum $\oplus_{i\in\mathbb{N}}e_{i0}K[\partial]$ of isomorphic right $\mathbb{I}_1$-modules. The right $\mathbb{I}_1$-module $e_{00}K[\partial]$ is a free right $K[\partial]$-module of rank 1. When we identify the right $K[\partial]$-modules $e_{00}K[\partial]$ and $P_1' := K[\partial]$, the right $\mathbb{I}_1$-module structure on the polynomial algebra P_1' is given by the rule: For $i \geq 0$, $\partial^i \cdot \partial = \partial^{i+1}$, $\partial^i \cdot \int = \partial^{i-1}$ $(i \geq 1)$ and $1 \cdot \int = 0$, $\partial^i \cdot H = \partial^i (i+1)$. So, $'\mathcal{C}_{\mathbb{I}_1} = \{a \in \mathbb{I}_1 \,|\, \ker(\cdot a_{P_1'}) = 0\}$.

The algebra $\mathbb{I}_1$ admits the involution $*$ over the field K: $\partial^* = \int$, $\int^* = \partial$ and $H^* = H$, i.e. it is a K-algebra *anti-isomorphism* ($(ab)^* = b^* a^*$) such that $a^{**} = a$. Therefore, the algebra $\mathbb{I}_1$ is *self-dual*, i.e. it is isomorphic to its opposite algebra $\mathbb{I}_1^{op}$. As a result, the left and right properties of the algebra $\mathbb{I}_1$ are the same. Clearly, $e_{ij}^* = e_{ji}$ for all $i, j \in \mathbb{N}$, and so $F^* = F$.

Lemma 6.1 *Suppose that $T \in \mathrm{Den}_l(R)$ and S be a multiplicative set of R such that $S \subseteq T$, $\mathrm{ass}_R(S) = \mathrm{ass}_R(T)$ and for each element $t \in T$ there exists an element $r \in R$ such that $rt \in S + \mathrm{ass}_R(T)$. Then $S \in \mathrm{Den}_l(R)$ and $S^{-1}R \simeq T^{-1}R$.*

Proof. (i) $S \in \mathrm{Ore}_l(R)$: Given elements $s \in S$ and $r \in R$. We have to show that $s'r = r's$ for some elements $s' \in S$ and $r' \in R$. Since $T \in \mathrm{Ore}_l(R)$ and $s \in S \subseteq T$, $tr = r_1 s$ for some elements $t \in T$ and $r_1 \in R$. By the assumption, $r_2 t = s_1 + a$ for some elements $r_2 \in R$, $s_1 \in S$ and $a \in \mathfrak{a} := \mathrm{ass}_R(T)$. Since $\mathrm{ass}_R(S) = \mathfrak{a}$, $s_2 a = 0$ for some $s_2 \in S$, and so $s_2 r_2 t = s_2 s_1 \in S$. Now, $s_2 s_1 \cdot r = s_2 r_2 t \cdot r = s_2 r_2 \cdot tr = s_2 r_2 \cdot r_1 s = s_2 r_2 r_1 \cdot s$. It suffices to take $s' = s_2 s_1 \in S$ and $r' = s_2 r_2 r_1 \in R$.

(ii) $S \in \mathrm{Den}_l(R, \mathfrak{a})$: Suppose that $rs = 0$ for some elements $r \in R$ and $s \in S$. Then $r \in \mathfrak{a} = \mathrm{ass}_R(S)$ since $s \in T$. Now, by (i), $S \in \mathrm{Den}_l(R, \mathfrak{a})$.

(iii) $S^{-1}R \simeq T^{-1}R$ (by Lemma 2.5.(3)). $\square$

Let Δ_1 be the subalgebra of the Weyl algebra A_1 generated by the elements H and ∂. The algebra Δ_1 is isomorphic to the skew Laurent polynomial ring $K[H][\partial, \sigma^{-1}]$ where $\sigma(H) = H - 1$. Let $A_1^0 := A_1 \backslash \{0\}$, $A_{1,\partial}^0 := A_{1,\partial} \backslash \{0\}$ and $\Delta_1^0 := \Delta_1 \backslash \{0\}$.

