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# Are Short Proofs Narrow? QBF Resolution is *not* so Simple

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The groundbreaking paper 'Short proofs are narrow – resolution made simple' by Ben-Sasson and Wigderson (J. ACM 2001) introduces what is today arguably *the* main technique to obtain resolution lower bounds: to show a lower bound for the width of proofs. Another important measure for resolution is space, and in their fundamental work, Atserias and Dalmau (J. Comput. Syst. Sci. 2008) show that lower bounds for space again can be obtained via lower bounds for width.

In this paper we assess whether similar techniques are effective for resolution calculi for quantified Boolean formulas (QBF). There are a number of different QBF resolution calculi like Q-resolution (the classical extension of propositional resolution to QBF) and the more recent calculi ∀Exp+Res and IR-calc. For these systems a mixed picture emerges. Our main results show that both the relations between size and width as well as between space and width drastically *fail* in Q-resolution, even in its weaker tree-like version. On the other hand, we obtain positive results for the expansion-based resolution systems ∀Exp+Res and IR-calc, however only in the weak tree-like models.

Technically, our negative results rely on showing width lower bounds together with simultaneous upper bounds for size and space. For our positive results we exhibit space and width-preserving simulations between QBF resolution calculi.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*Complexity of proof procedures*

General Terms: Algorithms, Theory

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## 1. INTRODUCTION

The main objective in *proof complexity* is to obtain precise bounds on the size of proofs in various formal systems; and this objective is closely linked to and motivated by foundational questions in computational complexity (Cook's program), first-order logic (separating theories of bounded arithmetic), and SAT solving. In particular, propositional resolution is one of the best studied and most important propositional proof systems, as it forms the backbone of modern SAT solvers based on conflict-driven clause learning (CDCL) [Marques-Silva et al. 2009]. Complexity lower bounds for resolution

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proofs directly translate into lower bounds on the performance of SAT solvers [Sabharwal 2005; Buss 2012].

What is arguably even more important than showing these actual bounds is to develop *general techniques* that can be applied to obtain lower bounds for important proof systems. A number of ingenious techniques have been designed to show lower bounds for the *size of resolution proofs*, among them feasible interpolation [Krajíček 1997], which applies to many further systems. In their pioneering paper Ben-Sasson and Wigderson [2001] showed that resolution size lower bounds can be elegantly obtained by showing lower bounds to the *width* of resolution proofs. Here, the size of a proof denotes the number of its clauses, and the width of a proof is the length of the biggest clause in it. Indeed, the discovery of this relation between width and size of resolution proofs was a milestone in our understanding of resolution, and today many if not most lower bounds for resolution are obtained via the size-width technique.

Another important measure for resolution is *space* [Esteban and Torán 2001], as it corresponds to memory requirements of solvers in the same way as resolution size relates to their running time. Informally, the space complexity for refuting a formula in resolution is the minimum number of clauses that must be kept in memory simultaneously to refute the formula. In their fundamental work Atserias and Dalmau [2008] demonstrated that also space is tightly related to width. Indeed, showing lower bounds for width serves again as the primary method to obtain space lower bounds. Since these discoveries the relations between resolution size, width, and space have been subject to intense research (cf. [Beyersdorff and Kullmann 2014]), and in particular sharp trade-off results between the measures have been obtained (cf. e.g. [Beame et al. 2012; Ben-Sasson and Nordström 2011; Nordström 2013]).

In this paper we initiate the study of width and space in resolution calculi for quantified Boolean formulas (QBF) and address the question whether similar relations between size, width and space as for classical resolution hold for QBF calculi. Quantified Boolean formulas are propositional formulas where each variable is quantified with either an existential or a universal quantifier. Before explaining our results we sketch recent developments in QBF proof complexity.

*QBF proof complexity* is a relatively young field studying proof systems for quantified Boolean logic. As in the propositional case, one of the main motivations for the field comes via its intimate connection to solving. Although QBF solving is at an earlier state than SAT solving, it offers great potential. Due to its PSPACE completeness, QBF allows for more succinct encodings and therefore QBF solving applies to further fields such as formal verification or planning [Rintanen 2007; Benedetti and Mangassarian 2008; Egly et al. 2017]. Each successful run of a solver on an unsatisfiable instance can be interpreted as a proof of unsatisfiability; and this connection turns proof complexity into the main theoretical tool to understand the performance of solving. As in SAT, many QBF solvers implement decision procedures that have resolution (and its variants) as their underlying proof system.

However, compared to SAT, the QBF picture is more complex as there exist two main solving approaches: (1) utilising ideas from conflict-driven clause learning (CDCL), e.g. in the QBF solver DepQBF [Lonsing and Biere 2010; Lonsing and Egly 2017], and (2) using expansion of universal variables, e.g. in the QBF solver RAReQS [Janota et al. 2016]. To model the strength of these QBF solvers, a number of resolution-based QBF proof systems have been developed. Q-resolution (Q-Res) by Kleine Büning, Karpinski, and Flögel [1995] forms the core of the CDCL-based systems. To capture further ideas from CDCL solving, Q-Res has been augmented to long-distance resolution [Zhang and Malik 2002; Balabanov and Jiang 2012], universal resolution [Van Gelder 2012], and their combinations [Balabanov et al. 2014]. Powerful proof systems for expansion-based solving were recently developed in the form of ∀Exp+Res [Janota and Marques-Silva 2015], and the stronger IR-calc and IRM-calc [Beyersdorff et al. 2014].

In this paper we concentrate on the three QBF resolution systems Q-Res, ∀Exp+Res, and IR-calc. This choice is motivated by the fact that Q-Res and ∀Exp+Res form the base systems for CDCL and expansion-based solving, respectively, and IR-calc unifies both approaches in a natural way, as it simulates both Q-Res and ∀Exp+Res [Beyersdorff et al. 2014]. Recent findings show that CDCL and expansion are indeed orthogonal paradigms as Q-Res and ∀Exp+Res are incomparable with respect to simulations [Beyersdorff et al. 2015].

Understanding which lower bound techniques are effective in QBF proof complexity is of paramount importance for progress in the field. By Beyersdorff et al. [2017] it was shown that the feasible interpolation technique of Krajíček [1997], transferring (monotone) circuit size lower bounds to proof size lower bounds, applies to all QBF resolution systems. Another successful transfer of a classical technique was obtained by Beyersdorff et al. [2017] for a game-theoretic characterisation of proof size in tree-like Q-Res.

### Our contributions

The central question we address here is whether *lower bound techniques via width*, which have revolutionised classical proof complexity, are also effective for QBF resolution systems.

Though space and width have not been considered in QBF before, these notions straightforwardly apply to QBF resolution systems. However, due to the ∀-reduction rule in Q-Res allowing removal of universal variables from clauses (under certain side conditions), it is relatively easy to enforce that universal literals accumulate in clauses of Q-Res proofs, thus always leading to large width, irrespective of size and space requirements (Lemma 3.6). This prompts us to consider *existential width* — counting only existential literals — as an appropriate width measure in QBF. This definition aligns both with Q-Res, which only resolves on existential variables, as well as with ∀Exp+Res and IR-calc, which like all expansion systems only operate on existential literals.

**1. Negative results.** Our main results show that the size-width relation of Ben-Sasson and Wigderson [2001] as well as the space-width relation of Atserias and Dalmau [2008] dramatically *fail* for Q-Res in the sense that there exist formulas requiring maximal (linear) width, but allowing for proofs of minimal (polynomial) size and minimal (constant) space. This even holds when considering the tighter existential width.

We first notice that the proof establishing the size-width result of Ben-Sasson and Wigderson [2001] almost fully goes through, except for some very inconspicuous step that fails in QBF (Proposition 4.1). But not only the technique fails: we prove that Tseitin transformations[1] of formulas expressing a natural completion principle[2] of Janota and Marques-Silva [2015] have small size and space, but require large existential width in tree-like Q-Res (Theorem 4.2), thus refuting the size-width relation for tree-like Q-Res as well as the space-width relation for general dag-like Q-Res.

As the number of variables in the formulas for the completion principle is quadratic in their refutation width, these formulas do not rule out size-width relations in general Q-Res. However, we show that a different set of formulas, hard for tree-like Q-Res [Janota and Marques-Silva 2015], provide counterexamples for size-width relations in full Q-Res (Theorem 4.9).

[1]Tseitin transformations are a standard technique to transform arbitrary propositional formulas into 3-CNFs by using additional variables. Here we use that fact that they produce constant-width formulas.
[2]The completion principle expresses a simple game between two players on a matrix, cf. Section 4.

Technically, our main contributions are width lower bounds for the above formulas, which we show by careful counting arguments. We complement these results by existential width lower bounds for parity-formulas of Beyersdorff et al. [2015], providing an optimal width separation between Q-Res and ∀Exp+Res (Theorem 5.6).

**2. Positive results and width-space-preserving simulations.** Though the negative picture above prevails, we prove some positive results for size-width-space relations for tree-like versions of the expansion resolution systems ∀Exp+Res and IR-calc. Proofs in ∀Exp+Res can be decomposed into two clearly separated parts: an expansion phase followed by a classical resolution phase. This makes it easy to transfer almost the full spectrum of the classical relations to ∀Exp+Res (Theorem 6.1).

To lift these results to IR-calc (Theorem 6.2), we show a series of careful space and width-preserving simulations between tree-like Q-Res, ∀Exp+Res, and IR-calc. In particular, we show the surprising result that tree-like ∀Exp+Res and tree-like IR-calc are polynomially equivalent (Lemma 5.3), thus providing a rare example of two proof systems that coincide in the tree-like, but are separated in the dag-like model [Beyersdorff et al. 2015]. The only other such example that we are aware of is regular resolution vs. full resolution (although this is perhaps slightly less natural as regular resolution is just a sub-system of resolution). In addition, our simulations provide a simpler proof for the simulation of tree-like Q-Res by ∀Exp+Res (Corollary 5.5), shown by Janota and Marques-Silva [2015] via a substantially more involved argument.

Our last positive result is a size-space relation in tree-like Q-Res (Theorem 6.2), which we show by a pebbling game analogous to the classical relation by Esteban and Torán [2001]. Not surprisingly, this only positive result for Q-Res avoids any reference to the notion of width.

We highlight that throughout this article we deal with QBF resolution systems that can only resolve on existential variables, a restriction that is crucial for some of our results. This condition holds for the base systems Q-Res and ∀Exp+Res as well as the stronger system IR-calc. To clarify the size-width relation for QBF resolution systems like QU-Res of Van Gelder [2012], which allow resolution steps on universal variables, remains an open problem (cf. also the discussion in Section 7).

As the bottom line we can say that QBF proof complexity is not just a replication of classical proof complexity: it shows quite different and interesting effects as we demonstrate here. Especially for lower bounds it requires new ideas and techniques. We remark that in this direction, a new and 'genuine QBF technique' based on strategy extraction was recently developed, showing lower bounds for Q-Res [Beyersdorff et al. 2015] and indeed much stronger systems [Beyersdorff et al. 2016; Beyersdorff and Pich 2016].

### Organisation of the paper

The remainder of this paper is organised as follows. We start by reviewing background information on classical and QBF resolution systems (Section 2), including definitions of size, space, and width together with their main classical relations (Section 3). In Section 4 we prove our main negative results on the failure of the transfer of the classical size-width and space-width results to QBF. Section 5 contains the simulations between tree-like versions of Q-Res, ∀Exp+Res, and IR-calc, paying special attention to width and space. This enables us to show in Section 6 the positive results for relations between size, width, and space in these systems. We conclude in Section 7 with a discussion and directions for future research.

## 2. NOTATIONS AND PRELIMINARIES

We assume familiarity with basic notions from logic, including propositional and quantified Boolean logic. We just review those concepts here that are subsequently needed, also setting the notation for later sections. For background information and a rigorous syntactic and semantic definition of the logics we refer to the monograph of Kleine Büning and Lettmann [1999].

*Quantified Boolean Formulas.* A literal is a Boolean variable or its negation. We say a literal $x$ is complementary to the literal $\neg x$ and vice versa. A *clause* is a disjunction ($\vee$) of literals and a *term* is a conjunction ($\wedge$) of literals. The empty clause is denoted by $\square$, and is semantically equivalent to false. A propositional formula in *conjunctive normal form* (CNF) is a conjunction of clauses. For a literal $l = x$ or $l = \neg x$, we write $\mathrm{var}(l)$ for $x$ and extend this notation to the set $\mathrm{var}(C)$ of variables of a clause $C$.

A partial assignment $\alpha$ for a set of variables $X$ is a partial function $\alpha : X \to \{0, 1\}$. We say that a variable $x$ is assigned a value in $\alpha$ if $x$ is in the domain of $\alpha$, denoted $x \in \mathsf{dom}(\alpha)$. We denote an assignment $b \in \{0, 1\}$ to a single variable $x$ by the notation $x/b$. A partial assignment $\alpha$ is specified as a set of such singleton assignments, eg $\{x_1/0, x_3/1\}$.

