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FACTORISATION THEOREMS FOR GENERALISED POWER SERIES

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Abstract. Fields of generalised power series (or Hahn fields), with coefficients in a field and exponents in a divisible ordered abelian group, are a fundamental tool in the study of valued and ordered fields and asymptotic expansions. The subring of the series with non-positive exponents appear naturally when discussing exponentiation, as done in transseries, or integer parts. A notable example is the ring of omnific integers inside the field of Conway's surreal numbers.

In general, the elements of such subrings do not have factorisations into irreducibles. In the context of omnific integers, Conway conjectured in 1976 that certain series are irreducible (proved by Berarducci in 2000), and that any two factorisations of a given series share a common refinement.

Here we prove a factorisation theorem for the ring of series with non-positive real exponents: every series is shown to be a product of irreducible series with infinite support and a factor with finite support which is unique up to constants. From this, we shall deduce a general factorisation theorem for series with exponents in an arbitrary divisible ordered abelian group, including omnific integers as a special case. We also obtain new irreducibility and primality criteria.

To obtain the result, we prove that a new ordinal-valued function, which we call *degree*, is a valuation on the ring of generalised power series with real exponents, and we formulate some structure results on the associated RV monoid.

1. Introduction

1.1. **Generalised power series.** A **generalised power series** (or Hahn-Mal'cev-Neumann series) is a formal sum $\sum_x k_x t^x$, with **coefficients** k_x in some given field K and **exponents** x in some given ordered abelian group $G = (G, +)$, such that its **support** $\{x \in G : k_x \neq 0\}$ is well ordered. $K((t^G))$, or $K((G))$ for short, denotes the set of all such series. $K((G))$ has a natural ring structure, and it is usually called **Hahn field**. An example is the field $\mathbb{C}((\mathbb{Z})) = \mathbb{C}((t^{\mathbb{Z}})) = \mathbb{C}((t))$ of Laurent series. Such fields appear when dealing with valued or ordered fields and asymptotic expansions (see e.g. [Hah95, Kap42]). Given a subset $A \subseteq G$, let $K((A)) = K((t^A)) \subseteq K((G))$ be the subset of the series with support contained in A, such as the ring of Taylor series $\mathbb{C}((\mathbb{N})) = \mathbb{C}((t^{\mathbb{N}})) = \mathbb{C}[[t]] \subseteq \mathbb{C}((\mathbb{Z}))$.

We shall prove some factorisation theorems for the ring of the series with non-positive exponents $K((G^{\leq 0}))$, and more generally for the rings of the form $Z + K((G^{<0}))$, where Z is some given subring of K. Among these is Conway's ring **Oz** of **omnific integers**, the canonical integer part of the field

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No surreal numbers, as it is of the form $\mathbf{Oz} = \mathbb{Z} + \mathbb{R}((\mathbf{No}^{<0}))$. These subrings appear naturally when dealing with transseries [DMM97] and with integer parts [She64, MR93].

Denote by $K(G)$ the subring of the series that are *finite* sums, and likewise by $K(A)$ the subset of those finite sums whose support is contained in A (e.g. $\mathbb{C}(\mathbb{N}) = \mathbb{C}(t^{\mathbb{N}}) = \mathbb{C}[t]$ is the usual ring of polynomials over $\mathbb{C}$).

1.2. **Integer parts.** Given an ordered field K, an **integer part** of K is a (discrete) subring $Z \subseteq K$ such that for any $k \in K$, there exists a unique $z \in Z$ such that $z \leq k < z + 1$. For any generalised power series field of the form $\mathbb{R}((G))$, the subring $\mathbb{Z} + \mathbb{R}((G^{<0}))$ is an integer part of $\mathbb{R}((G))$, and it is the unique one that is also "truncation closed".

Shepherdson proved that the non-negative part of an ordered ring is a model of Open Induction (the fragment of Peano's Arithmetic based on the induction scheme restricted to quantifier free formulas) if and only if the ring is the integer part of a real closed field. As corollary, $\mathbb{Z} + \mathbb{Q}^{\mathrm{rc}}(\mathbb{Q}^{<0})$ is a model of Open Induction, where $\mathbb{Q}^{\mathrm{rc}}$ is the field real algebraic numbers [She64] (this was the first example of a recursive model of a significant fragment of PA). Likewise, $\mathbb{Z} + \mathbb{R}((G^{<0}))$ is a model of Open Induction when $\mathbb{R}((G))$ is real closed, so if and only if G is divisible. Mourgues and Ressayre [MR93] proved conversely that all real closed fields have an integer part, and moreover, one can choose the integer part to be truncation closed inside an appropriate $\mathbb{R}((G))$.

1.3. **Conway's conjectures.** Shepherdson's ring $\mathbb{Z} + \mathbb{Q}^{\mathrm{rc}}(\mathbb{Q}^{<0})$ is a GCD domain, and no element is irreducible except for the integer primes (see for instance $t^{-1} = t^{-\frac{1}{2}}t^{-\frac{1}{2}} = t^{-\frac{1}{4}} \cdots$). The integer primes of $\mathbb{Z}$ are clearly prime even in the ring of omnific integers $\mathbf{Oz} = \mathbb{Z} + \mathbb{R}((\mathbf{No}^{<0}))$.

It is a natural question whether $\mathbf{Oz}$ has irreducible or prime elements outside of $\mathbb{Z}$. In $\mathbf{Oz}$, Conway conjectured that $1 + \sum_{n \in \mathbb{N}^*} t^{-\frac{1}{n}}$ (where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$) is irreducible [Con76], and Gonshor suggested that it should also be prime [Gon86]. Furthermore, even if $\mathbf{Oz}$ cannot be a GCD domain (Remark 8.3.9), Conway conjectured that $\mathbf{Oz}$ is a **common refinement domain** (or pre-Schreier domain) [Con76], namely whenever some $b \in \mathbf{Oz}$ divide a product $cd \in \mathbf{Oz}$, we can write $b = b_1 b_2$ so that b_1 divides c and b_2 divides d; equivalently, any two factorisations of the same omnific integers should have a common refinement.

Berarducci answered affirmatively to the first conjecture [Ber00] by showing that $1 + \sum_{n \in \mathbb{N}^*} t^{-\frac{1}{n}}$ and other similar series are irreducible in the ring $K((\mathbb{R}^{\leq 0}))$, which then implies they are also irreducible in $\mathbf{Oz}$. Pitteloud then proved that $1 + \sum_{n \in \mathbb{N}^*} t^{-\frac{1}{n}}$ is even prime in $K((\mathbb{R}^{\leq 0}))$ [Pit01], hence also in $\mathbf{Oz}$ by [BKK06]. A few other irreducibility and primality results, and some cases of uniqueness of the factorisation, appeared later [PS05, BKK06, LM17].

1.4. **A factorisation theorem for $K((\mathbb{R}^{\leq 0}))$.** In this paper, we prove a general factorisation theorem for rings of the form $Z + K((G^{<0}))$, where Z is a subring of K and G is a divisible ordered abelian group, including omnific integers as a special case. We first state the result in the special, but crucial case $Z = K$ and $G = \mathbb{R}$, namely for the ring $K((\mathbb{R}^{\leq 0}))$.

Theorem A (6.4.1)**.** *For all non-zero $b \in K((\mathbb{R}^{\leq 0}))$, there exist $c_1, \ldots, c_n \in K((\mathbb{R}^{\leq 0}))$ irreducible with infinite support and $p \in K(\mathbb{R}^{\leq 0})$ such that $b = p \cdot c_1 \cdot \cdots \cdot c_n$. Moreover, p is unique by multiplication by an element of $K^* := K \setminus \{0\}$.*

The further factorisations of the elements in $K(\mathbb{R}^{\leq 0})$ are well understood, as they form a GCD domain (indeed, they can be seen as polynomials with fractional powers [Rit27, Poo95, EP97]). Furthermore, the number of factors with infinite support is bounded (Remark 5.6.2). We remark that Theorem A says in particular that every element in $K(\mathbb{R}^{\leq 0})$ is **primal** in the ring $K((\mathbb{R}^{\leq 0}))$ (Corollary 6.3.9), namely, if $p \in K(\mathbb{R}^{\leq 0})$ divides a product $cd \in K((\mathbb{R}^{\leq 0}))$, then we can write $p = p_1 p_2$ so that p_1 divides c and p_2 divides d. Therefore, $K((\mathbb{R}^{\leq 0}))$ is a common refinement domain, and actually a GCD domain, if and only if every irreducible series is prime (Corollary 6.4.2).

1.5. **A factorisation theorem for omnific integers.** We now look at more general rings of the form $Z + K((G^{<0}))$, where Z is a subring of K and G is a divisible ordered abelian group. For the sake of presentation, we start with the special case of omnific integers. Note that the conclusion of Theorem A fails in the general case. For instance, the series $\sum_{n\in\mathbb{N}^*} t^{-\frac{1}{n}} \in \mathbb{Z} + K((\mathbb{R}^{<0}))$ is divisible by *all* elements of $\mathbb{Z}$. Similar issues appear when $G \neq \mathbb{R}$. We thus need weaken the notion of irreducibility.

Recall that the Hahn field $K((G))$ has a **natural valuation** $v : K((G))^* \to G$ (where $K((G))^* := K((G)) \setminus \{0\}$) which sends each $b \in K((G))^*$ to the least exponent $v(b)$ in its support. For $x \in G$, write $|x|$ for the absolute value of x, namely $|x| = x$ if $x \geq 0$ and $|x| = -x$ otherwise. Given $x, y \in G$, say that x is **infinitesimal** with respect to y, and write $x \prec y$, if $n \cdot |x| \leq |y|$ for all $n \in \mathbb{N}$, and that x is **comparable** with y, and write $x \asymp y$, if $|x| \leq n \cdot |y|$ and $|y| \leq n \cdot |x|$ for some $n \in \mathbb{N}$.

We say that $b \in Z + K((G^{<0}))$ is **coarsely irreducible** if whenever $b = cd$ for some $c, d \in Z + K((G^{<0}))$, we have one of $v(c) \prec v(b)$ or $v(d) \prec v(b)$ (for instance, $b = \sum_{n\in\mathbb{N}^*} t^{-\frac{x}{n}} \in Z + K((G^{<0}))$ is coarsely irreducible for any Z, G and $x \in G^{<0}$).

Furthermore, let $b \in Z + K((G^{<0}))$ with $b \notin Z$. Denote by $\mathrm{supp}(b)$ its support. For any $x \in G$ such that $v(b) \leq x \leq 0$ (in particular, for x in $\mathrm{supp}(b)$) call the **standard part** of x with respect to b the real number $\mathrm{st}_b(x) := \inf\{q \in \mathbb{Q} : |x| \leq q \cdot |v(b)|\}$. Note that $\mathrm{st}_b(x)$ is contained in the interval $[0, 1]$. We call **coarse support** of b, denoted by $\overline{\mathrm{supp}}(b)$, the set of all the real numbers of the form $\mathrm{st}_b(x)$ for $x \in \mathrm{supp}(b)$. [1]

We can now state the theorem for the ring of omnific integers $\mathbf{Oz}$. Recall that $\mathbf{Oz} = \mathbb{Z} + \mathbb{R}((\mathbf{No}^{<0}))$, and also $\mathbf{No} = \mathbb{R}((\mathbf{No}))$.

Theorem B (8.4.2)**.** *For all non-zero $b \in \mathbf{Oz}$, there exist*

- *$c_1, \ldots, c_n \in \mathbf{Oz}$ coarsely irreducible with infinite coarse support,*
- *$p \in \mathbf{Oz}$ with finite coarse support or with $v(p) \prec v(b)$,*

such that $b = p \cdot c_1 \cdot \cdots \cdot c_n$ and $v(c_1) \asymp \cdots \asymp v(c_n) \asymp v(b)$. Moreover, p is unique up to multiplication by a surreal number $d \in \mathbf{No}$ such that $\mathrm{supp}(d) \prec v(b)$ (namely $x \prec v(b)$ for all $x \in \mathrm{supp}(b)$).

Additionally, for each coarsely irreducible factor c_i, we may further assume that either c_i is irreducible, or c_i is divisible by *any* nonzero $d \in \mathbf{Oz}$ with $v(d) \prec v(c_i)$ (see Remark 8.3.7).

[1] The coarse support will actually be defined differently in the proofs. The definition presented here is equivalent up to multiplication by a non-zero real number, so the difference is immaterial for stating Theorems B and C.

1.6. **The general case $Z + K((G^{<0}))$.** For full generality, we must further weaken irreducibility. We say that $b \in Z + K((G^{<0}))$ is **coarsely irreducible up to monomials** if whenever $b = cd$ for some $c, d \in Z + K((G^{<0}))$, we have one of $v(c) \prec v(b)$, $|\overline{\operatorname{supp}}(c)| = 1$, $v(d) \prec v(b)$, $|\overline{\operatorname{supp}}(d)| = 1$. One example of such series is $\sum_{n\in\mathbb{N}^*} t^{q_n} \in K((\mathbb{Q}^{\leq 0}))$ where $(q_n : n \in \mathbb{N})$ is an increasing sequence of rational numbers converging to $-\sqrt{2}$.

Theorem C (8.4.1). *For all non-zero $b \in Z + K((G^{<0}))$, there exist*

- *$c_1, \dots, c_n \in Z + K((G^{<0}))$ coarsely irreducible up to monomials with infinite coarse support,*
- *$p \in Z + K((G^{<0}))$ with finite coarse support or with $v(p) \prec v(b)$,*

such that $b = p \cdot c_1 \cdot \dots \cdot c_n$ and $v(c_1) \asymp \dots \asymp v(c_n) \asymp v(b)$. Moreover, p is unique up to multiplication by an element $d \in K((G))$ such that either $\operatorname{supp}(d) \prec v(b)$ or $d \in Z + K((G^{<0}))$ and $|\overline{\operatorname{supp}}(d)| = 1$.

If moreover $\sup(\overline{\operatorname{supp}}(b))$ is in the image of st_b, then we may take $c_1, \dots, c_n$ coarsely irreducible, in which case p is unique up to multiplication by an element $d \in K((G))$ such that $\operatorname{supp}(d) \prec v(b)$.

Again, for each factor c_i, we may further assume one of the following three options: c_i is irreducible, or c_i is coarsely irreducible and divisible by *any* nonzero series $d \in Z + K((G^{<0}))$ with $v(d) \prec v(c_i)$, or c_i is divisible by *any* nonzero series $d \in Z + K((G^{<0}))$ with either $v(d) \prec v(c_i)$ or $|\overline{\operatorname{supp}}(d)| = 1$ and such that $v(d) > \operatorname{supp}(c_i)$ (see Remark 8.3.8).

Note that Theorem C includes Theorem B as a special case, as in **Oz**, the image of the map st_b is the full interval $[0, 1] \subseteq \mathbb{R}$ for any $b \in$ **Oz**. In fact, we shall prove Theorem C first, and deduce the case of **Oz** as a corollary.

1.7. **A new, finer valuation.** The main ingredient in Berarducci's proof of the existence of irreducibles in $K((\mathbb{R}^{\leq 0}))$, and later in Pitteloud's proofs of primality, is the ordinal-valued semi-valuation "v_J" (see [Ber00] for more details).

Here, the key ingredient is a new ordinal-valued valuation on $K((\mathbb{R}^{\leq 0}))$ which refines v_J. Call **order type** of a series $b \in K((\mathbb{R}^{\leq 0}))$, denoted by $\operatorname{ot}(b)$, the unique ordinal number isomorphic to its support as a (well) ordered set. Call **degree** of b, denoted by $\deg(b)$, the degree of its order type $\operatorname{ot}(b)$, namely the maximum α such that $\omega^\alpha \leq \operatorname{ot}(b)$, with the convention "$\omega^{-\infty} = 0$", so that $\deg(0) = -\infty$. We shall prove that the degree is a *valuation* (up to reversing the ordering of **On**).

Theorem D (3.4.5). *For all $b, c \in K((\mathbb{R}^{\leq 0}))$,*

- *$\deg(b + c) \leq \max\{\deg(b), \deg(c)\}$ (ultrametric inequality);*
- *$\deg(b \cdot c) = \deg(b) \oplus \deg(c)$ (multiplicativity);*

where $\oplus$ denotes Hessenberg's natural sum.

One of the main advantages of the degree over Berarducci's v_J is that deg is a *valuation* (namely we also have $\deg(b) = -\infty$ if and only if $b = 0$), whereas v_J is only a semi-valuation. Since v_J takes only values of the form ω^α, and $v_J(b) \leq \operatorname{ot}(b)$, it follows at once that $v_J(b) \leq \omega^{\deg(b)}$ for all series b.

For completeness, we also remark that there are other interesting valuations on $K((\mathbb{R}^{\leq 0}))$, and in some cases, on $K((G^{\leq 0}))$. Firstly, the natural valuation v sending each series to the minimum of its support. Moreover, the 'dual' map $\sup : K((\mathbb{R}^{\leq 0}))^* \to \mathbb{R}^{\leq 0}$ sending each series to the *supremum* of its support, which can be shown to be multiplicative (Proposition 3.5.1), as it was already implicit in [Ber00].

For an arbitrary group G, the maps v_J, deg and sup are generally not multiplicative ([Pit02], Remark 3.5.3). However, there is a suitable quotient of the map sup which is multiplicative even in the general case. This will not be used in the paper, but it provides an alternative, simpler proof that the ideal generated by the monomials t^x for $x \in G^{<0}$ is prime [Pit02] (Corollary A.2.4), so it is included in Appendix A.

1.8. **New criteria for irreducibility and primality.** A byproduct of the proof of Theorem A is a strengthening of Berarducci's criterion for irreducibility [Ber00, Thm. 10.5].

Theorem E (6.5.3)**.** *For all $b \in K((\mathbb{R}^{\leq 0}))$, if the order type of the support of b is of the form $\omega^{\omega^\alpha} + \beta$ with $\beta < \omega^{\omega^\alpha}$, and b is not divisible by t^x for any $x \in \mathbb{R}^{<0}$, then b is irreducible.*

For comparison, in [Ber00] β was only allowed to be 0 or 1, and 0 needed to be an accumulation point of the support. One gets, for instance, that the series $1 + \sum_{n \in \mathbb{N}^*} t^{-1-\frac{1}{n}}$ is irreducible. Such series is then irreducible in **Oz**. Likewise, we can extend the Pitteloud's primality result [Pit01].

Theorem F (6.6.12)**.** *For all $b \in K((\mathbb{R}^{\leq 0}))$, if the order type of the support of b is of the form $\omega + k$ with $k < \omega$, and b is not divisible by t^x for any $x \in \mathbb{R}^{<0}$, then b is prime.*

In [Pit01] the order type could only be ω or $\omega + 1$ and 0 had to be an accumulation point of the support. A further, more technical criterion for primality for series of degree 1 is given in Corollary 6.6.15, yielding for instance the primality of the series

$$b = b_1 t^{-\sqrt{2}} + b_2 t^{-1} + b_3$$

whenever b_1, b_2, b_3 have order type ω, are not divisible by t^x for any $x \in \mathbb{R}^{<0}$, and at least one of $b_1 - b_2$, $b_1 - b_2$ has infinite support. It also follows that $b \pm 1$ is prime in $K((\mathbb{R}^{\leq 0}))$, so $b \pm 1$ is prime in **Oz**.

1.9. **Structure of the proof.** At first, we shall prove Theorem D, namely that the degree on $K((\mathbb{R}^{\leq 0}))$ is a valuation, in Section 3. The proof relies on Berarducci's results related to the semi-valuation v_J.

We shall then build an auxiliary monoid "RV", and an auxiliary ring "$\widehat{\mathrm{RV}}$", by taking an appropriate quotient of $K((\mathbb{R}^{\leq 0}))$ via the degree in Section 4 (this mimics the construction of the RV group of valued fields). We shall then exploit a structure result on RV and $\widehat{\mathrm{RV}}$ to prove Theorem A and Theorems E, F in Sections 5, 6.

Once Theorem A is given, we shall prove in Section 7 a variant where $\mathbb{R}$ is replaced by an Archimedean group G using only some elementary theory of polynomials. In Section 8, this will lead to a proof of Theorem C with a reasonably straightforward argument in the style of [BKK06], and Theorem B will be simply a corollary.

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2. Preliminaries

We first fix the notations and the basic facts we need to formulate and prove our results.

2.1. Ordinal arithmetic.

The **order type** of a totally ordered set $A = (A, <)$, denoted by $\mathrm{ot}(A)$, is the equivalence class of A with respect to order similarity. A **well ordered** set is a totally ordered set such that every nonempty subset has a least element. An **ordinal number** is the order type of a well ordered set.

Given two ordinal numbers α, β, we say that $\alpha \leq \beta$ if there are two representatives A and B such that $A \subseteq B$ and such that the inclusion of A in B is increasing; we say that $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. Let $\mathbf{On}$ be the (proper) class of all ordinal numbers. The class $\mathbf{On}$ is *well ordered by* $\leq$. Let 0 be the least ordinal number, which corresponds to the order type of the empty set.

Given an ordinal α, its **successor** $S(\alpha)$ is the least ordinal β such that $\alpha < \beta$. Given a set A of ordinals, we let $\sup(A)$ be the least β such that $\alpha \leq \beta$ for all $\alpha \in A$. Ordinal arithmetic is then defined by induction on $\leq$:

- $\alpha + 0 := \alpha$ and $\alpha + \beta := \sup_{\gamma<\beta}(S(\alpha + \gamma))$;
- $\alpha \cdot 0 := 0$ and $\alpha \cdot \beta := \sup_{\gamma<\beta}(\alpha \cdot \gamma + \beta)$;
- $\alpha^0 := 1$ and $\alpha^\beta := \sup_{\gamma<\beta}(\alpha^\gamma \cdot \beta)$.

For the sake of notation, we also let $\mathbf{On}_\infty := \mathbf{On} \cup \{-\infty\}$, and we define $-\infty < \alpha$, $-\infty + \alpha = \alpha + (-\infty) := -\infty$, $-\infty \cdot \alpha = \alpha \cdot (-\infty) := -\infty$, $\alpha^{-\infty} := 0$ for all $\alpha \in \mathbf{On}_\infty$. Given an ordinal $\alpha \in \mathbf{On}$, we let the **degree** $\deg(\alpha)$ of α be the maximum $\beta \in \mathbf{On}_\infty$ such that $\omega^\beta \leq \alpha$.

