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# An Analytic Novikov Conjecture for Semigroups

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## Abstract

In this article we formulate a version of the analytic Novikov conjecture for semigroups rather than groups, and show that the descent argument from coarse geometry generalises effectively to this new situation.

## 1 Introduction

For the purposes of this article, a *semigroup* is a set  $P$  equipped with an associative binary operation  $P \times P \rightarrow P$ , such that we have a *unit element*  $e \in P$  where  $pe = ep = p$  for all  $p \in P$ , and the *left cancellation property* holds, that is to say  $pq = pr$  implies  $q = r$  for all  $p, q, r \in P$ . Note that the left cancellation property tells us that the unit element  $e$  is unique.

In [7], both the reduced and maximal  $C^*$ -algebras associated to a semigroup are defined, issues associated to amenability examined, and  $K$ -theory groups computed. The computations of  $K$ -theory groups lead to a natural question, namely whether a version of the Baum-Connes conjecture (see for instance [1]) could be formulated for semigroups.

In this paper, we make a first step towards such a conjecture, formulating an *analytic assembly map*  $\beta: K_n^P(EP) \rightarrow K_n C_r^*(P)$ , where  $C_r^*(P)$  is the reduced  $C^*$ -algebra of the semigroup  $P$ , and  $EP$  is the classifying space for free  $P$ -actions. We conjecture that this map is injective for torsion-free semigroups.

We also show that the descent argument from the coarse Baum-Connes conjecture, as explained for example in [12], or more generally in [11], still works in the semigroup case. Thus the analytic Novikov conjecture holds for semigroups where the space  $EP$  is a finite  $P$ -CW-complex, and has a compatible coarse structure where the coarse Baum-Connes conjecture is satisfied. We conclude the article by looking at some simple examples where the descent argument applies.

The descent argument works in the same way as it does for groups, but to carry it out we need to generalise parts of the general theory of equivariant homology for group actions to the semigroup case. These generalisations are fortunately mainly straightforward, and the details can be found in sections 4 and 5.

The broad strategy of the proof is as follows.

- We define a notion of generalised homology theory for  $P$ -spaces, where  $P$  is a semigroup.
- We prove that a natural transformation of generalised homology theories for  $P$ -spaces is an isomorphism for finite  $P$ -CW-complexes if it is an isomorphism for homogeneous  $P$ -spaces.
- We show that the  $K$ -homology of homotopy fixed-point defines a generalised homology theory for  $P$ -spaces, and that there is a natural transformation from  $P$ -equivariant  $K$ -homology to the  $K$ -homology of homotopy fixed point sets. We use the above argument to prove that this natural transformation is an isomorphism for finite  $P$ -CW-complexes.
- A commutative diagram relates the above transformation to the analytic assembly map and the coarse Baum-Connes conjecture, which lets us deduce the descent result.

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## 2 Semigroup Actions

Let  $P$  be a semigroup. Let  $X$  be a set. A *left  $P$ -action* on  $X$  is a map  $P \times X \rightarrow X$ , written  $(p, x) \mapsto px$ , such that  $p(qx) = (pq)x$  for all  $p, q \in P$  and  $x \in X$ .

Similarly, a *right  $P$ -action* on  $X$  is a map  $X \times P \rightarrow X$ , written  $(x, p) \mapsto xp$ , such that  $(xp)q = x(pq)$  for all  $p, q \in P$  and  $x \in X$ .

For a set  $X$  equipped with a left  $P$ -action, and a subset  $A \subseteq X$ , we write

$$pA = \{pa \mid a \in A\} \quad PA = \bigcup_{p \in P} pA.$$

Given sets  $X$  and  $Y$  with left  $P$ -actions, a map  $f: X \rightarrow Y$  is called *equivariant* if  $f(px) = pf(x)$  for all  $x \in X$  and  $p \in P$ . We similarly talk about equivariant maps between sets equipped with right  $P$ -actions.

A  $P$ -*space* is a topological space equipped with a continuous right  $P$ -action. Just as for groups acting on spaces, we distinguish certain types of  $P$ -spaces. Given  $P$ -spaces  $X$  and  $Y$ , we write  $\text{Map}_P(X, Y)$  to denote the set of all continuous equivariant maps from  $X$  to  $Y$ . It is a topological space, with the compact open topology.

**Definition 2.1** Let  $X$  be a  $P$ -space. Then we call  $X$ :

- *Free* if for all  $x \in X$ , there is an open neighbourhood  $U \ni x$  such that  $Up \cap U = \emptyset$  for all  $p \in P \setminus \{e\}$ , where  $e$  is the unit element of  $P$ .
- *Cocompact* if there is a compact subset  $K \subseteq X$  such that  $X = KP$ .

A *fundamental domain* in a  $P$ -space  $X$  is a subset  $D \subseteq X$  such that every element  $x \in X$  can be written uniquely as  $x = sp$ , where  $s \in D$  and  $p \in P$ .

We call an equivariant continuous map  $f: X \rightarrow Y$  *proper* if whenever  $Z \subseteq Y$  is cocompact, the inverse image  $f^{-1}[Z] \subseteq X$  is also cocompact.

