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CATEGORIES OF UNBOUNDED OPERATORS

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Abstract. In this article we introduce the concept of an LK^*-algebroid, which is defined axiomatically. The main example of an LK^*-algebroid is the category of all subspaces of a Hilbert space and closed (not necessarily bounded) linear operators. We prove that for any LK^*-algebroid there is a faithful functor that respects its structure and maps it into this main example.

1. Introduction

One of the nicest things about C^*-algebras, known since they were introduced (see [12]) is that any C^*-algebra can be representated as an algebra of bounded linear operators acting on a Hilbert space. In the classical algebraic approach to quantum field theory and statistical mechanics, as described in [11, 1] C^*-algebras are typically used to describe the algebra of observables and states, free from any particular model, and the deep mathematical theory of C^*-algebras (see for instance the books [8, 9, 10, 20]) is there to exploit.

However, as pointed out for instance in the introduction to [2], there is in principle a problem with the C^*-algebra approach. Specifically, C^*-algebras correspond to *bounded* linear operators on a Hilbert space, and most observables appearing in physical systems correspond to *unbounded* operators, such as differentiation, on a Hilbert space. One issue with handling unbounded operators algebraically is that a well-behaved unbounded linear operator does not have a domain equal to the whole subspace it acts on, but rather a dense subset.

This issue led to the introduction of partial $*$-algebras and various elaborations such as quasi-$*$-algebras, O^*-algebras and CQ^*-algebras; see for instance [2, 6, 5, 7]. One major area of interest in the study of these structures is the representation of one described axiomatically as a concrete partial algebra of not necessarily bounded operators on a Hilbert space.

As is the case for C^*-algebras, a partial $*$-algebra with suitable additional structure can always be represented in this way. Indeed, the proof of this largely follows the classical GNS construction for C^*-algebras (see [12]), but the partial multiplication means that the states used to construct the Hilbert space are replaced by more fiddly constructions called *biweights* (see [4, 3, 22]) which explicitly involve sesquilinear forms and behave slightly awkwardly for homomorphisms.

In this article we present an alternative construction. Instead of considering partial $*$-algebras, we remember the domains of unbounded operators on a Hilbert space and consider *algebroids*. Doing this carefully, and describing a fair amount of extra structure, allows us to come up with a representation theory for unbounded operators which again uses states rather than biweights. The constructions are based on work for C^*-categories in [13, 19].

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More specifically, we focus on algebroids where the morphism sets are locally convex vector spaces equipped with an A-bimodule structure, where A is a C^*-algebra, the objects are arranged in a lattice (inspired by looking at subspaces of a Hilbert space), a partially defined involution, and the condition that morphisms of the form xx^*, where $x \mapsto x^*$, have a positive spectrum. We call algebroids with this additional structure *LK^*-algebroids*.

We then look at examples of LK^*-algebroids. The primary example is the algebroid where the objects are all subspaces of a Hilbert space H, and the set of morphisms from a subspace U to a subspace V consists of all operators with domain U and image contained in V. We conclude by adapting the GNS construction to prove that for any LK^*-algebroid there is a faithful functor that respects its structure into the category of subspaces and operators on a Hilbert space.

2. Locally convex algebroids and further structures

In a small category $\mathcal{C}$, let us write $Ob(\mathcal{C})$ to denote the set of objects, $Hom(U,V)_\mathcal{C}$ to denote the set of morphisms from an object U to an object V, and $Mor(\mathcal{C}) = \bigcup_{U,V\in Ob(\mathcal{C})} Hom(U,V)_\mathcal{C}$ to denote the total set of morphisms.

Recall from [17] that a small category $\mathcal{C}$ is called a *complex algebroid* if each morphism set is a complex vector space, and composition between morphism spaces

$$Hom(V,W)_\mathcal{C} \times Hom(U,V)_\mathcal{C} \to Hom(U,W)_\mathcal{C}$$

is bilinear.

If V is a real or complex vector space, recall (see [21]) that a *seminorm* on V is a map $p\colon X \to \mathbb{R}$ such that

- $p(x) \geq 0$ for all $x \in X$.
- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and $\lambda \in \mathbb{C}$.
- $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

If V has a family $\{p_\alpha \mid \alpha \in A\}$ of seminorms, we equip V with the weakest topology under which each map $x \mapsto p_\alpha(x - x_0)$ is continuous, where $x_0 \in V$ and $\alpha \in A$. With this topology, V is a *topological vector space*, that is to say the operations of addition and scalar multiplication are continuous.

A topological vector space where the topology is defined by a family of seminorms is called a *locally convex space*. The main property of locally convex spaces that we will need in these notes is the *second geometric form of the Hahn-Banach theorem*; again, see [21] for a proof.

Theorem 2.1. *Let V be a real locally convex space. Let $A, B \subseteq V$ be convex, with A compact, B closed and $A \cap B = \emptyset$. Then there is a continuous linear map $\varphi\colon V \to \mathbb{R}$ and real numbers $\alpha, \beta \in \mathbb{R}$ such that*

$$\varphi(x) \leq \alpha < \beta \leq \varphi(y)$$

for all $x \in A$ and $y \in B$.

Definition 2.2. A complex algebroid $\mathcal{C}$ is called a *locally convex algebroid* if each morphism set is a locally convex vector space, and composition is continuous.