For all $\alpha \in \mathbf{On}$ there is a unique finite sequence $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq 0$ of ordinals such that

$$\alpha = \omega^{\beta_1} + \ldots + \omega^{\beta_n}.$$

Note that if $\alpha \neq 0$, then $\beta_1 = \deg(\alpha)$. The expression on the right-hand side is called **Cantor normal form** of α. Furthermore, $\mathbf{On}$ also admits different commutative operations called Hessenberg's **natural sum** $\oplus$ and **natural product** $\odot$. These can be defined rather easily using the Cantor normal form. Given $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \ldots + \omega^{\gamma_n}$ and $\beta = \omega^{\gamma_{n+1}} + \omega^{\gamma_{n+2}} + \ldots + \omega^{\gamma_{n+m}}$ in Cantor normal form, the natural sum $\alpha \oplus \beta$ is defined as $\alpha \oplus \beta := \omega^{\gamma_{\pi(1)}} + \omega^{\gamma_{\pi(2)}} + \ldots + \omega^{\gamma_{\pi(n+m)}}$, where π is a permutation of the integers $1, \ldots, n+m$ such that $\gamma_{\pi(1)} \geq \ldots \geq \gamma_{\pi(n+m)}$, and the natural product is defined by $\alpha \odot \beta := \bigoplus_{1\leq i\leq n} \bigoplus_{n+1\leq j\leq k+m} \omega^{\gamma_j \oplus \gamma_j}$. We extend these definitions to $\mathbf{On}_\infty$ by letting $\alpha \oplus (-\infty) := -\infty$, $\alpha \odot (-\infty) := -\infty$. Note that by definition $\omega^\alpha \odot \omega^\beta = \omega^{\alpha\oplus\beta}$.

Given an ordered set A and two subsets $B, C \subseteq A$, write $B < C$ if $x < y$ for all $x \in B$ and $y \in C$. Furthermore, if A is a subset of an ordered monoid, let $B + C := \{x + y : x \in B, y \in C\}$. We have the following.

Fact 2.1.1. Let A be a well ordered set. Then:

- for all $\alpha, \beta \in \mathbf{On}$, $\mathrm{ot}(A) = \alpha + \beta$ if and only if there are $B, C \subseteq A$ such that $B < C$, $\mathrm{ot}(B) = \alpha, \mathrm{ot}(C) = \beta$ and $A = B \cup C$;
- for all $B, C \subseteq A$, $\mathrm{ot}(B \cup C) \leq \mathrm{ot}(B) \oplus \mathrm{ot}(C)$ ([Ber00, Lem. 4.1]);
- if A is a subset of an ordered group, then for all $B, C \subseteq A$, $\mathrm{ot}(B + C) \leq \mathrm{ot}(B) \odot \mathrm{ot}(C)$ ([Ber00, Lem. 4.5]).

2.2. **Generalised power series.** Let K be a field and G be an ordered abelian group.

Definition 2.2.1. The **generalised power series field**, or **Hahn field**, $K((G))$ with coefficients in K and exponents in G is the set

$$K((G)) := \left\{ \sum_{x \in G} k_x t^x \,:\, k_x \in K,\ \{x \in G \,:\, k_x \neq 0\} \text{ is well-ordered by } < \right\}$$

where for $b = \sum_{x \in G} k_x t^x, c = \sum_{x \in G} l_x t^x \in K((G))$ we define

$$b + c := \sum_{x \in G} (k_x + l_x) t^x, \quad b \cdot c := \sum_{x \in G} \sum_{y+z=x} k_y l_z t^x, \quad \mathrm{supp}(b) := \{x \in G \,:\, k_x \neq 0\}.$$

The **group algebra** $K(G)$ is the subset

$$K(G) := \{p \in K((G)) \,:\, |\mathrm{supp}(p)| < \infty\}.$$

Notation 2.2.2. If A is a subset of G, we denote by $K((A))$ (or $K(A)$) the subset of $K((G))$ (resp. $K(G)$) of the series whose support is contained in A.

Note that $K((A))$ and $K(A)$ are clearly K-vector spaces. When A is a subset of G closed under sum, then $K((A))$ and $K(A)$ are closed under products, so they are (possibly nonunital) ring; if A also contains 0, then $K((A))$ and $K(A)$ are rings. In particular, $K((G^{\leq 0}))$ and $K(G^{\leq 0})$ are subrings of $K((G))$ (where $G^{\leq 0} = \{x \in G \,:\, x \leq 0\}$).

Definition 2.2.3. The **natural valuation** of $K((G))$ is the function $v : K((G))^* \to G$ defined by $v(b) := \min \mathrm{supp}(b)$ for $b \in K((G))^*$.

Fact 2.2.4. The natural valuation is indeed a valuation, namely for all $b, c \in K((G))^*$ we have

- $v(b + c) \geq \min\{v(b), v(c)\}$;
- $v(bc) = v(b) + v(c)$.

2.3. **Common refinement domains.** Let R be an integral domain.

Definition 2.3.1. An element $b \in R$ is **primal** (after [Coh68]) if for all $c, d \in R$, if b divides cd, then there are $b_1, b_2 \in R$ such that $b = b_1 b_2$ and b_1 divides c, b_2 divides d.

Definition 2.3.2. We say that R is a **common refinement domain**, or a **pre-Schreier domain** ([Zaf87]), if every element of R is primal.

Remark 2.3.3. For all irreducible $b \in R$, b is prime if and only if b is primal.

Definition 2.3.4. Given $b, c \in R$, we say that d is a **greatest common divisor** of b and c if d divides b and c, and for any other $e \in R$ dividing b and c, e divides d. When it exists, we denote by $\mathrm{GCD}(b, c)$ a choice of a greatest common divisor between b and c, which is defined up to a unit of R. We say that R is a **GCD domain** if all $b, c \in R$ have a greatest common divisor.

We recall the following facts about GCD domains.

Lemma 2.3.5. *Let R be a GCD domain. For all $b, c, d \in R$, $c \cdot \mathrm{GCD}(b, d)$ is a greatest common divisor of bc and cd.*

Proof. Let $b, c, d \in R$. Clearly, $c \cdot \mathrm{GCD}(\mathrm{b}, \mathrm{d})$ divides bc and cd, so it divides $\mathrm{GCD}(bc, cd)$. In particular, we can write $\mathrm{GCD}(bc, cd) = c \cdot e$ for some $e \in R$. It follows at once that e divides b and d, so it divides $\mathrm{GCD}(b, d)$. In turn, $\mathrm{GCD}(bc, cd)$ divides $c \cdot \mathrm{GCD}(b, d)$, as desired. □

Proposition 2.3.6. *Every GCD domain is a common refinement domain.*

Proof. Let R be a GCD domain. Let $b, c, d \in R$ be such that b divides cd. Let $b_1 = \mathrm{GCD}(b, c)$, and write $b = b_1 b_2$ for the appropriate $b_2 \in R$. It suffices to verify that b_2 divides d. Without loss of generality, we may divide b and c by b_1 and assume that $b_1 = \mathrm{GCD}(b, c)$ is a unit. Likewise, we may further assume that $\mathrm{GCD}(b, d)$ is also a unit. We then note that b divides both bc and cd, so it divides $\mathrm{GCD}(bc, cd)$, so it divides $c \cdot \mathrm{GCD}(b, d)$ by Lemma 2.3.5, so it divides c. Therefore, b itself is a unit, and the conclusion follows. □

For our purposes, we shall use the fact that $K(G)$ and $K(G^{\leq 0})$ are both GCD domains, so in particular common refinement domains.

Fact 2.3.7. For all fields K and ordered groups G, $K(G^{\leq 0})$ and $K(G)$ are GCD domains (see for instance [GP74, Thm. 6.4]).

3. The degree

3.1. **Order type, degree and supremum of a series.** Following [Ber00], we define the order type of a series as the order type of its support as a well ordered set. We also define the degree as the degree of the order type, and, for $G = \mathbb{R}$, the supremum as the supremum of the support. Recall that for $\alpha \in \mathbf{On}$, $\deg(\alpha)$ is the maximum $\beta \in \mathbf{On}_\infty$ such that $\omega^\beta \leq \alpha$ (see Subsection 2.1).

Definition 3.1.1. Given $b \in K((G))$, we define:

- the **order type** of b as $\mathrm{ot}(b) := \mathrm{ot}(\mathrm{supp}(b)) \in \mathbf{On}$;
- the **degree** of b as $\deg(b) := \deg(\mathrm{ot}(b)) = \deg(\mathrm{ot}(\mathrm{supp}(b))) \in \mathbf{On}_\infty$.

Given $b \in K((\mathbb{R}))$, we define the **supremum** of b as $\sup(b) := \sup(\mathrm{supp}(b))$.

Remark 3.1.2. For $G = \mathbb{R}$, the maps ot and deg take values among the *countable* ordinals, namely ordinals represented by countable sets. Indeed, each well ordered subset of $\mathbb{R}$ is necessarily countable. The set of countable ordinals is denoted by ω_1, which is itself an ordinal (the least uncountable one).

Conversely, every countable ordinal is represented by a well ordered subset of $\mathbb{R}$ (or even $\mathbb{R}^{\leq 0}$)), so every ordinal in ω_1 is the order type of some series in $K((\mathbb{R}))$ (or $K((\mathbb{R}^{\leq 0}))$). One can easily check that the same is true for the degree.

Proposition 3.1.3 ([Ber00, Remark 5.4])**.** *For all $b, c \in K((G))$ we have:*

- $\mathrm{ot}(b + c) \leq \mathrm{ot}(b) \oplus \mathrm{ot}(c)$;
- $\mathrm{ot}(b \cdot c) \leq \mathrm{ot}(b) \odot \mathrm{ot}(c)$;

Proof. Let $b, c \in K((G))$. Note that $\mathrm{supp}(b+c) \subseteq \mathrm{supp}(b) \cup \mathrm{supp}(c)$ and $\mathrm{supp}(bc) \subseteq \mathrm{supp}(b) + \mathrm{supp}(c)$. The inequalities then follow at once from Fact 2.1.1. □

Corollary 3.1.4. *For all $b, c \in K((G))$ we have:*

- $\deg(b + c) \leq \max\{\deg(b), \deg(c)\}$;
- $\deg(b \cdot c) \leq \deg(b) \oplus \deg(c)$.

Similar inequalities hold for the supremum.

Proposition 3.1.5. *For all $b, c \in K((\mathbb{R}))$ we have:*

- $\sup(b + c) \leq \max\{\sup(b), \sup(c)\}$;
- $\sup(b \cdot c) \leq \sup(b) + \sup(c)$.

Proof. Immediate from the definition of sum and product. □

3.2. **Truncations and principal series.** Fact 2.1.1 translates naturally to the following statement on series.

Proposition 3.2.1. *For all $b \in K((G))$ and $\alpha, \beta \in \mathbf{On}$, $\mathrm{ot}(b) = \alpha + \beta$ if and only if there are $c, d \in K((G))$ such that $\mathrm{supp}(c) < \mathrm{supp}(d)$, $\mathrm{ot}(c) = \alpha$, $\mathrm{ot}(d) = \beta$ and $b = c + d$.*

Proof. Let $A = \mathrm{supp}(b)$. By Fact 2.1.1, $\mathrm{ot}(A) = \alpha + \beta$ if and only if there are $B, C \subseteq A$ such that $B < C$, $\mathrm{ot}(B) = \alpha$, $\mathrm{ot}(C) = \beta$ and $A = B \cup C$. Write $b = \sum_{x \in G} k_x t^x$. Then clearly

$$b = \sum_{x \in G} k_x t^x = \sum_{x \in A} k_x t^x = \sum_{x \in B} k_x t^x + \sum_{x \in C} k_x t^x.$$

The conclusion then follows at once on setting $c = \sum_{x \in B} k_x t^x$, $d = \sum_{x \in C} k_x t^x$: indeed, $\mathrm{supp}(c) = B$, $\mathrm{supp}(d) = C$, so $\mathrm{supp}(c) < \mathrm{supp}(d)$, $\mathrm{ot}(c) = \alpha$, $\mathrm{ot}(d) = \beta$ and $b = c + d$. □

When a series b is written as a sum $b = c + d$ as in the above proposition, we shall call c and d truncations of b. We will use the following notation.

Definition 3.2.2. Given $b = \sum_x b_x t^x \in K((G))$ and $y \in G$, we let the **truncations** of b at y to be:

$$b_{\leq y} := \sum_{x \leq y} b_x t^x, \quad b_{< y} := \sum_{x < y} b_x t^x, \quad b_{\geq y} := \sum_{x \geq y} b_x t^x, \quad b_{> y} := \sum_{x > y} b_x t^x.$$

Note that the order type of a truncation of b is always at most the order type of b itself, and the same is true for the degree. A stronger inequality holds for the series of the following type.

Definition 3.2.3. An ordinal $\alpha \in \mathbf{On}$ is **(additively) principal** if $\alpha = \omega^\beta$ for some $\beta \in \mathbf{On}$; equivalently, if $\alpha = \beta + \gamma$ implies $\gamma = \alpha$ for all $\beta, \gamma \in \mathbf{On}$ and $\alpha \neq 0$.

A series $b \in K((G))$ is **weakly principal** if $\mathrm{ot}(b)$ is principal.

A series $b \in K((\mathbb{R}))$ is **principal** if it is weakly principal and $\sup(b) = 0$ (in particular, $b \in K((\mathbb{R}^{\leq 0}))$).

Proposition 3.2.4. *If $b \in K((\mathbb{R}^{\leq 0}))$ is weakly principal and $x < \sup(b)$, then* $\deg(b_{\leq x}) < \deg(b)$.

Proof. Let b, x be as in the hypothesis. Write $b = b_{<x} + b_{\geq x}$. Since $x < \sup(b)$, $b_{\geq x} \neq 0$, so $\mathrm{ot}(b) = \mathrm{ot}(b_{<x}) + \mathrm{ot}(b_{\geq x})$ implies $\mathrm{ot}(b_{<x}) < \mathrm{ot}(b)$. Since $\mathrm{ot}(b) = \omega^{\alpha}$ for some ordinal α, it follows that $\deg(b_{<x}) < \alpha = \deg(b)$, as desired. □

3.3. **Normal forms.** Given a series $b \in K((G^{\leq 0}))$, we use the Cantor Normal Form of $\mathrm{ot}(b)$ to give a canonical decomposition of b as a sum of weakly principal series, or in the case $G = \mathbb{R}$, as sums of principal series multiplied by monomials. This will be useful for proving Theorem D, as well as for the later Proposition 5.3.1.

Definition 3.3.1. Given $b \in K((G))$, we call the sum

$$b = b_1 + \cdots + b_n$$

the **weak normal form** of b when:

- b_1 is weakly principal for all $i = 1, \dots, n$;
- $\mathrm{ot}(b_1) \geq \cdots \geq \mathrm{ot}(b_n)$;
- $\mathrm{supp}(b_1) < \cdots < \mathrm{supp}(b_n)$.

Remark 3.3.2. Note that $\mathrm{ot}(b) = \mathrm{ot}(b_1) + \cdots + \mathrm{ot}(b_n)$; since each $\mathrm{ot}(b_i)$ is weakly principal, this sum is by definition the Cantor normal form of $\mathrm{ot}(b)$.

Proposition 3.3.3. *For all series $b \in K((G))$ there exists a unique weak normal form.*

Proof. Let $b \in K((G))$. There is a unique way of writing $\mathrm{ot}(b) = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ with $\beta_1 \geq \cdots \geq \beta_n$. By iterating Proposition 3.2.1, there are unique $b_1, \dots, b_n$ such that $b = b_1 + \cdots + b_n$, $\mathrm{ot}(b_i) = \omega^{\beta_i}$, and $\mathrm{supp}(b_i) < \mathrm{supp}(b_{i+1})$. Conversely, for any weak normal form $b = b_1 + \cdots + b_n$, the sum $\mathrm{ot}(b) = \mathrm{ot}(b_1) + \cdots + \mathrm{ot}(b_n)$ is the Cantor normal form of $\mathrm{ot}(b)$. Therefore, the weak normal form is unique. □

Corollary 3.3.4. *For all $b \in K((G))^*$ there exists $x \in G$ such that $b_{\geq x}$ is non-zero and weakly principal.*

Proof. Given $b \in K((G))^*$, it suffices to write $b = b_1 + \cdots + b_n$ in weak normal form and let $x = v(b_n)$. Then $b_{\geq x} = b_n$ is non-zero and weakly principal. □

When $G = \mathbb{R}$, we can also use principal series.

Definition 3.3.5. Given $b \in K((\mathbb{R}))$, we call the sum

$$b = b_1 t^{x_1} + \cdots + b_n t^{x_n}$$

the **normal form** of b when:

- $x_1 \leq \cdots \leq x_n$;
- b_i is principal for all $i = 1, \dots, n$;
- $\mathrm{ot}(b_1) \geq \cdots \geq \mathrm{ot}(b_n)$;
- $x_i + \mathrm{supp}(b_i) < x_{i+1} + \mathrm{supp}(b_{i+1})$ for all $i = 1, \dots, n-1$.

Proposition 3.3.6. *For all series $b \in K((\mathbb{R}^{\leq 0}))$ there exists a unique normal form.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$. By Proposition 3.3.3, we can write uniquely $b = b'_1 + \dots b'_n$ in weak normal form. It now suffices to define $x_i = \sup(\operatorname{supp}(b'_i))$ and $b_i = t^{-x_i} b'_i$. The conclusion then follows trivially. □

Remark 3.3.7. Proposition 3.3.6 shows that any series in $K((\mathbb{R}^{\leq 0}))$ can be intuitively thought as a series in "$P(\mathbb{R}^{\leq 0})$", where P is the set of principal series plus the element 0. However, P *is not a ring*, as it is not closed under sum (but it is under multiplication, as we shall see in a moment). P can be made into a ring by taking an appropriate quotient, which is the main motivation for defining the RV monoid in Section 4.

Note that the above proof uses not only that $\mathbb{R}$ is complete, but also that $\operatorname{supp}(b_i)$ is bounded from above for $b_i \in K((\mathbb{R}^{\leq 0}))$. In fact, arbitrary series in $K((\mathbb{R}))$ may not have a normal form, as for instance

$$1 + t + t^2 + t^3 + \dots .$$

3.4. **Multiplicativity of the degree.** By [Ber00], the order type is multiplicative on weakly principal series.

Theorem 3.4.1 ([Ber00, Corollary 9.9])**.** *If $b, c \in K((\mathbb{R}^{\leq 0}))$ are weakly principal, then* $\operatorname{ot}(bc) = \operatorname{ot}(b) \odot \operatorname{ot}(c)$.

In particular, bc is weakly principal and $\deg(bc) = \deg(b) \oplus \deg(c)$ for weakly principal b, c. We shall now prove that this is in fact true for all $b, c \in K((\mathbb{R}^{\leq 0}))$.

Lemma 3.4.2. *Let $b, c \in K((\mathbb{R}^{\leq 0}))$ and $x \in \mathbb{R}$. If b is principal and* $\operatorname{supp}(c) > x$, *then* $\deg((bc)_{\leq x}) < \deg(b) \oplus \deg(c)$.

Proof. Let b, c, x as in the hypothesis. Let y be such that $x < y < \operatorname{supp}(c)$ and write $b = d + e$ in way so that $\operatorname{supp}(d) \leq x - y < \operatorname{supp}(e)$. Then $\operatorname{supp}(ce) > y + (x - y) = x$. It follows that $(bc)_{\leq x} = (cd + ce)_{\leq x} = (cd)_{\leq x}$. Therefore,

$$\deg((bc)_{\leq x}) = \deg((cd)_{\leq x}) \leq \deg(cd) \leq \deg(c) \oplus \deg(d).$$

Now note that $x - y < 0$, so $\sup(\operatorname{supp}(d)) < 0$. Since b is principal, we must have $b \neq d$, so $e \neq 0$, so $\operatorname{ot}(e) > 0$. Since $\operatorname{ot}(b) = \operatorname{ot}(d) + \operatorname{ot}(e)$, it follows that $\operatorname{ot}(d) < \operatorname{ot}(b)$. On the other hand, $\operatorname{ot}(b)$ is of the form ω^α, so $\deg(d) < \deg(b)$. In turn, $\deg((bc)_{\leq x}) < \deg(b) \oplus \deg(c)$, as desired. □

Lemma 3.4.3. *Let $b, c \in K((\mathbb{R}^{\leq 0}))$ and $x \in \mathbb{R}$. If b is principal,* $\deg(c) > 0$ *and* $\operatorname{supp}(c) \geq x$, *then* $\deg((bc)_{\leq x}) < \deg(b) \oplus \deg(c)$.

Proof. Let b, c, x as in the hypothesis. Write $c = kt^x + c'$, where $k \in K$ and $\operatorname{supp}(c') > x$. Then $bc = bkt^x + bc'$. By Lemma 3.4.2, $\deg((bc')_{\leq x}) < \deg(b) \oplus \deg(c') \leq \deg(b) \oplus \deg(c)$. Moreover, $\deg(bkt^x) \leq \deg(b) < \deg(b) \oplus \deg(c)$. Therefore, $\deg((bc)_{\leq x}) < \deg(b) \oplus \deg(c)$. □

Proposition 3.4.4. *For all $b, c \in K((\mathbb{R}^{\leq 0}))$,* $\deg(b \cdot c) = \deg(b) \oplus \deg(c)$.

Proof. Let $b, c \in K((\mathbb{R}^{\leq 0}))$. We may assume that b, c are not 0, otherwise the conclusion is trivial. By Proposition 3.3.6, we can write $b = b_1 t^x + b'$ with b_1 principal, $\mathrm{supp}(b') \geq x$ and $\deg(b_1) = \deg(b) \geq \deg(b')$. Moreover, if $\deg(b) = 0$, we may further assume that $\mathrm{supp}(b') > x$. Similarly, we may write $c = c_1 t^y + c'$ with c_1 principal, $\mathrm{supp}(c') \geq y$, $\deg(c_1) = \deg(c) \geq \deg(c')$, and if $\deg(c) = 0$, then $\mathrm{supp}(c') > y$.