**Example 2.2** We call a subset  $S \subseteq P$  a *generating set* if every element of  $P$  is a product of elements of  $S$ . The *Cayley graph*  $\text{Cay}(P; S)$  is an oriented labelled graph with set of vertices  $P$ . There is an oriented edge from  $p$  to  $q$  if  $p = sq$ . There is a right  $P$ -action on the space  $\text{Cay}(P; S)$  defined by right-multiplication on the vertices, and extending to be linear on the edges. The  $P$ -space  $\text{Cay}(P; S)$  is cocompact if  $S$  is finite. The  $P$ -action on the vertices is free; it is also free on the edges if  $P$  contains no elements of order 2.

**Example 2.3** The *infinite join* (see [8]),  $P * P * P * \dots$  of countably many copies of the semigroup  $P$  is a free and weakly contractible  $P$ -space.

Let  $X$  and  $Y$  be metric spaces. Recall (see for instance [13]) that a (not necessarily continuous) map  $f: X \rightarrow Y$  is called a *coarse map* if:

- For all  $R > 0$  there exists  $S > 0$  such that if  $d(x, y) < R$ , then  $d(f(x), f(y)) < S$ .
- Let  $B \subseteq Y$  be bounded. Then  $f^{-1}[B] \subseteq X$  is also bounded.

A *coarse  $P$ -space* is a proper metric space  $X$  equipped with a  $P$ -action such that for each  $p \in P$ , the map  $p: X \rightarrow X$  is both coarse and continuous.

Note that for a generating set  $S$ , the Cayley graph  $\text{Cay}(P; S)$  is an example of a coarse  $P$ -space.

**Definition 2.4** Let  $P$  be a semigroup. We call an equivalence relation,  $\sim$ , on  $P$  a *right congruence* if whenever  $p \sim q$  and  $p, q, r \in P$ , we have  $pr \sim qr$ .

Observe that if we have a right congruence,  $\sim$ , we have a right  $P$ -action on the set of equivalence classes  $P/\sim$  defined by writing  $([p])q = [pq]$ , where  $[r]$  is the equivalence class containing an element  $r \in P$ . The quotient  $P/\sim$  can be considered a  $P$ -space with the discrete topology.

**Definition 2.5** A *homogeneous  $P$ -space* is a  $P$ -space  $X$  such that there is an equivariant homeomorphism  $X \rightarrow P/\sim$  for some right congruence  $\sim$ .

We now define a class of  $P$ -spaces of particular importance to us, called  $P$ -CW-complexes. Firstly, write

$$D^{n+1} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 \leq 1\}$$

and

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 = 1\}.$$

Note that  $S^n \subseteq D^{n+1}$ . An  $n$ -dimensional  $P$ -cell is a  $P$ -space of the form  $X \times D^n$ , where  $X$  is a homogeneous  $P$ -space, and  $P$  acts trivially on  $D^n$ .

Given a  $P$ -space  $Y$  and  $P$ -cell  $X \times D^n$  equipped with a continuous equivariant map  $f: X \times S^{n-1} \rightarrow Y$ , we can form a  $P$ -space

$$(X \times D^n) \cup_{X \times S^{n-1}} Y = \frac{(X \times D^n) \amalg Y}{\sim}$$

where  $(x, s) \sim f(x, s)$  if  $(x, s) \in X \times S^{n-1}$ .

We call the  $P$ -space  $(X \times D^n) \cup_{X \times S^{n-1}} Y$  the space obtained from  $Y$  by *attaching* the  $P$ -cell  $X \times D^n$  by the map  $f$ .

**Definition 2.6** A *finite  $P$ -CW-complex* is a  $P$ -space  $X$  together with a sequence of subspaces

$$X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$$

such that:

- Each inclusion  $X^i \hookrightarrow X^{i+1}$  is equivariant.
- The space  $X^0$  is a finite disjoint union of homogeneous  $P$ -spaces.
- The space  $X^k$  is equivariantly homeomorphic to the space obtained from  $X^{k-1}$  by attaching finitely many  $k$ -dimensional  $P$ -cells.

The above sequence  $X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$  is called a *CW-decomposition* of  $X$ . Assuming  $X^n \neq X^{n-1}$ , the number  $n$  is called the *dimension* of the cell decomposition.

Note that any finite  $P$ -CW-complex is cocompact. The following is fairly clear.

**Proposition 2.7** *Let  $X$  be a finite  $P$ -CW-complex. Then  $X$  is free if and only if it has a CW-composition in which, for all  $k$ , every  $k$ -dimensional  $P$ -cell takes the form  $P \times D^n$ .  $\square$*

### 3 The Coarse Baum-Connes Conjecture

Let  $X$  be a proper metric space. Recall that a Hilbert space  $H$  is called an  $X$ -module if the  $C^*$ -algebra of bounded linear operators  $\mathcal{L}(H)$  is equipped with a  $*$ -homomorphism  $\rho: C_0(X) \rightarrow \mathcal{L}(H)$ .

Let  $\mathcal{K}(H)$  be the  $C^*$ -algebra of compact operators on  $H$ . Then we call an  $X$ -module  $H$  *ample* if  $\overline{\rho[C_0(X)]H} = H$  and  $\rho(f) \in \mathcal{K}(H)$  implies  $f = 0$ .