Recall that a *lattice* is a partially ordered set, $(S, \leq)$, where any two elements $a, b \in S$ have a *least upper bound*, $a \vee b$, and a *greatest lower bound* $a \wedge b$.

Definition 2.3. We call an algebroid $\mathcal{C}$ a *lattice algebroid* if:

- The set of objects has a partial ordering, $\leq$, under which it is a lattice.

- If $U \leq V$ for objects U and V then there is a canonical monomorphism $i_{U,V} : U \hookrightarrow V$ such that $i_{V,W} i_{U,V} = i_{U,W}$ for all $U, V, W \in Ob(\mathcal{C})$.[1]
- Let $V' \leq V$ and let $x \in Hom(U,V)_{\mathcal{C}}$. Let $V' \leq V$. Then there is an object $x^{-1}[V'] \leq U$ and a morphism $x|_{x^{-1}[V']}$ such that $i_{V',V} x|_{x^{-1}[V']} = x i_{x^{-1}[V'],U}$.

In a lattice algebroid, let us write $U \vee V$ to denote the join of objects U and V, and $U \wedge V$ to denote their meet. In terms of arrows, we write $U \hookrightarrow V$ to denote the monomorphism $i_{U,V}$ when $U \leq V$. For morphisms $x \in Hom(U,V)_{\mathcal{C}}$ and $y \in Hom(U',V')_{\mathcal{C}}$, we write $x \leq y$, and call y an *extension* of x if $U \leq U'$, and we have an object W such that $V, V' \leq W$ and the morphisms

$$U \xrightarrow{x} V \hookrightarrow W$$

and

$$U \hookrightarrow U' \xrightarrow{y} V' \hookrightarrow W$$

are equal.

A lattice structure enables us to add morphisms in different morphism sets.

Definition 2.4. Let $\mathcal{C}$ be a lattice algebroid. Let $x \in Hom(U,V)_{\mathcal{C}}$ and $y \in Hom(U',V')_{\mathcal{C}}$. Then we define $x + y \in Hom(U \wedge U', V \vee V')_{\mathcal{C}}$ to be the sum of the morphisms

$$U \wedge U' \hookrightarrow U \xrightarrow{x} V \hookrightarrow V \vee V'$$

and

$$U \wedge U' \hookrightarrow U' \xrightarrow{y} V' \hookrightarrow V \vee V'.$$

We can also compose any two morphisms.

Definition 2.5. Let $x \in Hom(U,V)_{\mathcal{C}}$ and $y \in Hom(U',V')_{\mathcal{C}}$. Then we define the *product* yx to be the composite

$$x^{-1}[V \wedge U'] \xrightarrow{x|_{x^{-1}[V \wedge U']}} V \wedge U' \hookrightarrow U' \xrightarrow{y} V'.$$

One warning is that the above product is not, in general, associative. Therefore, when we can, we avoid it, sticking with the associative composition of morphisms at the category level.

Definition 2.6. Let $\mathcal{C}$ be a locally convex algebroid. A *partial involution* on $\mathcal{C}$ consists of:

- A set of distinguished objects, $Ob(\mathcal{C})_0$, called the *dense objects*. We write $Mor(\mathcal{C})_0 = \bigcup_{U \in Ob(\mathcal{C}_0), V \in Ob(\mathcal{C})} Hom(U,V)_{\mathcal{C}}$,
- A function $Mor(\mathcal{C})_0 \to Mor(\mathcal{C})_0$, written $x \mapsto x^*$

such that:

- For any object $U \in Ob(\mathcal{C})$, we have an object $V \in Ob(\mathcal{C})_0$ such that $U \leq V$.
- If $V \leq W$ and V is a dense object, then so is W, and for each object U, the map $Hom(U,V)_{\mathcal{C}} \to Hom(U,W)_{\mathcal{C}}$ defined by the formula $x \mapsto i_{V,W} x$ is a dense embedding.
- Let $x, y \in Hom(U,V)_{\mathcal{C}}$, where U is a dense object. Let $\alpha, \beta \in \mathbb{C}$. Then $\overline{\alpha} x^* + \overline{\beta} y^* \leq (\alpha x + \beta y)^*$.
- Let $x \in Hom(U,V)_{\mathcal{C}}$ and $y \in Hom(V,W)_{\mathcal{C}}$, where U and V are dense objects. Then $x^* y^* \leq (yx)^*$.
- Let $x \in Mor(\mathcal{C})_0$. Then $x = (x^*)^*$.

A locally convex lattice algebroid with an involution is called a locally convex $*$-algebroid.

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[1] A morphism, $i \in Hom(U,V)_{\mathcal{C}}$ is a *monomorphism* if it has the left-cancellation property, that is to say if $ix = iy$ for $x, y \in Hom(U',U)_{\mathcal{C}}$, then $x = y$.

Note that the lattice structure on $\mathcal{C}$ is needed to define the sum of two arbitrary morphisms, and so used to formulate the first of the above axioms.