Let $\alpha$ be any partial assignment. For a clause $C$, we write $C|_\alpha$ for the clause obtained by applying the partial assignment $\alpha$ to $C$. That is, we remove literals falisfied by $\alpha$ from $C$, and further, if some literal of $C$ is true under $\alpha$, then $C_\alpha$ is the tautological clause 1. For example, applying $\alpha = \{x_1/0\}$ to the clause $C = (x_1 \vee x_2 \vee x_3)$ yields $C|_\alpha = (x_2 \vee x_3)$, and applying $\alpha' = \{x_1/1\}$ to the same clause gives $C|_{\alpha'} = 1$. We say that a partial assignment $\alpha$ satisfies a clause $C$ if $C|_\alpha = 1$, and it satisfies a CNF formula $F$ if it satisfies each of the clauses of $F$.

Let $A, B$ be propositional formulas. We say that $A \models B$ holds, if any (partial) assignment which satisfies $A$ also satisfies $B$. Let $F$ be a CNF formula, and $x$ be a variable in $F$. Then $F|_{x/1}$ is a CNF formula obtained from $F$ by removing all clauses containing the literal $x$, and removing all occurrences of the literal $\neg x$. The CNF formula $F|_{x/0}$ is similarly defined.

We consider quantified Boolean Formulas (QBFs) in *closed prenex form* with a CNF matrix[3], i.e., we consider the form $\mathcal{Q}_1 x_1 \cdots \mathcal{Q}_n x_n \,.\, \phi$ where each $\mathcal{Q}_i$ is either $\exists$ or $\forall$, and $\phi$ is a quantifier-free CNF formula in the variables $x_1, \ldots, x_n$. Such formulas are succinctly denoted as $\mathcal{Q}\,\phi$, where $\phi$ is called the *matrix*, and $\mathcal{Q}$ is its *quantifier prefix*.

Given a variable $y$, either existentially quantified or universally quantified in $\mathcal{Q}\,\phi$, the *quantification level* of $y$ in $\mathcal{Q}\,\phi$, $\mathrm{lv}(y)$, is the number of alternations of quantifiers $y$ has on its left in the quantifier prefix of $\mathcal{Q}\,\phi$. Given a variable $y$, we will sometimes refer to the variables with quantification level lower than $\mathrm{lv}(y)$ as variables *left* of $y$; analogously the variables with quantification lever higher than $\mathrm{lv}(y)$ will be *right* of $y$.

The semantics of QBFs can be defined via a 2-player game between a universal and an existential player (cf. e.g. [Arora and Barak 2009]) or via an inductive truth definition, using that $\forall x.F$ is equivalent to $F|_{x/0} \wedge F|_{x/1}$ and $\exists x.F$ to $F|_{x/0} \vee F|_{x/1}$ (cf. [Kleine Büning and Lettmann 1999]).

### Resolution Calculi

*Resolution* (Res), introduced by Blake [1937] and Robinson [1965], is a refutational proof system for formulas in CNF. The lines in the Res proofs are clauses. Given a CNF formula $F$, Res can infer new clauses according to the resolution inference rule:

[3]Any QBF can be efficiently (in polynomial time) converted to an equivalent QBF in this form. See for instance [Arora and Barak 2009]

$$\frac{C \vee x \qquad D \vee \neg x}{C \vee D} \text{ (Res)}.$$

Here $C, D$ denote clauses and $x$ is a variable being resolved, called the *pivot* variable. The clauses $C \vee x$ and $D \vee \neg x$ are referred to as the hypotheses and $C \vee D$ is the conclusion (resolvent) of the resolution rule.

Let $F$ be an unsatisfiable CNF formula. A resolution proof (refutation) $\pi$ of $F$ is a sequence of clauses $D_1, \ldots, D_l$, where $D_l = \Box$, and each clause in the sequence is either from $F$ or is derived from some previous clauses of the sequence via the above resolution rule.

We say that a directed acyclic graph (dag) $G = (V, E)$ represents the refutation $\pi$ if $V = \{D_1, \ldots, D_l\}$, the source nodes are the clauses from $F$, internal nodes are the derived clauses, and the empty node $D_l$ is the unique sink. Furthermore, edges in $G$ are from the hypotheses to the conclusion for each resolution step. That is, each derived clause $D_i$ has incoming edges from $D_j$ and $D_k$ where the indices $j, k$ are less than $i$, and $D_i$ is the resolvent of $D_j$ and $D_k$. (Since a clause could be derived from more than one set of previous premises, there could be more than one graph representing $\pi$. Similarly, such a graph $G$ represents not just $\pi$, but any sequence corresponding to a topological sort of the nodes of $G$.) If there is a tree representing $\pi$, we call $\pi$ a tree-like resolution proof ($\mathsf{Res}_\mathsf{T}$) of $F$. In other words, in a tree-like resolution proof one cannot reuse the derived clauses. We call $\pi$ a regular resolution proof if in some representation $G$, on each directed path in $G$ no variable appears twice as a pivot variable. In what follows, we will refer to any graph $G$ representing $\pi$ (and having the desired property of being a tree, or not reusing pivots along a path, in the case of tree-like and regular proofs respectively) as the graph $G_\pi$ corresponding to $\pi$. This is a slight abuse of notation, but the intended meaning should be clear from the context.

*QBF resolution calculi. Q-resolution* (Q-Res) by Kleine Büning et al. [1995] is a resolution-like calculus that operates on QBFs in closed prenex form where the matrix is a CNF. The lines in Q-Res proofs are clauses. It uses the *resolution rule* (Res) with the side condition that the pivot variable is existential and provided that the resolvent clause is not a tautology. That is, from $C \vee x$ and $D \vee \neg x$, it can infer $C \vee D$ provided $x$ is an existential variable and there is no literal $\ell \in C$ whose negation $\neg \ell$ is in $D$.

In addition Q-Res has a *universal reduction rule* ($\forall$-Red) which allows dropping a universal variable literal from a clause provided the clause has no existential variable to the right of the reduced variable. Note that we also forbid tautological clauses in the input. This is to ensure the soundness of the system. For example, consider the true formula $\forall x.\ (x \vee \neg x)$. The $\forall$-Red rule on the formula derives the empty clause, which is unsound. The inference rules of Q-Res are given in Figure 1.

Similar to tree-like resolution we have tree-like Q-Res (denoted $\mathsf{Q\text{-}Res}_\mathsf{T}$). To be precise, if the underlying proof graph of a Q-Res proof is a tree (that is, no derived clause is used more than once), then we have a $\mathsf{Q\text{-}Res}_\mathsf{T}$ proof.

In addition to Q-Res we consider two further QBF resolution calculi that have been introduced to model *expansion-based QBF solving*. The basic idea used in expansion-based QBF solving is to first expand the universal variables and then apply resolution. For example, consider the QBF $\exists x \forall y \exists z. \phi(x, y, z)$. We can expand the universal variable $y$ and get $\exists x.(\exists z.\phi(x, 0, z)) \wedge (\exists z.\phi(x, 1, z))$. Observe that $z$ may depend on the universal variable $y$. Therefore while converting this to prenex form, we need two distinct copies of $z$. Doing so yields an equivalent formula $\exists x \exists z^{y/0} \exists z^{y/1}.\ \phi(x, 0, z^{y/0}) \wedge \phi(x, 1, z^{y/1})$. Here $z^{y/0}$ and $z^{y/1}$ are two fresh copies of $z$, which have been annotated by the reason for their creation. Syntactically, $z^{y/0}$ and $z^{y/1}$ are just new, distinct existential variables.

| | |
|---|---|
| $\overline{\phantom{x}C\phantom{x}}$ (Axiom) | $C$ is a clause in the input matrix. |
| $\dfrac{C_1 \cup \{x\} \qquad C_2 \cup \{\neg x\}}{C_1 \cup C_2}$ (Res) | Variable $x$ is existential. If $z \in C_1$, then $\neg z \notin C_2$. |
| $\dfrac{C \cup \{u\}}{C}$ ($\forall$-Red) | $u$ is a universal literal. If $x \in C$ is existential, then $\mathrm{lv}(x) < \mathrm{lv}(u)$. |

Fig. 1. The rules of Q-Res [Kleine Büning et al. 1995]

Inspired by the above idea, two calculi based on *instantiation* of universal variables were introduced: ∀Exp+Res by Janota and Marques-Silva [2015] and IR-calc by Beyersdorff et al. [2014]. Both calculi operate on clauses that comprise of only existential variables from the original QBF, which are additionally *annotated* by a substitution to some universal variables, e.g. $\neg x^{u_1/0, u_2/1}$. For any annotated literal $l^\sigma$, the substitution $\sigma$ must not make assignments to variables at a higher quantification level than $l$, i.e. if $u \in \mathsf{dom}(\sigma)$, then $u$ is universal and $\mathrm{lv}(u) < \mathrm{lv}(l)$. To preserve this invariant, we use the *auxiliary notation* $l^{[\sigma]}$, which for an existential literal $l$ and an assignment $\sigma$ to the universal variables filters out all assignments that are not permitted, i.e.

$$l^{[\sigma]} = l^{\{u/c \in \sigma \mid \mathrm{lv}(u) < \mathrm{lv}(l),\ c \in \{0,1\}\}}.$$

We say that an assignment is complete if its domain is the set of all universal variables. Likewise, we say that a literal $x^\tau$ is fully annotated if all universal variables $u$ with $\mathrm{lv}(u) < \mathrm{lv}(x)$ in the QBF are in $\mathsf{dom}(\tau)$, and a clause is fully annotated if all its literals are fully annotated.

The calculus ∀Exp+Res of Janota and Marques-Silva [2015] works with fully annotated clauses on which resolution is performed. This requires, apart from resolution, an *axiom download* rule that specifies, for an axiom clause $C$, what annotated clause can be used in the proof. The rules of ∀Exp+Res are shown in Figure 2.

$$\overline{\left\{ l^{[\tau]} \mid l \in C, l \text{ existential} \right\}} \text{ (Axiom)}$$

$C$ is a clause from the input matrix and $\tau$ is an assignment to all universal variables that falsifies all universal literals in $C$.

$$\frac{C_1 \vee x^\tau \qquad C_2 \vee \neg x^\tau}{C_1 \vee C_2} \text{ (Res)}$$

Fig. 2. The rules of ∀Exp+Res [Janota and Marques-Silva 2015]

We illustrate the axiom download step in ∀Exp+Res with an example: consider a QBF with the quantifier prefix $\exists e_1 \forall u_1 \exists e_2 \forall u_2 \exists e_3 \forall u_3$ and containing the clause $C = (e_1 \vee \neg e_2 \vee u_1 \vee e_3 \vee \neg u_3)$. Let $\tau = \{u_1/0, u_2/1, u_3/1\}$. Note that $\tau$ is an assignment to all universal variables, which falsifies all universal literals in $C$. Then in ∀Exp+Res the clause $(e_1 \vee \neg e_2^{u_1/0} \vee e_3^{u_1/0, u_2/1})$ can be downloaded from $C$ with respect to $\tau$. Likewise, under a different assignment we could download the clause as $(e_1 \vee \neg e_2^{u_1/0} \vee e_3^{u_1/0, u_2/0})$.

The resolution rule (Res) of ∀Exp+Res is just the propositional resolution rule. However, the pivot annotations need to match exactly. This makes sense, as different annotations syntactically lead to different variables.

In comparison to ∀Exp+Res, the system IR-calc by Beyersdorff et al. [2014] is more flexible. It uses 'delayed' expansion and can mix instantiation with resolution steps. Formally, IR-calc works with partial assignments on which we use auxiliary operations of *completion* and *instantiation*. For assignments $\tau$ and $\mu$, we write $\tau \circ \mu$ for the assignment $\sigma$ defined as $\sigma(x) = \tau(x)$ if $x \in \mathsf{dom}(\tau)$, otherwise $\sigma(x) = \mu(x)$ if $x \in \mathsf{dom}(\mu)$. The operation $\tau \circ \mu$ is called *completion* as $\mu$ provides values for variables not defined in $\tau$. For an assignment $\tau$ and an annotated clause $C$, the function $\mathsf{inst}(\tau, C)$ returns the annotated clause $\left\{l^{[\sigma\circ\tau]} \mid l^{\sigma} \in C\right\}$. The system IR-calc uses the rules depicted in Figure 3.

$$\frac{}{\left\{x^{[\tau]} \mid x \in C, x \text{ is existential}\right\}}\ \text{(Axiom)}$$

$C$ is a non-tautological clause from the input matrix.
$\tau = \{u/0 \mid u \text{ is universal in } C\}$, where the notation $u/0$ for literals $u$ is shorthand for $x/0$ if $u = x$ and $x/1$ if $u = \neg x$.

$$\frac{C_1 \vee x^{\tau} \qquad C_2 \vee \neg x^{\tau}}{C_1 \vee C_2}\ \text{(Res)} \qquad\qquad \frac{C}{\mathsf{inst}(\tau, C)}\ \text{(Instantiation)}$$

$\tau$ is a (partial) assignment to universal variables.