**Definition 3.1** Let  $H$  be an  $X$ -module, and let  $T \in \mathcal{L}(H)$ . Then:

- We call  $T$  *locally compact* if  $\rho(f)T, T\rho(f) \in \mathcal{K}(H)$  for all  $f \in C_0(X)$ .
- We call  $T$  *pseudolocal* if  $\rho(f)T - T\rho(f) \in \mathcal{K}(H)$  for all  $f \in C_0(X)$ .
- We define the *support* of  $T$ ,  $\text{Supp}(T) \subseteq X \times X$ , to be the set of pairs  $(x, y) \in X \times X$  such that for all open sets  $U \ni x$  and  $V \ni y$ , we have  $f \in C_0(U)$  and  $g \in C_0(V)$  such that  $\rho(f)T\rho(g) \neq 0$ .
- We call  $T$  *controlled* if the support  $\text{Supp}(T)$  is contained in a neighbourhood of the diagonal,  $\Delta_R = \{(x, y) \in X \times X \mid d(x, y) < R\}$ , for some  $R > 0$ .

**Definition 3.2** Let  $H$  be an ample  $X$ -module. Then we define  $D^*(X)$  to be the smallest  $C^*$ -subalgebra of  $\mathcal{L}(H)$  containing all pseudolocal and controlled operators.

We define  $C^*(X)$  to be the smallest  $C^*$ -subalgebra of  $\mathcal{L}(H)$  containing all locally compact and controlled operators.

Now,  $C^*(X)$  is a  $C^*$ -ideal in  $D^*(X)$ , so we have a short exact sequence

$$0 \rightarrow C^*(X) \rightarrow D^*(X) \rightarrow \frac{D^*(X)}{C^*(X)} \rightarrow 0.$$

Further, as shown in [6, 12], the  $K$ -theory group of the quotient,  $K_n\left(\frac{D^*(X)}{C^*(X)}\right)$ , is isomorphic to the  $K$ -homology group  $K_{n-1}(X)$ , and the  $K$ -theory group  $K_n C^*(X)$  does not depend on a particular choice of  $X$ -module. Thus, looking at the boundary maps in the long exact sequence of  $K$ -theory groups (see for example [14, 16]), we obtain a map

$$\alpha: K_*(X) \rightarrow K_* C^*(X)$$

called the *coarse assembly map*.

The *coarse Baum-Connes conjecture* asserts that this map is an isomorphism whenever the space  $X$  has bounded geometry and is uniformly contractible; we refer the reader again to [6, 12] for details, including precisely what the terms bounded geometry and uniformly contractible mean.

The coarse Baum-Connes conjecture is known to be true for a vast number of spaces, perhaps most notably bounded geometry and uniformly contractible spaces which can be uniformly embedded in Hilbert space (see [17]), but, as shown in [5], is false in general.

## 4 Equivariant Homology

**Definition 4.1** Let  $f, g: X \rightarrow Y$  be equivariant maps between  $P$ -spaces. A  $P$ -homotopy between  $f$  and  $g$  is an equivariant continuous map  $H: X \times [0, 1] \rightarrow Y$  such that  $H(-, 0) = f$  and  $H(-, 1) = g$ .

We call  $H$  a *proper  $P$ -homotopy* if  $H$  is a proper equivariant continuous map.

Above, the space  $X \times [0, 1]$  is given the  $P$ -action defined by the formula  $(x, t)p = (xp, t)$  where  $p \in P$ ,  $x \in X$  and  $t \in [0, 1]$ .

If a  $P$ -homotopy exists between maps  $f$  and  $g$ , we call them  $P$ -homotopic, and write  $f \simeq_P g$ . The notion of being  $P$ -homotopic is an equivalence relation. The equivalence relation of being properly  $P$ -homotopic is defined similarly.

A continuous equivariant map  $f: X \rightarrow Y$  is called a  $P$ -homotopy equivalence if there is a continuous equivariant map  $g: Y \rightarrow X$  such that  $g \circ f \simeq_P id_X$  and  $f \circ g \simeq_P id_Y$ . We write  $X \simeq_P Y$  when a  $P$ -homotopy equivalence  $X \rightarrow Y$  exists.

**Definition 4.2** A locally finite  $P$ -homology theory,  $h_*^P$ , graded over  $/Z$ , consists of a sequence of functors,  $h_n^P$  (where  $n \in \mathbb{Z}$ ), from the category of  $P$ -spaces and proper equivariant maps to the category of abelian groups satisfying the following axioms.