Definition 2.7. Let A be a unital C^*-algebra, and let $\mathcal{C}$ be a locally convex $*$-algebroid. An *A-bimodule structure* on $\mathcal{C}$ consists of continuous maps

$$A \times Mor(\mathcal{C}) \to \mathcal{C}, \quad Mor(\mathcal{C}) \times A \to \mathcal{C}$$

written simply $(a, x) \mapsto ax$ and $(x, a) \mapsto xa$ respectively, such that:

- Let $a, b \in A$ and $x \in Mor(\mathcal{C})$. Then $(ab)x = a(bx)$ and $x(ab) = (xa)b$.
- Let $\alpha, \beta \in \mathbb{C}$. Then $(\alpha a+\beta b)x = \alpha(ax)+\beta(bx) = (\alpha a)x+(\beta b)x$ and $x(\alpha a+\beta b) = x(\alpha a) + y(\beta b) = \alpha(xa) + \beta(xb)$.
- Let $x, y \in Hom(U, V)_{\mathcal{C}}$. Then $a(\alpha x + \beta y) = \alpha(ax) + \beta(by)$ and $(\alpha x + \beta y)a = \alpha(xa) + \beta(ya)$.
- If $x, y \in Mor(\mathcal{C})$ are composable and $a \in A$, then so are ax and y, and $a(xy) = (ax)y$.
- Let $x, y \in Mor(\mathcal{C})$ and $a \in A$. Then the morphisms xa and y are composable if and only if the morphisms x and ay are composable, and $(xa)y = x(ay)$.
- If $x, y \in Mor(\mathcal{C})$ are composable and $a \in A$, then so are x and ya, and $(xy)a = x(ya)$.
- If $x \in Mor(\mathcal{C})_0$ and $a \in A$, then $ax, xa \in Mor(\mathcal{C})_0$, and $(ax)^* = x^*a^*$.
- Let $1 \in A$ be the unit. Then $1x = x1 = x$.
- Let $a \in A$, and $U \leq V$ be objects in $\mathcal{C}$. Then $ai_{U,V} = i_{U,V}a$.

The definition ensures the following is valid for the more general addition and multiplication present in a locally convex $*$-algebroid.

Proposition 2.8. *Let $\mathcal{C}$ be a locally convex $*$-algebroid with an A-bimodule structure. Let $x, y \in Mor(\mathcal{C})$ and $a \in A$. Then*

$$a(x + y) = ax + ay, \quad (x + y)a = xa + ya$$

and

$$a(xy) = (ax)y, \quad (xa)y = x(ay), \quad (xy)a = x(ya).$$

Let $\mathcal{C}$ be a locally convex $*$-algebroid with an A-bimodule structure. We say a morphism $x \in Hom(U, V)_{\mathcal{C}}$ has *bounded inverse* if we have an element $a \in A$ such that $ax = xa = i_{U,V}$.

We define the *spectrum* of T, *Spectrum*(T), to be the set of all $\lambda \in \mathbb{C}$ such that the morphism $x - \lambda i_{U,V}$ does *not* have bounded inverse.

We call an element $x \in Hom(U, V)_{\mathcal{C}}$, where U is a dense object, *positive* if $x^* = x$, and $Spectrum(x) \subseteq [0, \infty)$.

Definition 2.9. Let A be a C^*-algebra. An *LK^*-algebroid* over A is a locally convex $*$-algebroid equipped with an A-bimodule structure such that for each $x \in Hom(U, V)_{\mathcal{C}}$ we have that xx^* is positive.

LK^*-algebroids have a fair amount of structure. An LK^*-functor is a functor between LK^*-algebroids that preserves all of this structure. Specifically, we have the following.

Definition 2.10. Let $\mathcal{A}$ and $\mathcal{B}$ be LK^*-algebroids over a C^*-algebra A. A function $\gamma\colon \mathcal{A} \to \mathcal{B}$ is called an *LK^*-functor* if:

- The map $\gamma\colon Ob(\mathcal{A}) \to Ob(\mathcal{B})$ is order-preserving, and takes dense objects to dense objects, with $\gamma(U \vee V) = \gamma(U) \vee \gamma(V)$ and $\gamma(U \wedge V) = \gamma(U) \wedge \gamma(V)$.
- For $U, V \in Ob(\mathcal{A})$ with $U \leq V$, we have $\gamma(i_{U,V}) = i_{\gamma(U),\gamma(V)}$.
- Let $V' \leq V$ and let $x \in Hom(U, V)_{\mathcal{A}}$. Let $V' \leq V$. Then $\gamma(x^{-1}[V']) = \gamma(x)^{-1}[\gamma(V')]$ and $\gamma(x|_{x^{-1}[V']}) = \gamma(X)|_{\gamma(x)^{-1}[\gamma(V')]}$.

- Each map $\gamma\colon\ Hom(U,V)_{\mathcal{A}} \to Hom(\gamma(U),\gamma(V))_{\mathcal{B}}$ is continuous and linear.
- For each morphism $T \in Mor(\mathcal{A})$ and $a \in A$, we have $\gamma(aT) = a\gamma(T)$, and $\gamma(Ta) = \gamma(T)a$.
- For any morphism $T \in Mor(\mathcal{A})_0$ we have $\gamma(T^*) = \gamma(T)^*$.

The structure we have defined ensures that LK^*-functors preserve the more general addition and multiplication of morphisms. Specifically, we have the following.