Lemma 6.2 *For each element $a \in \mathbb{I}_1 \backslash F$, there is a natural number i such that $\partial^i a \in \Delta_1^0$.*

Proof. By (2), $\partial^i a \in \Delta_1^0 + F$ for some i. Since $F = \cup_{j \geq 1} \ker(\partial_{\mathbb{I}_1}^i \cdot)$ (where $\partial_{\mathbb{I}_1}^i \cdot : \mathbb{I}_1 \to \mathbb{I}_1$, $u \mapsto \partial^i u$) and $\partial^j \Delta_1^0 \subseteq \Delta_1^0$, we can enlarge the natural number i such that $\partial^i a \in \Delta_1^0$. $\square$

The next proposition is the key step in finding the rings $Q_{l,cl}(\mathbb{I}_1)$ and $Q_{r,cl}(\mathbb{I}_1)$.

Proposition 6.3 *Let $'\mathcal{C} := {}'\mathcal{C}_{\mathbb{I}_1}$, $\pi : \mathbb{I}_1 \to \mathbb{I}_1/F \simeq A_{1,\partial}$, $r \mapsto r + F$, and $S = \pi^{-1}(A_{1,\partial}^0) = \mathbb{I}_1 \backslash F$. Then*

1. *$S \in \mathrm{Den}_l(\mathbb{I}_1, F)$ and $S^{-1}\mathbb{I}_1 \simeq Q(A_1)$.*
2. *$\Delta_1^0 \in \mathrm{Den}_l(\mathbb{I}_1, F)$ and ${\Delta_1^0}^{-1}\mathbb{I}_1 \simeq S^{-1}\mathbb{I}_1$.*
3. *$'\Delta_1^0 := {}'\mathcal{C}_{\mathbb{I}_1} \cap \Delta_1^0 \in \mathrm{Den}_l(\mathbb{I}_1, F)$ and ${'\Delta_1^0}^{-1}\mathbb{I}_n \simeq {\Delta_1^0}^{-1}\mathbb{I}_1$.*

Therefore, ${'\Delta_1^0}^{-1}\mathbb{I}_1 \simeq {\Delta_1^0}^{-1}\mathbb{I}_1 \simeq S^{-1}\mathbb{I}_1 \simeq Q(A_1)$.

Proof. 1. Since $S_\partial := \{\partial^i \,|\, i \in \mathbb{N}\} \in \mathrm{Den}_l(\mathbb{I}_1, F)$ and $S_\partial \subseteq S$, we have the inclusion $F = \mathrm{ass}(S_\partial) \subseteq \mathrm{ass}(S)$. In fact, $F = \mathrm{ass}(S)$ since the algebra $\mathbb{I}_1/F$ is a domain. Then $S \in \mathrm{Den}_l(\mathbb{I}_1, F)$, since $\pi(S) = A_{1,\partial}^0 \in \mathrm{Den}_l(A_{1,\partial}, 0)$, $S_\partial \subseteq S$ and $S_\partial \in \mathrm{Den}_l(\mathbb{I}_1, F)$. Now, it is obvious that $S^{-1}\mathbb{I}_1 \simeq \pi(S)^{-1}(\mathbb{I}_1/F) \simeq (A_{1,\partial}^0)^{-1} A_{1,\partial} \simeq Q(A_1)$.

2. The inclusion $S_\partial \subseteq \Delta_1^0$ implies that $F = \mathrm{ass}(S_\partial) \subseteq \mathrm{ass}(\Delta_1^0)$. The factor algebra $\mathbb{I}_1/F$ is a domain and $\pi|_{\Delta_1^0} : \Delta_1^0 \to \Delta_1^0$ is a bijection, hence $\mathrm{ass}(\Delta_1^0) \subseteq F$, and so $\mathrm{ass}(\Delta_1^0) = F$. By Lemma 6.2, the multiplicative set Δ_1^0 is dense in $S = \mathbb{I}_1 \backslash F$. By Lemma 6.1, $\Delta_1^0 \in \mathrm{Den}_l(\mathbb{I}_1, F)$ and $(\Delta_1^0)^{-1}\mathbb{I}_1 \simeq S^{-1}\mathbb{I}_1$.