- Let  $f, g: X \rightarrow Y$  be equivariant continuous maps that are properly  $P$ -homotopic. Then the maps  $f_*, g_*: h_n^P(X) \rightarrow h_n^P(Y)$  induced by the functor  $h_n^P$  are equal for all  $n$ .
- Let  $X = A \cup B$  be a  $P$ -space, where  $A, B \subseteq X$  are open, and  $PA \subseteq A$ ,  $PB \subseteq B$ . Consider the inclusions  $i: A \cap B \hookrightarrow A$ ,  $j: A \cap B \hookrightarrow B$ ,  $k: A \hookrightarrow X$  and  $l: B \hookrightarrow X$ . Let  $\alpha = (i_*, -j_*): h_n^P(A \cap B) \rightarrow h_n^P(A) \oplus h_n^P(B)$  and  $\beta = k_* + l_*: h_n^P(A) \oplus h_n^P(B) \rightarrow h_n^P(X)$ . Then we have natural maps  $\partial: h_n^P(X) \rightarrow h_{n-1}^P(A \cap B)$  fitting into a long exact sequence

$$\rightarrow h_n^P(A \cap B) \xrightarrow{\alpha} h_n^P(A) \oplus h_n^P(B) \xrightarrow{\beta} h_n^P(X) \xrightarrow{\partial} h_{n-1}^P(A \cap B) \rightarrow .$$

- $h_n(\emptyset) = \{0\}$  for all  $n$ .

We call the first of these axioms *homotopy invariance*. The long exact sequence in the second axiom is called the *Mayer-Vietoris sequence* associated to the decomposition  $X = A \cup B$ .

We can also talk about locally finite  $P$ -homology theories on subcategories of the category of  $P$ -spaces and proper equivariant maps, for instance on the category of free  $P$ -spaces.

**Lemma 4.3** Let  $X$  and  $Y$  be  $P$ -spaces, let  $f: X \times S^{n-1} \rightarrow Y$  be a proper equivariant continuous map, and let  $Z = (X \times D^n) \cup_{X \times S^{n-1}} Y$ . Then we have a natural long exact sequence

$$\rightarrow h_n^P(X \times S^{n-1}) \rightarrow h_n^P(X) \oplus h_n^P(Y) \rightarrow h_n^P(Z) \xrightarrow{\partial} h_{n-1}^P(X \times S^{n-1}) \rightarrow$$

Further, the map  $\partial$  arises from a Mayer-Vietoris sequence associated to a decomposition of  $Z$ .

**Proof:** Let  $\pi: (X \times D^n) \amalg Y \rightarrow Z$  be the quotient map. We can choose open sets  $U \subseteq \pi[X \times D^n]$  and  $V \supseteq \pi[Y]$  such that:

- $U \cup V = Z$ .
- $PU \subseteq U, PV \subseteq V$ .
- $U \simeq_P X \times D^n \simeq_P X$ .
- $V \simeq_P Y$ .
- $U \cap V \simeq_P X \times S^{n-1}$ .

Applying the Mayer-Vietoris sequence of the decomposition  $X = U \cup V$ , along with homotopy invariance, we get a long exact sequence

$$\rightarrow h_n^P(X \times S^{n-1}) \rightarrow h_n^P(X) \oplus h_n^P(Y) \rightarrow h_n^P(Z) \xrightarrow{\partial} h_{n-1}^P(X \times S^{n-1}) \rightarrow$$

□

**Definition 4.4** Let  $h_*^P$  and  $k_*^P$  be locally finite  $P$ -homology theories. A *natural transformation*  $\tau: h_*^P \rightarrow k_*^P$  is a sequence of natural transformations  $\tau: h_n^P \rightarrow k_n^P$  that preserves Mayer-Vietoris sequences.

**Lemma 4.5** Let  $X$  be a  $P$ -space, and let  $\tau: h_*^P \rightarrow k_*^P$  be a natural transformation between  $P$ -homology theories such that the maps  $\tau: h_n^P(X) \rightarrow k_n^P(X)$  are isomorphisms.

Then the maps  $\tau: h_n^P(X \times S^k) \rightarrow k_n^P(X \times S^k)$  are all isomorphisms.

**Proof:** Observe

$$X \times S^0 = X_1 \amalg X_2$$

where  $X_1$  and  $X_2$  are both equivariantly homeomorphic to  $X$ . Certainly  $X_1 \cap X_2 = \emptyset$ , so  $h_n^P(X_1 \cap X_2) = 0$  for all  $n$ , and the Mayer-Vietoris sequence tells us that  $h_n^P(X \times S^0) = h_n^P(X) \oplus h_n^P(X)$ . Similarly,  $k_n^P(X \times S^0) = k_n^P(X) \oplus k_n^P(X)$ .

It follows immediately that the map  $\tau: h_n^P(X \times S^0) \rightarrow k_n^P(X \times S^0)$  is an isomorphism.

Now suppose the map  $\tau: h_n^P(X \times S^{k-1}) \rightarrow k_n^P(X \times S^{k-1})$  is an isomorphism for all  $n$ . We can write  $S^k = A \cup B$ , where  $A \cong D^n, B \cong D^n$  and  $A \cap B \simeq S^{k-1}$ , so  $X \times A \simeq_P B \times A \simeq_P X$ . Then we have a commutative diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} h_n^P(X \times S^{k-1}) & \rightarrow & h_n^P(X) \oplus h_n(X) & \rightarrow & h_n^P(S^k \times X) & \rightarrow & h_{n-1}^P(S^{k-1} \times X) & \rightarrow & h_{n-1}^P(X) \oplus h_{n-1}^P(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ k_n^P(X \times S^{k-1}) & \rightarrow & k_n^P(X) \oplus k_n(X) & \rightarrow & k_n^P(S^k \times X) & \rightarrow & k_{n-1}^P(S^{k-1} \times X) & \rightarrow & k_{n-1}^P(X) \oplus h_{n-1}^P(X) \end{array}$$