Trivially, we have

$$bc = (b_1 t^x + b')(c_1 t^y + c') = b_1 c_1 t^{x+y} + b_1 c' t^x + b' c_1 t^y + b' c'.$$

After truncating both sides at $x + y$, we get

$$(bc)_{\leq x+y} = b_1 c_1 t^{x+y} + (b_1 c')_{\leq y} t^x + (b' c_1)_{\leq x} t^y + (b' c')_{\leq x+y}.$$

By Theorem 3.4.1, $\deg(b_1 c_1 t^{x+y}) = \deg(b_1 c_1) = \deg(b_1) \oplus \deg(c_1) = \deg(b) \oplus \deg(c)$. We claim that the other three summands on the right-hand side have smaller degree.

For the fourth summand, note that $\mathrm{supp}(b'c') \geq x + y$, so $(b'c')_{\leq x+y} \in K$, which means that its degree is at most 0. If $\deg(b) > 0$ or $\deg(c) > 0$, then $\deg(b) \oplus \deg(c) > 0$, and we are done. If instead $\deg(b) = \deg(c) = 0$, then actually $\mathrm{supp}(b'c') > x + y$, so $(b'c')_{\leq x+y} = 0$, which has degree $-\infty < 0 = \deg(b) \oplus \deg(c)$, and we are done.

For the second and third summand, we distinguish two cases. If $\deg(c') < \deg(c)$, then in fact $\deg((b_1 c')_{\leq y}) \leq \deg(b_1) \oplus \deg(c') < \deg(b_1) \oplus \deg(c')$, in which case we are done. By symmetry, the same holds for the third summand when $\deg(b') < \deg(b)$.

Now assume $\deg(c') = \deg(c)$. If $\deg(c) > 0$, then by Lemma 3.4.3 we have $\deg((b_1 c')_{\leq y}) < \deg(b_1) \oplus \deg(c') = \deg(b) \oplus \deg(c)$, and we are done. If $\deg(c) = 0$, then $\mathrm{supp}(c') > y$, so by Lemma 3.4.2 we have $\deg((b_1 c')_{\leq y}) < \deg(b_1) \oplus \deg(c') = \deg(b) \oplus \deg(c)$, and we are done. Again, by symmetry the same holds for the third summand when $\deg(b') = \deg(b)$, concluding the proof of the claim.

Therefore, by Corollary 3.1.4, $\deg((bc)_{\leq x+y}) = \deg(b_1 c_1 t^{x+y}) = \deg(b) \oplus \deg(c)$. Since in fact $\deg((bc)_{\leq x+y}) \leq \deg(bc) \leq \deg(b) \oplus \deg(c)$, it follows that $\deg(bc) = \deg(b) \oplus \deg(c)$, as desired. ☐

This finally implies Theorem D, which is the key tool in the proof of Theorem A.

Theorem 3.4.5 (Theorem D)**.** *For all $b, c \in K((\mathbb{R}^{\leq 0}))$,*

- $\deg(b + c) \leq \max\{\deg(b), \deg(c)\}$ *(ultrametric inequality);*
- $\deg(b \cdot c) = \deg(b) \oplus \deg(c)$ *(multiplicativity).*

Proof. It suffices to combine the inequalities of Corollary 3.1.4 with the conclusion of Proposition 3.4.4. ☐

3.5. **Multiplicativity of the supremum.** Another consequence of Berarducci's Theorem 3.4.1 is that the supremum is a valuation. This can be easily deduced from the fact that the ideal J generated by all the monomials t^x for $x \in \mathbb{R}^{<0}$ is prime [Ber00, Cor. 9.8]. Here we include a proof using the results presented in this section.

Proposition 3.5.1. *For all $b, c \in K((\mathbb{R}^{\leq 0}))^*$ we have:*

- $\sup(b + c) \leq \max\{\sup(b), \sup(c)\}$*;*

- $\sup(bc) = \sup(b) + \sup(c)$.

Proof. Let $b, c \in K((\mathbb{R}^{\leq 0}))^*$. Let $x = \sup(b)$, $y = \sup(c)$. Then we can write $b = t^x b'$, $c = t^y c'$ for some suitable $b', c' \in K((\mathbb{R}^{\leq 0}))$ such that $\sup(b') = \sup(c') = 0$. Note that $\sup(bc) = \sup(t^{x+y}b'c') = x + y + \sup(b'c')$. Therefore, without loss of generality, we may assume that $\sup(b) = \sup(c) = 0$. We need to prove that $\sup(bc) = 0$.

By Corollary 3.3.4, we can write $b = b' + b''$ and $c = c' + c''$ such that b'', c'' are non-zero weakly principal and $\mathrm{supp}(b') < \mathrm{supp}(b'')$, $\mathrm{supp}(c') < \mathrm{supp}(c'')$. In particular, $\sup(b'') = 0$ and $\sup(c'') = 0$, so they are in fact principal. Then

$$\sup(bc) = \sup(b'c' + b'c'' + b''c' + b''c'') = \sup(b''c'')$$

by Proposition 3.1.5. Therefore, it suffices to prove that $\sup(b''c'') = 0$. Without loss of generality, we may directly assume that b, c are principal.

Fix some $x < 0$. Then

$$(bc)_{\leq 2x} = (b_{\leq x}c_{\leq x} + b_{>x}c_{\leq x} + b_{\leq x}c_{>x} + b_{>x}c_{>x})_{\leq 2x}.$$

Since $\mathrm{supp}(b_{>x}c_{>x}) > 2x$, we have $(bc)_{\leq 2x} = (b_{\leq x}c_{\leq x} + b_{>x}c_{\leq x} + b_{\leq x}c_{>x})_{\leq 2x}$. Therefore,

$$\mathrm{ot}((bc)_{\leq 2x}) \leq \mathrm{ot}(b_{\leq x}c_{\leq x} + b_{>x}c_{\leq x} + b_{\leq x}c_{>x}) \leq \mathrm{ot}(b_{\leq x}c_{\leq x}) \oplus \mathrm{ot}(b_{>x}c_{\leq x}) \oplus \mathrm{ot}(b_{\leq x}c_{>x}).$$

Since b, c are principal, the latter three summands are strictly less than $\mathrm{ot}(b) \odot \mathrm{ot}(c)$; since the ordinal $\mathrm{ot}(b) \odot \mathrm{ot}(c)$ is principal, it follows that

$$\mathrm{ot}(bc_{\leq 2x}) < \mathrm{ot}(b) \odot \mathrm{ot}(c) = \mathrm{ot}(bc),$$

where the latter equality follows from Theorem 3.4.1. This implies that $bc \neq bc_{\leq 2x}$ for all $x < 0$, so in particular $\sup(bc) = 0$, as desired. □

Corollary 3.5.2. *If $b, c \in K((\mathbb{R}^{\leq 0}))$ are principal, then bc is principal.*

Proof. Let $b, c \in K((\mathbb{R}^{\leq 0}))$ be principal series. By Theorem 3.4.1, bc is weakly principal, and by Proposition 3.1.5, $\sup(bc) = \sup(b) + \sup(c) = 0$, so bc is principal. □

Remark 3.5.3. The maps deg and sup are not multiplicative in non-Archimedean groups (where sup is defined in some appropriate way, such as in Appendix A). Indeed, suppose that $x, y \in G^{\leq 0}$ are two elements such that $x \prec y$, namely $|x| \leq n \cdot |y|$ for all $n \in \mathbb{N}$, where $|x| = \max\{x, -x\}$. Then

$$b = t^y \cdot \sum_{n \in \mathbb{N}^*} t^{nx}, \quad c = 1 - t^x$$

are both series in $K((G^{\leq 0}))$, and $bc = t^y$. It follows that $\deg(bc) = 0 \neq \deg(b) \oplus \deg(c) = 1$ and $\sup(bc) = y < \sup(b) + \sup(c) = \sup(b)$.

4. The RV monoid

In this section, let R be a commutative ring, $(M, +)$ be an ordered commutative monoid, and $w : R \to M \cup \{\infty\}$ be a *semi-valuation*, namely a map such that for all $b, c \in R$ we have

- $w(b + c) \geq \min\{w(b), w(c)\}$ (ultrametric inequality),

- $w(bc) = w(b) + w(c)$ (multiplicativity),

where by convention $m + \infty = \infty + m = \infty$ and $m < \infty$ for all $m \in M \cup \{\infty\}$ (including $\infty < \infty$). The function w is a *valuation* when $w(b) = \infty$ if and only if $b = 0$.

All of the following constructions also work for non-unital rings, provided we further assume that $w(-b) = w(b)$ for all $b \in R$. In this case, the words "monoid" and "ring" should be replaced by "semi-group" and "non-unital ring" respectively.

4.1. **The RV monoid.** We now construct the monoid RV of the valued ring (R, w). The definition mimics the construction of the RV group of valued fields.

Definition 4.1.1. Given $b, c \in R$, we write $b \sim c$ if $w(b - c) > w(b)$.

It is almost immediate to verify that $\sim$ is an equivalence relation, and that it preserves multiplication and the semi-valuation w.

Proposition 4.1.2. *For all $b, c \in R$, if $b \sim c$, then $w(b) = w(c)$.*

Proof. By the ultrametric inequality, $w(b) = w(c + (b - c)) \geq \min\{w(c), w(b - c)\}$, so necessarily $w(b) \geq w(c)$. Symmetrically, $w(c) = w(b + (c - b)) \geq \min\{w(b), w(c - b)\} = w(b)$. Therefore, $w(b) = w(c)$. □

Corollary 4.1.3. *The relation $\sim$ is an equivalence relation.*

Proof. Let $b, c, d \in R$. Clearly, $b \sim b$, as $w(b - b) = \infty > w(b)$. Moreover, if $b \sim c$, then $w(b) = w(c)$ by Proposition 4.1.2, so $w(c - b) = w(b - c) > w(c)$, that is $c \sim b$. Finally, if $b \sim c$ and $c \sim d$, then again $w(c) = w(d) = w(b)$, and $w(b - d) \geq \min\{w(b - c), w(c - d)\} > w(b)$, so $b \sim d$. □

Definition 4.1.4. Let RV be the quotient $R_{/\sim}$, and let $\mathrm{rv} : R \to \mathrm{RV}$ be the natural quotient map. Given $b \in R$ we define $w(\mathrm{rv}(b)) := w(b)$.

Proposition 4.1.5. *For all $b, c, d \in R$, if $c \sim d$, then $bc \sim bd$.*

Proof. Let $b, c, d \in R$ with $c \sim d$. Then $w(bc - bd) = w(b) + w(c - d) > w(b) + w(c) = w(bc)$, so $bc \sim bd$. □

This immediately implies that the following product is well defined.

Definition 4.1.6. Given $B, C \in \mathrm{RV}$, we let $B \cdot C := \mathrm{rv}(bc)$, where $\mathrm{rv}(b) = B$, $\mathrm{rv}(c) = C$.

Proposition 4.1.7. $(\mathrm{RV}, \cdot)$ *is a commutative monoid.*

Proof. Let $B, C, D \in \mathrm{RV}$. Pick some $b, c, d \in R$ such that $B = \mathrm{rv}(b), C = \mathrm{rv}(c), D = \mathrm{rv}(d)$. It follows at once from the definition that $B \cdot C = \mathrm{rv}(bc) = \mathrm{rv}(cb) = C \cdot B$ and that $B \cdot (C \cdot D) = \mathrm{rv}(b(cd)) = \mathrm{rv}((bc)d) = (B \cdot C) \cdot D$. Moreover, $B \cdot C = \mathrm{rv}(bc) = \mathrm{rv}(cb) = C \cdot B$, and $\mathrm{rv}(1)$ is the neutral element, as $\mathrm{rv}(b) \cdot \mathrm{rv}(1) = \mathrm{rv}(b \cdot 1) = \mathrm{rv}(b)$.. □

Remark 4.1.8. For all $B, C \in \mathrm{RV}$ we have $w(B \cdot C) = w(B) + w(C)$, and $w(B) = \infty$ if and only if $B = \mathrm{rv}(0)$.

4.2. **Modules in RV.** We shall now verify that RV also carries a natural notion of (partial) sum. To define the sum, we first partition RV into subsets of constant valuation.

Definition 4.2.1. Given any m in the monoid M, let $\mathrm{RV}_m := \{B \in \mathrm{RV} : w(B) = m\} \cup \{\mathrm{rv}(0)\}$.

Remark 4.2.2. For all $m, n \in M$, $\mathrm{RV}_m \cdot \mathrm{RV}_n \subseteq \mathrm{RV}_{m+n}$.

Proposition 4.2.3. *For all $b, c, d \in R$, if $c \sim d$ and $w(b + c) = w(b)$, then $b + c \sim b + d$, and moreover $w(c) = w(b + d) = w(b)$.*

Proof. Let $b, c, d \in R$, with $c \sim d$ and $w(b+c) = w(b)$. Note in particular that $w(c) = w((b+c)-b) \geq \min\{w(b + c), w(b)\} = w(b + c)$. Then $w((b + c) - (b + d)) = w(c - d) > w(c) \geq w(b + c)$, so $b + c \sim b + d$. □

This implies that the following partial sum is well defined.

Definition 4.2.4. Given $m \in M$ and $B, C \in \mathrm{RV}_m$, we define

$$B + C := \begin{cases} \mathrm{rv}(b + c) & \text{if } w(b + c) = w(b), \\ \mathrm{rv}(0) & \text{otherwise,} \end{cases}$$

where $B = \mathrm{rv}(b)$ and $C = \mathrm{rv}(c)$.

We shall verify that each RV_m is an abelian group with respect to the above sum, and moreover that the product is distributive over the sum.

Lemma 4.2.5. *For all $m \in M$ and all $b_1, \dots, b_n \in R$ either zero or of value m, we have*

$$(((\mathrm{rv}(b_1) + \mathrm{rv}(b_2)) + \mathrm{rv}(b_3)) + \dots) + \mathrm{rv}(b_n) = \begin{cases} \mathrm{rv}(b_1 + \dots + b_n) & \text{if } w(b_1 + \dots + b_n) = m, \\ \mathrm{rv}(0) & \text{otherwise.} \end{cases}$$

Proof. We prove the conclusion by induction on n, the base case $n = 2$ being the definition of sum. Suppose that $n > 2$ and that the conclusion true for $n-1$. Let $b_1, \dots, b_n \in R$ be zero or of valuation m, and let $c = b_1 + \dots + b_{n-1}$. We distinguish two cases.

Suppose $w(c) \neq m$. Then by inductive hypothesis $((\mathrm{rv}(b_1)+\mathrm{rv}(b_2))+\dots)+\mathrm{rv}(b_{n-1}) = \mathrm{rv}(0)$. In turn, $((\mathrm{rv}(b_1)+\mathrm{rv}(b_2))+\dots)+\mathrm{rv}(b_n) = \mathrm{rv}(b_n)$. On the other hand, $b_n \sim c+b_n$, so $\mathrm{rv}(b_n) = \mathrm{rv}(c+b_n)$, proving the desired conclusion.

Suppose now that $w(c) = m$. Then by inductive hypothesis $(\mathrm{rv}(b_1) + \mathrm{rv}(b_2)) + \dots + \mathrm{rv}(b_{n-1}) = \mathrm{rv}(c)$. In turn, $(\mathrm{rv}(b_1)+\mathrm{rv}(b_2))+\dots+\mathrm{rv}(b_n) = \mathrm{rv}(c)+\mathrm{rv}(b_n)$. By definition, if $w(c+b_n) = m$, then $\mathrm{rv}(c) + \mathrm{rv}(b_n) = \mathrm{rv}(c + b_n) = \mathrm{rv}(b_1 + \dots + b_n)$, otherwise $\mathrm{rv}(c) + \mathrm{rv}(b_n) = \mathrm{rv}(0)$, as desired. □

Proposition 4.2.6. *For all $m \in M$, $(\mathrm{RV}_m, +)$ is an abelian group.*

Proof. Let $m \in M$ and $B, C, D \in \mathrm{RV}_m$. Choose $b, c, d \in R$ such that $B = \mathrm{rv}(b), C = \mathrm{rv}(c), D = \mathrm{rv}(d)$. If $w(b+c) = m$, then $B+C = \mathrm{rv}(b+c) = \mathrm{rv}(c+b) = C+B$, otherwise $B+C = \mathrm{rv}(0) = C+B$, so the sum is commutative. Moreover, by Lemma 4.2.5, if $w(b + c + d) = m$ we have

$$B + (C + D) = \mathrm{rv}(b + c + d) = \mathrm{rv}(d + b + c) = D + (B + C) = (B + C) + D,$$

otherwise $B + (C + D) = \mathrm{rv}(0) = D + (B + C) = (B + C) + D$. Therefore, the sum is associative.

By definition, $B + \mathrm{rv}(0) = \mathrm{rv}(0) + B = B$, so $\mathrm{rv}(0)$ is the neutral element. Finally, $\mathrm{rv}(-b)$ is the inverse of B, as $w(\mathrm{rv}(-b)) = m$ and $B + \mathrm{rv}(-b) = \mathrm{rv}(0)$. Therefore, $(\mathrm{RV}_m, +)$ is an abelian group. □

Proposition 4.2.7. *For all $B \in \mathrm{RV}$, $m \in M$ and $C, D \in \mathrm{RV}_m$, we have $B\cdot(C+D) = B\cdot C + B\cdot D$.*

Proof. Let $C, D \in \mathrm{RV}_m$, $B \in \mathrm{RV}_n$ for some $m, n \in M$. Note that $B \cdot C, B \cdot D \in \mathrm{RV}_{m+n}$. Pick $b, c, d \in R$ such that $B = \mathrm{rv}(b), C = \mathrm{rv}(c), D = \mathrm{rv}(d)$.

Assume first that $w(c + d) = w(c)$. Then $w(bc + bd) = w(b) + w(c + d) = w(b) \oplus w(c) = w(bc)$. Therefore, $\mathrm{rv}(bc) + \mathrm{rv}(bd) = \mathrm{rv}(bc + bd) = \mathrm{rv}(b) \cdot \mathrm{rv}(c + d)$. Now suppose that $w(c + d) \neq w(c)$. Then $w(bc + bd) = w(b) + w(c + d) \neq w(b) \oplus w(c) = w(bc)$. Therefore, $\mathrm{rv}(bc) + \mathrm{rv}(bd) = \mathrm{rv}(0) = \mathrm{rv}(b) \cdot \mathrm{rv}(c + d)$. □

Recall that $\mathrm{RV}_m \cdot \mathrm{RV}_n \subseteq \mathrm{RV}_{m+n}$ for all $m, n \in M$. It follows at once that RV_0 is closed under multiplication and that $\mathrm{RV}_0 \cdot \mathrm{RV}_m \subseteq \mathrm{RV}_m$ for all $m \in M$.

Corollary 4.2.8. $(\mathrm{RV}_0, +, \cdot)$ *is a ring.*

Corollary 4.2.9. *For all $m \in M$, RV_m is an RV_0-module.*

In turn, the ring RV_0 is isomorphic to what one would call the "residue ring" of (R, w), in analogy with how one defines the residue field of a valued field.

Proposition 4.2.10. *Let $\mathcal{O} = \{b \in R : w(b) \geq 0\}$, $\mathcal{M} = \{b \in R : w(b) > 0\}$. Let $\pi : \mathcal{O} \to \mathrm{RV}_0$ be defined as $\pi(b) = \mathrm{rv}(b)$ if $b \notin \mathcal{M}$ and $\pi(b) = \mathrm{rv}(0)$ otherwise. Then $\pi : \mathcal{O} \to \mathrm{RV}_0$ is a surjective ring homomorphism with kernel $\mathcal{M}$.*

Proof. Let $b, c \in \mathcal{O}$. If $b \in \mathcal{M}$ or $c \in \mathcal{M}$, then clearly $\pi(b + c) = \pi(b) + \pi(c)$, so we may assume $b, c \in \mathcal{O} \setminus M$. Suppose $b + c \notin \mathcal{M}$. Then $w(b + c) = w(b) = w(c) = 0$, so by definition $\pi(b + c) = \mathrm{rv}(b + c) = \mathrm{rv}(b) + \mathrm{rv}(c) = \pi(b) + \pi(c)$. If otherwise $b + c \in \mathcal{M}$, then $w(b + c) = w(b) = w(c) = 0$, so by definition $\pi(b + c) = \mathrm{rv}(0) = \mathrm{rv}(b) + \mathrm{rv}(c) = \pi(b) + \pi(c)$. Therefore, π is a ring homomorphism. Clearly, π is surjective and its kernel is $\mathcal{M}$. □

Definition 4.2.11. We call RV_0 the **residue ring** of (R, w).

4.3. **Embedding RV into a ring.** Since the multiplication of RV is distributive over the sum defined in each RV_m, it is possible to embed RV into a ring. In fact, there is a free ring $\widehat{\mathrm{RV}}$ containing RV, with the universal property that any map from RV to a ring preserving sum and product factors through $\widehat{\mathrm{RV}}$. We construct $\widehat{\mathrm{RV}}$ explicitly as follows.

Definition 4.3.1. We let $(\widehat{\mathrm{RV}}, +)$ be the direct sum $\bigoplus_{m\in M}(\mathrm{RV}_m, +)$. We denote its elements as $\sum_{m\in M} B_m$, where $B_m \in \mathrm{RV}_m$ for all $m \in M$, and $B_m \neq 0$ for only finitely many values of m.

For $\sum_m B_m, \sum_m C_m \in \widehat{\mathrm{RV}}$, we define

$$\left(\sum_m B_m\right) \cdot \left(\sum_m C_m\right) := \sum_m \left(\sum_{n+o=m} B_n \cdot C_o\right).$$

Notation 4.3.2. RV embeds naturally into $\widehat{\mathrm{RV}}$ by sending $B \in \mathrm{RV}$ into the sum $\sum_m B_m$ having $B_m = B$ when $m = w(B)$ and $B_m = 0$ otherwise. It is immediate from the definition that such embedding preserves sums and products, so we shall identify RV with its isomorphic copy in $\widehat{\mathrm{RV}}$.

Proposition 4.3.3. $(\widehat{\mathrm{RV}}, +, \cdot)$ *is a ring.*

Proof. Immediate from the definitions and from the properties of sum and product in RV. □

Remark 4.3.4. By construction, $\widehat{\mathrm{RV}}$ is a *graded* ring.