3. The inclusion $S_\partial \subseteq {}'\Delta_1^0$ implies that $F = \mathrm{ass}(S_\partial) \subseteq \mathrm{ass}('\Delta_1^0)$. The algebra $\mathbb{I}_1/F$ is a domain and $\pi|_{'\Delta_1^0} : {}'\Delta_1^0 \to {}'\Delta_1^0$ is a bijection, hence $\mathrm{ass}('\Delta_1^0) \subseteq F$, and so $\mathrm{ass}('\Delta_1^0) = F$. The right A_1-module P_1' is a simple one. By [10], for every nonzero element a of A_1, $\ker(\cdot a_{P_1'})$ is a finite dimensional vector space. In particular, this is the case for all elements $a \in \Delta_1^0$ (since $\Delta_1^0 \subseteq A_1^0$). Since $\cap_{i \geq 1} K[\partial]\partial^i = 0$, for each element $a \in \Delta_1^0$, we have $\ker(\cdot a_{P_1'}) \cap \mathrm{im}(\partial_{P_1'}^i) = 0$ for some $i = i(a) \geq 1$, i.e. $\ker(\cdot(\partial^i a)_{P_1'}) = 0$. Therefore, $\partial^i a \in {}'\mathcal{C}_{\mathbb{I}_1} \cap \Delta_1^0 = {}'\Delta_1^0$ (since $F_{K[\partial]} \simeq K[\partial]^{(\mathbb{N})}$). So, $'\Delta_1^0$ is dense in Δ_1^0 and $\mathrm{ass}('\Delta_1^0) = \mathrm{ass}(\Delta_1^0) = F$. By Lemma 6.1, $'\Delta_1^0 \in \mathrm{Den}_l(\mathbb{I}_1, F)$ and $('\Delta_1^0)^{-1}\mathbb{I}_1 \simeq ('\Delta_1^0)^{-1}\mathbb{I}_1$. $\square$

Corollary 6.4 *For each element $a \in \mathbb{I}_1 \backslash F$ there is a natural number i such that $\partial^i a \in {}'\Delta_1^0$.*

Proof. This was proven in the proof of Theorem 6.3. $\square$

Theorem 6.5 *1. ${}'Q_{l,cl}(\mathbb{I}_1) \simeq Q(A_1)$ is a division ring and $\mathrm{ass}({}'\mathcal{C}_{\mathbb{I}_1}) = F$.*

2. $Q'_{r,cl}(\mathbb{I}_1) \simeq Q(A_1)$ is a division ring and $\mathrm{ass}_r({}'\mathcal{C}_{\mathbb{I}_1}) = F$.

Proof. 1. By Proposition 6.3.(3), ${}'\Delta_1^0 \in \mathrm{Den}_l(\mathbb{I}_1, F)$. Since ${}'\Delta_1^0 \subseteq {}'\mathcal{C} = {}'\mathcal{C}_{\mathbb{I}_1}$ and the algebra $\mathbb{I}_1/F$ is a domain, we must have $\mathrm{ass}({}'\mathcal{C}) = F = \mathrm{ass}({}'\Delta_1^0)$. The ideal F is a prime ideal since $\mathbb{I}_1/F$ is a domain. By Corollary 6.4, the set ${}'\overline{\mathcal{C}} := \pi({}'\mathcal{C})$ is dense in $\mathcal{C}_{\mathbb{I}_1/F} = A^0_{1,\partial}$. By Theorem 1.1.(3), ${}'Q_{l,cl}(\mathbb{I}_1) \simeq Q_{l,cl}(A_{1,\partial}) \simeq Q(A_1)$.