By the five lemma, we see the map  $\tau: h_n^P(X \times S^k) \rightarrow k_n^P(X \times S^k)$  is an isomorphism for all  $n$ . The desired result now follows by induction □



**Theorem 4.6** *Let  $\tau: h_*^P \rightarrow k_*^P$  be a natural transformation of  $P$ -homology theories such that  $\tau: h_m^P(X) \rightarrow k_m^P(X)$  is an isomorphism whenever  $X$  is a homogeneous  $P$ -space. Then  $\tau: h_m^P(Z) \rightarrow k_m^P(Z)$  is an isomorphism whenever  $Z$  is a finite  $P$ -CW-complex.*

**Proof:** Let  $Z$  be a finite  $P$ -CW-complex. Then we have subsets

$$Z^0 \subseteq Z^1 \subseteq \dots \subseteq Z^n = Z$$

where  $Z^0$  is a finite disjoint union of homogeneous  $P$ -spaces, and  $Z^k$  is equivariantly homeomorphic to the space obtained from  $Z^{k-1}$  by attaching finitely many  $k$ -dimensional  $P$ -cells.

Certainly, the map  $\tau: h_m^P(Z^0) \rightarrow k_m^P(Z^0)$  is an isomorphism for all  $m$ .

Let  $Y$  be a  $P$ -space such that the map  $\tau: h_m^P(Y) \rightarrow k_m^P(Y)$  is an isomorphism for all  $m$ . Suppose we have an attaching map  $f: X \times S^{n-1} \rightarrow Y$ , for a homogeneous  $P$ -space  $X$ . Let  $Y' = (X \times D^n) \cup_{X \times S^{n-1}} Y$ . Then it follows by lemma 4.3, lemma 4.5 and the five lemma that the map  $\tau: h_m^P(Y') \rightarrow k_m^P(Y')$  is an isomorphism for all  $m$ .

But this proves the desired result by induction.  $\square$

The following is proved similarly.

**Theorem 4.7** *Let  $\tau: h_*^P \rightarrow k_*^P$  be a natural transformation of  $P$ -homology theories such that  $\tau: h_m^P(P) \rightarrow k_m^P(P)$  is an isomorphism for all  $m$ . Then  $\tau: h_m^P(Z) \rightarrow k_m^P(Z)$  is an isomorphism whenever  $Z$  is a free finite  $P$ -CW-complex.  $\square$*

## 5 Homotopy Fixed Point Sets

Let  $E$  be a  $P$ -space. Given an equivariant continuous map  $f: X \rightarrow Y$ , we have an induced map  $f_*: \text{Map}_P(E, X) \rightarrow \text{Map}_P(E, Y)$  defined by the formula  $f_*(g)(\lambda) = f(g(\lambda))$  where  $g \in \text{Map}_P(E, X)$  and  $\lambda \in E$ .

**Proposition 5.1** *Let  $E$  be a finite free  $P$ -CW-complex. Let  $X$  be (non-equivariantly) weakly contractible. Then the space  $\text{Map}_P(E, X)$  is also weakly contractible.*

**Proof:** Let  $E$  be a 0-dimensional free  $P$ -CW-complex. Then  $E$  is a disjoint union of finitely many, say  $k$ , copies of  $P$ . Hence

$$\text{Map}_P(E, X) = \text{Map}_P(P, X)^k \cong X^k$$

which is weakly contractible since  $X$  is.

More generally, if  $E$  is a free  $P$ -CW-complex, it is obtained by attaching finitely many cells to a 0-dimensional  $P$ -CW-complex. So to prove the result, it suffices to show, by induction, that if  $Y$  is a free  $P$ -space where  $\text{Map}_P(Y, X)$

is weakly contractible, and  $E$  is a space obtained by attaching a free  $P$ -cell to  $Y$ , then  $\text{Map}_P(E, X)$  is weakly contractible.

Let  $f: P \times S^{n-1} \rightarrow Y$  be a continuous equivariant map, and let  $E = (P \times D^n) \cup_{P \times S^{n-1}} Y$ . It is a standard example from algebraic topology (see for example [15]) that the inclusion  $S^{n-1} \rightarrow D^n$  is a cofibration. Hence, the inclusion  $P \times S^{n-1} \rightarrow P \times D^n$  has the homotopy extension property for  $P$ -homotopies, and so the push-out  $Y \rightarrow E$  also has this property. Now, the cofibration  $Y \rightarrow E$  has cofibre

$$E/Y \cong \frac{P \times D^n}{P \times S^{n-1}} \cong P \times S^n.$$

Hence we have a *fibration*  $\text{Map}_P(E, X) \rightarrow \text{Map}_P(Y, X)$ , with fibre  $\text{Map}_P(P \times S^n, X)$ . Pick  $f_0 \in \text{Map}_P(Y, X)$ . Then the fibre is the inverse image of  $f_0$ , that is the set of  $P$ -equivariant maps  $f: E \rightarrow X$  which restrict to  $f_0$  on  $Y$ , which by definition of  $E$  is

$$\text{Map}_P\left(\frac{E}{Y}, X\right) \cong \text{Map}_P(P \times S^n, X).$$

Now, by hypothesis, the space  $\text{Map}_P(Y, X)$  is weakly contractible. We know that the space  $X$  is weakly contractible, and  $\text{Map}_P(P \times S^n, X) = \text{Map}(S^n, X)$ , so the homotopy groups of  $\text{Map}(S^n, X)$  are all zero, as those of the space  $X$  are all zero. Therefore, by the long exact sequence of homotopy groups associated to a fibration, the homotopy groups of the space  $\text{Map}_P(E, X)$  are all zero, that is to say  $\text{Map}_P(E, X)$  is weakly contractible, and we are done.  $\square$