Fig. 3. The rules of IR-calc [Beyersdorff et al. 2014]

Unlike ∀Exp+Res, in an axiom download step in IR-calc the assignment $\tau$ sets values to all universal variables in the clause being downloaded, but not to other universal variables. For example, consider the same QBF quantifier prefix and clause $C$ described above while discussing ∀Exp+Res. For $\tau = \{u_1/0, u_3/1\}$, IR-calc downloads the following clause: $(e_1 \vee \neg e_2^{u_1/0} \vee e_3^{u_1/0})$. Note that the universal variable $u_2$ does not belong to the domain of $\tau$, but $\tau$ falsifies all universal variables in $C$.

The resolution rule in IR-calc is exactly as in ∀Exp+Res. Again, pivot annotations need to match in both parent clauses.

To enable further resolution steps, the system IR-calc allows to extend the annotations in the instantiation rule, which uses the function inst discussed above. For instance, in the preceding example, $(e_1 \vee \neg e_2^{u_1/0} \vee e_3^{u_1/0})$ can be further instantiated by $\tau = \{u_2/0\}$ to $(e_1 \vee \neg e_2^{u_1/0} \vee e_3^{u_1/0, u_2/0})$.

*Simulations.* Given two proof systems $P$ and $Q$ for the same language (the set of propositional tautologies TAUT, or the set of true quantified Boolean formulas QBF), $P$ *p-simulates* $Q$ (denoted $Q \leq_p P$) if each $Q$-proof can be transformed in polynomial time into a $P$-proof of the same formula. Two systems are called *p-equivalent* if they p-simulate each other.

Beyersdorff et al. [2014] have shown that IR-calc p-simulates both Q-Res and ∀Exp+Res, while Beyersdorff et al. [2015] show that Q-Res and ∀Exp+Res are incomparable, i.e., IR-calc is exponentially stronger than both Q-Res and ∀Exp+Res. However, ∀Exp+Res can p-simulate Q-Res$_{\mathsf{T}}$ [Janota and Marques-Silva 2015].

## 3. SIZE, WIDTH, AND SPACE IN RESOLUTION CALCULI

The purpose of the section is twofold: first to review the measures size, width, and space and their relations in classical resolution; and second to explain how to apply these measures to QBF resolution systems. While this is straightforward for size and space, we need a more elaborate discussion on what constitutes a good notion of width for QBF resolution systems.

### 3.1. Defining size, width, and space for resolution

For a CNF $F$, $|F|$ denotes the number of clauses in it. We extend the same notation to QBFs with a CNF matrix.

For $P$ one of the resolution calculi Res, Q-Res, ∀Exp+Res, IR-calc, let $\pi \vdash_{\mathsf{P}} F$ (resp. $\pi \vdash_{\mathsf{P_T}} F$) denote that $\pi$ is a $P$-proof (tree-like $P$-proof, respectively) of the formula $F$. For a proof $\pi$ of $F$ in system $P$, its size $|\pi|$ is defined as the number of clauses in $\pi$. The **size** complexity $S(\vdash_{\mathsf{P}} F)$ of deriving $F$ in $P$ is defined as $\min\{|\pi| \ : \ \pi \vdash_{\mathsf{P}} F\}$. The tree-like size complexity, denoted $S(\vdash_{\mathsf{P_T}} F)$, is $\min\{|\pi| \ : \ \pi \vdash_{\mathsf{P_T}} F\}$.

A second complexity measure is the minimal **width**. The width of a clause $C$ is the number of literals in $C$, denoted $w(C)$. The width of a CNF $F$, denoted $w(F)$, is the maximum width of a clause in $F$, i.e., $w(F) = \max\{w(C) \ : \ C \in F\}$. The width $w(\pi)$ of a proof $\pi$ is defined as the maximum width of any clause appearing in $\pi$, i.e., $w(\pi) = \max\{w(C) \ : \ C \in \pi\}$. The width $w(\vdash_{\mathsf{P}} F)$ of refuting a CNF $F$ in $P$ is defined as $\min\{w(\pi) \ : \ \pi \vdash_{\mathsf{P}} F\}$. Again the same notation extends to quantified CNFs.

Note that for width in any calculus, whether the proof is tree-like or not is immaterial, since a proof can always be made tree-like by duplication without increasing the width. We therefore drop the T subscript when talking about proof width.

The third complexity measure for resolution calculi is **space**. For classical resolution, this measure was first defined by Esteban and Torán [2001]. In the literature, it is also called clause space, to distinguish it from variable space or total space (see for example, [Ben-Sasson 2002]). We consider only clause space in this paper, and so we call it just space. Informally, space is the minimal number of clauses that must be kept simultaneously in memory to refute a formula. Instead of viewing a proof $\pi$ as a dag, we view it as a sequence $\sigma$ of CNF formulas $\sigma = F_0, F_1, \ldots, F_s$, where $F_0 = \emptyset$, $\Box \in F_s$, and each $F_{i+1}$ is obtained from $F_i$ by either erasing some clause, or by downloading an axiom, or by adding a resolvent of clauses in $F_i$. In the latter case, one of the clauses used in the resolution may also simultaneously be deleted. The space used by this proof is the maximum number of clauses in any $F_i$, i.e., $\mathit{CSpace}(\sigma) = \max\{|F_i| \mid i \in [s]\}$. A straightforward way of representing a proof $\pi = D_1, \ldots, D_l$ in this way is to set $F_i = \{D_j \mid j \leq i\}$; this proof will have space $l$. But there could be other ways of representing $\pi$ that are more economical in space.

The space used by a proof is precisely the number of pebbles required to pebble the proof dag (cf. also the survey by Nordström [2013]), and we here use the pebbling number as the formal definition of the space used by the proof. We first define the pebbling game on graphs.

*Definition* 3.1. (Pebbling Game) Let $G = (V, E)$ be a connected directed acyclic graph with a unique sink $s$, where every vertex of $G$ has at most 2 incoming edges. The aim of the game is to put a pebble on the sink of the graph following this set of rules:

(1) A pebble can be placed on any source vertex, that is, on a vertex with no incoming edge.
(2) A pebble can be removed from any vertex.

(3) A pebble can be placed on an internal vertex provided all vertices with an incoming edge to it are pebbled. In this case, instead of placing a new pebble on it, one can shift a pebble along an incoming edge to the vertex.

The minimum number of pebbles needed to pebble the unique sink following the above rules is said to be the *pebbling number* of $G$.

Consider the proof graph $G_\pi$ corresponding to a Q-Res proof $\pi$ of a false QBF $\mathcal{F}$. In $G_\pi$ clauses are the vertices and edges go from the hypotheses to the conclusion of inference rules (i.e., $\forall$-Red, resolution steps). Clearly $G_\pi$ is a dag with initial clauses as sources and the empty clause as the unique sink. Also each vertex in $G_\pi$ is at most 2 incoming edges. Hence the pebbling game is well defined on $G_\pi$.

We now define the space required to refute a false QBF $\mathcal{F}$ as the minimum number of pebbles needed to play the pebble game on the graph of a Q-Res proof of $\mathcal{F}$.

*Definition* 3.2. (Space in Q-Res) For a false QBF $\mathcal{F}$ in prenex form we set

$$CSpace(\vert\!\overline{\phantom{x}}_{\textsf{Q-Res}}\, \mathcal{F}) = \min\left\{k : \exists \text{ Q-Res proof } \pi \text{ of } \mathcal{F},\ G_\pi \text{ can be pebbled with } k \text{ pebbles}\right\}.$$

The analogous definition is used for tree-like proofs:

$$CSpace(\vert\!\overline{\phantom{x}}_{\textsf{Q-Res}_T}\, \mathcal{F}) = \min\{k : \exists \text{ Q-Res}_T \text{ proof } \pi \text{ of } \mathcal{F},\ G_\pi \text{ can be pebbled with } k \text{ pebbles}\}.$$

### 3.2. Relations between size, width, and space in classical resolution

We now state some of the main relations between size, width, and space for classical resolution. We start with the foundational size-width relations of Ben-Sasson and Wigderson [2001].

THEOREM 3.3 (BEN-SASSON AND WIGDERSON [2001]). *For all unsatisfiable CNFs $F$ in $n$ variables the following holds:*

$$S(\vert\!\overline{\phantom{x}}_{\textit{Res}_T}\, F) \ \geq\ 2^{w\left(\vert\overline{\phantom{x}}_{\textit{Res}}\, F\right) - w(F)}, \quad \textit{and}$$
$$S(\vert\!\overline{\phantom{x}}_{\textit{Res}}\, F) \ =\ \exp\left(\Omega\left(\frac{\left(w\left(\vert\overline{\phantom{x}}_{\textit{Res}}\, F\right) - w(F)\right)^2}{n}\right)\right).$$

Space complexity was introduced by Esteban and Torán [2001] and relations between space, size and width are explored (cf. also [Kullmann 1999; Beyersdorff and Kullmann 2014]), establishing the size-space relation for tree-like resolution:

THEOREM 3.4 (ESTEBAN AND TORÁN [2001]). *For all unsatisfiable CNFs $F$ the following relation holds:* $S(\vert\!\overline{\phantom{x}}_{\textit{Res}_T}\, F) \geq 2^{CSpace\left(\vert\overline{\phantom{x}}_{\textit{Res}_T}\, F\right)} - 1$.

The fundamental relation between space and width for full resolution was obtained by Atserias and Dalmau [2008].

THEOREM 3.5 (ATSERIAS AND DALMAU [2008]). *For all unsatisfiable CNFs $F$ the following relation holds:* $w(\vert\!\overline{\phantom{x}}_{\textit{Res}}\, F) \leq CSpace(\vert\!\overline{\phantom{x}}_{\textit{Res}}\, F) + w(F) - 1$.

A more direct proof was given recently by Filmus et al. [2015] and shows that $w(\vert\!\overline{\phantom{x}}_{\textsf{Res}}\, F) \leq CSpace(\vert\!\overline{\phantom{x}}_{\textsf{Res}}\, F) + w(F) - 3$.

### 3.3. Existential width: What is the right width notion for QBF?

We wish to explore the possibility of a similar approach as used by Ben-Sasson and Wigderson [2001] to prove an analogue of Theorem 3.3 when dealing with QBFs.

The following simple example shows that the relationships in Theorem 3.3 and Theorem 3.5 do not carry over for the system Q-Res. For $n \in \mathbb{N}$, let $[n]$ denote $\{1, 2, \ldots, n\}$.

Consider the following false QBF $\mathcal{F}_n$ over $2n + 1$ variables:

$$\begin{aligned} \mathcal{F}_n = &\forall u_1 \ldots u_n \exists e_0 \exists e_1 \ldots e_n. \\ C_0: \quad & (e_0) \wedge \\ \text{For } i \in [n], D_i: \quad & (\neg e_{i-1} \vee u_i \vee e_i) \wedge \\ D_{n+1}: \quad & (\neg e_n) \end{aligned}$$

PROPOSITION 3.6. *$S(\vdash_{\text{Q-Res}_T} \mathcal{F}_n) = O(n)$ and $CSpace(\vdash_{\text{Q-Res}_T} \mathcal{F}_n) = O(1)$, but $w(\vdash_{\text{Q-Res}} \mathcal{F}_n) = \Omega(n)$.*

PROOF SKETCH. For the upper bounds consider the following proof. For $i \in [n]$, let $C_i = (u_1 \vee \cdots \vee u_i \vee e_i)$. For $i \in [n]$ in sequence, resolving $C_{i-1}$ and $D_i$ on variable $e_{i-1}$ gives $C_i$. Resolving $C_n$ and $D_{n+1}$ on variable $e_n$ gives the clause $U = (u_1 \vee \cdots \vee u_n)$. Finally, applying $\forall$-Red on the clause $U$ yields the empty clause in $n$ more steps. The proof is depicted in Figure 4.