Proposition 2.11. *Let $\gamma\colon \mathcal{A} \to \mathcal{B}$ be an LK^*-functor. Then for all $x,y \in Mor(\mathcal{A})$, we have $\gamma(x+y) = \gamma(x)+\gamma(y)$ and $\gamma(xy) = \gamma(x)\gamma(y)$.*

The following notion is slightly more generaral, though the above proposition still holds.

Definition 2.12. Let $\mathcal{A}$ an LK^*-algebroid over a C^*-algebra A, and $\mathcal{B}$ be an LK^*-algebroids over a C^*-algebra B. A pair (γ,θ), where $\theta\colon A \to B$ is an $*$-homomorphism, and $\gamma\colon \mathcal{A} \to \mathcal{B}$ is a function, is called an *LK^*-functor* if:

- The map $\gamma\colon\ Ob(\mathcal{A}) \to Ob(\mathcal{B})$ is order-preserving, and takes dense objects to dense objects, with $\gamma(U \vee V) = \gamma(U) \vee \gamma(V)$ and $\gamma(U \wedge V) = \gamma(U) \wedge \gamma(V)$.
- For $U,V \in Ob(\mathcal{A})$ with $U \leq V$, we have $\gamma(i_{U,V}) = i_{\gamma(U),\gamma(V)}$.
- Let $V' \leq V$ and let $x \in Hom(U,V)_{\mathcal{A}}$. Let $V' \leq V$. Then $\gamma(x^{-1}[V']) = \gamma(x)^{-1}[\gamma(V')]$ and $\gamma(x|_{x^{-1}[V']}) = \gamma(X)|_{\gamma(x)^{-1}[\gamma(V')]}$.
- Each map $\gamma\colon\ Hom(U,V)_{\mathcal{A}} \to Hom(\gamma(U),\gamma(V))_{\mathcal{B}}$ is continuous and linear.
- For each morphism $T \in Mor(\mathcal{A})$ and $a \in A$, we have $\gamma(aT) = \theta(a)\gamma(T)$, and $\gamma(Ta) = \gamma(T)\theta(a)$.
- For any morphism $T \in Mor(\mathcal{A})_0$ we have $\gamma(T^*) = \gamma(T)^*$.

3. Examples

Let H be a Hilbert space. Let $U \subseteq H$ be a subset, and let $T\colon U \to V \subseteq H$ be a linear map (not in general bounded); we call a not necessarily bounded linear map an *operator*.

Let $x \in H$, and define $\varphi_x\colon U \to \mathbb{C}$ by the formula $\varphi_x(y) = \langle x, Ty\rangle$. Set

$$V' = \{x \in H \mid \varphi_y \text{ is continuous}\}.$$

Then one can show using the Hahn-Banach theorem (see [21]) that if U is dense, then there is a unique operator $T^*\colon V' \to U' \subseteq H$ such that $\langle T^*x, y\rangle = \langle x, Ty\rangle$ for all $x \in W$ and $y \in V$. We call T^* the *adjoint* of T.

Recall that we call an operator T *closed* if the *graph* $\mathbb{G}r(T) = \{(u,Tu) \mid u \in U\}$ is a closed subset of $H \oplus H$. This does not in general imply that U is a closed subset of H; if this were true, by the closed graph theorem, the operator T would be bounded. See chapter 10 of [14] for details, where the following is also shown.

Proposition 3.1. *Let $T\colon U \to V$ be a closed operator. Then the above domain of the adjoint, V', is a dense subset of H.*

In particular, in this case, we can form the second adjoint $(T^*)^*$. It turns out that $(T^*)^* = T$; again, see [14] for details. The definition of LK^*-categories was motivated by the following result, which is now straightforward to verify.

Proposition 3.2. *Let H be a Hilbert spaces. Let $\mathcal{U}(H)$ be the category where the set of objects is the collection of linear subspaces of H, and the morphisms are closed operators between them. Let us define a locally convex topology on the space $Hom(U,V)_{\mathcal{U}(H)}$ by the family of seminorms*

$$p_M(T) = \sup\{\langle u, Tv\rangle \mid (u,v) \in M\},$$

where M is a subset of $V \times U$ such that the above supremum is finite.

Define a partial ordering on $Ob(\mathcal{U}(H))$ by taking subsets, and a lattice structure by writing $U \wedge V = U \cap V$ and $U \vee V = U + V$. If $U \subseteq V$, let $i_{U,V}$ be inclusion map $U \hookrightarrow V$. If $T \in Hom(U,V)_{\mathcal{U}(H)}$, and $V' \subseteq V$, let $T^{-1}[V'] = \{u \in U \mid T(u) \in V'\}$, and let $T|_{T^{-1}[V']} \colon T^{-1}[V'] \to V'$ be defined by restricting T to this set.

Call $U \in Ob(\mathcal{U}(H))$ a dense object if U is a dense subset of H. If $T \in Hom(U,V)_{\mathcal{C}}$ for some subspace V, define T^* to be the above adjoint.

Finally, let $\mathcal{B}(H)$ denote the C^*-algebra of bounded linear operators on H. Then $\mathcal{U}(H)$ is an LK^*-algebroid over $\mathcal{B}(H)$.

The locally convex topology described above is typical in the literature on topological algebras of operators; see for example [15].