4.4. **Some examples.** If $R = K$ is a *field*, then the residue ring RV_0 is in fact the residue field by Proposition 4.2.10. Let $\mathrm{RV}^* := \mathrm{RV} \setminus \{\mathrm{rv}(0)\}$. Then the monoid RV^* is a group sitting in the exact sequence $1 \to \mathrm{RV}_0^\times \to \mathrm{RV}^* \to M \to 0$, where $\mathrm{RV}_0^\times$ is the multiplicative group of the residue field and M is the value group. Moreover, each RV_n is a one-dimensional RV_0-vector space. RV is often isomorphic to $\mathrm{RV}_0^\times \times M$, in which case the ring $\widehat{\mathrm{RV}}$ is the group ring $\mathrm{RV}_0(M)$. Many other valued rings exhibit a similar structure (for instance, for $R = \mathbb{Z}$ and $w = v_p$ the p-adic valuation, RV_0 is the finite field $\mathbb{F}_p$ and $\widehat{\mathrm{RV}}$ is $\mathbb{F}_p(\mathbb{N})$).

For the sake of example, we present a list of residue rings RV_0, monoids $\mathrm{RV}^* = \mathrm{RV} \setminus \{\mathrm{rv}(0)\}$, modules RV_m and rings $\widehat{\mathrm{RV}}$ for various valuations in Table 4.4.1. In the table, v_J is Berarducci's order value and J is the ideal generated by the monomials t^x for $x \in \mathbb{R}^{<0}$. The first four rows follow immediately from the definitions. The fifth row, along with the definitions of P, P_α and $\widehat{\mathrm{P}}$, will be discussed and proved in Sections 5, 6. For the last row, see Remark 6.6.5.

(R, w)	RV_0	RV^*	RV_m	$\widehat{\mathrm{RV}}$
$(\mathbb{Q}, v_p)$	$\mathbb{F}_p$	$\mathbb{F}_p^\times \times \mathbb{Z}$	$\mathbb{F}_p$	$\mathbb{F}_p(\mathbb{Z})$
$(\mathbb{Z}, v_p)$	$\mathbb{F}_p$	$\mathbb{F}_p^\times \times \mathbb{N}$	$\mathbb{F}_p$	$\mathbb{F}_p(\mathbb{N})$
$(K((\mathbb{R}^{\leq 0})), v)$	K	$K^\times \times \mathbb{R}$	K	$K(\mathbb{R}^{\leq 0})$
$(K((\mathbb{R}^{\leq 0})), \sup)$	$K((\mathbb{R}^{\leq 0}))_{/J}$	$(K((\mathbb{R}^{\leq 0}))_{/J})^* \times \mathbb{R}$	$K((\mathbb{R}^{\leq 0}))_{/J}$	$K((\mathbb{R}^{\leq 0}))_{/J}(\mathbb{R}^{\leq 0})$
$(K((\mathbb{R}^{\leq 0})), \deg)$	$K(\mathbb{R}^{\leq 0})$ (5.1.1)	RV^*	$\mathrm{P}_\alpha \otimes_K K(\mathbb{R}^{\leq 0})$ (5.3.1)	$\widehat{\mathrm{P}}(\mathbb{R}^{\leq 0})$ (6.1.3)
$(K((\mathbb{R}^{\leq 0})), v_J)$	K	P (6.6.5)	P_α (for $m = \omega^\alpha$)	$\widehat{\mathrm{P}}$

TABLE 4.4.1. Rings $\widehat{\mathrm{RV}}$ arising from various valued rings.

5. EXISTENCE OF THE FACTORISATION IN $K((\mathbb{R}^{\leq 0}))$

During this section, we shall work with the ring $K((\mathbb{R}^{\leq 0}))$ and with the valuation deg (so the monoid is ω_1 with Hessenberg's natural sum). In particular, the constructions of Section 4 are applied with $R = K((\mathbb{R}^{\leq 0}))$, $w = \deg$, $M = (\omega_1, \oplus)$, and since deg actually satisfies the inequality $\deg(b + c) \leq \max\{\deg(b), \deg(c)\}$ rather than $w(b + c) \geq \min\{w(b), w(c)\}$, the constructions are carried out with the *reverse* ordering on $M = \omega_1$.

Therefore, the equivalence relation $\sim$, the monoid RV and the modules RV_α (for $\alpha \in M = \omega_1$) shall always refer to the ones obtained from the ring $R = K((\mathbb{R}^{\leq 0}))$ with the valuation $w = \deg$ (we will not use the ring $\widehat{\mathrm{RV}}$ here).

5.1. **The residue ring of generalised power series.** First, we check that the residue ring of $K((\mathbb{R}^{\leq 0}))$ is in fact (an isomorphic copy of) the subring of the series with degree at most 0, namely $K(\mathbb{R}^{\leq 0})$.

Proposition 5.1.1. *The residue ring* RV_0 *is the image* $\mathrm{rv}(K(\mathbb{R}^{\leq 0}))$*, and the restriction* $\mathrm{rv}_{\restriction K(\mathbb{R}^{\leq 0})} : K(\mathbb{R}^{\leq 0}) \to \mathrm{RV}_0$ *is a ring isomorphism.*

Proof. Immediate by Proposition 4.2.10, as $\mathcal{O} = \{b \in K((\mathbb{R}^{\leq 0})) : \deg(b) \leq 0\} = K(\mathbb{R}^{\leq 0})$ and $\mathcal{M} = \{b \in K((\mathbb{R}^{\leq 0})) : \deg(b) < 0\} = \{0\}$. □

Notation 5.1.2. With a slight abuse of notation, we shall identify RV_0 with $K(\mathbb{R}^{\leq 0})$. In particular, we shall also say that each RV_α is a $K(\mathbb{R}^{\leq 0})$-module.

5.2. **The submonoid of principal elements.** By Theorem 3.4.1, weakly principal series form a multiplicative subset of $K((\mathbb{R}^{\leq 0}))$, and in fact *principal* series form a multiplicative subset (Corollary 3.5.2). Its image through the map rv is then a submonoid of RV.

Definition 5.2.1. Given $B \in \mathrm{RV}$, we say that B is **principal** if $B = \mathrm{rv}(b)$ for some principal $b \in K((\mathbb{R}^{\leq 0}))$. We denote by P the subset of RV consisting of $\mathrm{rv}(0)$ and of all principal elements of RV. Given $\alpha \in \omega_1$, we define $\mathrm{P}_\alpha := \mathrm{P} \cap \mathrm{RV}_\alpha$.

Example 5.2.2. Under the identification $K(\mathbb{R}^{\leq 0}) = \mathrm{RV}_0$, we have that $\mathrm{P}_0 = K$. Indeed, a series $p \in K(\mathbb{R}^{\leq 0})$ is principal if and only if $p \in K^*$.

Proposition 5.2.3. P *is a multiplicatively closed subset of* RV*.*

Proof. Recall that if $b, c \in K((\mathbb{R}^{\leq 0}))$ are principal, then bc is principal by Corollary 3.5.2. Therefore, if $B, C \in \mathrm{P}$, then $B = \mathrm{rv}(b), C = \mathrm{rv}(c)$ for some principal $b, c \in K((\mathbb{R}^{\leq 0}))$, so $B \cdot C = \mathrm{rv}(bc)$ is principal as well. □

We shall now verify that the intersection of P with each RV_α is also a K-module.

Proposition 5.2.4. *For all* $b, c \in K((\mathbb{R}^{\leq 0}))$*, if* b, c *are principal and* $\deg(b+c) = \deg(b)$*, then* $b+c$ *is principal.*

Proof. Let $b, c \in K((\mathbb{R}^{\leq 0}))$. If one of b, c is zero, the conclusion is trivial, so we may assume that $b, c \in K((\mathbb{R}^{\leq 0}))^*$. Assume that $\deg(b) = \deg(b+c) = \alpha$ for some $\alpha \in \omega_1$. Note in particular that $\deg(c) \leq \alpha$. If $\alpha = 0$, then $b, c \in K^*$, so $bc \in K^*$, so bc is principal, as desired. Then assume $\alpha > 0$.

Fix some $x \in \mathbb{R}^{<0}$. Since b, c are principal, we have $\sup(b) = \sup(c) = 0$, so $\deg(b_{\leq x}), \deg(c_{\leq x}) < \alpha$ (Proposition 3.2.4). Therefore, $\deg((b+c)_{\leq x}) = \deg(b_{\leq x} + c_{\leq x}) < \alpha$. Since $\deg(b+c) = \alpha$, it follows that $(b+c)_{\leq x} \neq b+c$, so $\sup(b+c) > x$. Since x was arbitrary, it follows that $\sup(b+c) = 0$.

It remains to verify that $\mathrm{ot}(b+c) = \omega^\alpha$. Note that by assumption we have $\mathrm{ot}(b+c) \geq \omega^\alpha$. Moreover, $0 \notin \mathrm{supp}(b+c)$, as 0 is neither in $\mathrm{supp}(b)$ nor in $\mathrm{supp}(c)$. Therefore, $\mathrm{supp}(b+c)$ is the union of all sets $\mathrm{supp}((b+c)_{\leq x}))$ for $x \in \mathbb{R}^{<0}$. It follows that $\mathrm{ot}(b+c)$ is the supremum of $\mathrm{ot}((b+c)_{\leq x})$ for $x \in \mathbb{R}^{<0}$. But $\deg(b_{\leq x} + c_{\leq x}) < \alpha$ for all $x \in \mathbb{R}^{<0}$, so $\mathrm{ot}(b+c) \leq \omega^\alpha$, so $\mathrm{ot}(b+c) = \omega^\alpha$, as desired. □

Corollary 5.2.5. *For all $\alpha \in \omega_1$, P_α is an additive subgroup of RV_α.*

Proof. Let $B, C \in \mathrm{P}_\alpha$ for some given $\alpha \in \omega_1$. Pick $b, c \in K((\mathbb{R}^{\leq 0}))$ such that $B = \mathrm{rv}(b), C = \mathrm{rv}(c)$. If $\deg(b+c) = \deg(b)$, then $B + C = \mathrm{rv}(b+c)$, and since $b+c$ is principal by Proposition 5.2.4, $B + C$ is principal and in RV_α, so $B + C \in \mathrm{P}_\alpha$. On the other hand, if $\deg(b+c) \neq \deg(b)$, then $B + C = \mathrm{rv}(0) \in \mathrm{P}_\alpha$. Therefore, for each $\alpha \in \omega_1$, P_α is closed under sum. Finally, if $B \in \mathrm{P}_\alpha$ is of the form $B = \mathrm{rv}(b)$ for some principal b, then $-b$ is also principal, so $\mathrm{rv}(-b) \in \mathrm{P}_\alpha$ and $B + \mathrm{rv}(-b) = \mathrm{rv}(0)$. □

Corollary 5.2.6. *For all $\alpha \in \omega_1$, P_α is a K-module.*

Proof. Recall that if $b \in K((\mathbb{R}^{\leq 0}))$ is principal, then so is $k \cdot b$ for any $k \in K^*$, hence $K \cdot \mathrm{P}_\alpha \subseteq \mathrm{P}_\alpha$. □

5.3. **Decomposing the modules.** We shall now verify that each RV_α is in fact the scalar extension of P_α from K to $K(\mathbb{R}^{\leq 0})$. In other words, RV_α is the tensor product $\mathrm{P}_\alpha \otimes_K K(\mathbb{R}^{\leq 0})$.

Proposition 5.3.1. *For all $\alpha \in \omega_1$, $\mathrm{RV}_\alpha = \mathrm{P}_\alpha \otimes_K K(\mathbb{R}^{\leq 0})$.*

Proof. Let $\alpha \in \omega_1$. First, we verify that if $B_1, \ldots, B_n$ are K-linearly independent elements of P_α, then they are also $K(\mathbb{R}^{\leq 0})$-linearly independent in RV_α. Take a choice of such B_i's, and let $p_1, \ldots, p_n \in K(\mathbb{R}^{\leq 0})$ be such that

$$B_1 \cdot p_1 + \cdots + B_n \cdot p_n = 0.$$

Let $x_1 < \cdots < x_m$ be an enumeration of the real numbers appearing in the supports of the series p_i, so that we can write $p_i = \sum_{j=1}^m k_{ij} t^{x_j}$ for some $k_{ij} \in K$. We then have

$$B_1 \cdot p_1 + \cdots + B_n \cdot p_n = \left(\sum_{i=1}^n B_i \cdot k_{i1} \right) \cdot t^{x_1} + \cdots + \left(\sum_{i=1}^n B_i \cdot k_{im} \right) \cdot t^{x_m} = 0.$$

Pick some $b_j \in K((\mathbb{R}^{\leq 0}))$ such that $\mathrm{rv}(b_j) = \sum_{i=1}^n B_i \cdot k_{ij}$. Since each B_i is principal, we may assume that b_j is either 0 or principal of degree α. Moreover, we may further assume that $\mathrm{supp}(b_j) \geq x_{j-1} - x_j$ for all $j = 2, \ldots, m$. Then $b = b_1 \cdot t^{x_1} + \cdots + b_m \cdot t^{x_m}$ is written in normal form. On the other hand, by Lemma 4.2.5 we must have $\deg(b) \neq \alpha$. It follows at once that $b_1 = \cdots = b_n = 0$.

Therefore, $\sum_{i=1}^n B_i \cdot k_{ij} = \mathrm{rv}(b_j) = 0$ for all j. Since $B_1, \ldots, B_n$ are K-linearly independent, it follows that $k_{ij} = 0$ for all i and j, so $p_1 = \cdots = p_n = 0$. This shows that $B_1, \ldots, B_n$ are $K(\mathbb{R}^{\leq 0})$-linearly independent. This implies that $\mathrm{P}_\alpha \otimes_K K(\mathbb{R}^{\leq 0}) \subseteq \mathrm{RV}_\alpha$.

To conclude, we verify that every element in RV_α is a sum of principal elements in P_α multiplied by series in $K(\mathbb{R}^{\leq 0})$. Let $B \in \mathrm{RV}_\alpha$. Pick some $b \in K((\mathbb{R}^{\leq 0}))$ such that $B = \mathrm{rv}(b)$. By Proposition 3.3.6, we can write $b = b_1 t^{x_1} + \ldots b_n t^{x_n}$ with $b_1, \ldots, b_n$ principal. Without loss of generality, we may further assume $\deg(b_1) = \cdots = \deg(b_n) = \alpha$, as we may simply discard the terms with lower degree. Then by Lemma 4.2.5

$$B = \mathrm{rv}(b) = \mathrm{rv}(b_1 t^{x_1} + \ldots b_n t^{x_n}) = \mathrm{rv}(b_1) \cdot t^{x_1} + \cdots + \mathrm{rv}(b_n) \cdot t^{x_n},$$

proving the claim. □

5.4. **The maximal divisor of finite support for** RV**.** Thanks to the presentation of each RV_α as a scalar extension of P_α by $K(\mathbb{R}^{\leq 0})$, we can verify that each element of RV_α has a maximal divisor in $K(\mathbb{R}^{\leq 0})$.

Definition 5.4.1. Given $B, C \in \mathrm{RV}$, we say that B **divides** C, and write $B \mid C$, if $C = B \cdot D$ for some $D \in \mathrm{RV}$.

Remark 5.4.2. For all $p, q \in K(\mathbb{R}^{\leq 0}) = \mathrm{RV}_0$, p divides q in the sense of Definition 5.4.1 if and only if p divides q in the ring $K(\mathbb{R}^{\leq 0})$.

Proposition 5.4.3. *For all $B \in \mathrm{RV}$, there exists a $p \in K(\mathbb{R}^{\leq 0})$ such that for all $q \in K(\mathbb{R}^{\leq 0})$, $q \mid B$ if and only if $q \mid p$.*

Proof. Let $B \in \mathrm{RV}$ and $q \in K(\mathbb{R}^{\leq 0})$. If $B = 0$, the conclusion is trivial on taking $p = 0$, so assume otherwise. Let $\alpha = \deg(B)$, and choose any basis $\{C_i : i \in I\}$ of P_α as a K-linear vector space. By Proposition 5.3.1, every $B' \in \mathrm{RV}_\alpha$ can be written in a unique way as

$$B' = \sum_{i \in I} q'_i \cdot C_i$$

where $q'_i \in K(\mathbb{R}^{\leq 0})$. Moreover, the set of indices $i \in I$ such that $q'_i \neq 0$ is finite. Say in particular that $B = \sum_{i \in I} q_i \cdot C_i$.

Suppose that $q \mid B$ for some $q \in K(\mathbb{R}^{\leq 0})$. Then $B = q \cdot B'$ for some $B' \in \mathrm{RV}_\alpha$, so

$$B = \sum_{i \in I} q_i \cdot C_i = q \cdot \left(\sum_{i \in I} q'_i \cdot C_i \right).$$

Therefore, $q_i = q \cdot q'_i$ for all $i \in I$, so $q \mid q_i$ for all $i \in I$. Conversely, if $q \mid q_i$ for all $i \in I$, say $q_i = q \cdot q'_i$, then clearly

$$B = \sum_{i \in I} (q \cdot q'_i) \cdot C_i = q \cdot \left(\sum_{i \in I} q'_i \cdot C_i \right),$$

so $q \mid B$.

It now suffices to let p be the greatest common divisor of the set $\{q_i : q_i \neq 0\}$, which exists since the set is finite and non-empty and $K(\mathbb{R}^{\leq 0})$ is a GCD-domain. ☐

Notation 5.4.4. For all $B \in \mathrm{RV}$, let $p(B)$ be the unique element of $K(\mathbb{R}^{\leq 0})$ satisfying the conclusion of Proposition 5.4.3, namely $q \mid B$ if and only if $q \mid p(B)$ for all $q \in K(\mathbb{R}^{\leq 0})$, such that the coefficient of t^x in $p(B)$ for $x = \sup(p(B))$ is 1. Let also $p(0) := 0$. Note in particular that $p(B) \mid B$, and moreover, when $p \in K(\mathbb{R}^{\leq 0}) = \mathrm{RV}_0$, we clearly have $p(p) = k \cdot p$ for some $k \in K^*$.

5.5. **The maximal divisor of finite support for** $K((\mathbb{R}^{\leq 0}))$**.** The existence of the maximal divisor of finite support can be lifted from the quotient RV to the ring $K((\mathbb{R}^{\leq 0}))$.

Proposition 5.5.1. *For all $b \in K((\mathbb{R}^{\leq 0}))$, there exists a $p \in K(\mathbb{R}^{\leq 0})$ such that for all $q \in K(\mathbb{R}^{\leq 0})$, $q \mid b$ if and only if $q \mid p$.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$ and $q \in K(\mathbb{R}^{\leq 0})$. We reason by induction on $\deg(b)$. If $\deg(b) \leq 0$, then $b \in K(\mathbb{R}^{\leq 0})$, so the conclusion is trivial on taking $p = b$.

Suppose now that $\deg(b) > 0$. Let $B = \mathrm{rv}(b)$. Write $B = p(B) \cdot B'$ and $B' = \mathrm{rv}(b')$ for suitable $B' \in \mathrm{RV}$ and $b' \in K((\mathbb{R}^{\leq 0}))$. Then $b \sim p(B) \cdot b'$, so we can write $b = p(B) \cdot b' + c$ with $c \in K((\mathbb{R}^{\leq 0}))$ satisfying $\deg(c) < \deg(b)$.

If $q \mid p(B)$ and $q \mid c$, then clearly $q \mid b$. Conversely, assume that $q \mid b$. Note in particular that $q \mid \mathrm{rv}(b) = B$, so $q \mid p(B)$ by definition of $p(B)$. In turn, $q \mid (b - p(B)) \cdot b' = c$. Therefore, $q \mid b$ if and only if $q \mid p(B)$ and $q \mid c$. By inductive hypothesis, there exists a $p' \in K(\mathbb{R}^{\leq 0})$ such that $q \mid c$ if and only if $q \mid p'$. The conclusion follows on letting p be the greatest common divisor of $p(B)$ and p'. □

Notation 5.5.2. For all $b \in K((\mathbb{R}^{\leq 0}))$, let $p(b)$ be the unique element of $K(\mathbb{R}^{\leq 0})$ satisfying the conclusion of Proposition 5.5.1, namely $q \mid b$ if and only if $q \mid p(b)$ for all $q \in K(\mathbb{R}^{\leq 0})$, such that the coefficient of t^x in $p(b)$ for $x = \sup(p(b))$ is 1. Let also $p(0) := 0$. Note that $p(b) \mid b$, and moreover, when $p \in K(\mathbb{R}^{\leq 0}) = \mathrm{RV}_0$, we clearly have $p(p) = k \cdot p$ for some $k \in K^*$ (in fact, this new definition of $p(p)$ coincides with the one of Notation 5.4.4).

Proposition 5.5.3. *For all $b, c \in K((\mathbb{R}^{\leq 0}))$, $p(b)p(c) \mid p(bc)$.*

Proof. Let $b, c \in K((\mathbb{R}^{\leq 0}))$. Write $b = p(b) \cdot b'$ and $c = p(c) \cdot c'$. Then clearly $bc = (p(b)p(c)) \cdot b'c'$, so $p(c)p(c) \mid bc$, so by definition $p(b)p(c) \mid p(bc)$. □

5.6. **The factorisation.** We can now factor the series in $K((\mathbb{R}^{\leq 0}))$ by induction on the degree.

Proposition 5.6.1. *For all non-zero $b \in K((\mathbb{R}^{\leq 0}))$, there exist irreducible series $c_1, \ldots, c_n \in K((\mathbb{R}^{\leq 0}))$ with infinite support such that*

$$b = p(b) \cdot c_1 \cdot \cdots \cdot c_n.$$

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))^*$. Write $b = p(b) \cdot b'$ for a suitable $b' \in K((\mathbb{R}^{\leq 0}))^*$. Note that $p(b')$ is necessarily 1. We work by induction on $\deg(b)$. If $\deg(b) = 0$, then $b' \in K^*$, and we are done.

Assume $\deg(b) > 0$. If b' is irreducible, then we are done. Otherwise, $b' = c \cdot d$ for some $c, d \in K((\mathbb{R}^{\leq 0}))$ not in K. Since $p(b') = 1$, we have $p(c) = p(d) = 1$ by Proposition 5.5.3, so c, d cannot be in $K(\mathbb{R}^{\leq 0})$, namely $\deg(c), \deg(d) > 0$. Since $\deg(b') = \deg(c) \oplus \deg(d)$, it follows that $\deg(c) < \alpha$, $\deg(d) < \alpha$.