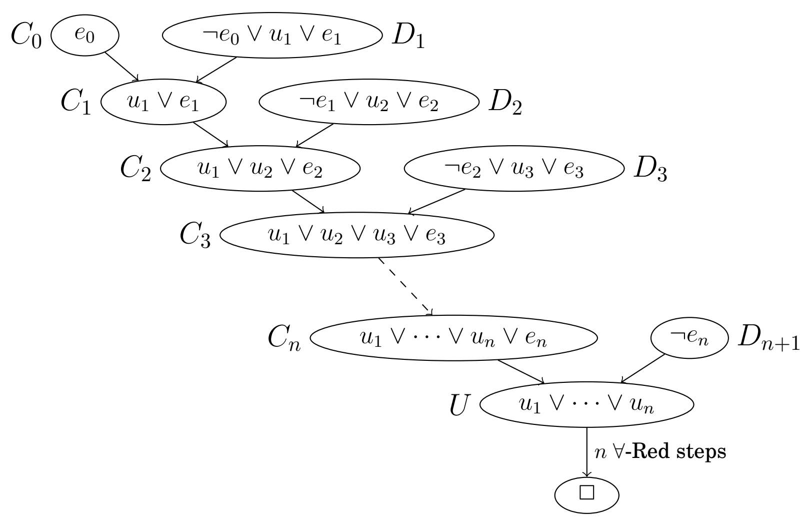



Fig. 4. Proof of Proposition 3.6: A Q-Res$_T$ refutation of the false QBF $\mathcal{F}_n$.

This is a tree-like proof of size $O(n)$. Further, each resolution step involves an axiom clause, so at each step we need to pebble just two clauses, and so the space requirement is $O(1)$.

Concerning the width lower bound, by the order of quantification in $\mathcal{F}_n$, every existential literal in $\mathcal{F}_n$ blocks any $\forall$-reduction. Therefore, in any refutation, when a $\forall$-reduction is first used, the clause $C$ has only universal variables. At this point, the empty clause is derivable from $C$ by a series of $\forall$-reductions. Note that if any clause is dropped from $\mathcal{F}_n$, the resulting QBF is no longer false. Thus any refutation must use all clauses. Hence $C$ must have all universal variables in it; it must be $(u_1 \vee \cdots \vee u_n)$ as all $u_i$ variables have been accumulated, without being reduced. Then clause $C$ has width $n$. $\Box$

Noting that $w(\mathcal{F}_n) = 3$, Proposition 3.6 implies that the relationships from Theorem 3.3 and Theorem 3.5 do not hold for Q-Res and Q-Res$_T$.

As the above example illustrates, it is easy to enforce that universal variables are accumulated in a clause, thus leading to large width. Hence the following question naturally arises: can we obtain size-width or space-width relations by using the tighter measure of only counting existential variables?

This aligns with the situation in the expansion systems ∀Exp+Res and IR-calc, where clauses contain only existential variables. In this respect, it is worth noting that the above example indeed does not demonstrate the failure of the size-width relationship in expansion-based calculi. For instance, in ∀Exp+Res, a tree-like refutation could download the existential variables of axioms annotated with $u_i/0$ for $i \in [n]$, and generate the empty clause in $O(n)$ steps with width just 2 at the leaves and 1 at the internal nodes. More formally, consider the assignment $\tau$ which assigns 0 to all universal variables of $\mathcal{F}_n$. In ∀Exp+Res, we can download the following clauses, with respect to $\tau$:

$$\begin{aligned} C_0^\tau: &\quad (e_0^{u_1/0,\ldots,u_n/0}) \\ \text{For } i \in [n], D_i^\tau: &\quad (\neg e_{i-1}^{u_1/0,\ldots,u_n/0} \vee e_i^{u_1/0,\ldots,u_n/0}) \\ D_{n+1}^\tau: &\quad (\neg e_n^{u_1/0,\ldots,u_n/0}). \end{aligned}$$

Now, the ∀Exp+Res proof of $\mathcal{F}_n$ is straightforward: for $i \in \{0, 1, \ldots, n\}$, let $E_i^\tau$ be the unit clause $(e_i^{u_1/0,\ldots,u_n/0})$. Note that $E_0^\tau$ has been downloaded as $C_0^\tau$. For $i \in [n]$, in sequence, resolve $E_{i-1}^\tau$ and $D_i^\tau$ on variable $e_{i-1}^{u_1/0,\ldots,u_n/0}$ to derive $E_i^\tau$. Finally resolve $E_n^\tau$ and $D_{n+1}^\tau$ on variable $e_n^{u_1/0,\ldots,u_n/0}$ to derive the empty clause. Clearly, the size and width of this proof are $O(n)$ and $O(1)$ respectively.

Thus, to get a consistent and interesting width measure for QBF calculi, we consider the notion of **existential width** that just counts the number of existential literals. This approach is justified also for Q-Res as the calculus can only resolve on existential variables, and rules out the easy counterexamples above. Formally we define it as follows.

*Definition* 3.7. The *existential width* of a clause $C$ is the number of existential literals in $C$; we denote it by $w_\exists(C)$. Using $w_\exists$ instead of $w$, we obtain the existential width of a formula $w_\exists(F)$, of a proof $w_\exists(\pi)$, and of refuting a false QBF $w_\exists(\vdash_{\mathsf{P}} \mathcal{F})$.

For the expansion systems ∀Exp+Res and IR-calc the notions of existential width and width of a proof coincide. (In particular, distinct annotations of the same existential variable in a single clause are counted as distinct literals.) Hence we can drop the $\exists$ subscript in width of proofs in these systems. However, for the width of the input clauses from the QBF under consideration, there is still a difference between the two measures $w$ and $w_\exists$, as the QBF may contain universal literals.

## 4. NEGATIVE RESULTS: SIZE-WIDTH AND SPACE-WIDTH RELATIONS FAIL IN Q-RES

In this section we show that in the Q-Res proof system, even replacing width by existential width, the relations to size or space as in classical resolution (Theorems 3.3 and 3.5) no longer hold for both tree-like and general proofs.

Firstly, we point out where the technique of Ben-Sasson and Wigderson [2001] fails. A crucial ingredient of their proof is the following statement: if a clause $A$ can be derived from $F|_{x/1}$ in width $w$, then the clause $A \vee \neg x$ can be derived from $F$ in width $w + 1$ (possibly using a weakening rule at the end). We show that the statement no longer holds in Q-Res.

PROPOSITION 4.1. *There are false QBFs $F_n$, with an existential variable $b$ quantified at the innermost level, such that the QBF $F_n|_{b/1}$ is false and has a small existential-*

*width proof, but to derive $\neg b$ from $F_n$ requires large existential width in Q-Res. In fact, $F_n$ itself requires large existential width to refute in Q-Res.*

PROOF. The QBF $F_n$ is constructed by taking the conjunction of two QBFs with distinct variables. The first QBF is a very simple one: $\exists a \forall u \exists b.\ (a \vee u \vee \neg b) \wedge (\neg a)$. It is true, but if $b$ is set to 1, it becomes false. The second QBF is a false QBF of the form $\exists \vec{x} G_n(\vec{x})$, where $G_n$ are polynomial-size unsatisfiable CNF formulas over the $\vec{x}$ variables, such that $G_n$ needs large width in classical resolution. One such example is the CNF formula described by Bonet and Galesi [1999], that we denote as $BG_n$. $BG_n$ is an unsatisfiable 3-CNF formula over $O(n^2)$ variables with $w(\vdash_{\mathsf{Res}} BG_n) = \Omega(n)$. Now define $F_n$ as:

$$\exists \vec{x} \exists a \forall u \exists b.\ (a \vee u \vee \neg b) \wedge (\neg a) \wedge BG_n(\vec{x}).$$

Note that the clauses $(a \vee u \vee \neg b) \wedge (\neg a)$ contain a contradiction if and only if $b = 1$. Thus $F_n|_{b/1}$ can be refuted with existential width 1 using just these two clauses: a $\forall$-Red on $(a \vee u)$ yields $a$ which can be resolved with $\neg a$.

Let us now see how we can derive $\neg b$ from $F_n$. From clauses $a \vee u \vee \neg b$ and $\neg a$ we can derive $u \vee \neg b$, but now we cannot $\forall$-reduce $u$ as it is blocked by $b$. Therefore we need to expose the contradiction in $BG_n$, derive the empty clause and then use weakening to obtain $\neg b$. Since all the variables in $BG_n$ are existential, Q-Res degenerates to classical resolution, requiring (existential) width $\Omega(n)$.

Since setting $a = b = 0$ satisfies the first part of the QBF, and since the two parts of the QBF have disjoint variables, the only way to refute $F_n$ is to expose the contradiction in $BG_n$, and as discussed above, this requires (existential) width $\Omega(n)$. □

The example in the proof of Proposition 4.1 can be made 'less degenerate' by interleaving more existential and universal variables disjoint from $\vec{x}$ and putting them in the first QBF. All we need is that $b$ is quantified existentially at the end, the first QBF is true as a whole but false if $b = 1$, and this latter QBF can be refuted in Q-Res with small existential width.

We now show that it is not just the technique of Ben-Sasson and Wigderson [2001] that fails for Q-Res. No other technique will work either, because the relation from Theorem 3.3 between size and existential width itself fails to hold. The same example also shows that the relation from Theorem 3.5 between space and existential width also fails to hold.

We first give an example where the relation for tree-like proofs fails. For this we use formulas $CR_n$ describing a natural completion principle, introduced by Janota and Marques-Silva [2015].[4] The formula $CR_n$ is as follows:

$$\begin{array}{rll} & CR_n = \exists x_{1,1} \dots x_{n,n}\ \forall z\ \exists a_1 \dots a_n \exists b_1 \dots b_n. & \\ C_{i,j}: & (x_{i,j} \vee z \vee a_i), & i,j \in [n] \\ D_{i,j}: & (\neg x_{i,j} \vee \neg z \vee b_j), & i,j \in [n] \\ A: & \bigvee_{i \in [n]} \neg a_i & \\ B: & \bigvee_{i \in [n]} \neg b_i. & \end{array}$$

[4]These formulas are called $CR_n$ in [Janota and Marques-Silva 2015]; we use the same name.

$CR_n$ is constructed from a principle called the *completion principle*. Consider two sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, and depict their cross product $A \times B$ as in Table I.

Table I. Completion Principle.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | $a_1$ | $\dots$ | $a_1$ | $a_2$ | $a_2$ | $\dots$ | $a_2$ | $\dots$ | $\dots$ | $a_n$ | $a_n$ | $\dots$ | $a_n$ |
| $b_1$ | $b_2$ | $\dots$ | $b_n$ | $b_1$ | $b_2$ | $\dots$ | $b_n$ | $\dots$ | $\dots$ | $b_1$ | $b_2$ | $\dots$ | $b_n$ |

The following two-player game is played on Table I. In the first round, player 1 deletes exactly one cell from each column. In the second round, player 2 chooses one of the two rows. Player 2 wins if the chosen row contains either the complete set $A$ or the set $B$; otherwise player 1 wins. The completion principle states that player 2 has a winning strategy. The false QBF $CR_n$ expresses the notion that player 1 has a winning strategy. For each column $\begin{bmatrix} a_i \\ b_j \end{bmatrix}$ of the table (denote this the $(i,j)^{th}$ column), there is a Boolean variable $x_{i,j}$. Let $x_{i,j} = 0$ denote that player 1 'deletes $b_j$' (i.e., keeps $a_i$) from the $(i,j)^{th}$ column, and $x_{i,j} = 1$ denotes that player 1 keeps $b_j$ in the $(i,j)^{th}$ column. There is a variable $z$ to denote the choice of player 2: $z = 0$ means 'choose top row'. The Boolean variables $a_i$, $b_j$, for $i, j \in [n]$ encode that for the chosen values of all the $x_{k,\ell}$, and the row chosen via $z$, at least one copy of the element $a_i$ and $b_j$ respectively is kept. (eg $(x_{i,j} \wedge z) \Rightarrow b_j$).

It is known that $CR_n$ has a proof of size $O(n^2)$ in Q-Res, and even in Q-Res$_\mathsf{T}$ [Mahajan and Shukla 2016]. However, $CR_n$ has large existential width (i.e., $w_\exists(CR_n) = n$), and for our next result we need a formula with constant initial existential width. To achieve this we proceed similarly as in the Tseitin transformations, i.e., we introduce $2n + 2$ new existential variables (i.e., $\vec{y}, \vec{p}$) at the innermost level in $CR_n$, and replace the two large clauses in $CR_n$ by any CNF formula which preserves their satisfiability. Let $CR'_n$ denote the modified formula

$$CR'_n = \exists x_{1,1} \dots x_{n,n}\ \forall z\ \exists a_1 \dots a_n \exists b_1 \dots b_n \exists y_0 \dots y_n \exists p_0 \dots p_n.$$

$$C_{i,j}: \quad (x_{i,j} \vee z \vee a_i), \qquad i, j \in [n] \tag{1}$$

$$D_{i,j}: \quad (\neg x_{i,j} \vee \neg z \vee b_j), \qquad i, j \in [n] \tag{2}$$

$$\neg y_0 \wedge \bigwedge_{i \in [n]} (y_{i-1} \vee \neg a_i \vee \neg y_i) \wedge y_n \tag{3}$$

$$\neg p_0 \wedge \bigwedge_{i \in [n]} (p_{i-1} \vee \neg b_i \vee \neg p_i) \wedge p_n. \tag{4}$$

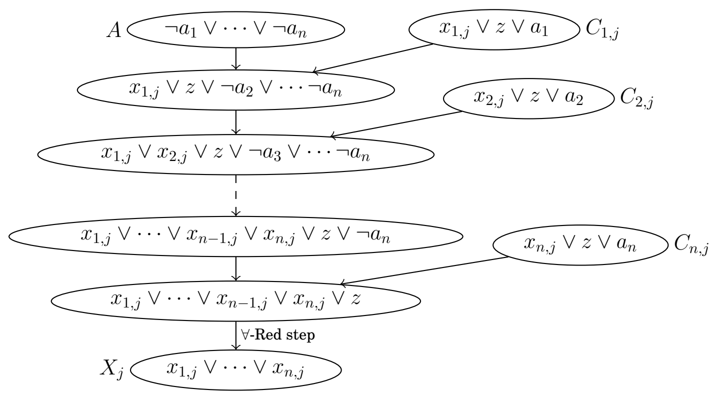
Note that $CR'_n$ has $O(n^2)$ variables and $w_\exists(CR'_n) = 3$.