2. Applying the involution $*$ to statement 1 and using the fact that $A_1^* \simeq A_1$ we obtain statement 2: $Q'_{r,cl}(\mathbb{I}_1) = {}'Q_{l,cl}(\mathbb{I}_1)^* \simeq Q(A_1)^* \simeq Q(A_1^*) \simeq Q(A_1)$. $\square$

Let $\nabla_1 := \Delta_1^* = K[H][\int, \sigma]$ where $\sigma(H) = H - 1$ and $\nabla_1^0 := \nabla_1 \backslash \{0\}$. Applying the involution $*$ of the algebra $\mathbb{I}_1$ to Proposition 6.3 and using the facts that $\mathcal{C}'_{\mathbb{I}_1} = ({}'\mathcal{C}_{\mathbb{I}_1})^*$, $F^* = F$, $S^* = S$ and $A_1^* \simeq A_1$ (since the Weyl algebra A_1 is isomorphic to its dual; in particular $Q(A_1^*) \simeq Q(A_1)$) we obtain the next proposition.

Proposition 6.6 *Let $S = \mathbb{I}_1 \backslash F$. Then*

1. *$S \in \mathrm{Den}_r(\mathbb{I}_1, F)$ and $\mathbb{I}_1 S^{-1} \simeq Q(A_1)$.*
2. *$\nabla_1^0 \in \mathrm{Den}_r(\mathbb{I}_1, F)$ and $\mathbb{I}_1 {\nabla_1^0}^{-1} \simeq \mathbb{I}_1 S^{-1}$.*
3. *$\nabla'_1 := \mathcal{C}'_{\mathbb{I}_1} \cap \nabla_1^0 \in \mathrm{Den}_r(\mathbb{I}_1, F)$ and $\mathbb{I}_n {\nabla'_1}^{-1} \simeq \mathbb{I}_1 {\nabla_1^0}^{-1}$.*

Therefore, $\mathbb{I}_n {\nabla'_1}^{-1} \simeq \mathbb{I}_1 {\nabla_1^0}^{-1} \simeq \mathbb{I}_1 S^{-1} \simeq Q(A_1)$.

Descriptions of the sets ${}'\mathcal{C}_{\mathbb{I}_1}$ and $\mathcal{C}'_{\mathbb{I}_1}$. We are going to give explicit descriptions of the sets ${}'\mathcal{C}_{\mathbb{I}_1}$ and $\mathcal{C}'_{\mathbb{I}_1}$ (Theorem 6.7). They have a sophisticated structure. Let $\Gamma := \{a = a_0 + \sum_{i \geq 1} \int^i a_i + f \,|\, a_0 \neq 0, \text{ all } a_i \in K[H], f \in F\}$. In the proof of Theorem 6.7.(1), it is shown that ${}'\mathcal{C}_{\mathbb{I}_1} \subseteq \cup_{i \geq 1} \partial^i \Gamma$ and, for each element $a \in \Gamma$, $\partial^i a \in {}'\mathcal{C}_{\mathbb{I}_1}$ for some $i = i(a) \geq 0$. Then map

$$d : \Gamma \rightarrow \mathbb{N}, \quad a \mapsto d(a) := \min\{i \in \mathbb{N} \,|\, \partial^i a \in {}'\mathcal{C}_{\mathbb{I}_1}\} \tag{3}$$

is called the *left regularity degree function* and the natural number $d(a)$ is called the *left regularity degree* of a. For each element $a \in \Gamma$, $d(a)$ can be found in finitely many steps, see the proof of Theorem 6.7.(1) where the explicit expression (4) is given for $d(a)$.

Theorem 6.7 *1. ${}'\mathcal{C}_{\mathbb{I}_1} = \{\partial^{d(a)+i} a \,|\, i \in \mathbb{N}, a \in \Gamma\}$.*

2. $\mathcal{C}'_{\mathbb{I}_1} = {}'\mathcal{C}^*_{\mathbb{I}_1}$.

Before giving a proof of Theorem 6.7, we introduce some definitions. For each element $a = a_0 + \sum_{i \geq 1} \int^i a_i + f \in \Gamma$, the elements $l(a) := a_0$ and $a_F := f$ are called the *leading term* and the *F-term* of a, respectively. The *size* $s(f)$ of the element f is equal to -1 if $f = 0$, and to $\min\{m \in \mathbb{N} \,|\, f \in \oplus_{i,j=0}^m K e_{ij}\}$. Then $s(a) := s(a_F)$ is called the *size* of a. Let $\mathcal{R}(a_0) := \{i \in \mathbb{N} \,|\, a_0(i+1) = 0\}$, the set of roots of the polynomial $a_0(H+1)$ that are natural numbers. Let $r(a)$ be the maximal element in the set $R(a) := \{i \in \mathcal{R}(a_0) \,|\, i > s(a)\}$. If $R(a) = \emptyset$ then $r(a) := \emptyset$.