Definition 3.3. Let A be a C^*-algebra, and let B be a sub-algebra of A. We call a subalgebroid, $\mathcal{D}$, of an LK^*-category, $\mathcal{C}$, over A, a *sub-LK^*-algeboid over B* if:

- The greatest lower and least upper bounds of any two objects in $\mathcal{D}$ are also in $\mathcal{D}$.
- Let $U \leq V$, where $U, V \in Ob(\mathcal{D})$. Then $i_{U,V} \in Hom(U,V)_{\mathcal{D}}$.
- Let $x \in Hom(U,V)_{\mathcal{D}}$, and let $V' \in Ob(\mathcal{D})$ be such that $V' \leq V$. Then $x^{-1}[V'] \in Ob(\mathcal{D})$ and $x|_{x^{-1}[V']} \in Hom(x^{-1}[V'], V')_{\mathcal{D}}$.
- Each morphism set $Hom(U,V)_{\mathcal{D}}$ is a B-bimodule, with operations inherited from the A-bimodule $Hom(U,V)_{\mathcal{C}}$.
- Let $U \in Ob(\mathcal{D})$. Then there is an object $V \in Ob(\mathcal{D})$ such that $U \leq V$ and V is a dense object in $\mathcal{C}$.
- Let $x \in Hom(U,V)_{\mathcal{D}}$, where U is a dense object in $\mathcal{C}$. Then $x^* \in Mor(\mathcal{D})$.

Certainly, a sub-LK^*-algebroid over B of an LK^*-algebroid is itself an LK^*-algebroid with its inherited structure.

Example 3.4. Let H be a Hilbert space, let $U \subseteq H$ be a dense subspace, let V be another subspace , and let $D \colon U \to V$ be a closed operator. Then we write $\mathcal{U}^*(D)$ to denote the smallest sub-LK^*-algebroid over $\mathbb{C}$ of $\mathcal{U}(H)$ that contains the objects U and V and the operator $D \in Hom(U,V)_{\mathcal{U}^*(D)}$.

Recall (see [13]) that an algebroid $\mathcal{C}$ is called a *C^*-category* if each morphism set is a Banach algebra, and

- Composition of morphisms satisfies the inequality
$$\|xy\| \leq \|x\| \cdot \|y\|, \quad x \in Hom(V,W)_{\mathcal{C}}, \quad y \in Hom(U,V)_{\mathcal{C}}.$$
- There are conjugate linear maps $Hom(U,V)_{\mathcal{C}} \to Hom(V,U)_{\mathcal{C}}$, written $x \mapsto x^*$ such that $(xy)^* = y^*x^*$ if x and y are composable morphisms, and $(x^*)^* = x$ for any morphism x.
- The C^*-identity $\|xx^*\| = \|x\|^2$ holds for any morphism x.
- If $x \in Hom(U,V)_{\mathcal{C}}$, then the composite xx^* is a positive element of the C^*-algebra $Hom(V,V)_{\mathcal{C}}$.

We call a C^*-category *additive* if there is a 0 object, 0, and for any two objects U and V there is a biproduct (in the sense of category theory; see for example [16, 23]) $U \oplus V$. As shown in [18], any C^*-category $\mathcal{C}$ has an *additive completion* $\mathcal{C}_{\oplus}$. Objects of the additive completion $\mathcal{C}_{\oplus}$ are formal strings

$$U_1 \oplus U_2 \oplus \cdots \oplus U_m, \quad U_i \in Ob(\mathcal{C}).$$

Let us write

$$(U_1 \oplus U_2 \oplus \cdots \oplus U_m) \vee (V_1 \oplus V_2 \oplus \cdots \oplus V_n) = U_1 \oplus U_2 \oplus \cdots \oplus U_m \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

and let $(U_1 \oplus U_2 \oplus \cdots \oplus U_m) \wedge (V_1 \oplus V_2 \oplus \cdots \oplus V_n)$ be the largest string $W_1 \oplus \cdots \oplus W_r$ such that

- $W_i = U_{a_i} = V_{b_i}$ for some a_i and b_i.
- If $i \leq j$, then $a_i \leq a_j$ and $b_i \leq b_j$.

If no such string exists, we set $(U_1 \oplus U_2 \oplus \cdots \oplus U_m) \wedge (V_1 \oplus V_2 \oplus \cdots \oplus V_n) = 0$. The following is then straightforward to check.

Proposition 3.5. *Let $\mathcal{C}$ be a C^*-category. For $U, V \in Ob(\mathcal{C}_\oplus)$, write $U \leq V$ if $U \vee V = U$. Then, with the above lattice structure on the objects, $\mathcal{C}_\oplus$ is an LK^*-category in which every object is dense.*

We conclude our examples by looking at how to make new LK^*-categories out of old ones. The first construction is fairly obvious.