We can use these formulas to refute the size-width and space-width relations in Q-Res$_\mathsf{T}$.

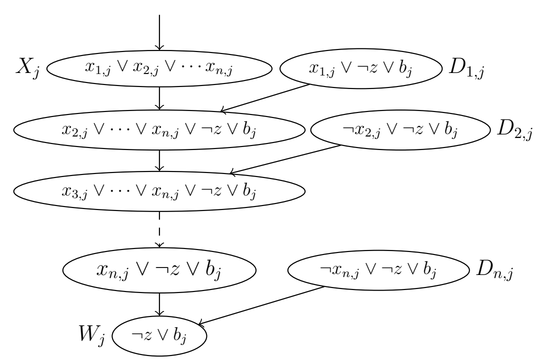
THEOREM 4.2. *For the above family of QBFs $CR'_n$ holds $S(\vdash_{\text{Q-Res}_\mathsf{T}} CR'_n) = n^{O(1)}$, $w_\exists(CR'_n) = 3$, $CSpace(\vdash_{\text{Q-Res}_\mathsf{T}} CR'_n) = O(1)$, and $w_\exists(\vdash_{\text{Q-Res}} CR'_n) \geq n$.*

PROOF. The clauses of $CR'_n$, as described above, are partitioned into $4$ groups. For $i \in [4]$, we call an initial clause $C$ a type-($i$) clause if it belongs to the $i^{\text{th}}$ group. It is clear that from the type-(3) clauses of $CR'_n$, we can derive the large clause $A = \bigvee_{i \in [n]} \neg a_i$ of $CR_n$ in $n+1$ resolution steps. Similarly we can derive the large clause $B = \bigvee_{i \in [n]} \neg b_i$ of $CR_n$ from the type-(4) clauses in $n+1$ steps. The proof refuting $CR_n$ uses each of these large clauses $n$ times; see below. Thus $S(\vdash_{\text{Q-Res}_\mathsf{T}} CR'_n) \leq S(\vdash_{\text{Q-Res}_\mathsf{T}} CR_n) + O(n^2) = O(n^2)$.

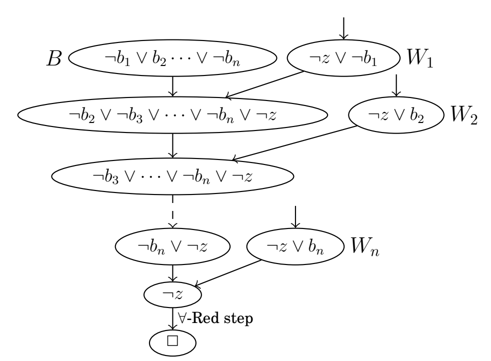
We briefly sketch the refutation of $CR_n$ of Mahajan and Shukla [2016] to analyse its space requirement. The fragment $W_j$ starts with clause $A$, successively resolves it with clauses from $C_{*,j}$ to get $z \vee x_{1,j} \vee \cdots \vee x_{n,j}$, eliminates $z$ through a $\forall$-reduction to get $X_j = (x_{1,j} \vee \cdots \vee x_{n,j})$, then successively resolves $X_j$ with clauses from $D_{*,j}$ to get $W_j = \neg z \vee b_j$. It is easy to see that $O(1)$ space suffices to construct this fragment. The overall proof starts with the clause $B$, successively resolves it with $W_1, W_2, \ldots, W_n$ (reusing the space to construct successive $W_j$'s), and finally gets $\neg z$ which is eliminated through a $\forall$-reduction. Again $O(1)$ space suffices. Refer to Figure 5.




(a) A Q-Res$_\mathsf{T}$ derivation of $X_j$ from axiom clauses $A$ and $C_{*,j}$




(b) A Q-Res$_\mathsf{T}$ derivation of $W_j = (\neg z \vee b_j)$ from the derived clause $X_j$ and axiom clauses $D_{*,j}$




(c) Deriving the empty clause from the derived clauses $W_j$ and the axiom clause $B$

Fig. 5. A Q-Res$_\mathsf{T}$ refutation of $CR_n$ from [Mahajan and Shukla 2016].

Finally, we show that $CR'_n$ needs large existential width to refute, i.e., $w_\exists(\vdash_{\textsf{Q-Res}} CR'_n) \geq n$.

Let $\pi$ be a proof in Q-Res, $\pi \vdash_{\textsf{Q-Res}} CR'_n$. List the clauses of $\pi$ in sequence, $\pi = \{D_0, D_1, \ldots, D_s = \Box\}$, where each clause in the sequence is either a clause from $CR'_n$, or is derived from clause(s) preceding it in the sequence using resolution or $\forall$-Red. There must be at least one universal reduction step in $\pi$, since all the initial clauses are necessary for refuting $CR'_n$, some of them contain universal variables, and the only way to remove a universal variable in Q-Res is by $\forall$-Red. Let $t$ be the least index such that in the clause $D_t$, a $\forall$-Red step has been performed on the only universal variable. Without loss of generality, let the universal literal be the positive literal $z$; the argument for $\neg z$

is identical. As the existential variables $\vec{a}, \vec{b}, \vec{y}$, and $\vec{p}$ all block the universal variable $z$, none of them is present in the clause $D_t$. We use this fact to show that $w_\exists(D_t) \geq n$. Our strategy is to associate some set with each clause in $\pi$ in a specific way, and use the set size to bound existential width. More formally, we associate a set $\sigma$ with each clause in $\pi$, and show that the cardinality of $\sigma$ is large for the clause $D_t$. We further argue that $D_t$ can have a large $\sigma$ set only if its existential width is large.

We associate the following sets with the literals of $CR'_n$ and the clauses of $\pi$.

$$\begin{array}{lrcl}
 & \sigma(z) & = & \emptyset = \sigma(\neg z) \\
\forall i \in [n] & \sigma(a_i) & = & [n] \setminus \{i\} = \{1, \ldots, n\} \setminus \{i\} \\
\forall i \in [n] & \sigma(x_{i,j}) = \sigma(\neg a_i) & = & \{i\} \\
\forall i \in [n] & \sigma(\neg y_i) & = & [n] \setminus [i] = \{i+1, \ldots, n\} \\
\forall i \in [n] & \sigma(y_i) & = & [i] = \{1, \ldots, i\} \\
\forall j \in [n] & \sigma(b_j) & = & [n] \setminus \{j\} = \{1, \ldots, n\} \setminus \{j\} \\
\forall j \in [n] & \sigma(\neg x_{i,j}) = \sigma(\neg b_j) & = & \{j\} \\
\forall j \in [n] & \sigma(\neg p_j) & = & [n] \setminus [j] = \{j+1, \ldots, n\} \\
\forall j \in [n] & \sigma(p_j) & = & [j] = \{1, \ldots, j\} \\
\forall D \in \pi & \sigma(D) & = & \bigcup_{l \in D} \sigma(l).
\end{array}$$

The intuition of defining $\sigma$ in such a way is simple: for all the initial clauses, we want the cardinality of the set $\sigma$ to be large. Observe that for all clauses $C \in CR'_n$, $\sigma(C) = [n]$.

Secondly, we want that as long as no $\forall$-Red step has been used, every resolution step must preserve the cardinality of $\sigma$. Observe that for variables $v$ in $\vec{a}$, $\vec{b}$, $\vec{p}$, $\vec{y}$, the sets $\sigma(v)$ and $\sigma(\neg v)$ form a partition of $[n]$. This helps us in achieving our second goal as follows: for $CR'_n$, we show that any resolution step, before a $\forall$-Red step, must use only one of the variables $\vec{a}$, $\vec{b}$, $\vec{p}$, and $\vec{y}$ as a pivot variable. Since the resolvent clause of a resolution rule contains all the literals from the hypothesis except the literals corresponding to the pivot variables, and the literals corresponding to the pivot variables form a partition of $[n]$, the second goal follows.

Finally, we want to show that the existential width of the clause $D_t$ is large. Observe that we have a singleton set $\sigma$ for the literals $x_{i,j}$, and $\neg x_{i,j}$. We show that the clause $D_t$ contains only the literals corresponding to the $x_{i,j}$ variables (along with the only universal variable being resolved), and since $D_t$ has a large set (this follows from our second goal), it must have many $x_{i,j}$ variables.

For $D \in \pi$, let $\pi_D$ be the sub-dag of $\pi$, rooted at $D$. Consider the sub-dag $\pi_{D_t}$ of $\pi$. We have the following observations:

OBSERVATION 4.3. *$\pi_{D_t}$ contains at least one type-(1) clause as a source; this is because $z \in D_t$, and the only initial clauses containing $z$ are the type-(1) clauses.*

OBSERVATION 4.4. *$\pi_{D_t}$ does not contain any clause of type-(2) : as $z \in D_t$, we know that $\neg z \notin D_t$. Therefore if some type-(2) clause is present in this sub-dag, the only way to remove $\neg z$ is via $\forall$-Red. This reduction will take place before the reduction on $D_t$, contradicting our choice of index $t$. We also conclude that the literal $\neg z$ cannot appear anywhere in $\pi_{D_t}$.*

OBSERVATION 4.5. *$\pi_{D_t}$ does not contain any type-(4) clause: we know that $D_t$ does not contain $\vec{p}$ and $\vec{b}$ variables (because they block $z$). Any use of type-(4) clauses introduces $\vec{p}$ variables and possibly $\neg b$ literals. Removing $\vec{p}$ variables introduces $\neg b$ literals. But $\neg b$ can be removed only by resolving with $b$, which is only in type-(2) clauses. We have already seen that type-(2) clauses are not present in $\pi_{D_t}$.*

OBSERVATION 4.6. *No clause in $\pi_{D_t}$ contains a literal $\neg x_{i,j}$, since $\neg x_{i,j}$ are introduced only in type-(2) clauses which were already ruled out.*

OBSERVATION 4.7. *For any clause $C$ derived solely from type-(3) clauses, $\sigma(C) = [n]$. This is true for type-(3) clauses by definition of $\sigma$. Using only these clauses, the only resolution step possible is with a $y$ variable as pivot. The claim can be verified by induction on depth: since $\sigma(y_i)$ and $\sigma(\neg y_i)$ partition $[n]$, $[n] \setminus \sigma(y_i)$ and $[n] \setminus \sigma(\neg y_i)$ also partition $[n]$.*

We show that all clauses in $\pi_{D_t}$ that are descendants of some type-(1) clause have large sets associated with them. In particular, we show:

CLAIM 4.8. *Every clause $D$ in $\pi_{D_t}$ such that $\pi_D$ contains a type-(1) clause has $\sigma(D) = [n]$.*

Deferring the proof briefly, we continue with our argument. From Claim 4.8, we conclude that $\sigma(D_t) = [n]$. Recall that the variables $\vec{a}, \vec{b}, \vec{y}, \vec{p}$ and the literals $\neg x_{i,j}$ are not present in $D_t$. The only literals left are positive $x_{i,j}$. These literals are associated with singleton sets, and the variables $x_{i,j}$ for different values of $j$ give the same singleton set. So we conclude that for each $i \in [n]$, there must be some $x_{i,j} \in D_t$. Hence $w_\exists(D_t) \geq n$.

It remains to establish the claimed set size.

PROOF OF CLAIM 4.8. We proceed by induction on the depth of descendants of type-(1) clauses in $\pi_{D_t}$. The base case is a type-(1) clause itself and follows from the definition of $\sigma$.