The following immediately follows from the above by looking at mapping fibres.

**Corollary 5.2** *Let  $E$  be a finite free  $P$ -CW-complex. Let  $X$  and  $Y$  be  $P$ -spaces, and let  $f: X \rightarrow Y$  be an equivariant map that is (non-equivariantly) a weak equivalence. Then the induced map  $f_*: \text{Map}_P(E, X) \rightarrow \text{Map}_P(E, Y)$  is a weak equivalence.*  $\square$

**Definition 5.3** We define a *classifying space for free  $P$ -actions*,  $EP$ , to be a free  $P$ -space that is weakly contractible.

**Proposition 5.4** *A classifying space  $EP$  always exists. If we can choose  $EP$  to be a finite free  $P$ -CW-complex, then it is unique up to  $P$ -homotopy equivalence.*

**Proof:** By example 2.3, a free and weakly contractible  $P$ -space  $EP$  exists.

Suppose that we can choose  $EP$  to be a finite free  $P$ -CW-complex. Let  $X$  be another free and weakly contractible  $P$ -CW-complex. Then by proposition 5.1, the spaces  $\text{Map}_P(EP, X)$  and  $\text{Map}_P(X, EP)$  are weakly contractible. In particular, they are non-empty, so we have continuous equivariant maps  $f: EP \rightarrow X$  and  $g: X \rightarrow EP$ .

Similarly, the spaces  $\text{Map}_P(EP, EP)$  and  $\text{Map}_P(X, X)$  are weakly contractible, so the sets  $\pi_0 \text{Map}_P(EP, EP)$  and  $\pi_0 \text{Map}_P(X, X)$  are trivial. Since

$g \circ f, id_{EP} \in \text{Map}_P(EP, EP)$ , they must be  $P$ -homotopic. Similarly,  $f \circ g, id_P \in \text{Map}_P(X, X)$  are  $P$ -homotopic. In other words, the composites  $g \circ f$  and  $f \circ g$  are both  $P$ -homotopic to identity maps, and we are done.  $\square$

Note that if  $G$  is a group with torsion, then the classifying space  $BG$  is never a finite  $CW$ -complex, and so the universal cover  $EG$  is never a finite  $G$ - $CW$ -complex. The same is true in the semigroup world, since if a semigroup has torsion, it contains a finite subgroup, which of course is a subgroup with torsion.

**Definition 5.5** Let  $X$  be a  $P$ -space. We define the *homotopy fixed point set* of  $X$  to be the space  $X^{hP} = \text{Map}_P(EP, X)$ .

By the above, if  $EP$  is a finite free  $P$ - $CW$ -complex, then up to homotopy, the space  $X^{hP}$  does not depend on which version of the space  $EP$  we have chosen. Further, by proposition 5.1 and corollary 5.2, if  $X$  is weakly contractible, then so is  $X^{hP}$ , and if  $f: X \rightarrow Y$  is a weak equivalence, then so is  $f_*: X^{hP} \rightarrow Y^{hP}$ .

Now, let  $X$  be a cocompact coarse  $P$ -space. Then the  $C^*$ -algebra  $C_0(X)$  is equipped with a left  $P$ -action defined by writing  $(pf)(x) = f(xp)$  for all  $f \in C_0(X)$ ,  $p \in P$  and  $x \in X$ .

Let  $H$  be a Hilbert space equipped with a left  $P$ -action by isometries. Let  $v_p: H \rightarrow H$  be the isometry associated to the element  $p \in P$ . Then the  $C^*$ -algebra  $\mathcal{L}(H)$  comes equipped with a left  $P$ -action by homomorphisms defined by writing

$$pT = v_p T v_p^* \quad p \in P, T \in \mathcal{L}(H).$$

We call  $H$  an *equivariant  $X$ -module* if it comes equipped an equivariant  $*$ -homomorphism  $\rho: C_0(X) \rightarrow \mathcal{L}(H)$ .

Now, observe in the case of an ample equivariant  $X$ -module that the semigroup  $P$  acts on the  $C^*$ -algebras  $C^*(X)$  and  $D^*(X)$  on the left by  $*$ -homomorphisms. We can therefore form homotopy fixed point sets  $C^*(X)^{hP}$  and  $D^*(X)^{hP}$ . If  $EP$  is a finite  $P$ - $CW$ -complex, then the space  $EP$  is cocompact, and these sets are  $C^*$ -algebras, with addition, multiplication and involution defined pointwise, and the norm defined by taking the supremum

$$\|f\| = \sup\{\|f(x)\| \mid x \in EP\}$$

for  $g \in C^*(X)^{hP}$  or  $g \in D^*(X)^{hP}$ . Further,  $C^*(X)^{hP}$  is a  $C^*$ -ideal in  $D^*(X)^{hP}$ , so we can form the quotient  $D^*(X)^{hP}/C^*(X)^{hP}$ .