By inductive hypothesis, c and d can be written as products of irreducible series of positive degree. Therefore, b' is also a product of irreducible series of positive degree, that is to say, with infinite support, as desired. □

Remark 5.6.2. The number of factors with infinite support of a series $b \in K((\mathbb{R}^{\leq 0}))^*$ can be bounded in terms of $\deg(b)$ only. Indeed, the number of such factors cannot exceed the number of terms in the Cantor Normal Form of $\deg(b)$.

6. Uniqueness of the factor with finite support

As in the previous section, we shall work with the ring $K((\mathbb{R}^{\leq 0}))$ and the valuation deg, so the equivalence relation $\sim$, the monoid RV, the modules RV_α and the ring $\widehat{\mathrm{RV}}$ shall always refer to the ones obtained from the ring $K((\mathbb{R}^{\leq 0}))$ with the valuation deg, and we shall identify RV_0 with $K(\mathbb{R}^{\leq 0})$.

6.1. **The subring of principal elements.** Just as the principal elements in RV form the submonoid P, we can easily verify that the principal elements in $\widehat{\mathrm{RV}}$ forma subring of $\widehat{\mathrm{P}}$.

Definition 6.1.1. An element $\sum_\alpha B_\alpha$ of $\widehat{\mathrm{RV}}$ is **principal** if each B_α is either zero or principal (namely in P). We denote by $\widehat{\mathrm{P}}$ the set of all principal elements of $\widehat{\mathrm{RV}}$.

Proposition 6.1.2. *$\widehat{\mathrm{P}}$ is a subring of $\widehat{\mathrm{RV}}$.*

Proof. Immediate from the fact that P is closed under sum and product. □

Proposition 6.1.3. $\widehat{\mathrm{RV}} = \widehat{\mathrm{P}} \otimes_K K(\mathbb{R}^{\leq 0})$. *In particular,* $\widehat{\mathrm{RV}} = \widehat{\mathrm{P}}(\mathbb{R}^{\leq 0})$.

Proof. Note that as a $K(\mathbb{R}^{\leq 0})$-module, $\widehat{\mathrm{RV}}$ is the direct sum of the $K(\mathbb{R}^{\leq 0})$-modules RV_α. Likewise, as a K-module, P is the direct sum of the K-modules P_α. Since $\mathrm{RV}_\alpha = \mathrm{P}_\alpha \otimes_K K(\mathbb{R}^{\leq 0})$, the conclusion follows at once. □

Remark 6.1.4. The ring $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}}$ can be written as $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}} = \mathrm{Frac}(\widehat{\mathrm{P}})(\mathbb{R}^{\leq 0})$. In particular, it is a GCD domain.

6.2. **Divisibility in $\widehat{\mathrm{RV}}$.** From now on, we shall talk about divisibility *in the ring* $\widehat{\mathrm{RV}}$.

Definition 6.2.1. Given $B, C \in \widehat{\mathrm{RV}}$, we say that B **divides** C, and write $B \mid C$, if $C = B \cdot D$ for some $D \in \widehat{\mathrm{RV}}$.

Divisibility in $\widehat{\mathrm{RV}}$ is an extension of the notion of divisibility in Definition 5.4.1, so there is no risk of ambiguity, thanks to the following observation.

Proposition 6.2.2. *For all $B, C \in \widehat{\mathrm{RV}}^*$, if $B \cdot C \in \mathrm{RV}^*$, then $B, C \in \mathrm{RV}^*$.*

Proof. Given a $B = \sum_\alpha B_\alpha \in \widehat{\mathrm{RV}}^*$, let $\deg^-(B)$ be the smallest α such that $B_\alpha \neq 0$, and $\deg^+(B)$ be the largest α such that $B_\alpha \neq 0$. Clearly, $\deg^-(B) \leq \deg^+(B)$, and $\deg^-(B) = \deg^+(B)$ if and only if $B \in \mathrm{RV}^*$ (in which case they are both equal to $\deg(B)$).

We claim that for all $B, C \in \widehat{\mathrm{RV}}^*$, $\deg^-(B \cdot C) = \deg^-(B) \oplus \deg^-(C)$, and likewise $\deg^+(B \cdot C) = \deg^+(B) \oplus \deg^+(C)$. Let $B, C \in \widehat{\mathrm{RV}}^*$, and write $B \cdot C = \sum_\alpha D_\alpha$. By definition,

$$D_\alpha = \left(\sum_{\beta \oplus \gamma = \alpha} B_\beta \cdot C_\gamma \right).$$

It follows at once that $D_\alpha = 0$ for all $\alpha < \deg^-(B) \oplus \deg^-(C)$, while $D_\alpha = B_\beta \cdot C_\gamma \neq 0$ for $\beta = \deg^-(B)$, $\gamma = \deg^-(C)$ and $\alpha = \beta \oplus \gamma$. Therefore, $\deg^-(B \cdot C) = \deg^-(B) \oplus \deg^-(C)$. Likewise, $D_\alpha = 0$ for all $\alpha > \deg^+(B) \oplus \deg^+(C)$, while $D_\alpha = B_\beta \cdot C_\gamma \neq 0$ for $\beta = \deg^+(B)$, $\gamma = \deg^+(C)$ and $\alpha = \beta \oplus \gamma$, hence $\deg^+(B \oplus C) = \deg^+(B) \oplus \deg^+(C)$.

Now given $B, C \in \widehat{\mathrm{RV}}^*$, if $B \cdot C \in \mathrm{RV}$, then $\deg^-(B \cdot C) = \deg^-(B) \oplus \deg^-(C) = \deg^+(B \cdot C) = \deg^+(B) \oplus \deg^+(C)$. It follows at once that $\deg^-(B) = \deg^+(B)$ and $\deg^-(C) = \deg^+(C)$, so $B, C \in \mathrm{RV}^*$. □

Corollary 6.2.3. *For all $B, C \in \mathrm{RV}$, B divides C in the sense of Definition 5.4.1 if and only if B divides C in the ring $\widehat{\mathrm{RV}}$. In particular, for all $p, q \in K(\mathbb{R}^{\leq 0})$, p divides q in $\widehat{\mathrm{RV}}$ if and only if p divides q in $K(\mathbb{R}^{\leq 0})$.*

Remark 6.2.4. The maps $\deg^-$, $\deg^+$ are valuations. More precisely, one can easily verify that $\deg^-(B+C) \geq \min\{\deg^-(B), \deg^-(C)\}$ and $\deg^+(B+C) \leq \max\{\deg^+(B), \deg^+(C)\}$, provided one defines $\deg^-(0) = +\infty$, $\deg^+(0) = -\infty$. More generally, given any semi-valued ring (R, w), one can define similar semi-valuations w^- and w^+ on the associated ring $\widehat{\mathrm{RV}}$. However, we will not make any further use of such valuations in the rest of the paper.

Lemma 6.2.5. *For all $B \in \mathrm{RV}$, $C = \sum_\alpha C_\alpha \in \widehat{\mathrm{RV}}$, $B \mid C$ if and only if $B \mid C_\alpha$ for all $\alpha \in \omega_1$.*

Proof. Let $B \in \mathrm{RV}$, $C = \sum_\alpha C_\alpha \in \widehat{\mathrm{RV}}$. Clearly, we may assume $B \neq 0$, so let $\beta = \deg(B)$.

If $B \mid C$, then $C = B \cdot C'$ for some $C' = \sum_\alpha C'_\alpha$. By definition of product, $B \cdot C' = \sum_\alpha \sum_{\beta \oplus \gamma = \alpha} B \cdot C'$. Therefore, we have $C_{\alpha \oplus \beta} = B \cdot C'_\alpha$ for all α, and $C_\gamma = 0$ for all ordinal γ that cannot be written in the form $\alpha \oplus \beta$. In turn, $B \mid C_\alpha$ for all α.

Conversely, suppose $B \mid C_\alpha$ for all α. Then for all α we must have $C_{\alpha \oplus \beta} = B \cdot C'_\alpha$ for some $C'_\alpha \in \mathrm{RV}_\alpha$, and if γ is an ordinal not of the form $\gamma = \alpha \oplus \beta$, then $C_\gamma = 0$. It follows at once that $B \cdot \sum_\alpha C'_\alpha = C$, so $B \mid C$, as desired. □

6.3. **Primality of the series in $K(\mathbb{R}^{\leq 0})$.** We shall now prove that the series in $K(\mathbb{R}^{\leq 0})$ are primal in $\widehat{\mathrm{RV}}$ and in turn in $K((\mathbb{R}^{\leq 0}))$.

Lemma 6.3.1. *For all $p \in K(\mathbb{R}^{\leq 0})$, $B \in \mathrm{RV}$, $C \in \mathrm{P}^*$, if $p \mid B \cdot C$, then $p \mid B$.*

Proof. Let $p \in K(\mathbb{R}^{\leq 0})$, $B \in \mathrm{RV}$, $C \in \mathrm{P}^*$. Clearly, we may assume $B \neq 0$. Let $\beta = \deg(B)$, $\gamma = \deg(C)$, $\alpha = \beta \oplus \gamma = \deg(B \cdot C)$. Let $\{D_i : i \in I\}$ be a K-linear basis of P_γ. Note that $\{B \cdot D_i : i \in I\}$ is K-linearly independent, so it can be completed to a K-linear basis of P_α by adding some further elements $\{E_j : j \in J\}$. By Proposition 5.3.1, B can be written uniquely as a sum

$$B = \sum_{i \in I} q_i \cdot D_i$$

with $q_i \in K(\mathbb{R}^{\leq 0})$. Therefore,

$$B \cdot C = \sum_{i \in I} q_i \cdot (D_i \cdot C)$$

is the unique representation of $B \cdot C$ in the basis $\{D_i\} \cup \{E_j\}$. It follows at once that if $p \mid B \cdot C$, then $p \mid q_i$ for all $i \in I$, hence $p \mid B$. □

Lemma 6.3.2. *For all $p \in K(\mathbb{R}^{\leq 0})$, $B \in \widehat{\mathrm{RV}}$, $C \in \widehat{\mathrm{P}}^*$, if $p \mid B \cdot C$, then $p \mid B$.*

Proof. Let $p \in K(\mathbb{R}^{\leq 0})$, $B \in \widehat{\mathrm{RV}}$, $C \in \widehat{\mathrm{P}}^*$. Clearly, we may assume $B \neq 0$. Write $B = \sum_\alpha B_\alpha$, $C = \sum_\alpha C_\alpha$. Let β and γ be the maximum ordinals such that respectively $B_\beta \neq 0, C_\gamma \neq 0$. We shall prove the conclusion by induction on β.

If $B \cdot C = \sum_\alpha D_\alpha$, then clearly the maximum ordinal α such that $D_\alpha \neq 0$ is $\alpha = \beta \oplus \gamma$, and $D_\alpha = B_\beta \cdot C_\gamma$. By Lemma 6.2.5, if $p \mid B \cdot C$, then in particular $p \mid D_\alpha$, so $p \mid B_\beta \cdot C_\gamma$. By Lemma 6.3.1, $p \mid B_\beta$.

We now replace B with $B' = B - B_\beta$. If $B' = 0$, we are done. Otherwise, note that $p \mid B' \cdot C$. Moreover, if $B' = \sum_\alpha B'_\alpha$, then clearly the maximum β' such that $B'_{\beta'} \neq 0$ is strictly less than β. By inductive hypothesis, $p \mid B'$, so $p \mid B$, as desired. □

Lemma 6.3.3. *K is relatively algebraically closed in* $\mathrm{Frac}(\widehat{\mathrm{P}})$.

Proof. Let $B, C \in \widehat{\mathrm{P}}$ be such that $\frac{B}{C}$ is algebraic over K, namely there is a polynomial $p(X) = X^d + k_{d-1}X^{d-1} + \cdots + k_0 \in K[X]^*$ such that

$$p\left(\frac{B}{C}\right) = \left(\frac{B}{C}\right)^d + k_{d-1}\left(\frac{B}{C}\right)^{d-1} + \cdots + k_0 = 0.$$

Assume that d is minimal with this property. In particular, $p(X)$ is irreducible in $K[X]$. Now rewrite the above equation as

$$B^d + k_{d-1}B^{d-1}C + \cdots + k_1 BC^{d-1} + k_0 C^d = 0.$$

It follows at once that $\deg(B) = \deg(C) = \beta$ for some $\beta \in \omega_1$. On writing $B = \sum_\alpha B_\alpha$, $C = \sum_\alpha C_\alpha$, let $b, c \in K((\mathbb{R}^{\leq 0}))$ be two series such that $\mathrm{rv}(b) = B_\beta$ and $\mathrm{rv}(c) = C_\beta$. Note in particular that b, c satisfy $\deg(B - \mathrm{rv}(b)) < \beta$, $\deg(C - \mathrm{rv}(c)) < \beta$. This immediately implies that

$$\mathrm{rv}(b)^d + k_{d-1}\,\mathrm{rv}(b)^{d-1}\,\mathrm{rv}(c) + \cdots + k_1\,\mathrm{rv}(b)\,\mathrm{rv}(c)^{d-1} + k_0\,\mathrm{rv}(c)^d = 0.$$

In turn, by Lemma 4.2.5 this means that

$$\deg(b^d + k_{d-1}b^{d-1}c + \cdots + k_1 bc^{d-1} + k_0 c^d) < \beta.$$

Now write $p(X) = \prod_{i=1}^d (X - \zeta_i)$ for some $\zeta_i \in K^{\mathrm{alg}}$. Note that the definition of degree is independent of the field of the coefficients, so it can be naturally extended from $K((\mathbb{R}^{\leq 0}))$ to $K^{\mathrm{alg}} \cdot K((\mathbb{R}^{\leq 0})) = K^{\mathrm{alg}}((\mathbb{R}^{\leq 0}))$ while remaining multiplicative. Therefore, there is some $i = 1, \ldots, d$ such that $\deg(b - \zeta_i c) < \beta$.

In particular, there must exist some $x \in \mathrm{supp}(b) \cup \mathrm{supp}(c)$ such that $b - \zeta_i c$ has coefficient 0 at x. Therefore, $\zeta_i = \frac{b_x}{c_x}$, where b_x is the coefficient of b at x and c_x is the coefficient of c at x. This implies that $\zeta_i \in K$. Since $p(X)$ is monic irreducible in $K[X]$, we must have $p(X) = X - \zeta_i$. Therefore, $B = \zeta_i C$, so $\frac{B}{C} \in K$, which means that K is relatively algebraically closed in $\mathrm{Frac}(\widehat{\mathrm{P}})$, as desired. □

Lemma 6.3.4. *For all $p \in K(\mathbb{R}^{\leq 0})$ and $p_1, p_2 \in \widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}}$, if $p = p_1 p_2$, then there is some $B \in$* $\mathrm{Frac}(\widehat{\mathrm{P}})$ *such that $p_1 \cdot B \in K(\mathbb{R}^{\leq 0})$, $p_2 \cdot B^{-1} \in K(\mathbb{R}^{\leq 0})$.*

Proof. Let $p \in K(\mathbb{R}^{\leq 0})$, $p_1, p_2 \in \widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}} = \mathrm{Frac}(\widehat{\mathrm{P}})(\mathbb{R}^{\leq 0})$. Clearly, we may assume that $p \neq 0$. For the sake of notation, let $L = \mathrm{Frac}(\widehat{\mathrm{P}})$.

There is a finite set of negative real numbers $x_1, \ldots, x_n$ which are $\mathbb{Z}$-linearly independent, and such that $p, p_1, p_2 \in L(H)$, where $H = \mathbb{N}x_1 + \cdots + \mathbb{N}x_n$. Therefore, $K(H) \cong K[X_1, \ldots, X_n]$ and $\mathrm{Frac}(\widehat{\mathrm{P}})(H) \cong L[X_1, \ldots, X_n]$, with isomorphisms sending t^{x_i} to the variable X_i. In particular, $K(H)$ and $L(H)$ are unique factorisation domains, with groups of units K^* and L^* respectively. Moreover, since K is relatively algebraically closed in L by Lemma 6.3.3, each irreducible element of $K(H)$ remains irreducible in $L(H)$.

Let us write $p = q_1 \cdot \ldots \cdot q_m$ where $q_1, \ldots, q_m$ are irreducible elements of $K(H)$. It follows at once that p_1 is a product of some of the factors $q_1, \ldots, q_m$ and some invertible element $B \in L(H)$. But then $B \in \mathrm{Frac}(\widehat{\mathrm{P}})$, so in fact $p_1 \in B \cdot K(\mathbb{R}^{\leq 0})$. Likewise, $p_2 \in C \cdot K(\mathbb{R}^{\leq 0})$ for some $C \in \mathrm{Frac}(\widehat{\mathrm{P}})$.

To conclude, note that $p \in B \cdot C \cdot K(\mathbb{R}^{\leq 0})$. It follows at once that $B \cdot C \in K$, so in particular $C \in B^{-1} \cdot K$, so $p_2 \in B^{-1} \cdot K(\mathbb{R}^{\leq 0})$, as desired. □

Remark 6.3.5. Lemma 6.3.4 says in particular that for all $p, q \in K(\mathbb{R}^{\leq 0})$, p divides q in the ring $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}}$ if and only if p divides q in the ring $K(\mathbb{R}^{\leq 0})$. Indeed, if $q = p \cdot q'$ for some $q' \in \widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}}$, then for some $B \in \mathrm{Frac}(\widehat{\mathrm{P}})$ we have $p \cdot B \in K(\mathbb{R}^{\leq 0})$, $q' \cdot B^{-1} \in K(\mathbb{R}^{\leq 0})$. This immediately implies that $B \in K^*$, so in particular $q' \in K(\mathbb{R}^{\leq 0})$, hence p divides q in $K(\mathbb{R}^{\leq 0})$.

Corollary 6.3.6. *Every $p \in K(\mathbb{R}^{\leq 0})$ is primal in $\widehat{\mathrm{RV}}$.*

Proof. Let $p \in K(\mathbb{R}^{\leq 0})$. Since $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}} = \mathrm{Frac}(\widehat{\mathrm{P}})(\mathbb{R}^{\leq 0})$ is a GCD domain, we know that there are $q_1, q_2, B', C' \in \widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}}$ such that $p = p_1 p_2$ and $B = p_1 \cdot B'$, $C = p_2 \cdot C'$. By Lemma 6.3.4, we may further assume that $p_1, p_2 \in K(\mathbb{R}^{\leq 0})$.

We can now write B', C' as fractions $B' = \frac{M}{D}$, $C' = \frac{N}{E}$ for some $M, N \in \widehat{\mathrm{RV}}$ and $D, E \in \widehat{\mathrm{P}}$. But then $p_1 \mid B \cdot D$, $p_2 \mid C \cdot E$. Since $D, E \in \widehat{\mathrm{P}}$, it follows by Lemma 6.3.2 that $p_1 \mid B$, $p_2 \mid C$, as desired. □

Corollary 6.3.7. *For all $B, C \in \mathrm{RV}^*$, $p(BC) = p(B)p(C)$.*

Proof. Let $B, C \in \mathrm{RV}^*$. We know already that $p(B)p(C) \mid p(BC)$. We claim that $p(BC) \mid p(B)p(C)$. Recall that by definition $p(BC) \mid BC$, so by Proposition 6.3.6 we can write $p(BC) = p_1 p_2$ for some $p_1, p_2 \in K(\mathbb{R}^{\leq 0})$ such that $p_1 \mid B$, $p_2 \mid C$. But then $p_1 \mid p(B)$, $p_2 \mid p(C)$, so $p(BC) = p_1 p_2 \mid p(B)p(C)$. Therefore, $p(BC) = k \cdot p(B)p(C)$ for some $k \in K^*$. By comparing the coefficients we deduce that $p(BC) = p(B)p(C)$, as desired. □

Proposition 6.3.8. *For all $b, c \in K((\mathbb{R}^{\leq 0}))^*$, $p(bc) = p(b)p(c)$.*

Proof. Let $b, c \in K((\mathbb{R}^{\leq 0}))^*$. We reason by induction on $\deg(b)$ and $\deg(c)$. We already know that $p(b)p(c) \mid p(bc)$. We claim that $p(bc) \mid p(b)p(c)$. After dividing b and c by $p(b)$ and $p(c)$, we may assume that $p(b) = p(c) = 1$, so our claim reduces to proving that $p(bc) \in K^*$.

Let q be the greatest common divisor between $p(bc)$ and $p(\mathrm{rv}(b))$. By definition of $p(\mathrm{rv}(b))$, we can write $b = p(\mathrm{rv}(b)) \cdot b' + d$ where $b', d \in K((\mathbb{R}^{\leq 0}))$ are such that $\deg(d) < \deg(b)$. Since $q \mid p(bc) \mid bc$ and $q \mid p(\mathrm{rv}(b))$, we must have $q \mid d \cdot c$. By inductive hypothesis, this means that $q \mid p(d)p(c) = p(d)$. Therefore, $q \mid d$. In turn, $q \mid b$, which means that $q \mid p(b) = 1$, so $q \in K^*$. Therefore, $p(bc)$ and $p(\mathrm{rv}(b))$ are coprime.

By symmetry, $p(bc)$ and $p(\mathrm{rv}(c))$ are also coprime. On the other hand, $p(bc) \mid p(\mathrm{rv}(bc))$, and $p(\mathrm{rv}(bc)) = p(\mathrm{rv}(b)) \cdot p(\mathrm{rv}(c))$ by Corollary 6.3.7. Since $p(bc)$ is coprime with both $p(\mathrm{rv}(b))$ and $p(\mathrm{rv}(c))$, we must have $p(bc) \in K^*$, proving the claim.

Therefore, for all $b, c \in K((\mathbb{R}^{\leq 0}))^*$, $p(bc) = k \cdot p(b)p(c)$ for some $k \in K^*$. By comparing the coefficients we deduce that $p(bc) = p(b)p(c)$, as desired. □

Corollary 6.3.9. *Every $p \in K(\mathbb{R}^{\leq 0})$ is primal in $K((\mathbb{R}^{\leq 0}))$.*

Proof. Let $p \in K(\mathbb{R}^{\leq 0})$ and $b, c \in K((\mathbb{R}^{\leq 0}))^*$. By Proposition 6.3.8, $p \mid bc$ if and only if $p \mid p(b)p(c)$. Since $K(\mathbb{R}^{\leq 0})$ is a common refinement domain, there are $p_1, p_2 \in K(\mathbb{R}^{\leq 0})$ such that $p = p_1 p_2$ and $p_1 \mid p(b) \mid b$, $p_2 \mid p(c) \mid c$, as desired. □

6.4. **Uniqueness of the factor with finite support.** It now follows at once that if a series $b \in K((\mathbb{R}^{\leq 0}))$ factors into a product of one series of finite support and other irreducible series of infinite support, the factor of finite support is unique up to multiplication by an element of K^*.