For the inductive step, let $D$ be obtained by resolving $(E \vee r)$ and $(F \vee \neg r)$. There are two cases to consider: both are descendants of some type-(1) clauses, or only one of them, say $(E \vee r)$, is a descendant of a type-(1) clause. In the former case, by the induction hypothesis, $\sigma(E \vee r) = [n]$ and $\sigma(F \vee \neg r) = [n]$. In the latter case, $\sigma(E \vee r) = [n]$ by induction hypothesis, and $\sigma(F \vee \neg r) = [n]$ from the observations above. ($(F \vee \neg r)$ is not a descendant of any type-(1) clause. But it belongs to $\pi_{D_t}$ which has only type-(1) and type-(3) clauses. So it must be a descendant of only type-(3) clauses, and hence has $[n]$ associated with it.)

Thus in both cases, we have $\sigma(E \vee r) = \sigma(F \vee \neg r) = [n]$. So we have $\sigma(E) \supseteq [n] \setminus \sigma(r)$ and $\sigma(F) \supseteq [n] \setminus \sigma(\neg r)$. Observe that the pivot variable $r$ can only be either an $\vec{a}$ or a $\vec{y}$ variable. Thus $\sigma(r)$ and $\sigma(\neg r)$ are disjoint, and hence $\sigma(E) \cup \sigma(F) = [n]$. Thus $\sigma(D) = \sigma(E) \cup \sigma(F) = [n]$ as claimed. □

This completes the proof of the theorem. □

Since tree-like space is at least as large as space, Theorem 4.2 also rules out the space-width relation for general dag-like Q-Res proofs. However, observe that Theorem 4.2 cannot be used to show that the size-existential-width relationship for general dag-like proofs fails in Q-Res, because the QBFs $CR'_n$ have $O(n^2)$ variables. However, we show via another example that the relation fails to hold in Q-Res as well. This example cannot be used for proving Theorem 4.2 because it is known to be hard for $\text{Q-Res}_\text{T}$ [Janota and Marques-Silva 2015]. (Janota and Marques-Silva [2015] show the hardness for ∀Exp+Res, which implies hardness for $\text{Q-Res}_\text{T}$, as ∀Exp+Res p-simulates $\text{Q-Res}_\text{T}$.)

THEOREM 4.9. *There is a family of false QBFs $\phi'_n$ in $O(n)$ variables such that $S(\vdash_{\text{Q-Res}} \phi'_n) = n^{O(1)}$, $w_\exists(\phi'_n) = 3$, and $w_\exists(\vdash_{\text{Q-Res}} \phi'_n) = \Omega(n)$.*

PROOF. Consider the following formulas $\phi_n$, also introduced by Janota and Marques-Silva [2015]:

$$\phi_n = \exists e_1 \forall u_1 \exists c_1 c_2 \dots \exists e_n \forall u_n \exists c_{2n-1} c_{2n}.$$
$$\bigwedge_{i\in[n]} \big( (\neg e_i \vee c_{2i-1}) \wedge (\neg u_i \vee c_{2i-1}) \wedge (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i}) \big) \wedge \big( \bigvee_{i\in[2n]} \neg c_i \big)$$

We know from [Janota and Marques-Silva 2015] that $\phi_n$ have polynomial-size proofs in Q-Res (but require exponential-size proofs in Q-Res$_\mathsf{T}$). However, in order to prove Theorem 4.9, we need a formula with constant initial width. To achieve this we consider quantified Tseitin transformations of $\phi_n$, i.e., we introduce $2n+1$ new existential variables $x_i$ at the innermost quantification level in $\phi_n$, and replace the only large clause in $\phi_n$ by any CNF formula that preserves satisfiability. Let $\phi'_n$ denote the modified formula:

$$\phi'_n = \exists e_1 \forall u_1 \exists c_1 c_2 \dots \exists e_n \forall u_n \exists c_{2n-1} c_{2n} \exists x_0 \dots x_{2n}.$$
$$\bigwedge_{i\in[n]} \big( (\neg e_i \vee c_{2i-1}) \wedge (\neg u_i \vee c_{2i-1}) \wedge (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i}) \big) \wedge \tag{5}$$
$$\neg x_0 \wedge \bigwedge_{i\in[2n]} (x_{i-1} \vee \neg c_i \vee \neg x_i) \wedge x_{2n}. \tag{6}$$

Note that $w_\exists(\phi'_n) = 3$.

We refer to the clauses in (6) as $x$-clauses. It is clear that from the $x$-clauses, we can derive the large clause of $\phi_n$ in $2n+1$ resolution steps and get back $\phi_n$. Thus $S(\vdash_{\text{Q-Res}} \phi'_n) \leq S(\vdash_{\text{Q-Res}} \phi_n) + 2n + 1 = n^{O(1)}$.

We now show that $\phi'_n$ needs large existential width. We follow the same strategy used in proving Theorem 4.2.

Let $\pi$ be a proof in Q-Res, $\pi \vdash_{\text{Q-Res}} \phi'_n$. List the clauses of $\pi$ in sequence, $\pi = \{D_0, D_1, \dots, D_s = \Box\}$, where each clause in the sequence is either a clause from $\phi'_n$, or is derived from clause(s) preceding it in the sequence using resolution or $\forall$-Red. There must be at least one universal reduction step in $\pi$, since all the initial clauses are necessary for refuting $\phi'_n$, some of them contain universal variables, and the only way to remove a universal variable in Q-Res is by $\forall$-Red. Let $i$ be the least index such that the clause $D_i$ is obtained by $\forall$-Red on $D_j$ for some $0 < i$. Since all $x$ variables block all $u$ variables, $D_j$ and $D_i$ cannot contain any $x$ variables. We use this fact to show that $w_\exists(D_i) = \Omega(n)$. Our strategy is to associate some set with each clause in $\pi$ in a specific way, and use the set size to bound existential width.

We associate the following sets with the literals of $\phi'_n$ and the clauses of $\pi$.

$$\begin{array}{lrcl}
 & \sigma(x_0) & = & \emptyset \\
\forall i \in [2n] & \sigma(x_i) & = & [i] = \{1, 2, \dots, i\} \\
 & \sigma(\neg x_0) & = & [2n] \\
\forall i \in [2n] & \sigma(\neg x_i) & = & [2n] \setminus [i] = \{i+1, \dots, 2n\} \\
\forall i \in [n] & \sigma(e_i) = \sigma(u_i) = \sigma(\neg c_{2i}) = \sigma(c_{2i-1}) & = & \{2i\} \\
\forall i \in [n] & \sigma(\neg e_i) = \sigma(\neg u_i) = \sigma(\neg c_{2i-1}) = \sigma(c_{2i}) & = & \{2i-1\} \\
\forall D \in \pi & \sigma(D) & = & \bigcup_{l \in D} \sigma(l).
\end{array}$$

Note that for any literal $\ell$, $\sigma(\ell)$ and $\sigma(\neg \ell)$ are disjoint. The intuition of defining $\sigma$ this way is as in the proof of Theorem 4.2.

For $D \in \pi$, let $\pi_D$ be the sub-dag of $\pi$, rooted at $D$.

CLAIM 4.10. *$\pi_{D_i}$ contains at least one $x$-clause (axiom clause of type-(6)).*

PROOF. The parent $D_j$ of node $D_i$ contains a universal variable which is then removed through $\forall$-Red to get $D_i$. The universal variables appear only in clauses of type-(5), but are blocked by the $c$-variables in every clause where they appear. Thus, before a reduction is permitted, a $c$-variable must be eliminated by resolution. Since all $c$-variables appear only positively in type-(5) clauses, some $x$-clause must be used in the resolution. $\square$

We show that all clauses in $\pi_{D_i}$ that are descendants of some $x$-clause have large sets associated with them. In particular, we show:

CLAIM 4.11. *Every clause $D$ in $\pi_{D_i}$ such that $\pi_D$ contains an $x$-clause has $\sigma(D) = [2n]$.*

Deferring the proof briefly, we continue with our argument. From Claim 4.11, we conclude that $\sigma(D_i) = [2n]$. Recall that none of the $x$ variables belongs to $D_i$. All other literals are associated with singleton sets, so $D_i$ must contains at least $2n$ literals in order to be associated with the complete set $[2n]$. Since Q-Res proofs prohibit a variable and its negation in the same clause, at most $n$ of the literals in $D_i$ can be universal variables. Thus $D_i$ has at least $n$ existential literals, hence $w_\exists(D_i) = \Omega(n)$.

It remains to establish the claimed set size.

PROOF OF CLAIM 4.11. We proceed by induction on the depth of descendants of $x$-clauses in $\pi_{D_i}$. The base case is an $x$-clause itself and follows from the definition of $\sigma$.

For the inductive step, let $D$ be obtained by resolving $(E \vee z)$ and $(F \vee \neg z)$. There are two cases to consider:

**Case 1:** Both $(E \vee z)$ and $(F \vee \neg z)$ are descendants of $x$-clauses (not necessarily the same $x$-clause). Then by induction, $\sigma(E \vee z) = \sigma(F \vee \neg z) = [2n]$. So $\sigma(E) \supseteq [2n] \setminus \sigma(z)$ and $\sigma(F) \supseteq [2n] \setminus \sigma(\neg z)$. Since $\sigma(z)$ and $\sigma(\neg z)$ are disjoint, $\sigma(E) \cup \sigma(F) = [2n]$. Thus $\sigma(D) = \sigma(E) \cup \sigma(F) = [2n]$ as claimed.

**Case 2:** Exactly one of $(E \vee z)$ and $(F \vee \neg z)$ is a descendant of an $x$-clause. Without loss of generality, let $F \vee \neg z$ be the descendant. Then $E \vee z$ is either a type-(5) clause or is derived solely from type-(5) clauses using resolution. However, observe that the only clauses derivable solely from type-(5) clauses via resolution, without creating tautologies as mandated in Q-Res, are of the form $(c_{2i-1} \vee c_{2i})$ for some $i$. It follows that $z$ is not an $x$ variable. Hence $\sigma(z)$ and $\sigma(\neg z)$ are distinct singleton sets. Further, $z$ cannot be a $u$ variable either, since resolution on universal variables is not permitted in Q-Res.

Now note that for any type-(5) clause $C$, $\sigma(C) = \{2i-1, 2i\}$ for the appropriate $i$. Similarly, $\sigma(c_{2i-1} \vee c_{2i}) = \{2i-1, 2i\}$. So if $E \vee z$ is one of these clauses, then $\sigma(E \vee z) = \sigma(z) \cup \sigma(\neg z)$ and $\sigma(E) = \sigma(\neg z)$. Further, as in Case 1, by induction we know that $\sigma(F \vee \neg z) = [2n]$ and $\sigma(F) \supseteq [2n] \setminus \sigma(\neg z)$. Hence, $\sigma(E \vee F) = [2n]$ as claimed. $\square$

This completes the proof of the theorem. $\square$

The above counterexamples are provided by formulas that require small size, but large existential width. We will now illustrate via another example that also *large size and large width* can occur. These examples are very natural formulas based on the parity function, which have recently been used by Beyersdorff et al. [2015] to show exponential size lower bounds for Q-Res, and indeed a separation between Q-Res and ∀Exp+Res. We will later use these formulas in Section 5 to also show a separation for width between Q-Res and ∀Exp+Res.

Let $\mathrm{xor}(o_1, o_2, o)$ be the set of clauses expressing $o \equiv o_1 \oplus o_2$; that is, $\{\neg o_1 \vee \neg o_2 \vee \neg o,\ o_1 \vee o_2 \vee \neg o,\ \neg o_1 \vee o_2 \vee o,\ o_1 \vee \neg o_2 \vee o\}$. In [Beyersdorff et al. 2015], the QBF QPARITY$_n$ is defined as follows:

$$\exists x_1 \cdots \exists x_n\, \forall z\, \exists t_2 \cdots \exists t_n.\ \mathrm{xor}(x_1, x_2, t_2) \cup \bigcup_{i=3}^{n} \mathrm{xor}(t_{i-1}, x_i, t_i) \cup \{z \vee t_n, \neg z \vee \neg t_n\}.$$

The $x_i$ variables act as the input for the parity function, and the $t_i$ variables are defined inductively to calculate $\textsc{Parity}(x_1, \ldots, x_i)$.

We now complement the exponential size lower bound of Beyersdorff et al. [2015] by a width lower bound.

THEOREM 4.12. $w_\exists(\vdash_{\textsf{Q-Res}} \mathrm{Q}\textsc{Parity}_n) \geq n$.

PROOF. In the formula $\mathrm{Q}\textsc{Parity}_n$, the contradiction occurs semantically because of the clauses $z \vee t_n$, $\neg z \vee \neg t_n$ asserting $z \neq t_n$ (along with the fact that the values of $x$ variables uniquely determine the values of all $t$ variables, in particular, $t_n$). Thus, at least one of these clauses must be used in any proof, necessitating a $\forall$-reduction. In Q-Res we cannot reduce $z$ while any of the $t$ variables are present; and due to the restrictions in Q-Res we cannot resolve any descendants of $z \vee t_n$ with any descendants of $\neg z \vee \neg t_n$ until there is at least one $\forall$-reduction.