Let us write

$$K_n^{hP}(X) = K_{n+1} \left( \frac{D^*(X)^{hP}}{C^*(X)^{hP}} \right).$$

By the open mapping theorem, the quotient map  $D^*(X) \rightarrow D^*(X)/C^*(X)$  has a continuous section (though not one that is a  $*$ -homomorphism). This implies that we have a natural isomorphism

$$K_n^{hP}(X) \cong K_{n+1} \left( \frac{D^*(X)}{C^*(X)} \right)^{hP}.$$

**Proposition 5.6** *The sequence of functors  $K_*^{hP}$  is a locally finite  $P$ -homology theory.*

**Proof:** Let  $U(X)$  be the stable unitary group of the  $C^*$ -algebra  $D^*(X)/C^*(X)$ . Then the groups  $K_{n-1}(X)$  and  $K_{n-1}^{hP}(X)$  are the homotopy groups of  $U(X)$  and  $U(X)^{hP}$  respectively.

By proper homotopy-invariance of  $K$ -homology, the inclusions  $i_0, i_1: X \rightarrow X \times [0, 1]$  defined by the formulae  $i_0(x) = (x, 0)$  and  $i_1(x) = (x, 1)$  respectively, induce weak equivalences  $U(X) \rightarrow U(X \times [0, 1])$ . By corollary 5.2, these maps both induce weak equivalences  $U(X)^{hG} \rightarrow U(X \times [0, 1])^{hG}$ , and so isomorphisms  $K_n^{hP}(X) \rightarrow K_n^{hP}(X \times [0, 1])$ . Proper  $P$ -homotopy-invariance of the functors  $K_n^{hP}$  now follows.

Let  $X = A \cup B$  be a  $P$ -space, where  $A, B \subseteq X$  are open, and  $PA \subseteq A$ ,  $PB \subseteq B$ . Then by looking at Mayer-Vietoris sequences in  $K$ -homology, we have a weak fibration sequence

$$U(A \cap B) \rightarrow U(A) \vee U(B) \rightarrow U(X)$$

and so, by corollary 5.2, a weak fibration sequence

$$U(A \cap B)^{hP} \rightarrow U(A)^{hP} \vee U(B)^{hP} \rightarrow U(X)^{hP}.$$

The existence of Mayer-Vietoris sequences for the sequence of functors  $K_*^{hP}$  now also follows.  $\square$

## 6 Semigroup $C^*$ -algebras and assembly

Let  $P$  be a semigroup. Let  $l^2(P)$  be the Hilbert space with an orthonormal basis indexed by  $P$ , that is to say we have an orthonormal basis  $\{e_p \mid p \in P\}$ .

Given  $p \in P$ , we have an isometry  $v_p: l^2(P) \rightarrow l^2(P)$  defined by the formula  $v_p(e_q) = e_{pq}$ . Note that for this to be an isometry, we need the left-cancellation property. The semigroup  $P$  acts on the space  $l^2(P)$  by the formula  $pw = v_p(w)$  where  $p \in P$  and  $w \in l^2(P)$ .

The following definition comes from [7].

**Definition 6.1** *The reduced semigroup  $C^*$ -algebra,  $C_r^*(P)$ , is the smallest  $C^*$ -subalgebra of the space of bounded linear operators  $\mathcal{L}(l^2(P))$  that contains the set of isometries  $\{v_p \mid p \in P\}$ .*

Note that reduced group  $C^*$ -algebras are an obvious special case.

More generally, let  $H$  be a Hilbert space equipped with a right  $P$ -action by isometries. Let  $v_p: H \rightarrow H$  be the isometry associated to the element  $p \in P$ . Then  $C_r^*(P)$  is isomorphic to the smallest  $C^*$ -subalgebra of  $\mathcal{L}(H)$  containing the set of isometries  $\{v_p \mid p \in P\}$ .

As noted in the previous section, the  $C^*$ -algebra  $\mathcal{L}(H)$  has a left  $P$ -action defined by writing  $pT = v_p T v_p^*$   $p \in P$ ,  $T \in \mathcal{L}(H)$ .

Let  $X$  be a free cocompact  $P$ -space. Then we can write  $X = U_1 \cup \dots \cup U_n$ , where each  $U_i$  is an open  $P$ -space that is equivariantly homeomorphic to  $P \times W_i$  for some space  $W_i$  with trivial  $P$ -action.

Now, let  $H$  be an equivariant  $X$ -module, meaning  $H$  is a Hilbert space with a  $P$ -action and an equivariant  $*$ -homomorphism  $\rho: C_0(X) \rightarrow \mathcal{L}(H)$ . Then on each set  $E_i$ ,  $H$  restricts to an equivariant  $U_i$ -module through composing the  $*$ -homomorphism  $\rho$  with the inclusion  $C_0(U_i) \hookrightarrow C_0(X)$ .