Theorem 6.4.1 (Theorem A)**.** *For all non-zero $b \in K((\mathbb{R}^{\leq 0}))$, there exist $c_1, \dots, c_n \in K((\mathbb{R}^{\leq 0}))$ irreducible with infinite support and $p \in K(\mathbb{R}^{\leq 0})$ such that $b = p \cdot c_1 \cdot \dots \cdot c_n$. Moreover, p is unique up to multiplication by an element of K^*.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$. The existence of the desired factorisation is the conclusion of Proposition 5.6.1, so we only need to check for uniqueness.

Suppose that $b = p \cdot c_1 \cdot \dots \cdot c_n$ for some $c_1, \dots, c_n \in K((\mathbb{R}^{\leq 0}))$ irreducible with infinite support and $p \in K(\mathbb{R}^{\leq 0})$. Since each c_i is irreducible, we have $p(c_i) = 1$. Moreover, $p(p) = k \cdot p$ for some $k \in K^*$. Therefore,

$$p(b) = p(p) \cdot p(c_1) \cdot \dots \cdot p(c_n) = p(p) = k \cdot p,$$

and the conclusion follows. □

Corollary 6.4.2. *$K((\mathbb{R}^{\leq 0}))$ is a common refinement domain (in fact, a GCD domain) if and only if every irreducible series with infinite support is prime.*

Proof. If $K((\mathbb{R}^{\leq 0}))$ is a common refinement domain, then every irreducible series is prime. Suppose now that every irreducible series with infinite support is prime. Then for every $b \in K((\mathbb{R}^{\leq 0}))$ the factorisation $b = p \cdot c_1 \cdot \dots \cdot c_n$ is unique up to reordering the factors and to multiplication by elements of K^*. It follows that a series $d \in K((\mathbb{R}^{\leq 0}))$ divides b if and only if it is a product of some of factors $c_1, \dots, c_n$ and a factor of p. Therefore, given two series $b, c \in K((\mathbb{R}^{\leq 0}))$, their greatest common divisor is the greatest common divisor of $p(b)$ and $p(c)$, multiplied by the irreducible series with infinite support appearing in both factorisations. □

6.5. **A broader criterion for irreducibility.** A byproduct of Theorem 6.4.1 is a strengthening of Berarducci's criterion for irreducibility [Ber00, Thm. 10.5].

Lemma 6.5.1. *For all $b \in K((\mathbb{R}^{\leq 0}))$, if $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))}$ is irreducible and $p(b) = 1$, then b is irreducible.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$ as in the hypothesis, and suppose that $b = cd$ for some $c, d \in K((\mathbb{R}^{\leq 0}))$. Then $\mathrm{rv}(b) = \mathrm{rv}(c) \cdot \mathrm{rv}(d)$. By Corollary 6.3.7, we can divide both sides by $p(\mathrm{rv}(b)) = p(\mathrm{rv}(c)) \cdot p(\mathrm{rv}(d))$. By the hypothesis, one of $\frac{\mathrm{rv}(c)}{p(\mathrm{rv}(c))}, \frac{\mathrm{rv}(d)}{p(\mathrm{rv}(d))}$ is a unit. Therefore, one of $c, d \in K((\mathbb{R}^{\leq 0}))$ has finite support. Since $p(b) = 1$, it follows that one of c, d is a unit. □

Corollary 6.5.2. *For all $b \in K((\mathbb{R}^{\leq 0}))$, if $\deg(b)$ is a principal ordinal and $p(b) = 1$, then b is irreducible.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$ as in the hypothesis. Then any factorisation of $\mathrm{rv}(b)$ has a factor of degree 0, so $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))}$ is irreducible. By Lemma 6.5.1, b is irreducible. □

Theorem 6.5.3 (Theorem E)**.** *For all $b \in K((\mathbb{R}^{\leq 0}))$, if the order type of the support of b is of the form $\omega^{\omega^{\alpha}} + \beta$ with $\beta < \omega^{\omega^{\alpha}}$, and b is not divisible by t^x for any $x \in \mathbb{R}^{<0}$, then b is irreducible.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$, α, β as in the hypothesis. Since $\deg(b) = \omega^\alpha$ is additively principal, it suffices to prove that $p(b) = 1$ by Corollary 6.5.2.

By Proposition 3.3.6, we can write $b = b't^y + b''$ with b' principal of order type ω^{ω^α} and b'' of order type β with $\mathrm{supp}(b'') > y$. In particular, $\deg(b'') < \omega^\alpha = \deg(b')$. It follows that $\mathrm{rv}(b) = \mathrm{rv}(b't^x) = t^y \cdot \mathrm{rv}(b')$. Since $\mathrm{rv}(b')$ is principal, by Proposition 5.3.1 we get $p(\mathrm{rv}(b)) = t^y$.

Since $p(b) \mid p(\mathrm{rv}(b)) = t^y$, $p(b)$ must be of the form t^x for some $x \geq y$. By assumption, b is not divisible by t^x unless $x = 0$. Therefore, $x = 0$, so $p(b) = 1$, as desired. □

6.6. **A broader criterion for primality.** We can also obtain some improvements on Pitteloud's criterion for primality in [Pit01]. First, we translate Pitteloud's work in our language.

Definition 6.6.1 ([Ber00, Def. 5.2])**.** Let J be the ideal of $K((\mathbb{R}^{\leq 0}))$ generated by the monomials t^x for $x \in \mathbb{R}^{<0}$ (namely, the ideal of the series $b \in K((\mathbb{R}^{\leq 0}))$ with $\sup(b) < 0$). Given $b \in K((\mathbb{R}^{\leq 0}))$, we define $v_J(b)$ as follows:

$$v_J(b) := \begin{cases} 0 & \text{if } b \in J; \\ 1 & \text{if } b \in J + K; \\ \min\{\mathrm{ot}(c) \,:\, c \in K((\mathbb{R}^{\leq 0})) \text{ with } b - c \in J + K\} & \text{otherwise.} \end{cases}$$

Proposition 6.6.2. *A series $b \in K((\mathbb{R}^{\leq 0}))^*$ is principal if and only if* $\mathrm{ot}(b) = v_J(b)$.

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))^*$. We distinguish three cases.

Case $b \in J$. In this case, b is not principal, as $\sup(b) < 0$, and $\mathrm{ot}(b) > 0 = v_J(b)$.

Case $b \in J + K \setminus J$. We have $0 \in \mathrm{supp}(b)$. It follows that $\mathrm{ot}(b) = \alpha + 1$ for some ordinal α. Therefore, b is principal if and only if $\mathrm{ot}(b) = 1 = v_J(b)$.

Case $b \notin J + K$. Let $c \in K((\mathbb{R}^{\leq 0}))$. Note that $b - c \in J + K$ if and only if there are $x \in \mathbb{R}^{<0}$ and $k \in K$ such that $c_{\geq x} = b_{\geq x} + k$. Since $\mathrm{ot}(c_{\geq x}) \leq \mathrm{ot}(c)$, we conclude that $v_J(b)$ is the minimum of $\mathrm{ot}(b_{\geq x} + k)$ for $x \in \mathbb{R}^{<0}$ and $k \in K$.

If b is principal, then $\mathrm{ot}(b_{\geq x}) = \mathrm{ot}(b)$ for any $x \in \mathbb{R}^{<0}$, and $\mathrm{ot}(b_{\geq x} + k) = \mathrm{ot}(b) + 1$ for any $k \in K^*$, so $v_J(b) = \mathrm{ot}(b)$. If b is not principal, $\mathrm{ot}(b) = \omega^\alpha + \beta$ with $0 < \beta \leq \omega^\alpha$. Write $b = b' + b''$ with $\mathrm{supp}(b') < \mathrm{supp}(b'')$ and $\mathrm{ot}(b'') = \beta$. By construction, $\sup(b - b'') = \sup(b') < 0$, so $v_J(b) \leq \mathrm{ot}(b'') = \beta < \mathrm{ot}(b)$. □

Definition 6.6.3 ([Pit01, p. 1209])**.** Given $\alpha \in \omega_1$, let J_{ω^α} be the K-vector space $J_{\omega^\alpha} := \{b \in K((\mathbb{R}^{\leq 0})) \,:\, v_J(b) < \omega^\alpha\}$. Moreover, write $b \mid c \mod J_{\omega^\alpha}$ if there exists $d \in K((\mathbb{R}^{\leq 0}))$ such that $c \equiv bd \mod J_{\omega^\alpha}$.

Lemma 6.6.4. *Let $b, c \in K((\mathbb{R}^{\leq 0}))$ be two principal series, with* $\deg(c) = \alpha$*. Then* $\mathrm{rv}(b) = \mathrm{rv}(c)$ *if and only if $b - c \in J_{\omega^\alpha}$.*

Proof. If $\mathrm{rv}(b) = \mathrm{rv}(c)$, then $\deg(b - c) < \alpha$, so $\deg(b) = \alpha$ and $v_J(b - c) \leq \omega^{\deg(b-c)} < \omega^\alpha$, hence $b - c \in J_{\omega^\alpha}$. If $\mathrm{rv}(b) \neq \mathrm{rv}(c)$, then $b - c$ is principal of degree $\max\{\deg(b), \deg(c)\}$: indeed, this is trivial if $\deg(b) \neq \alpha$, and it follows from Proposition 5.2.4 if $\deg(b) = \alpha$. Therefore, $v_J(b - c) = \mathrm{ot}(b - c) = \omega^\alpha$, hence $b - c \notin J_{\omega^\alpha}$. □

Remark 6.6.5. Every element in the space $J_{\omega^{\alpha+1}}/J_{\omega^\alpha}$ can be represented as the class $b+J_{\omega^\alpha}$ for some principal series $b \in K((\mathbb{R}^{\leq 0}))$. By Lemma 6.6.4, it follows at once that the K-vector space P_α can be alternatively presented as the quotient $J_{\omega^{\alpha+1}}/J_{\omega^\alpha}$. On the other hand, the quotient $J_{\omega^{\alpha+1}}/J_{\omega^\alpha}$ is also the module RV_m for the semi-valuation $w = v_J$ and for $m = \omega^\alpha$ (the verification is left to the reader). In particular, P is the RV monoid of the semi-valuation v_J.

Lemma 6.6.6. *For all $B, C \in \mathrm{RV}^*$, if $B \cdot C \in \mathrm{P}$ (or $\widehat{\mathrm{P}}$), then $B, C \in \mathrm{P}$ (resp. $\widehat{\mathrm{P}}$).*

Proof. Recall that $\widehat{\mathrm{RV}} = \widehat{\mathrm{P}}(\mathbb{R}^{\leq 0})$ by Proposition 6.1.3. Then clearly if $B, C \in \widehat{\mathrm{RV}}^*$ are such that $B \cdot C \in \widehat{\mathrm{P}}$, we must have $B, C \in \widehat{\mathrm{P}}$. If moreover $B \cdot C \in \mathrm{P} \subseteq \mathrm{RV}$, then by Proposition 6.2.2, $B, C \in \mathrm{RV} \cap \widehat{\mathrm{P}} = \mathrm{P}$. $\square$

Corollary 6.6.7. *Let $b, c \in K((\mathbb{R}^{\leq 0}))$ be two principal series, with $\deg(c) = \alpha$. Then $b \mid c \mod J_{\omega^\alpha}$ if and only if $\mathrm{rv}(b) \mid \mathrm{rv}(c)$.*

Proof. Suppose that $b \mid c \mod J_{\omega^\alpha}$, namely that $c \equiv bd \mod J_{\omega^\alpha}$ for some $d \in K((\mathbb{R}^{\leq 0}))$. Note that for any $x \in \mathbb{R}^{<0}$, $bd_{\geq x} \equiv bd \mod J$, so $c \equiv bd_{\geq x} \mod J_{\omega^\alpha}$ as in fact $J \subseteq J_{\omega^\alpha}$. If x is sufficiently close to 0, then $d_{\geq x} = d' + k$ for some principal series d' and $k \in K$. We replace d with $d' + k$, so that bd is principal. Then $\mathrm{rv}(c) = \mathrm{rv}(bd)$ by Lemma 6.6.4, so $\mathrm{rv}(b) \mid \mathrm{rv}(c)$.

Conversely, if $\mathrm{rv}(b) \mid \mathrm{rv}(c)$, then there exists $d \in K((\mathbb{R}^{\leq 0}))$ such that $\mathrm{rv}(c) = \mathrm{rv}(bd)$. Then $\mathrm{rv}(d)$ is principal by Lemma 6.6.6, so we may assume that d is principal. Then $c \equiv bd \mod J_{\omega^\alpha}$ by Lemma 6.6.4, so $b \mid c \mod J_{\omega^\alpha}$. $\square$

Thanks to the above translation, we can reinterpret the key step in Pitteloud's proof as a statement about primality in $\widehat{\mathrm{RV}}$.

Proposition 6.6.8 ([Pit01, Prop. 3.2])**.** *Let $a, b, c, d \in K((\mathbb{R}^{\leq 0}))$ be such that $v_J(a) = \omega$ and assume that $a^k b = c^l d \mod J_{v_J(a^k b)}$ with $k, l > 0$. Then either $a \mid c \mod J_{v_J(c)}$ or $a \mid d \mod J_{v_J(d)}$.*

Corollary 6.6.9. *For all $B \in \mathrm{P}_1$ and $C, D \in \mathrm{P}$, if $B \mid C \cdot D$, then $B \mid C$ or $B \mid D$.*

Proof. Let B, C, D as in the hypothesis. The conclusion is trivial for $B = 0$, so assume $B \neq 0$. Then $B = \mathrm{rv}(a)$ for some principal $a \in K((\mathbb{R}^{\leq 0}))$ of degree 1, and in particular with $v_J(a) = \omega$. Write $C = \mathrm{rv}(c)$, $D = \mathrm{rv}(d)$ with $c, d \in K((\mathbb{R}^{\leq 0}))$ principal.

Assume $B \mid C \cdot D$. By Corollary 6.6.7, this means that $a \mid cd \mod J_{v_J(cd)}$, so that there exists b such that $ab \equiv cd \mod J_{v_J(cd)}$. Note that we must have $v_J(ab) = v_J(cd)$. By Proposition 6.6.8, $a \mid c \mod J_{v_J(c)}$ or $a \mid d \mod J_{v_J(c)}$. By Corollary 6.6.7, this means that $B \mid C$ or $B \mid D$. $\square$

Corollary 6.6.10. *Every $B \in \mathrm{P}_1$ is prime in $\widehat{\mathrm{P}}$ and in $\widehat{\mathrm{RV}}$.*

Proof. Let $B \in \mathrm{P}_1$ and $C, D \in \widehat{\mathrm{RV}}$ as in the hypothesis. Suppose first that $C, D \in \widehat{\mathrm{P}}$. Write $C = \sum_\alpha C_\alpha$, $D = \sum_\alpha D_\alpha$. Let $\beta = \deg(C)$, $\gamma = \deg(D)$. Then clearly B divides $C_\beta \cdot D_\gamma$. By Corollary 6.6.9, $B \mid C_\beta$ or $B \mid D_\gamma$. Assume we are in the first case. Then B divides $(C - C_\alpha) \cdot D = \sum_{\alpha<\beta} C_\alpha \cdot D$. By induction on β and γ, either $B \mid D$, or $B \mid (C - C_\alpha)$, hence $B \mid C$, proving the conclusion.

For the general case of $C, D \in \widehat{\mathrm{RV}}$, it suffices to recall that $\widehat{\mathrm{RV}} = \widehat{\mathrm{P}}(\mathbb{R}^{\leq 0})$. $\square$

It is now easy to lift the above result to primality in $K((\mathbb{R}^{\leq 0}))$.

Lemma 6.6.11. *For all irreducible $b \in K((\mathbb{R}^{\leq 0}))$, if $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))}$ is prime, then b is prime.*

Proof. Let b as in the hypothesis, and let $c, d \in K((\mathbb{R}^{\leq 0}))$ be such that $b \mid cd$. We shall prove that $b \mid c$ or $b \mid d$ by induction on $\deg(cd)$. Write $\mathrm{rv}(b) = p \cdot B$ where $p = p(\mathrm{rv}(b))$. By assumption, B is prime, so in particular, $B \mid \mathrm{rv}(c)$ or $B \mid \mathrm{rv}(d)$.

Suppose that $B \mid \mathrm{rv}(c)$. Then $\mathrm{rv}(b) = p \cdot B \mid p \cdot \mathrm{rv}(c)$. Write $p \cdot c = b \cdot e + f$ so that $\deg(f) < \deg(c)$. Then $b \mid fd$. By induction, $b \mid d$, in which case we are done, or $b \mid f$, in which case, $b \mid p \cdot c$. Assume to be in the latter case.

Since b is irreducible, $p(b) = 1$, so by Theorem 6.4.1, it follows at once that $b \mid c$, as desired. □

Theorem 6.6.12 (Theorem F)**.** *For all $b \in K((\mathbb{R}^{\leq 0}))$, if the order type of the support of b is of the form $\omega + k$ with $k < \omega$, and b is not divisible by t^x for any $x \in \mathbb{R}^{<0}$, then b is prime.*

Proof. Let b as in the hypothesis. By the assumption on the order type of b, $\mathrm{rv}(b)$ is weakly principal, so $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))}$ is principal. Therefore, $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))} \in \mathrm{P}_1$, so it is prime by Corollary 6.6.10. Since b is irreducible by Theorem 6.5.3, b is prime by Lemma 6.6.11. □

We present a further primality criterion that follows easily from Corollary 6.6.10.

Lemma 6.6.13. *Let R be an integral domain and G be an Archimedean ordered abelian group. Let $b = b_1 t^{x_1} + \dots + b_n t^{x_n} \in R(G^{\leq 0})$ (with $x_1 < \dots < x_n$) be irreducible in $R(G^{\leq 0})$. If b is irreducible in $\mathrm{Frac}(R)(G^{\leq 0})$, and b_1 is prime in R, then b is prime in $R(G^{\leq 0})$.*

Proof. Suppose that $b \mid cd$ for some $c, d \in R(G^{\leq 0})$. Since b is irreducible in $\mathrm{Frac}(R)(G^{\leq 0})$, and the latter is a GCD domain, we may assume that $b \mid c$ or $b \mid d$ in the ring $\mathrm{Frac}(R)(G^{\leq 0})$. Without loss of generality, we may assume to be in the former case. We claim that $b \mid c$ in the ring $R(G^{\leq 0})$.

By assumption, $b^{-1}c$ is a series in $\mathrm{Frac}(R)(G^{\leq 0})$. Let $\varepsilon = b_1^{-1} t^{-x_1} b - 1 \in \mathrm{Frac}(R)(G)$, so that $b = b_1 t^{x_1} \cdot (1 + \varepsilon)$ and $v(\varepsilon) = x_2 - x_1 > 0$. We have

$$b^{-1} = b_1^{-1} t^{-x_1} \cdot \frac{1}{1 + \varepsilon} = b_1^{-1} t^{-x_1} \cdot \left(1 - \varepsilon + \varepsilon^2 - \dots\right).$$

Since G is Archimedean, there exists some $n \in \mathbb{N}$ such that $v(c \cdot t^{-x_1} \cdot \varepsilon^n) > 0$. It follows at once that the denominators in $b^{-1}c$ are of the form b_1^m with $m < n$.

In turn, $b \mid b_1^n c$ in the ring $R(G^{\leq 0})$. Now write $be = b_1^n c$ with $e \in R(G^{\leq 0})$. Since b_1 is prime, and b is irreducible, b_1^n must divide e. Therefore, b divides c, as desired. □

Corollary 6.6.14. *If $B \in \mathrm{RV}_1$ is irreducible in both $\widehat{\mathrm{RV}}$ and $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}} = \mathrm{Frac}(\widehat{\mathrm{P}})(\mathbb{R}^{\leq 0})$, then B is prime in $\widehat{\mathrm{RV}}$.*

Proof. Let $B \in \mathrm{RV}_1$ be as in the hypothesis. Then $B = B_1 t^{x_1} + \dots + B_n t^{x_n}$ for some $B_i \in \mathrm{P}_1$ and $x_1 < \dots < x_n$. By Corollary 6.6.10, B_1 is prime, so by Lemma 6.6.13, B is prime in $\widehat{\mathrm{RV}}_1$. □

Corollary 6.6.15. *For all irreducible $b \in K((\mathbb{R}^{\leq 0}))$ of degree 1, if $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))}$ is irreducible in $\widehat{\mathrm{RV}}$ and $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}} = \mathrm{Frac}(\widehat{\mathrm{P}})(\mathbb{R}^{\leq 0})$, then b is prime.*

Proof. Write $\frac{\mathrm{rv}(b)}{p(\mathrm{rv}(b))} = B_1 t^{x_1} + \cdots + B_{n-1} t^{x_{n-1}} + B_n$ with $x_1 < \cdots < x_n$. By Corollary 6.6.10, $\mathrm{rv}(b_1)$ is prime. Therefore, by Corollary 6.6.14, $\mathrm{rv}(b)$ is prime, and by Lemma 6.6.11, b is prime as well. □

An easy example is the following: for any principal series $b_1, b_2, b_3 \in K((\mathbb{R}^{\leq 0}))$ of degree 1 such that $\mathrm{rv}(b_1) \neq \mathrm{rv}(b_i)$ for some $i = 2, 3$, the series

$$b_1 t^{-\sqrt{2}} + b_2 t^{-1} + b_3$$

has image through rv that is irreducible in both $\widehat{\mathrm{RV}}$ and $\widehat{\mathrm{P}}^{-1} \cdot \widehat{\mathrm{RV}} = \mathrm{Frac}(\widehat{\mathrm{P}})(\mathbb{R}^{\leq 0})$, so it is irreducible, and it is prime by Corollary 6.6.15.

7. Factorisation in $K((G^{\leq 0}))$ with G Archimedean

We now assume that G be an *Archimedean* divisible ordered abelian group. Without loss of generality, we may simply assume that G is some divisible subgroup of $(\mathbb{R}, +)$. In particular, we may assume that $K((G^{\leq 0}))$ is a subring of $K((\mathbb{R}^{\leq 0}))$.