Consider a smallest Q-Res proof, and assume without loss of generality that a first (lowest) $\forall$-reduction happens on the positive literal $z$. Therefore before this $\forall$-reduction step we have essentially a resolution proof $\pi$ from $\Gamma = \mathrm{xor}(x_1, x_2, t_2) \cup \bigcup_{i=3}^{n} \mathrm{xor}(t_{i-1}, x_i, t_i) \cup \{t_n \vee z\}$. The clause $D$ that occurs in $\pi$ immediately before the $\forall$-reduction must only contain variables from $\{x_1, \ldots, x_n\}$ apart from the literal $z$, else the reduction is blocked.

We now use the following observation.

CLAIM 4.13. *Suppose $x_1 \oplus \cdots \oplus x_n \vDash C$ for some clause $C$. Then $C$ is either a tautology or $C$ contains all variables $x_1, \ldots, x_n$.*

PROOF OF CLAIM 4.13. Suppose the clause $C$ is not a tautology, but for some nonempty set $I \subset [n]$, none of the variables $x_i$ with $i \in I$ appears in $C$. Since $C$ is a non-tautological clause, there is exactly one partial assignment $\alpha$ falsifying $C$. By setting the variables $x_i$, $i \in I$, appropriately, we can increase $\alpha$ to an assignment satisfying $x_1 \oplus \cdots \oplus x_n$, but still falsifying $C$. Hence $x_1 \oplus \cdots \oplus x_n \nvDash C$. $\square$

Any assignment to the $x$ variables satisfying $x_1 \oplus \cdots \oplus x_n$ has a unique extension to $z$ and the $t$ variables satisfying all clauses of the formula $\mathrm{Q}\textsc{Parity}_n$. This extension necessarily has $t_n = x_1 \oplus \cdots \oplus x_n = 1$ and $z = 0$. Since it satisfies all axioms, by soundness of resolution, it also satisfies $D$.

This, along with Claim 4.13, implies that $D$ is either a tautology or has all $x$ variables. Since it cannot be a tautology (it appears in the proof, and besides, at the very least it has the variable $z$), it must have all $x$ variables, and hence has existential width $n$. $\square$

## 5. SIMULATIONS: PRESERVING SIZE, WIDTH, AND SPACE ACROSS CALCULI

After these strong negative results, ruling out size-width and space-width relations in Q-Res and Q-Res$_\mathsf{T}$, we aim to determine whether any positive results hold in the expansion systems ∀Exp+Res and IR-calc. Before we can do this we need to relate the measures of size, width, and space across the three calculi Q-Res, ∀Exp+Res, IR-calc. Of course, such a comparison in terms of refined simulations is also interesting in its own as it determines the relative strength of the different proof systems. As size corresponds to running time, and space to memory consumption of QBF solvers, such a comparison yields interesting insights into the power of QBF solvers using CDCL vs. expansion techniques.

It is known that IR-calc p-simulates ∀Exp+Res and Q-Res [Beyersdorff et al. 2014], and that ∀Exp+Res p-simulates Q-Res$_\mathsf{T}$ [Janota and Marques-Silva 2015]. We revisit these proofs, with special attention to the width parameter, and also obtain simulating proofs that are tree-like if the original proof is tree-like. The relationships we establish are stated in the following theorem:

THEOREM 5.1. *For all false QBFs $\mathcal{F}$, the following relations hold:*

(1) $\frac{1}{2}S(\vdash_{\textit{IR}_T\textit{-calc}} \mathcal{F}) \leq S(\vdash_{\forall\textit{Exp+Res}_T} \mathcal{F}) \leq S(\vdash_{\textit{IR}_T\textit{-calc}} \mathcal{F}) \leq 3S(\vdash_{\textit{Q-Res}_T} \mathcal{F})$.
(2) $w(\vdash_{\textit{IR-calc}} \mathcal{F}) = w(\vdash_{\forall\textit{Exp+Res}} \mathcal{F}) \leq w_\exists(\vdash_{\textit{Q-Res}} \mathcal{F})$.
(3) $CSpace(\vdash_{\forall\textit{Exp+Res}_T} \mathcal{F}) = CSpace(\vdash_{\textit{IR}_T\textit{-calc}} \mathcal{F}) \leq CSpace(\vdash_{\textit{Q-Res}_T} \mathcal{F})$.

These results follow from Proposition 5.2 and Lemmas 5.3, 5.4 that are stated and established below.

PROPOSITION 5.2 (BEYERSDORFF ET AL. [2014]). *Any proof in ∀Exp+Res of size $S$, width $W$, and space $C$ can be efficiently converted into a proof in IR-calc of size at most $2S$, width $W$, and space $C$. If the proof in ∀Exp+Res is tree-like, so is the resulting IR-calc proof.*

PROOF. In IR-calc, when an axiom is downloaded, the existential literals in it are annotated partially. However in ∀Exp+Res, the annotations are *complete*; all universal variables at a lower level than a literal appear in its annotation. To convert a proof $\pi$ in ∀Exp+Res to one in IR-calc, all that is needed is to follow up each axiom-download with an instantiation that completes the annotations as in $\pi$. This introduces at most one extra step per leaf but does not increase width. Also observe that the space required has not changed: to instantiate a clause we can reuse the same space. □

LEMMA 5.3. *∀Exp+Res$_T$ p-simulates IR$_T$-calc while preserving its width, size, and space.*

PROOF. Recall the main reason why IR$_T$-calc proofs differ from those in ∀Exp+Res$_T$: axioms are downloaded with partial rather than complete annotations, and annotations can be extended at any stage by the inst operation.

The idea is to systematically transform an IR$_T$-calc proof, proceeding downwards from the top where we have the empty clause, and modifying annotations as we go down, so that when all leaves have been modified the resulting proof is in fact an ∀Exp+Res$_T$ proof. This crucially requires that we start with a tree-like proof; if the underlying graph is not a tree, we cannot always find a way of modifying the annotations that will work for all descendants.

Let $\pi$ be an IR$_T$-calc proof of a false QBF $\mathcal{F}$. Without loss of generality, we can assume that every resolution node has, as parent, an instantiation node. (If it does not, we introduce the dummy inst$(\emptyset, *)$ node between it and its parent.) Since the proof is tree-like, we can also collapse contiguous instantiation nodes into a single instantiation node. Thus, as we move down a path from the root, nodes are alternately instantiation and resolution nodes. We consider each resolution node and its parent instantiation node to be at the same level.

Starting from the top, which we call level zero, we transform $\pi$ to another proof $\pi'$ in IR$_T$-calc maintaining the following invariants: after the $i^{th}$ step, all the instantiated clauses up to level $i$ are fully annotated and the instantiating assignments are complete. Thus the instantiation steps become redundant. This further implies that after the last level (when we reach the axiom farthest from the top), the resulting proof is in fact a ∀Exp+Res$_T$ proof.

— **At level** $0$**:** The node at this level must be a resolution producing the empty clause, followed by a dummy instantiation with the empty assignment. Thus the clauses at this level are already fully annotated, but the instantiating assignment is far from complete. Pick an arbitrary complete assignment, say $\sigma$, and instantiate the empty clause with $\sigma$. Clearly the invariants hold now.

— Assume that the invariants holds after processing all nodes at level $i-1$.

— **At level $i$:** Let $D$ be an instantiated clause at level $i-1$, obtained by instantiating some clause $C$ by an assignment $\sigma$. That is, $D = \mathsf{inst}(C,\sigma)$. By the induction hypothesis, $D$ is fully annotated and $\sigma$ is complete. Let $C$ be obtained by resolving $E = (G \vee x^\tau)$ and $F = (H \vee \neg x^\tau)$. We need to make $E$ and $F$ fully annotated. Let $E = \mathsf{inst}(I,\beta_1)$ and $F = \mathsf{inst}(J,\beta_2)$ in $\pi$. Replace $E$ by $E' = \mathsf{inst}(I,\beta_1 \circ \sigma)$ and $F$ by $F' = \mathsf{inst}(J,\beta_2 \circ \sigma)$. As $\sigma$ is complete, both $\beta_1 \circ \sigma$ and $\beta_2 \circ \sigma$ are complete, and hence both $E'$ and $F'$ are fully annotated. The resolution step is now performed on $x^{\tau'}$, where $\tau' = \tau \circ \sigma$ is the resulting annotation on $x$. It is easy to see that the resolvent of $E'$ and $F'$ is $D$, so the intermediate instantiation step going from $C$ to $D$ becomes redundant.

It is clear that the simulation preserves width. It also does not increase size: we may introduce dummy instantiation nodes to make the proof 'alternating', but after the transformation, all instantiations — dummy and actual — are eliminated completely. It is also clear that the simulation preserves the space needed, since the structure of the proof is preserved. □

The simulation in Lemma 5.3 exhibits an interesting phenomenon: while it shows that the tree-like versions of ∀Exp+Res and IR-calc are p-equivalent, it was shown by Beyersdorff et al. [2015] that in the dag-like versions, IR-calc is exponentially stronger than ∀Exp+Res. Thus ∀Exp+Res and IR-calc provide a rare example in proof complexity of two systems that coincide in the tree-like model, but are separated in the dag-like model.

LEMMA 5.4. *$\mathsf{IR}_\mathsf{T}$-calc p-simulates Q-$\mathsf{Res}_\mathsf{T}$ while preserving space and existential width exactly and size upto a factor of 3. That is,* $S(\vdash_{\mathsf{IR_T\text{-}calc}} \mathcal{F}) \leq 3S(\vdash_{\mathsf{Q\text{-}Res_T}} \mathcal{F})$, $\mathit{CSpace}(\vdash_{\mathsf{IR_T\text{-}calc}} \mathcal{F}) \leq \mathit{CSpace}(\vdash_{\mathsf{Q\text{-}Res_T}} \mathcal{F})$, *and* $w(\vdash_{\mathsf{IR\text{-}calc}} \mathcal{F}) \leq w_\exists(\vdash_{\mathsf{Q\text{-}Res}} \mathcal{F})$.

PROOF. We use the same simulation as given by Beyersdorff et al. [2014]. This simulation was originally for dag-like proof systems, but here we check that it also works for tree-like systems, and we observe that space and existential width are preserved.

Let $C_1,\ldots,C_k$ be a Q-$\mathsf{Res}_\mathsf{T}$ proof. We translate the clauses into clauses $D_1,\ldots,D_k$, which will form the skeleton of a proof in IR-calc.

— For an axiom $C_i$ in Q-$\mathsf{Res}_\mathsf{T}$ we introduce the same clause $D_i$ by the axiom rule of IR-calc, i.e., we remove all universal variables and add annotations.

— If $C_i$ is obtained via ∀-reduction from $C_j$, then $D_i = D_j$; we make no change.

— Consider now the case that $C_i$ is derived by resolving $C_j$ and $C_k$ with pivot variable $x$. Then $D_j = x^\tau \vee K_j$ and $D_k = \neg x^\sigma \vee K_k$. It is shown by Beyersdorff et al. [2014] that the annotations $\tau$ and $\sigma$ are not contradictory; in fact, no annotations in the two clauses are contradictory. So if we define $D'_j = \mathsf{inst}(\sigma, D_j)$ and $D'_k = \mathsf{inst}(\tau, D_k)$, then the annotations of $x$ in $D'_j$ and $\neg x$ in $D'_k$ match, and we can resolve on this literal. Define $D'_i$ as the resolvent of $D'_j$ and $D'_k$. We can perform a further instantiation to obtain $D_i = \mathsf{inst}(\eta, D_i)$, where $\eta$ is the set of all assignments to universal variables appearing anywhere in $D'_i$. $D_i$ has no more literals than $C_i$. For details, see [Beyersdorff et al. 2014].

Note that to complete this skeleton into a proof, we only add instantiation rules. Thus, if the original proof was tree-like, so is the new proof. If the original proof has size $S$, the new proof has size at most $4S$, since each resolution may now be preceded by two instantiations and followed by one instantiation. However, this is an overcount, since we are counting two instantiations per edge, and contiguous instantiations can be collapsed. That is, every instantiation following a resolution step can be merged with the instantiation preceding the next resolution and need not be counted separately. The only exception is at the root, where there is nothing to collapse it with. However, at the root, the instantiation itself is redundant and can be discarded. Thus we obtain a new proof of size at most $3S$.

Further, if the original proof had existential width $w$, then the new proof has width $w$ since each $D_i$ has at most (annotated versions of) the existential literals of $C_i$.