Proposition 3.6. *Let A be a C^*-algebra. Let $\mathcal{C}$ and $\mathcal{D}$ be LK^*-categories over A. Then we have an LK^*-category $\mathcal{C} \oplus \mathcal{D}$ over A where*

- *$Ob(\mathcal{C} \oplus \mathcal{D})$ is the set of formal pairs $U \oplus U'$ where $U \in Ob(\mathcal{C})$ and $U' \in Ob(\mathcal{D})$.*
- *$Hom(U \oplus U', V \oplus V')_{\mathcal{C}\oplus\mathcal{D}} = Hom(U, V)_{\mathcal{C}} \oplus Hom(U', V')_{\mathcal{D}}$.*

Proof. Say $U \oplus U' \leq V \oplus V'$ if $U \leq U'$ and $V \leq V'$. Then we have a lattice structure defined by writing

$$U \oplus U' \vee V \oplus V' = (U \vee V) \oplus (U' \vee V'), \quad U \oplus U' \wedge V \oplus V' = (U \wedge V) \oplus (U' \wedge V').$$

We can write $i_{U\oplus U', V\oplus V'} = i_{U,V} \oplus i_{U',V'}$. If $x \in Hom(U, V)$ and $y \in Hom(U', V')$, with $W \leq V$ and $W' \leq V'$, we can define

$$(x \oplus y)^{-1}[W \oplus W'] = x^{-1}[W] \oplus y^{-1}[W'], \quad (x \oplus y(x \oplus y)^{-1}[W \oplus W'] = x|_{x^{-1}[W]} \oplus y|_{y^{-1}[W']}.$$

Call $U \oplus U'$ dense if U is dense in $\mathcal{C}$ and U' is dense in $\mathcal{D}$. Given $x \oplus y \in Hom(U \oplus U', V \oplus V')_{\mathcal{C}\oplus\mathcal{D}}$, define $(x \oplus y)^* = x^* \oplus y^*$.

Then the required axioms are easy to check. □

The following is similar.

Proposition 3.7. *Let A and B be C^*-algebras. Let $\mathcal{C}$ be an LK^*-category over A, and $\mathcal{D}$ be an LK^*-categories over B. Then we have an LK^*-category $\mathcal{C} \oplus \mathcal{D}$ over $A \oplus B$ where*

- *$Ob(\mathcal{C} \oplus \mathcal{D})$ is the set of formal pairs $U \oplus U'$ where $U \in Ob(\mathcal{C})$ and $U' \in Ob(\mathcal{D})$.*
- *$Hom(U \oplus U', V \oplus V')_{\mathcal{C}\oplus\mathcal{D}} = Hom(U, V)_{\mathcal{C}} \oplus Hom(U', V')_{\mathcal{D}}$.*

Finally, let V and W be locally convex vector spaces over $\mathbb{C}$ with topologies defined by the families of seminorms $\{p_a \mid a \in A\}$ and $\{q_b \mid b \in B\}$ respectively. Recall (see for example [21]) that we can define a family of seminorms $\{p_a \otimes q_b \mid a \in A,\ b \in B\}$ on the tensor product $V \otimes W$ by the formula

$$p_a \otimes q_b(x) = \inf\{\max_{j=1}^{n} p_a(u_j) \cdot q_b(v_j) \mid x = \sum_{j=1}^{n} u_j \otimes v_j\}.$$

We call the locally convex topology on $V \otimes W$ defined by this set of seminorms the *projective topology.*

Proposition 3.8. *Let $\mathcal{C}$ be an LK^*-category over a C^*-algebra A. Let B be another C^*-algebra. Then we have an LK^*-category $\mathcal{C} \otimes B$ over $A \otimes B$ with objects, dense objects and lattice structure the same as in $\mathcal{C}$, and morphism sets*

$$Hom(U, V)_{\mathcal{C}\otimes B} = Hom(U, V)_{\mathcal{C}} \otimes B$$

equipped with the projective topology.

Proof. We can define an $A \otimes B$-bimodule structure on a morphism set $Hom(U,V)_{\mathcal{C}\otimes B}$ by writing

$$(a \otimes b)(x \otimes b') = ax \otimes bb', \quad (x \otimes b')(a \otimes b) = xa \otimes b'b.$$

If $U \in Ob(\mathcal{C})_0$, and $x \in Hom(U,V)_{\mathcal{C}}$, $b \in B$, we define $(x \otimes b)^* = x^* \otimes b^*$. The required axioms are easy to check. □

Example 3.9. *Consider the unbounded operator $\frac{d}{dx}$ on $L^2(\mathbb{R})$. Then we define the LK^*-algebroid of differential operators on $\mathbb{R}$, $\Psi(\mathbb{R})$, to be the tensor product*

$$\mathcal{U}^*\left(\frac{d}{dx}\right) \otimes C_0(\mathbb{R}).$$

The above is a foundation for further examples of LK^*-algebroids of differential operators.

4. States and representations

Definition 4.1. Let $\mathcal{C}$ be an LK^*-algebroid over a C^*-algebra A. Then a *representation* of $\mathcal{C}$ is an LK^*-functor $(\rho, \theta)\colon (\mathcal{C}, A) \to (\mathcal{U}(H), \mathcal{B}(H))$ for some Hilbert space H. We call ρ *faithful* if it is injective on each morphism set.

The major result of this section is that any LK^*-algebroid has a faithful representation. This is a generalisation of the corresponding result for C^*-algebras (see [12]) through the well-known GNS construction. Indeed, the proof is conceptually very similar to the C^*-algebra result, and its generalisation to C^*-categories in [13, 19]

First of all, let $V \in Ob(\mathcal{C})$. Let M_V be the direct limit of the locally convex vector spaces $Hom(U,V)_{\mathcal{C}}$, where $U \in Ob(\mathcal{C})_0$, and if $U' \leq U$, we can identify $x \in Hom(U,V)_{\mathcal{C}}$ with $xi_{U',U} \in Hom(U',V)_{\mathcal{C}}$. This limit makes sense, and is a vector space because of the lattice algebroid structure.