7.1. **Irreducibility up to monomials.** If we apply Theorem 6.4.1 to a series in $K((G^{\leq 0}))$, the factors appearing may not be in $K((G^{\leq 0}))$. For instance, let $(q_n \in \mathbb{Q})_{n \in \mathbb{N}}$ be an increasing sequence of rational numbers converging to $-\sqrt{2}$, and let $b = \sum_{n \in \mathbb{N}} t^{q_n}$. Theorem 6.4.1 then yields the factorisation

$$b = t^{-\sqrt{2}} \cdot \sum_{n \in \mathbb{N}} t^{q_n + \sqrt{2}},$$

where the exponent $-\sqrt{2}$ is unique. Therefore, if we are working in $G = \mathbb{Q}$, we cannot hope to have a conclusion as strong as the one of Theorem 6.4.1. We can still produce a meaningful statement by weakening the notion of irreducibility.

Definition 7.1.1. Given $b \in K((G^{\leq 0}))^*$, we say that b is **irreducible up to monomials** if $b = cd$ for some $c, d \in K((G^{\leq 0}))$ implies $\mathrm{ot}(c) = 1$ or $\mathrm{ot}(d) = 1$.

Remark 7.1.2. For all $b \in K((G^{\leq 0}))^*$, b is irreducible if and only if b is irreducible up to monomials and $\sup(b) = 0$. Indeed, if b is irreducible, then clearly it is irreducible up to monomials and $\sup(b) = 0$. Conversely, suppose b is irreducible up to monomials and $\sup(b)$. Then for any factorisation $b = cd$ we have $\mathrm{ot}(c) = 1$ or $\mathrm{ot}(d) = 1$. But $\sup(b) = \sup(c) + \sup(d) = 0$, so $\sup(c) = \sup(d) = 0$, hence $c \in K^*$ or $d \in K^*$, which means that b is irreducible.

Remark 7.1.3. For all $b, c \in K((G^{\leq 0}))^*$, if b divides c in the ring $K((\mathbb{R}^{\leq 0}))$, then b divides c in the ring $K((G^{\leq 0}))$. In fact, if $bd = c$ for some $d \in K((\mathbb{R}^{\leq 0}))$, then we have $d = cb^{-1} \in K((G))$, so necessarily $d \in K((G^{\leq 0}))$. For the same reason, for all $b, c \in K(G)^*$, b divides c in $K(G)$ if and only if b divides c in $K(\mathbb{R})$.

7.2. **Almost divisibility.** We also introduce the following more technical notions, which are only used in this section.

Definition 7.2.1. Given $b, c \in K((\mathbb{R}^{\leq 0}))$, we say that b **almost divides** c if b divides $t^x c$ for some $x \in \mathbb{R}$. Given $p \in K(\mathbb{R}^{\leq 0})$, we say that p is **monic** if $0 \notin \mathrm{supp}(p - 1)$, namely if the coefficient of p at the exponent $x = 0$ is 1.

Remark 7.2.2. For all $p, q \in K(\mathbb{R}^{\leq 0})^*$, p almost divides q if and only if p divides q in the ring $K(\mathbb{R})$.

Lemma 7.2.3. *For all $p \in K(\mathbb{R}^{\leq 0})$, there exists a unique monic $p_G \in K(G^{\leq 0})$ such that for all $q \in K(G^{\leq 0})$, q almost divides p if and only if q almost divides p_G.*

Proof. Let $p \in K(\mathbb{R}^{\leq 0})$, $q \in K(G^{\leq 0})$. Clearly, we may assume that $p \neq 0$. Since G is divisible, we may choose a complement $H \subseteq \mathbb{R}$ such that $\mathbb{R} = G \oplus H$ (as non-ordered groups).

There is a finite set of real numbers $x_1, \ldots, x_n \in G$, $y_1, \ldots, y_m \in H$ which are $\mathbb{Z}$-linearly independent, and such that $p \in K(G' \oplus H')$, where $G' = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$ and $H' = \mathbb{Z}y_1 + \ldots \mathbb{Z}y_n$. Note that $K(G' \oplus H')$ is isomorphic to the ring of Laurent polynomials $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}]$, with the isomorphism sending t^{x_i} to X_i and t^{y_i} to Y_i. In particular, $K(G')$ and $K(G' \oplus H')$ are unique factorisation domains, with group of units generated by K^* and $t^{x_1}, \ldots, t^{x_n}, t^{y_1}, \ldots, t^{y_m}$. Moreover, every irreducible element of $K(G')$ is irreducible in $K(G' \oplus H')$ as well.

Let us write $p = u \cdot q_1 \cdot \ldots \cdot q_m$ where $q_1, \ldots, q_m$ are irreducible elements of $K(G' \oplus H')$ and u is a unit. After multiplying each q_i by a unit, we may further assume that either q_i is in $K(G')$, or no product of q_i by a unit falls in $K(G')$. Let p_G be the product of the factors q_i lying in $K(G')$. Clearly, p_G divides p. By construction, all irreducible divisors of $\frac{p}{p_G}$ are not in $K(G')$, so any divisor of $\frac{p}{p_G}$ in $K(G')$ is necessarily a unit. Finally, after multiplication by a unit, we may assume that $p_G \in K(G^{\leq 0})$ and also that p_G is monic.

Since $K(G')$ is a GCD-domain, we can write $q = q_1 q_2$ for some $q_1, q_2 \in K(G)$ such that q_1 divides p_G (in $K(G)$) and q_2 is coprime with p_G (in $K(G)$). Now suppose that q almost divides p, namely that q divides p in $K(G)$. Then q_2 almost divides $\frac{p}{p_G}$. Since $q_2 \in K(G')$, q_2 must be a unit, so q almost divides p_G. Conversely, if q almost divides p_G, then clearly it almost divides p.

Finally, if p' is another monic series in $K(G^{\leq 0})$ such that q almost divides p if and only if q almost divides p', then clearly p' almost divides p_G and p_G almost divides p'. Since they are both monic, we have $p_G = p'$, as desired. □

Corollary 7.2.4. *For all $b \in K((\mathbb{R}^{\leq 0}))$ and $q \in K(G^{\leq 0})$, q almost divides b if and only if q almost divides $p(b)_G$.*

7.3. **Factorisation.** In turn, we can obtain a conclusion as in Theorem 6.4.1 by adapting Proposition 6.3.8 and then repeating the same argument.

Proposition 7.3.1. *For all $p, q \in K(\mathbb{R}^{\leq 0})$, $(pq)_G = p_G q_G$.*

Proof. Let $p, q \in K(\mathbb{R}^{\leq 0})$, and $r \in K(G^{\leq 0})$. We first observe that $p_G q_G$ almost divides pq, so $p_G q_G$ almost divides pq. For the converse, since $K(G^{\leq 0})$ is a GCD-domain, we can write $(pq)_G = r_1 r_2$ for some $r_1, r_2 \in K(G^{\leq 0})$ such that r_1 divides p and r_2 divides q. Then r_1 divides p_G and r_2 divides q_G, so $(pq)_G$ divides $p_G q_G$. Since $(pq)_G$ and $p_G q_G$ are both monic, we may conclude that $(pq)_G = p_G q_G$, as desired. □

Corollary 7.3.2. *For all $b, c \in K((\mathbb{R}^{\leq 0}))$, $p(bc)_G = p(b)_G p(c)_G$.*

Proposition 7.3.3. *For all non-zero $b \in K((G^{\leq 0}))$, there exist $c_1, \ldots, c_n \in K((G^{\leq 0}))$ irreducible over monomials and with infinite support and $p \in K(G^{\leq 0})$ such that $b = p \cdot c_1 \cdot \cdots \cdot c_n$. Moreover, p is unique up to multiplication by a series $d \in K((G^{\leq 0}))$ such that $\mathrm{ot}(d) = 1$.*

If moreover $\sup(b) \in G$*, then we may take* $c_1, \ldots, c_n$ *irreducible, in which case* p *is unique up to multiplication by an element of* K^**.*

Proof. Let $b \in K((\mathbb{R}^{\leq 0}))$ be a non-zero series. For the existence of the factorisation, we proceed as in the proof of Proposition 5.6.1. Write $b = p_G(b) \cdot b'$ for a suitable $b' \in K((G^{\leq 0}))^*$. Note that $p_G(b')$ is necessarily 1. We work by induction on $\deg(b)$. If $\deg(b) = 0$, then b' is of the form kt^x for $k \in K$ and $x \in G$, and we are done.

Assume $\deg(b) > 0$. If b' is irreducible over monomials, then we are done. Otherwise, $b' = c \cdot d$ for some $c, d \in K((G^{\leq 0}))$ not of the form kt^x and with $\mathrm{ot}(c), \mathrm{ot}(d) > 1$. Since $p_G(b') = 1$, c, d are not in $K(G^{\leq 0})$, hence $\deg(c), \deg(d) > 0$. Since $\deg(b') = \deg(c) \oplus \deg(d)$, it follows that $\deg(c) < \alpha$, $\deg(d) < \alpha$. Note moreover that $p_G(c) = p_G(d) = 1$ by Proposition 7.3.1.

By inductive hypothesis, c and d can be written as products of series irreducible over monomials with infinite support. Therefore, b' is also a product of series irreducible over monomials with infinite support.

For the uniqueness of the factor p, suppose that $b = p \cdot c_1 \cdot \cdots \cdot c_n$ is a factorisation of b with $c_1, \ldots, c_n$ irreducible over monomials with infinite support and $p \in K(G^{\leq 0})$. It then suffices to note that

$$p_G(b) = p_G(p) \cdot p_G(k) \cdot p_G(c_1) \cdot \cdots \cdot p_G(c_n) = p_G(p) = kt^x \cdot p$$

for some $k \in K$, $x \in G$, as desired.

Finally, suppose $x = \sup(b) \in G$. Then we may write $b = p_G(b) \cdot t^x \cdot b'$ for a suitable $b' \in K((G^{\leq 0}))^*$. Note that $p_G(b') = 1$ and $\sup(b') = 0$. By the previous conclusion, we can write $b' = c_1 \cdot \cdots \cdot c_n$ with $c_1, \ldots, c_n \in K((G^{\leq 0}))$ irreducible over monomials and with infinite support. On the other hand, $\sup(b') = \sup(c_1) + \cdots + \sup(c_n) = 0$, so $\sup(c_1) = \cdots = \sup(c_n)$. Therefore, $c_1, \ldots, c_n$ are irreducible (Remark 7.1.2). We then get the desired factorisation on taking $p = p_G(b) \cdot t^x$. For the uniqueness of p, recall that p is unique up to multiplication by a series $d \in K((G^{\leq 0}))$ such that $\mathrm{ot}(d) = 1$. Since $\sup(p)$ is necessarily x, p is in fact unique up to multiplication by a series $d \in K((G^{\leq 0}))$ such that $\mathrm{ot}(d) = 1$ and $\sup(d) = 0$, which means that $d \in K^*$, as desired. □

8. Omnific integers and other rings of the form $Z + K((G^{<0}))$

We shall now prove the general version of Theorem 6.4.1, namely Theorem C, from which we shall deduce the corollary Theorem B for omnific integers.

8.1. **Factorisation with arbitrary ring of constants.** Let Z be a subring of K and G a divisible subgroup of $\mathbb{R}$.

Notation 8.1.1. Given $b \in K((\mathbb{R}^{\leq 0}))$, let $\mu(b)$ be the coefficient of b at 0.

Remark 8.1.2. The map $\mu : K((\mathbb{R}^{\leq 0})) \to K$ is a *ring homomorphism.* For all $b \in K((\mathbb{R}^{\leq 0}))$, $\mu(b) \in Z$ if and only if $b \in Z + K((\mathbb{R}^{<0}))$. Moreover, if $\mu(b) \neq 0$, then $\mu(b)$ divides b, while if $\mu(b) = 0$, then any element of Z divides b. Indeed, in the former case, $\mu(b)^{-1}b \in 1 + K((\mathbb{R}^{<0})) \subseteq Z + K((\mathbb{R}^{<0}))$, while in the latter case, for any $z \in Z^*$, $z^{-1}b \in K((\mathbb{R}^{<0})) \subseteq Z + K((\mathbb{R}^{<0}))$.

An immediate consequence of the above remark is that if a series $b \in Z + K((\mathbb{R}^{<0}))$ satisfies $\mu(b) = 0$, then b is never irreducible unless Z is a field. To account for this, we weaken the notion of irreducibility.

Definition 8.1.3. Given $b \in Z + K((G^{<0}))$, we say that b is **coarsely irreducible** if $b = cd$ for some $c, d \in Z + K((G^{<0}))$ implies $c \in Z$ or $d \in Z$.

Remark 8.1.4. A series $b \in Z + K((G^{<0}))$ is coarsely irreducible if and only if it is irreducible in $K((G^{\leq 0}))$, and a coarsely irreducible $b \in Z + K((G^{<0}))$ is irreducible in $Z + K((G^{<0}))$ if and only if $\mu(b)$ is a unit in Z.

Moreover, if $b \in Z + K((G^{<0}))$ is coarsely irreducible, we either have $\mu(b) = 0$, in which case b is divisible by *any* element of Z, or $\mu(b) \neq 0$, in which case we may write $b = \mu(b)b'$ with $b' \in Z + K((G^{<0}))$ irreducible.

Proposition 8.1.5. *For all non-zero $b \in Z + K((G^{<0}))$, there exist $c_1, \dots, c_n \in Z + K((G^{<0}))$ irreducible up to monomials with infinite support and $p \in Z + K(G^{<0})$ such that $b = p \cdot c_1 \cdot \dots \cdot c_n$. Moreover, p is unique up to multiplication by a series $d \in K((G^{\leq 0}))$ such that* $\mathrm{ot}(d) = 1$.

If moreover $\sup(b) \in G$, *then we may take $c_1, \dots, c_n$ coarsely irreducible, in which case p is unique up to multiplication by an element of K^*.*

Proof. Let $b \in Z+K((G^{<0}))$ be a non-zero series. By Proposition 7.3.3, we can write $b = p \cdot c_1 \cdot \dots \cdot c_n$ for some $c_1, \dots, c_n \in K((G^{\leq 0}))$ irreducible up to monomials with infinite support and $p \in K(G^{\leq 0})$, and p is unique up to multiplication by a series $d \in K((G^{\leq 0}))$ such that $\mathrm{ot}(d) = 1$. When $\sup(b) \in G$, we may further assume that $c_1, \dots, c_n$ are irreducible in $K((G^{\leq 0}))$, in which case p is unique up to multiplication by an element of K^*. Thus, we only need to prove that we may take $p, c_1, \dots, c_n$ in $Z + K((G^{<0}))$. We distinguish two cases.

If $\sup(b) < 0$, then $\sup(p) < 0$ or $\sup(c_i) < 0$ for some i. For simplicity, say that $\sup(p) < 0$. It then suffices to choose any $x \in G^{\leq 0}$ sufficiently close to 0 and replace p with $t^{-x}p$ and c_i with $t^{\frac{x}{n}}c_i$. After the transformation, all factors are in $K((G^{<0})) \subseteq Z + K((G^{<0}))$, and we reach the desired conclusion.

If $\sup(b) = 0$, then $\sup(p) = \sup(c_1) = \dots = \sup(c_n) = 0$. Suppose that $\mu(c_i) = 0$ for some i, for simplicity say $i = 1$. We then replace p with $\mu(p)^{-1}p$ and each c_i with $\mu(c_i)^{-1}c_i$ when $\mu(c_i) \neq 0$, and c_1 by $c_1 \cdot \mu(p) \cdot \prod_{\mu(c_i)\neq 0} \mu(c_i)$. After this transformation, we clearly still have $b = p \cdot c_1 \cdot \dots \cdot c_n$, while each c_i is either in $K((G^{<0}))$ or in $1 + K((G^{<0}))$, and likewise p is in $1 + K((G^{<0}))$, so all the factors are in $Z + K((G^{<0}))$. If instead $\mu(c_i) \neq 0$ for all i, we replace each c_i with $\mu(c_i)^{-1}c_i$ and p with $p \cdot \prod_i \mu(c_i)$. Again, we clearly still have $b = p \cdot c_1 \cdot \dots \cdot c_n$, while each c_i is in $1 + K((G^{<0})) \subseteq Z + K((G^{<0}))$, and $\mu(p) = \mu(b) \in Z$, so $p \in Z + K((G^{<0}))$, as desired. □

8.2. **The Archimedean valuation on G.** Let Z be a subring of K and G be any divisible ordered abelian group. As in any ordered group, we can define the Archimedean valuation.

Definition 8.2.1. Given $g, h \in G^*$, we say that

- g is **dominated by** h, written $g \preceq h$, if $|g| \leq n \cdot |h|$ for some $n \in \mathbb{N}$;
- g is **comparable with** h, written $g \asymp h$, if $g \preceq h$ and $h \preceq g$;

- g is **strictly dominated by** h, or **infinitesimal with respect to** h, written $g \prec h$, if $g \preceq h$ and $g \not\asymp h$ (equivalently, if $n \cdot |g| \leq |h|$ for all $n \in \mathbb{N}$).

Note that $\preceq$ is a total quasi-order, and that $\asymp$ is an equivalence relation.

Definition 8.2.2. Let $\Sigma = \Sigma(G) := G^*_{/\asymp}$ be the **Archimedean value set** of G. We denote by $\mathrm{ord} : G^* \to \Sigma$, and call **Archimedean valuation** of G, the quotient map $G \to \Sigma$. We order Σ by saying that $\mathrm{ord}(g) \leq \mathrm{ord}(h)$ if and only if $g \preceq h$. For the sake of notation, we also set $\mathrm{ord}(0) := -\infty$ and say that $-\infty < \sigma$ for all $\sigma \in \Sigma \cup \{-\infty\}$.

Remark 8.2.3. The function ord is a group valuation, in the sense that it satisfies the ultrametric inequality $\mathrm{ord}(g + h) \leq \max\{\mathrm{ord}(g), \mathrm{ord}(h)\}$ for all $g, h \in G$.

8.3. **Coarse irreducibility.** We use the Archimedean valuation to further extend the notion of coarse irreducibility and to introduce a few other coarse notions.

Notation 8.3.1. Given $b = \sum_x k_x t^x \in K((G^{\leq 0}))$ and $\sigma \in \Sigma(G)$, we write

- $\mu_\sigma(b) := \sum_{\mathrm{ord}(x)<\sigma} k_x t^x$;
- $M_\sigma(b) := \sum_{\mathrm{ord}(x)\leq\sigma} k_x t^x$.

Remark 8.3.2. The maps μ_σ and M_σ are ring homomorphisms.

We associate to each comparability class $\sigma \in \Sigma$ a few distinguished subgroups of G.

Notation 8.3.3. Given $\sigma \in \Sigma$, we shall denote by G_σ the subgroup $G_\sigma := \{g : \mathrm{ord}(g) \leq \sigma\}$, and by I_σ the subgroup $I_\sigma := \{g : \mathrm{ord}(g) < \sigma\} \subseteq G_\sigma$. Moreover, we let H_σ be a complement of I_σ in G_σ, and $\pi_\sigma : G_\sigma \to H_\sigma$ the natural projection. Let $Z_\sigma := Z + K((I_\sigma^{<0}))$ and $K_\sigma := K((I_\sigma))$.

For all $\sigma \in \Sigma$, I_σ is the maximal proper convex subgroup of G_σ. In particular, H_σ is naturally an ordered group, and $G_\sigma = H_\sigma \oplus I_\sigma$ as an ordered group, namely the order on G_σ is the lexicographic order on $H_\sigma \oplus I_\sigma$. Moreover, H_σ is Archimedean, so it can be embedded into $(\mathbb{R}, +)$. In view of this, for each σ we shall identify H_σ with a subgroup of $(\mathbb{R}, +)$.

Proposition 8.3.4. *For all* $\sigma \in \Sigma$, $Z + K((G_\sigma^{<0})) = Z_\sigma + K_\sigma((H_\sigma^{<0}))$.

Proof. Let $b = \sum_{x\in G_\sigma} k_x t^x \in Z + K((G_\sigma^{<0}))$. Recall that $G_\sigma = H_\sigma \oplus I_\sigma$ as an ordered group. Therefore,

$$\sum_{x\in G_\sigma} k_x t^x = \sum_{x\in H_\sigma} \left(\sum_{y\in I_\sigma} k_{x+y} t^y \right) t^x = \sum_{x\in H_\sigma} k'_x t^x,$$

where each coefficient k'_x is in $K_\sigma = K((I_\sigma))$. Clearly, since $b \in Z + K((G_\sigma^{<0}))$, we must have $k'_0 \in Z_\sigma = Z + K((I_\sigma^{<0}))$, so $b \in Z_\sigma + K_\sigma((H_\sigma^{<0}))$.

Conversely, let $b = \sum_{x\in H_\sigma} k'_x t^x \in Z_\sigma + K_\sigma((H_\sigma^{<0}))$. Then

$$\sum_{x\in H_\sigma} k'_x t^x = \sum_{x\in H_\sigma} \left(\sum_{y\in I_\sigma} l_{xy} t^y \right) t^x = \sum_{x\in G_\sigma} k_x t^x,$$

where $k_x = l_{yz}$ for the unique $y \in H_\sigma$ and $z \in I_\sigma$ such that $x = y + z$, and it is immediate to verify that indeed $b \in Z + K((G_\sigma^0))$, as desired. □

Definition 8.3.5. Given $b \in Z + K((G^{<0}))$ with $b \notin Z$, we let the **coarse support** of b be $\overline{\mathrm{supp}}(b) := \{\pi_\sigma(h) : h \in \mathrm{supp}(b)\} \subseteq H_\sigma$, where $\sigma = \mathrm{ord}(v(b))$. We also let $\overline{\mathrm{supp}}(0) := \emptyset$, $\overline{\mathrm{ot}}(0) := 0$, $\overline{\deg}(0) := -\infty$.

Moreover, we say that b is **coarsely irreducible** if $b = cd$ implies one of $v(c) \prec v(b)$, $v(d) \prec v(b)$ for any $c, d \in Z + K((G^{<0}))$. We say that b is **coarsely irreducible up to monomials** if $b = cd$ implies one of $v(c) \prec v(b)$, $|\overline{\mathrm{supp}}(c)| = 1$, $v(d) \prec v(b)$, $|\overline{\mathrm{supp}}(d)| = 1$ for any $c, d \in Z + K((G^{<0}))$.