For each element $a \in \mathbb{I}_1$, let $\mathcal{K}_a := \ker(\cdot a_{P'_1})$.

Proof of Theorem 6.7. Let ${}'\mathcal{C} := {}'\mathcal{C}_{\mathbb{I}_1}$, $P' = P'_1$ and $P'_{\leq i} := \oplus_{j=0}^i K\partial^i$ for all $j \geq 0$. Similarly, the vector space $P'_{<i}$ is defined.

1. (i) ${}'\mathcal{C} \subseteq \cup_{i \geq 0} \partial^i \Gamma$: Let $a \in {}'\mathcal{C}$. The element a is a unique sum (2). It suffices to show that there is $i \leq 0$ such that $a_i \neq 0$ (since then $a = \partial^j \gamma$ where $j = \min\{i \leq 0 \,|\, a_i \neq 0\}$ and $\gamma \in \Gamma$; this follows from the equalities $\partial^k \int^k = 1$ and $\partial^k F = F$ for all $k \geq 1$). Suppose that $a_i = 0$ for all

$i \leq 0$, i.e. $a = \sum_{i\geq 1} \int^i a_i + f$ where $f = a_F$. Fix a natural number n such that $n > s(a)$. Then $\cdot a : P_{\leq n} \to P_{\leq n-1}$, $p \mapsto ap$, and so $\mathcal{K}_a \neq 0$, a contradiction.

Till the end of the proof let $a = a_0 + \sum_{i\geq 1} \int^i a_i + f \in \Gamma$. For all $i \geq s(a)$, $aP'_{\leq i} \subseteq P'_{\leq i}$ and $\partial^i \cdot a \equiv \partial^i a_0(i+1) \mod P'_{\leq i-1}$. Now, the statement (ii) is obvious.

(ii) $'\Gamma := \Gamma \cap {'\mathcal{C}} = \{a \in \Gamma \,|\, r(a) = \emptyset \text{ and } \ker(\cdot a_{P'_{\leq s(a)}}) = 0\}$. For each element $'\Gamma$, we set $d(a) := 0$. Clearly, $\partial^i a \subseteq {'\mathcal{C}}$ for all $a \in {'\Gamma}$ and $i \geq 0$.

Till the end of the proof we assume that $a \in \Gamma \backslash {'\Gamma}$. By the statement (ii), there are two cases:
(a) $r(a) \neq \emptyset$ (i.e. $R(a) \neq \emptyset$), and
(b) $r(a) = \emptyset$ (i.e. $R(a) = \emptyset$) and $\ker(\cdot a'_{P_{\leq s(a)}}) \neq 0$.

In the case (a), $\mathcal{K}_a \subseteq P'_{\leq r(a)}$, and so $\partial^i s \in {'\mathcal{C}}$ for all $i > r(a)$. In the case (b), $\mathcal{K}_a \subseteq P'_{\leq s(a)}$, and so $\partial^i s \in {'\mathcal{C}}$ for all $i > s(a)$. This proves that the function d is well-defined (see (3)) and that ${'\mathcal{C}}_{\mathbb{I}_1} = \{\partial^{d(a)+i} a \,|\, i \in \mathbb{N}, a \in \Gamma\}$, in view of the statements (i) and (ii). Then

$$d(a) = \begin{cases} \min\{i \,|\, 0 \leq i \leq r(a)+1,\ \mathcal{K}_a \cap (\bigoplus_{j=i}^{r(a)+1} K\partial^j) = 0\} & \text{in the case (a),} \\ \min\{i \,|\, 0 \leq i \leq s(a)+1,\ \mathcal{K}_a \cap (\bigoplus_{j=i}^{s(a)+1} K\partial^j) = 0\} & \text{in the case (b).} \end{cases} \quad (4)$$

2. Statement 2 is obvious. $\square$

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