Similarly, if $V \leq V'$, we identify $x \in Hom(U,V)_{\mathcal{C}}$ with $i_{V,V'}x \in Hom(U,V')_{\mathcal{C}}$. Similarly, if $U' \leq U$, we can identify $x \in Hom(U,V)_{\mathcal{C}}$ with $xi_{U',U} \in Hom(U',V)_{\mathcal{C}}$. The following therefore makes sense.

Definition 4.2. Let $\mathcal{C}$ be an LK^*-algebroid over a C^*-algebra A. Let $V \in Ob(\mathcal{C})_0$. Then a *state* on V is a continuous linear map $\sigma\colon M_V \to \mathbb{C}$ such that $\sigma(1_V) = 1$ for the identity $1_V \in Hom(V,V)_{\mathcal{C}}$, and $\sigma(p) \geq 0$ if $p \in M_V$ is positive.

In particular, note that $\sigma(xx^*) \geq 0$ for all $x \in M_V$.

Proposition 4.3. *Let $x, y \in M_U$. Then $\sigma(xy^*) = \overline{\sigma(yx^*)}$.*

Proof. Let $\lambda \in \mathbb{C}$. Then we know that

$$0 \leq \sigma((x+\lambda y)(x+\lambda y)^*) = \sigma(xx^*) + |\lambda|^2\sigma(yy^*) + \lambda\sigma(yx^*) + \overline{\lambda}\sigma(xy^*).$$

Now the sum $\sigma(xx^*) + |\lambda|^2\sigma(yy^*)$ is a real number so the sum $\lambda\sigma(yx^*) + \overline{\lambda}\sigma(xy^*)$ is also real. Taking $\lambda = 1$, and $\lambda = i$, we see, respectively, that

$$Im\,\sigma(yx^*) = -Im\,\sigma(xy^*), \quad Re\,\sigma(yx^*) = Re\,\sigma(xy^*).$$

The result now follows. □

Proposition 4.4. *Let $x, y \in M_U$. Then*

$$|\sigma(xy^*)|^2 \leq \sigma(xx^*)\sigma(yy^*).$$

Proof. The result is obvious if $\sigma(xy^*) = 0$. So let $\sigma(xy^*) \neq 0$. Let $\lambda \in \mathbb{R}$, and define

$$\alpha = \frac{\lambda\sigma(xy^*)}{|\sigma(xy^*)|}.$$

By the above $\sigma((x+\alpha y)(x+\alpha y)^*) \geq 0$, so

$$\lambda^2 \sigma(yy^*) + 2\lambda|\sigma(xx^*)| + \sigma(xx^*) \geq 0$$

for all $\lambda \in \mathbb{R}$. Consideration of the descriminant of this quadratic yields the desired result. $\square$

For a state σ on V, set

$$N_V = \{x \in M_V \mid \sigma(xx^*) = 0\}$$

and let $\pi \colon M_V \to M_V/N_V$ be the quotient map. Then by the above two propositions, we have an inner product on the space M_V/N_N defined by the formula

$$\langle \pi(x), \pi(y) \rangle = \sigma(yx^*).$$

We can complete the quotient space M_V/N_V to obtain a Hilbert space H_V.

Lemma 4.5. *Let $\mathcal{C}$ be an LK^*-algebroid over a C^*-algebra A. Let $V \in Ob(\mathcal{C})_0$. Let σ be a state on V. Then for all $x \in M_V$ and $a \in A$ we have*

$$\|\pi(xa)\| \leq \|\pi(x)\| \cdot \|a\|.$$

Proof. Set

$$b = \frac{aa^*}{\|a\|^2}.$$

Then $\|b\| = 1$, so $1 - b$ is positive. Hence, by functional calculus on the C^*-algebra A, we have $c \in A$ such that $c^2 = 1 - b$. Observe

$$(cx)(cx)^* = x(1-b)x^*$$

so

$$\sigma(x(1-b)x^*) \geq 0$$

from which it follows that

$$\sigma(xx^*) \geq \sigma(xbx^*) = \sigma\left(\frac{xaa^*x^*}{\|a\|^2}\right)$$

and the result follows. $\square$

Similarly

$$\|\pi(ax)\| \leq \|a\| \cdot \|\pi(x)\|.$$

It follows that we have a representation $\theta \colon A \to \mathcal{B}(H_V)$ defined by writing

$$\theta(a)(\pi(x)) = \pi(ax).$$

Theorem 4.6. *Let $\mathcal{C}$ be an LK^*-category over a C^*-algebra A, and let $U \in Ob(\mathcal{C})_0$. Let σ be a state on U. Then there is a representation $\rho \colon \mathcal{C} \to \mathcal{U}(H)$ for some Hilbert space H, and an element $u \in H$ such that $\|u\| = 1$, and*

$$\sigma(x) = \langle u, \rho(x)u \rangle$$

for all $x \in M_U$.