Remark 8.3.6. Let $b \in Z + K((G^{<0}))$, with $b \notin Z$, and $\sigma = \mathrm{ord}(v(b))$. Then the above coarse definitions are just their non-coarse counterparts, but read in $K_\sigma((H_\sigma^{\leq 0}))$ rather than $Z + K((G_\sigma^{\leq 0}))$. More precisely, given $b \in Z + K((G^{<0}))^*$ and $\sigma = \mathrm{ord}(v(b))$:

- the coarse support of b is the support of b in $K_\sigma((H_\sigma^{\leq 0}))$;
- the coarse order type of b is the order type of b in $K_\sigma((H_\sigma^{\leq 0}))$;
- the coarse degree of b is the degree of b in $K_\sigma((H_\sigma^{\leq 0}))$;
- b is coarsely irreducible if and only if b is irreducible in $K_\sigma((H_\sigma^{\leq 0}))$;
- b is coarsely irreducible up to monomials if and only if b is irreducible up to monomials in $K_\sigma((H_\sigma^{\leq 0}))$.

Note moreover that if G is Archimedean, then $G = H_\sigma$ and $K = K_\sigma$, so the above definition of coarsely irreducibility coincides with Definition 8.1.3.

Remark 8.3.7. Let $b \in Z + K((G^{<0}))$ be coarsely irreducible, and let $\sigma = \mathrm{ord}(v(b))$. By Remark 8.1.4, b is irreducible if and only if $\mu_\sigma(b)$ is a unit. In particular, if $\mu_\sigma(b) \neq 0$, then we can write $b = \mu_\sigma(b) \cdot b'$ with $b' \in Z + K((G^{<0}))$ *irreducible*. If otherwise $\mu_\sigma(b) = 0$, then b is divisible by *any* non-zero series $c \in Z + K((G^{<0}))$ such that $v(c) \prec v(b)$. In this sense, we argue coarse irreducibility is as close as we can get to irreducibility.

Remark 8.3.8. Let $b = \sum_x b_x t^x \in Z + K((G^{<0}))$ be coarsely irreducible up to monomials, and let $\sigma = \mathrm{ord}(v(b))$. Let $r := \sup(\overline{\mathrm{supp}}(b)) \in \mathbb{R}$.

If $r \in H_\sigma$, define

$$\lambda(b) := \sum_{\pi_\sigma = r} b_x t^x.$$

By Proposition 8.3.4, $\lambda(b)$ is the coefficient of the monomial t^r seen in the ring $K_\sigma((H_\sigma^{\leq 0}))$. It follows at once that for any monomial c of such ring, or equivalently, for any $c \in Z + K((G^{<0}))$ with either $v(c) \prec v(b)$ or $|\overline{\mathrm{supp}}(c)| = 1$, c divides b if and only if c divides $\lambda(b)$. In particular, we can write $b = \lambda(b) \cdot b'$ with $b' \in Z + K((G^{<0}))$ *coarsely irreducible*.

If $r \notin H_\sigma$, then the monomials dividing b in the ring $Z_\sigma + K_\sigma((H_\sigma^{<0}))$ are precisely those of the form $k_\sigma t^s$ with $s \leq r$. Such monomials are the series $c \in Z + K((G^{<0}))$ with $v(c) \prec v(b)$ or $|\overline{\mathrm{supp}}(c)| = 1$ such that $v(c) > \mathrm{supp}(b)$. Therefore, b is divisible by *any* non-zero series $c \in Z + K((G^{<0}))$ with $v(c) \prec v(b)$ or $|\overline{\mathrm{supp}}(c)| = 1$ and such that $v(d) > \mathrm{supp}(b)$. Again, we argue that coarse irreducibility up to monomials is as close as we can get to irreducibility.

Remark 8.3.9. **Oz** is not a GCD domain. This is fairly easy to verify using the tools of this section. Take the omnific integers

$$b = \sum_{n \in \mathbb{N}^*} \omega^{\frac{1}{n}}, \quad c = \sum_{n \in \mathbb{N}^*} \omega^{\frac{2}{n}}.$$

By [Ber00, Thm. 10.5] (or its generalisation Theorem 6.5.3), combined with Remark 8.3.6, both b and c are coarsely irreducible. Moreover, their coarse supports are disjoint. It follows that if $d \in \mathbf{Oz}$ divides both b and c, then $v(d) \prec v(b)$ and $v(d) \prec v(c)$.

On the other hand, any $d \in \mathbf{Oz}$ such that $v(d) \prec v(b)$ divides b. Since $2 \cdot v(b) = v(c)$, any such d satisfies $v(d) \prec v(c)$, hence $d \mid c$. Therefore, $d \in \mathbf{Oz}$ divides both b and c *if and only if* $v(d) \prec v(b)$. This implies that there is no greatest common divisor of b and c: for any d that divides both b and c and is not a unit, d^2 also divides both b and c, but d^2 does not divide d, so d is not a greatest common divisor.

8.4. **Factorisation theorems.** With the above dictionary, it is immediate to generalise our previous theorems.

Theorem 8.4.1 (Theorem C)**.** *For all non-zero $b \in Z + K((G^{<0}))$, there exist*

- *$c_1, \dots, c_n \in Z + K((G^{<0}))$ coarsely irreducible up to monomials with infinite coarse support,*
- *$p \in Z + K((G^{<0}))$ with finite coarse support or $v(p) \prec v(b)$,*

such that $b = p \cdot c_1 \cdot \dots \cdot c_n$ and $v(c_1) \asymp \dots \asymp v(c_n) \asymp v(b)$. Moreover, p is unique up to multiplication by an element $d \in K((G))$ such that $\mathrm{supp}(d) \prec v(b)$ or $d \in Z + K((G^{<0}))$ and $|\overline{\mathrm{supp}}(d)| = 1$.

If moreover $\sup(\overline{\mathrm{supp}}(b)) \in H_{\mathrm{ord}(v(b))}$, then we may take $c_1, \dots, c_n$ coarsely irreducible, in which case p is unique up to multiplication by an element $d \in K((G))$ such that $\mathrm{supp}(d) \prec v(b)$.

Proof. Let $b \in Z + K((G^{<0}))$ be a non-zero series. The conclusion is trivial for $b \in Z$, so assume $b \notin Z$. Let $\sigma = \mathrm{ord}(v(b))$, so that $b \in Z_\sigma + K_\sigma((H_\sigma^{<0}))$. By Proposition 8.1.5, there are $c_1, \dots, c_n \in Z_\sigma + K_\sigma((H_\sigma^{<0}))$ irreducible up to monomials with infinite support and $p \in Z_\sigma + K_\sigma(H_\sigma^{<0})$ such that $b = p \cdot c_1 \cdot \dots \cdot c_n$. This implies that for all i, $v(c_i) \notin I_\sigma$, so in particular $v(c_i) \asymp v(b)$, hence c_i has infinite coarse support in $Z + K((G^{<0}))$, and it is coarsely irreducible up to monomials in $Z + K((G^{<0}))$. Note that if c_i is also coarsely irreducible in $Z_\sigma + K_\sigma((H_\sigma^{<0}))$, then it is coarsely irreducible in $Z + K((G^{<0}))$. As to the factor p, we note that if $v(p) \notin I_\sigma$, then in fact $v(p) \asymp v(b)$, so p has finite coarse support, and otherwise we have $v(p) \prec v(b)$.

For the uniqueness of p, let $b = p \cdot c_1 \cdot \dots \cdot c_n$ be a factorisation as in the conclusion. Since $v(b) = v(p) + v(c_1) + \dots + v(c_n)$, we have $v(b) \leq v(p)$ and $v(b) \leq v(c_i)$ for all i. In particular, $p, c_1, \dots, c_n \in Z + K((G_\sigma^{\leq 0}))$. By Proposition 8.1.5, p is unique up to multiplication by a series $d \in K_\sigma((H_\sigma)) \subseteq K((G))$ of the form $d = kt^x$ with $k \in K_\sigma^*$ and $x \in H_\sigma$. But for such a series $d = kt^x$, either $x = 0$, so $\mathrm{supp}(d) \subseteq I_\sigma$, which means that $\mathrm{supp}(d) \prec v(b)$, or $x < 0$, in which case $d \in K_\sigma((H_\sigma^{<0})) \subseteq Z_\sigma + K_\sigma((H_\sigma^{<0})) = Z + K((G^{<0}))$ and $\overline{\mathrm{supp}}(d) = \{x\}$.

Finally, suppose $\overline{\mathrm{supp}}(b)$ has supremum in $H_{\mathrm{ord}(v(b))}$. Then by Proposition 8.1.5 we may further assume that $c_1, \dots, c_n$ are coarsely irreducible in $Z_\sigma + K_\sigma((H_\sigma^{<0}))$, so coarsely irreducible in $Z + K((G^{<0}))$. If $b = p \cdot c_1 \cdot \dots \cdot c_n$ is a factorisation as in the conclusion, then p is unique up to multiplication by a series $d \in K_\sigma^* \subseteq K((G))$. For such a series d, $\mathrm{supp}(d) \subseteq I_\sigma$, so $\mathrm{supp}(d) \prec v(b)$, as desired. □

In turn, we obtain the desired factorisation theorem for the ring $\mathbf{Oz}$ of omnific integers. We do not define the ring $\mathbf{Oz}$ here. It suffices to know that the field of surreal numbers $\mathbf{No}$ is of the form $\mathbb{R}((G))$ where G is itself an isomorphic copy of the additive group of $\mathbf{No}$, and $\mathbf{Oz} = \mathbb{Z} + \mathbb{R}((G^{<0}))$.

In particular, each group H_σ is *complete*, namely $H_\sigma = \mathbb{R}$. Therefore, for all $b \in \mathbf{Oz}^*$, $\sup(\overline{\operatorname{supp}}(b))$ is an element of $H_{\operatorname{ord}(v(b))} = \mathbb{R}$.

Corollary 8.4.2 (Theorem B)**.** *For all non-zero $b \in \mathbf{Oz}$, there exist*

- *$c_1, \dots, c_n \in \mathbf{Oz}$ coarsely irreducible with infinite coarse support,*
- *$p \in \mathbf{Oz}$ with finite coarse support or $v(p) \prec v(b)$,*

such that $b = p \cdot c_1 \cdot \dots \cdot c_n$ and $v(c_1) \asymp \dots \asymp v(c_n) \asymp v(b)$. Moreover, p is unique up to multiplication by a surreal number $d \in \mathbf{No}$ such that $\operatorname{supp}(d) \prec v(b)$*.*

Appendix A. Coarse multiplicativity of sup

Using the tools of Section 8, we can give an alternative and simpler proof of a theorem of Pitteloud stating that the ideal generated by the monomials t^x for $x \in G^{<0}$ is prime in $K((G^{\leq 0}))$ for any abelian ordered group G [Pit01]. We obtain this by showing that the function sup of Definition 3.1.1 can be extended to $K((G^{\leq 0}))$, and that it results in a function very similar to a valuation. In what follows, let G be an ordered abelian group (possibly a proper class to account for the case of omnific integers).

A.1. A completion of G.

The only technical obstacle in defining sup is giving an appropriate completion of G in which sup has a well defined meaning. We choose a definition that works well when G is a proper class, so that the following arguments can also be applied to $\mathbf{Oz}$.

Definition A.1.1. Given two nonempty sub*sets* (not proper classes) A, B of G, we write:

- $A \leq_{\mathrm{cof}} B$ if for all $x \in A$ and $u \in G$ with $u < x$ there exists $y \in B$ such that $u < y$;
- $A \equiv_{\mathrm{cof}} B$ if $A \leq_{\mathrm{cof}} B$ and $B \leq_{\mathrm{cof}} A$.
- $A <_{\mathrm{cof}} B$ if $A \leq_{\mathrm{cof}} B$ but $B \not\leq_{\mathrm{cof}} A$.

Proposition A.1.2. *The relation $\leq_{\mathrm{cof}}$ is a total quasi-order on the class of subsets of G.*

Proof. Let A, B, C be nonempty subsets of G. Clearly, $A \leq_{\mathrm{cof}} A$. Suppose $A \leq_{\mathrm{cof}} B \leq_{\mathrm{cof}} C$. Pick some $x \in A$ and $u \in G$ with $u < x$. Then there is $y \in B$ such that $u < y$, so there is $z \in C$ such that $u < z$. Therefore, $A \leq_{\mathrm{cof}} C$, so $\leq_{\mathrm{cof}}$ is a quasi-order.

For totality, suppose that $A \not\leq_{\mathrm{cof}} B$. Then there exists $x \in A$ and $u \in G$ such that $u < x$ and $y \leq u$ for all $y \in B$. But then for all $y \in B$, $w \in G$, if $w \leq y$, then $w \leq u < x$, so $B \leq_{\mathrm{cof}} A$. □

Remark A.1.3. Clearly, if $A \subseteq B$, then $A \leq_{\mathrm{cof}} B$.

Definition A.1.4. Let $\operatorname{Sup}(G)$ be the class of the $\equiv_{\mathrm{cof}}$-equivalence classes of the nonempty subsets of G. Given a nonempty subset $A \subseteq G$, denote by $\sup(A) \in \operatorname{Sup}(G)$ its $\equiv_{\mathrm{cof}}$-equivalence class.

Given two nonempty subsets $A, B \subseteq G$, we define:

- $\sup(A) \leq \sup(B)$ if $A \leq_{\mathrm{cof}} B$;
- $\sup(A) + \sup(B) := \sup(A + B)$.

Proposition A.1.5. $(\operatorname{Sup}(G), +, \leq)$ *is an ordered commutative monoid.*

Proof. Since $\leq_{\text{cof}}$ is a quasi-order and $\operatorname{Sup}(G)$ is the class of the $\equiv_{\text{cof}}$-equivalence classes, $\leq$ is a well defined total order on $\operatorname{Sup}(G)$.

For the sum, let $A, B, C \subseteq G$ be nonempty sets. Suppose that $A \leq_{\text{cof}} B$. Pick some $x \in A + C$ and some $u \in G$ with $u < x$. Write $x = y + z$ with $y \in A$ and $z \in C$. Then $u - z < x$, so there is $w \in B$ such that $u - z < w$, hence $u < w + z$. Since $w + z \in B + C$, we have proved $A + C \leq_{\text{cof}} B + C$. Since the sum of two sets is commutative, it also follows that $C + A \leq_{\text{cof}} C + B$.

It follows at once that the sum $+$ in $\operatorname{Sup}(G)$ is well defined and commutative, and that $A \leq_{\text{cof}} B$ implies $A + C \leq_{\text{cof}} B + C$. Moreover, $(\operatorname{Sup}(G), +)$ is a monoid, as

$$(\sup(A) + \sup(B)) + \sup(C) = \sup(A + B) + \sup(C) = \sup(A + B + C) = \\ \sup(A) + (\sup(B + C)) = \sup(A) + (\sup(B) + \sup(C)),$$

and $\sup(A) + \sup(\{0\}) = \sup(A + \{0\}) = \sup(A)$. □

Notation A.1.6. Given $x \in G$, let $\iota(x) := \sup(\{x\})$. Note that ι is clearly a group homomorphism.

Proposition A.1.7. *For all non-empty $A \subseteq G$ and $x \in G$, $\sup(A) \leq \iota(x)$ if and only if $A \leq x$.*

Proof. Let $A \subseteq G$ be nonempty and $x \in G$. Suppose $A \not\leq_{\text{cof}} \{x\}$. Then there exists $y \in A$ and $u \in G$ with $u < y$ such that $x \leq u < y$. Therefore, $A \not\leq x$. Conversely, suppose $A \not\leq x$. Then there exists $y \in A$ such that $x < y$, while clearly for no $z \in \{x\}$ we have $x < z$, hence $\{x\} \not\leq_{\text{cof}} A$. □

Corollary A.1.8. *The map $\iota : G \to \operatorname{Sup}(G)$ is an ordered group embedding.*

Proposition A.1.9. *For all $\xi < \zeta \in \operatorname{Sup}(G)$ there exists $x \in G$ such that $\xi \leq \iota(x) < \zeta$.*

Proof. Let $A, B \subseteq G$ be such that $\sup(A) < \sup(B)$. Then there is some $u \in G$ such that $u < y$ for some $y \in B$, but $x \leq u$ for all $x \in A$. It follows that $\sup(A) \leq \iota(u) < \sup(B)$. □

With a slight abuse of notation, we shall identify G with its isomorphic image $\iota(G)$.

Definition A.1.10. Given $b \in K((G^{\leq 0}))^*$, let $\sup(b) := \sup(\operatorname{supp}(G)) \in \operatorname{Sup}(G)$.

A.2. **Coarse multiplicativity.** To capture the "almost" multiplicativity of sup, we use domination from Definition 8.2.1. We remark that those notions are well defined even when G is not divisible. In the proof, we shall use the notations of Subsection 8.2.

Definition A.2.1. Given $\xi, \zeta \in \operatorname{Sup}(G)$, we say that ξ is **coarsely equal** to ζ, denoted by $\xi \sim \zeta$, if for all $x, y \in G$, if $\xi \leq x, y \leq \zeta$ or $\zeta \leq x, y \leq \xi$, then $x - y \prec x$.

Remark A.2.2. For all $\xi \in \operatorname{Sup}(G)^{\leq 0}$, if $\xi \sim 0$, then actually $\xi = 0$. In fact, suppose by contradiction that $\xi \neq 0$. Then there exists $x \in G$ such that $\xi \leq x < 0$. This implies that $x - 0 \prec 0$, so $x = 0$, a contradiction.

Proposition A.2.3. *For all $b, c \in K((G^{\leq 0}))^*$ we have*

- $\sup(b + c) \leq \max\{\sup(b), \sup(c)\}$ *(ultrametric inequality);*
- $\sup(bc) \sim \sup(b) + \sup(c)$ *(coarse multiplicativity).*

Proof. Let $b, c \in K((G^{\leq 0}))^*$. For the sake of notation, let $\xi = \sup(b)$, $\zeta = \sup(c)$ and $\eta = \sup(bc)$. Since $\operatorname{supp}(b+c) \subseteq \operatorname{supp}(b) \cup \operatorname{supp}(c)$, it follows at once that $\eta \leq \max\{\xi, \zeta\}$. Likewise, since $\operatorname{supp}(bc) \subseteq \operatorname{supp}(b) + \operatorname{supp}(c)$, we must have $\eta \leq \xi + \zeta$. Now take any $x, y \in G$ such that $\eta \leq x \leq y \leq \xi + \zeta$. We wish to prove that $x - y \prec x$. Let $\sigma = \operatorname{ord}(x)$.

Recall that M_σ is a ring homomorphism, so $M_\sigma(bc) = M_\sigma(b)M_\sigma(c)$. Since $x \leq \xi, \zeta \leq 0$ we must have $M_\sigma(b), M_\sigma(c) \neq 0$. The support of $M_\sigma(b)$ is contained in the support of b, so $\sup(M_\sigma(b)) \leq \xi$. Conversely, since $M_\sigma(b) \neq 0$, for every $z \in \operatorname{supp}(b)$, there is some $w \in \operatorname{supp}(M_\sigma(b))$ such that $z \leq w$, so $\xi \leq \sup(M_\sigma(b))$, hence $\xi = \sup(M_\sigma(b))$. Likewise, $\zeta = \sup(M_\sigma(c))$, and since $M_\sigma(bc) = M_\sigma(b)M_\sigma(c) \neq 0$, $\eta = \sup(M_\sigma(bc))$.

Therefore, we may directly assume that $b = M_\sigma(b)$, $c = M_\sigma(c)$, or in other words, $b, c \in K((G^{\leq 0}_\sigma)) = Z_\sigma + K_\sigma((H^{\leq 0}_\sigma))$. In particular, $bc = M_\sigma(bc)$ as well. Let $r, s, u \in \mathbb{R}$ be the supremums of respectively b, c, bc in the ring $Z_\sigma + K_\sigma((H^{\leq 0}_\sigma))$. Since H_σ is archimedean, by Proposition 3.5.1 we have $u = r + s$.

On the other hand, for all $z \in \operatorname{supp}(bc)$, $z \leq x$, so $u \leq \pi_\sigma(x)$. Likewise, there are $z \in \operatorname{supp}(b)$, $w \in \operatorname{supp}(c)$ such that $y \leq z + w$, so $\pi_\sigma(y) \leq r + s$. It follows at once that $\pi_\sigma(x - y) = 0$, namely $\operatorname{ord}(x - y) < \sigma = \operatorname{ord}(x)$, namely $x - y \prec x$, as desired. □

Corollary A.2.4 ([Pit02])**.** *The ideal J generated by the monomials t^x for $x \in G^{<0}$ is prime in $K((G^{\leq 0}))$.*

Proof. Let $b, c \in K((G^{\leq 0}))$ be such that $bc \in J$. By definition bc is divisible by some t^x with $x < 0$. In particular, $\sup(bc) < 0$. By Proposition A.2.3 and Remark A.2.2, $\sup(b) < 0$ or $\sup(c) < 0$. Without loss of generality, assume $\sup(b) < 0$. Then there exists $y \in G$ such that $\sup(b) \leq y < 0$. It follows that b is divisible by t^y, so $b \in J$. Therefore, J is prime. □

Remark A.2.5. The function sup can be transformed into an actual valuation by quotienting $\operatorname{Sup}(G)^{\leq 0}$ by $\sim$, and the quotient is naturally an ordered monoid.

Indeed, the equivalence classes of the coarse equality $\sim$ on $\operatorname{Sup}(G)$ are clearly convex. Moreover, for $\xi, \zeta, \eta \in \operatorname{Sup}(G)^{\leq 0}$, if $\zeta \sim \eta$, then $\xi + \zeta \sim \xi + \eta$. To check this, say that $\xi + \zeta \leq x, y \leq \xi + \eta$. Then, after unravelling the definition of sum, $x, y \leq u + w$ for some $u, w \in G$ such that $u \leq \xi$ and $w \leq \eta$. In turn, $\zeta \leq x - u, y - u \leq \eta$, so $x - y \prec x - u \preceq x$.

It follows at once that $\operatorname{Sup}(G)^{\leq 0}_{/\sim}$ is an ordered commutative monoid, and the composition $\sup : K((G^{\leq 0})) \to \operatorname{Sup}(G)^{\leq 0} \to \operatorname{Sup}(G)^{\leq 0}_{/\sim}$ is a valuation.

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