Regarding space, observe that simulating axiom download and $\forall$-Red do not require additional space. At the resolution step, the simulation first performs additional instantiations. But instantiation does not need additional space. So the space bound remains the same. □

As a by-product, these simulations enable us to give an easy and elementary proof of the simulation of Q-Res$_\mathsf{T}$ by ∀Exp+Res, shown by Janota and Marques-Silva [2015] via a more involved argument. In fact, our result improves upon the simulation of Janota and Marques-Silva [2015] as we show that even *tree-like* ∀Exp+Res suffices to p-simulate Q-Res$_\mathsf{T}$.

COROLLARY 5.5 (JANOTA AND MARQUES-SILVA [2015]). *∀Exp+Res$_T$ p-simulates Q-Res$_T$.*

PROOF. By Lemma 5.3, ∀Exp+Res$_\mathsf{T}$ p-simulates IR$_\mathsf{T}$-calc, which in turn p-simulates Q-Res$_\mathsf{T}$ by Lemma 5.4. □

Using again the width lower bound for $\mathrm{QP}\textsc{arity}_n$ (Theorem 4.12) we can show that item 2 of Theorem 5.1 cannot be improved, i.e., we obtain an optimal width separation between Q-Res and ∀Exp+Res.

THEOREM 5.6. *There exist false QBFs $\psi_n$ with $w_\exists(\vdash_{\textit{Q-Res}} \psi_n) = \Omega(n)$, but $w(\vdash_{\forall\textit{Exp+Res}} \psi_n) = O(1)$.*

PROOF. We use the $\mathrm{QP}\textsc{arity}_n$ formulas, which by Theorem 4.12 require existential width $n$ in Q-Res. To get the separation it remains to show $w(\vdash_{\forall\mathsf{Exp+Res}} \mathrm{QP}\textsc{arity}_n) = O(1)$. For this we use the following ∀Exp+Res proofs of $\mathrm{QP}\textsc{arity}_n$ of Beyersdorff et al. [2015]: the formulas $\mathrm{QP}\textsc{arity}_n$ have exactly one universal variable $z$, which we expand in both polarities 0 and 1. This does not affect the $x_i$ variables, but creates different copies $t_i^{z/0}$ and $t_i^{z/1}$ of the existential variables right of $z$. Using the clauses of $\mathrm{xor}(t_{i-1}, x_i, t_i)$, we can inductively derive clauses representing $t_i^{z/0} = t_i^{z/1}$. This lets us derive a contradiction using the clauses $t_n^{z/0}$ and $\neg t_n^{z/1}$.

Clearly, this proof only contains clauses of constant width, giving the result. □

## 6. POSITIVE RESULTS: SIZE, WIDTH, AND SPACE IN TREE-LIKE QBF CALCULI

We are now in a position to show some positive results on size-width and size-space relations for QBF resolution calculi. However, most of these results only apply to rather weak tree-like proof systems.

### 6.1. Relations in the expansion calculi ∀Exp+Res and IR-calc

We first observe that for ∀Exp+Res almost the full spectrum of relations from classical resolution remains valid.

THEOREM 6.1. *For all false QBFs $\mathcal{F}$, the following relations hold:*

(*1*) $S\left(\vdash_{\forall\textit{Exp+Res}_T} \mathcal{F}\right) \geq 2^{w\left(\vdash_{\forall\textit{Exp+Res}} \mathcal{F}\right) - w_\exists(\mathcal{F})}$.

(2) $S\left(\vdash_{\forall Exp+Res_T} \mathcal{F}\right) \geq 2^{CSpace\left(\vdash_{\forall Exp+Res_T} \mathcal{F}\right)} - 1.$

(3) $CSpace\left(\vdash_{\forall Exp+Res_T} \mathcal{F}\right) \geq CSpace\left(\vdash_{\forall Exp+Res} \mathcal{F}\right) \geq w\left(\vdash_{\forall Exp+Res} \mathcal{F}\right) - w_{\exists}(\mathcal{F}) + 1.$

PROOF. This theorem follows from the analogous statements for classical resolution. We just describe how to apply those results to ∀Exp+Res.

We know that in ∀Exp+Res$_T$ proofs, leaves corresponds to the expanded clauses from $\mathcal{F}$. The expanded clauses contain only existential (annotated) literals and no universal literals. Let $\mathcal{G}$ be the QBF obtained after expanding $\mathcal{F}$ based on all possible assignments of universal variables. Clearly, $\mathcal{G}$ contains no universal variables and hence can be treated as a propositional CNF formula (all variables are only existentially quantified). That is, if $G$ is the matrix of clauses in $\mathcal{G}$, then $\mathcal{G}$ asserts that $G$ is satisfiable. Also, $w(G) = w(\mathcal{G}) = w_{\exists}(\mathcal{F})$.

Refutations of $\mathcal{F}$ in ∀Exp+Res (respectively, ∀Exp+Res$_T$) are precisely refutations (resp. tree-like refutations) of $G$ in classical resolution; the size, space and width are exactly the same, by definition. That is, $S(\vdash_{\mathsf{Res}_T} G) = S(\vdash_{\forall\mathsf{Exp+Res}_T} \mathcal{F})$, $w(\vdash_{\mathsf{Res}} G) = w(\vdash_{\forall\mathsf{Exp+Res}} \mathcal{F})$, $CSpace(\vdash_{\mathsf{Res}} G) = CSpace(\vdash_{\forall\mathsf{Exp+Res}} \mathcal{F})$, and $CSpace(\vdash_{\mathsf{Res}_T} G) = CSpace(\vdash_{\forall\mathsf{Exp+Res}_T} \mathcal{F})$. Now the Theorem follows by applying Theorems 3.3, 3.4, and 3.5, on $G$. □

We remark that as in item 3 from Theorem 6.1, lower bounds in terms of width for *total space*, which not only counts the number of pebbled clauses, but also the literals in it, cf. [Bonacina et al. 2016], can also be transferred. In fact, Bonacina [2016] shows that in propositional resolution, total space is at least width squared, and the same holds for ∀Exp+Res – total space is at least square of existential width – as we directly transfer the propositional bounds to that system.

By the equivalence of ∀Exp+Res$_T$ and IR$_T$-calc with respect to all the three measures size, width, and space (Theorem 5.1) we can immediately transfer all results from Theorem 6.1 to IR$_T$-calc.

THEOREM 6.2. *For all false QBFs $\mathcal{F}$, the following relations hold:*

(1) $S\left(\vdash_{IR_T\text{-}calc} \mathcal{F}\right) \geq 2^{w\left(\vdash_{IR\text{-}calc} \mathcal{F}\right) - w_{\exists}(\mathcal{F})}.$

(2) $S\left(\vdash_{IR_T\text{-}calc} \mathcal{F}\right) \geq 2^{CSpace\left(\vdash_{IR_T\text{-}calc} \mathcal{F}\right)} - 1.$

(3) $CSpace\left(\vdash_{IR_T\text{-}calc} \mathcal{F}\right) \geq w\left(\vdash_{IR\text{-}calc} \mathcal{F}\right) - w_{\exists}(\mathcal{F}) + 1.$

#### 6.2. The size-space relation in tree-like Q-resolution

We finally return to Q-Res. Most relations were already ruled out in Section 4 for both Q-Res and Q-Res$_T$. The only relation that we can still show to hold is the classical size-space relation (Theorem 3.4), which we transfer from Res$_T$ to Q-Res$_T$.

In classical resolution, this relationship was obtained using pebbling games [Esteban and Torán 2001]. We observe that the same approach works for Q-Res$_T$ as well, giving the analogous relationship. That is, we show:

THEOREM 6.3. *For a false QBF $\mathcal{F}$,*

$$S(\vdash_{Q\text{-}Res_T} \mathcal{F}) \geq 2^{CSpace\left(\vdash_{Q\text{-}Res_T} \mathcal{F}\right)} - 1.$$

PROOF. The proof is almost identical to the proof for classical resolution by Esteban and Torán [2001]. We give a brief sketch.

Let $S(\vdash_{\text{Q-Res}_\text{T}} \mathcal{F}) = s$. Consider a tree-like Q-Res$_\text{T}$ proof $\pi$ of $\mathcal{F}$ (i.e., $\pi \vdash_{\text{Q-Res}_\text{T}} \mathcal{F}$), of size $s$, and let $T$ be the underlying proof-tree.

In contrast to classical resolution, a proof graph in Q-Res may have unary nodes corresponding to $\forall$-reductions. In particular, for a proof in Q-Res$_\text{T}$, there may be paths corresponding to series of $\forall$-reductions. Once the lower end of such a path is pebbled, the same pebble can be slid up to the top of the path; no additional pebbles are needed. So without loss of generality we work with the tree $T'$ obtained by shortcutting all paths containing unary nodes.

Let $d_c(T)$ be the depth of the biggest complete binary tree that can be embedded in $T'$ or in $T$. (We say that a graph $G_1$ is embeddable in a graph $G_2$ if a graph isomorphic to $G_2$ can be obtained from $G_1$ by adding vertices and edges or subdividing edges of $G_1$.) Clearly, $2^{d_c(T)+1} - 1 \le s$.

By induction on $|T'|$, we can show that $d_c(T) + 1$ pebbles suffice to pebble $T'$. Hence, by the argument given above, $d_c(T) + 1$ pebbles suffice to pebble $T$ as well. Now, by Definition 3.2, we obtain $CSpace(\vdash_{\text{Q-Res}_\text{T}} \mathcal{F}) \le d_c(T) + 1$. Hence

$$2^{CSpace(\vdash_{\text{Q-Res}_\text{T}} \mathcal{F})} - 1 \le 2^{d_c(T)+1} - 1 \le s = S(\vdash_{\text{Q-Res}_\text{T}} \mathcal{F})$$

as claimed. □

## 7. CONCLUSION

Our results show that the success story of width in resolution needs to be rethought when moving to QBF. Indeed, the question arises: is width a central parameter in QBF resolution? Is there another parameter that plays a similar role as classical width for understanding QBF resolution size and space?

Our findings almost completely uncover the picture for size, space, and width for the most basic and arguably most important QBF resolution systems Q-Res, ∀Exp+Res, and IR-calc. We showed that for the width measure, which counts both the universal and existential variables, the size-width relation as in resolution fails in tree-like Q-Res as well as in general Q-Res (Proposition 3.6). We also introduce a tighter width measure, i.e., existential width, which only counts the existential variables and showed that the size-width relation fails, even for this tighter measure, for both the tree-like Q-Res (Theorem 4.2) as well as for general Q-Res (Theorem 4.9).

One question prompted by these results is whether one can define an even tighter width measure for which we can obtain positive results for Q-Res. For instance, such a measure could attach a weight to the existential variables, and, intuitively, the left-most existential block should receive the highest weight. However, our results above point to a negative answer also here.

In particular, consider QBFs of the form $\mathcal{Q}_1 X_1, \cdots \mathcal{Q}_n X_n.\ F$, where $\mathcal{Q}_i \in \{\exists, \forall\}$, with $\mathcal{Q}_1 = \exists, \mathcal{Q}_i \ne \mathcal{Q}_{i+1}$, and $X_i$ are pairwise disjoint sets of variables. $F$ is a CNF formula over variables $X_1 \cup \cdots \cup X_n$. Define the *first-block existential width* for a clause $C$ (over variables $X_1 \cup \cdots \cup X_n$) to be the number of existential literals in $C$ from the first existential block (i.e., from $X_1$). We denote this measure by $\mathrm{w}_{\exists_1}(C)$.

For the false QBF $CR'_n$ from Theorem 4.2 we have $S(\vdash_{\text{Q-Res}_\text{T}} CR'_n) = n^{O(1)}$, $\mathrm{w}_{\exists_1}(CR'_n) = O(1)$, but $\mathrm{w}_{\exists_1}(\vdash_{\text{Q-Res}} CR'_n) \ge n$. This holds because any tree-like Q-Res proof $\pi$ must contain a clause $D_t$ where the first $\forall$-Red step is performed, and we already showed in Theorem 4.2 that $D_t$ must contains at least $n$ distinct existential variables $x_{i,j}$. Obviously, $x_{i,j}$ belong to the first existential block of $CR'_n$. Thus Theorem 4.2 shows that the size-width relation with even the width measure $\mathrm{w}_{\exists_1}$ fails in tree-like Q-Res.

The most immediate open question arising from our investigation is whether size-width relations hold for general dag-like ∀Exp+Res or IR-calc proofs. The issue here is that in the classical size-width relation of Ben-Sasson and Wigderson [2001] the number of variables enters the formula in a crucial way. For the instantiation calculi it is not clear what should qualify as the right count for this as different annotations of the same existential variable are formally treated as distinct variables (which is also clearly justified by the semantic meaning of expansions).

For further research it will also be interesting whether size-width or space-width relations apply to any of the stronger QBF resolution systems QU-Res [Van Gelder 2012], LD-Q-Res [Balabanov and Jiang 2012], or IRM-calc [Beyersdorff et al. 2014]. However, we conjecture that the negative picture also prevails for these systems.

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