Proof. Let $U \leq W$. Define a state on V by $\sigma_W(x) = \sigma(x|_{x^1[W]})$ if $x \in M_W$. Then we can form a Hilbert space H_W by the above process. Let $H = \lim_{U \leq W} H_W$. We have a representation $\theta \colon A \to \mathcal{B}(H)$ defined as above.

By the second axiom for the partial involution, for any dense object V, $\pi[M_V]$ is a dense subset of H. Let $\rho(V) = \pi[M_V]$. Let $x \in Hom(V,W)_{\mathcal{C}}$. Then we have an operator $M_x \colon \pi(M_V) \to \pi(M_W)$ defined by the formula

$$M_x \pi(y) = \pi(xy).$$

Let $(u_n, M_x u_n) \to (u, v)$ as $n \to \infty$, with respect to the norm on $H_V \oplus H_W$ defined by the inner product. Set $u_n = \pi(y_n)$, $u = \pi(y)$, and $v = \pi(z)$. Then we have

$$(\pi(y_n), \pi(xy_n)) \to (\pi(y), \pi(z))$$

as $n \to \infty$, that is to say

$$\sigma((y_n - y)(y_n - y)^*) \to 0, \quad \sigma((xy_n - z)(xy_n - z)^*) \to 0$$

as $n \to \infty$.

From the first of these and proposition 4.4, we see that $\sigma((xy_n - xy)(xy_n - xy))^*) \to 0$ as $n \to \infty$. Combining this with the second of the above limits, we see that $xy - z \in N_V$, and hence that (u, v) belongs to the graph of M_x. In other words, we have shown that M_x is a closed operator.

Let $1_U \in Hom(U, U)_{\mathcal{C}}$ be the identity. Set $u = \pi(1_U)$. Then

$$\|u\|^2 = \sigma(1) = 1$$

and

$$\langle u, \rho(x)u \rangle = \langle \pi(1_U), \pi(x1_U) \rangle = \sigma(x).$$

It is now routine to check that (ρ, θ) is a represenation of $\mathcal{C}$. □

Lemma 4.7. *Let $\mathcal{C}$ be an LK^*-category, and let U be a dense object. Let $x \in M_U$, $x \neq 0$. Then we have a state, σ, on U such that $\sigma(xx^*) > 0$.*

Proof. Let M_U^+ be the set of positive elements of the locally convex vector space M_U. Let $M_U^{\mathbb{R}}$ be the smallest real vector space containing M_U^+. Then M_U^+ is a closed convex subspace of $M_U^{\mathbb{R}}$. Hence, by the second geometric form of the Hahn-Banach theorem, we have a continuous linear map $\varphi\colon M_U^{\mathbb{R}} \to \mathbb{R}$ and real numbers $\alpha, \beta \in \mathbb{R}$ such that

$$\varphi(-xx^*) < \beta < \varphi(y)$$

for all $y \in M_U^+$.

Taking $y = 0$, we see that $\beta < 0$, so $\varphi(xx^*) > 0$. Suppose $\varphi(y) < 0$ for some $y \in M_U^+$. Let $\lambda = \frac{\beta}{\varphi(y)} > 0$. Then λy is positive, and $\varphi(\lambda y) = \beta$, which contradicts the above inequality. Therefore $\varphi(y) \geq 0$ for all $y \in M_U^+$.

Let $\psi(z) = \frac{1}{\varphi(1)}\varphi(z)$ for $z \in M_U^{\mathbb{R}}$. Then $\psi(y) \geq 0$ if y is positive, $\psi(1) = 1$, and $\psi(xx^*) > 0$.

Extend ψ to a complex linear functional $\tilde{\psi}\colon M_U^{\mathbb{R}} + iM_U^{\mathbb{R}} \to \mathbb{C}$ by the formula

$$\tilde{\psi}(u + iv) = \psi(u) + i\psi(v).$$

Then by the Hahn-Banach theorem there is a continuous linear extension $\sigma\colon M_U \to \mathbb{C}$ of $\tilde{\psi}$. By construction, σ is a state with $\sigma(xx^*) > 0$. □

Theorem 4.8. *Let $\mathcal{C}$ be an LK^*-category over a C^*-algebra A. Then we have a faithful representation (ρ, θ).*

Proof. Pick $V \in Ob(\mathcal{C})_0$. Let σ be a state on V. Then by the above, we have a representation $(\rho_\sigma, \theta_\sigma)$ on a Hilbert space H, and a vector $u \in H$ such that

$$\sigma(x) = \langle u, \rho(x)u \rangle$$

for all $x \in M_V$.

Let Σ be the set of states on U. Let

$$\rho_V = \oplus_{\sigma \in \Sigma} \rho_\sigma, \quad \theta_V = \oplus_{\sigma \in \Sigma} \theta_\sigma.$$

Let $x \in Hom(U, V)_{\mathcal{C}}$. Then by the above lemma, we have a state σ with $\sigma(xx^*) > 0$. It follows that $\rho(x) \neq 0$.

Hence, if we define

$$\rho = \oplus_{V \in Ob(\mathcal{C})_0} \rho_V, \quad \theta = \oplus_{V \in Ob(\mathcal{C})_0} \theta_V,$$

then (ρ, θ) is a faithful representation, and we are done. □

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