**Definition 6.2** We call an equivariant  $X$ -module  $H$   $P$ -adequate if  $X$  is a finite union of open  $P$ -spaces  $U$ , where:

- Each  $U$  is an open  $P$ -space that is equivariantly homeomorphic to  $P \times W$  for some space  $W$  with trivial  $P$ -action.
- The restriction of  $H$  to  $U$  is equivariantly isomorphic to a  $U$ -module of the form  $l^2(P) \otimes H'_W$ , where  $H'_W$  is an ample  $W$ -module. Here the  $*$ -homomorphism  $C_0(U) \rightarrow \mathcal{L}(l^2(P) \otimes H'_W)$  is the composite

$$C_0(U) \rightarrow C_0(P \times W) \rightarrow \mathcal{L}(l^2(P) \otimes H'_W)$$

defined in the obvious way.

**Definition 6.3** Let  $H$  be a  $P$ -adequate  $X$ -module. We define  $D_P^*(X)$  to be the smallest  $C^*$ -subalgebra of  $\mathcal{L}(H)$  containing all pseudolocal, controlled operators that are fixed under the action of  $P$ . We define  $C_P^*(X)$  to be the smallest  $C^*$ -subalgebra of  $\mathcal{L}(H)$  containing all locally compact, controlled operators that are fixed under the action of  $P$ .

**Theorem 6.4** Let  $P$  be a semigroup. Let  $X$  be a cocompact free coarse  $P$ -space. Then the  $C^*$ -algebras  $C_r^*(P)$  and  $C_P^*(X)$  are Morita equivalent.

**Proof:** Let  $H$  be a  $P$ -adequate equivariant  $X$ -module. Decompose  $X$  into a finite union of  $n$  open  $P$ -sets,  $U$ , as in definition 6.2. Let  $T: H \rightarrow H$  be  $P$ -fixed, locally compact and controlled.

Then, by definition of locally compact, and the  $P$ -action, when restricted to  $U$ ,  $T$  can be considered to be a  $P \times P$  matrix of compact operators. Overall,  $T$  is an  $n \times n$  matrix  $(T_{ij})$  where each  $T_{ij}$  is a  $P \times P$  matrix of compact operators. Since  $T$  is translation-invariant under  $P$ , so is each  $T_{ij}$ .

Since  $T$  is also controlled, we see that  $T \in C_r^*(\Gamma) \otimes \mathcal{K}$ . More generally, taking limits, if  $T \in C_P^*(X)$ , then  $T \in C_r^*(\Gamma) \otimes \mathcal{K}$ .

Conversely, any operator of the form  $v_P \otimes k$ , where  $p \in P$  and  $k$  is compact, is both controlled and locally compact. Any element  $M_n(C_r^*(\Gamma) \otimes \mathcal{K})$  is a limit of sums of such operators, and so

$$M_n(C_r^*(\Gamma) \otimes \mathcal{K}) \subseteq C_P^*(X).$$

The desired result now follows.  $\square$

The above argument was essentially made for groups in lemma 5.14 of [12]. Although the cited argument was problematic, it works here as we have restricted ourselves to free actions.

Now, let  $X$  be any  $P$ -space. Then we have a short exact sequence

$$0 \rightarrow C_P^*(X) \rightarrow D_P^*(X) \rightarrow \frac{D_P^*(X)}{C_P^*(X)} \rightarrow 0.$$

By the above, when  $X$  is cocompact, we can identify the  $K$ -theory groups  $K_*(C_P^*(X))$  and  $K_*C_r^*(P)$ . Thus, looking at the boundary maps in the long exact sequence of  $K$ -theory groups (see for example [14, 16]), we obtain a map

$$\beta: K_{*+1} \left( \frac{D_P^*(X)}{C_P^*(X)} \right) \rightarrow K_*C_r^*(P)$$

called the *analytic assembly map*.

This assembly map is a generalisation of the corresponding map for groups; see for example [12].

**Definition 6.5** Let  $X$  be a coarse  $P$ -space. Then we define the  $P$ -equivariant  $K$ -homology groups of  $X$  by writing

$$K_n^P(X) = K_{n+1} \left( \frac{D_P^*(X)}{C_P^*(X)} \right).$$

**Definition 6.6** We say a torsion-free semigroup  $P$  satisfies the *analytic Novikov conjecture* if we have a cocompact classifying space  $EP$  such that the map

$$\beta: K_n^P(EP) \rightarrow K_nC_r^*(P)$$

is injective.

We restrict our attention to torsion-free semigroups, since, in the case of groups, the map  $\beta$  is not in general injective for groups with torsion, though it *is* conjectured to be rationally injective. However, all of our arguments here are for torsion-free semigroups.

Further, the equivariant  $K$ -homology groups  $K_n^P(EP)$  are not necessarily easy to compute. The issue is that, unlike the group case, even if  $X$  is a free  $P$ -space, we do not have the formula  $K_n^P(X) \cong K_n(X/P)$ , which holds in the classical case. We go into further detail on this issue in the final section when we look at examples.

## 7 Descent

The *descent argument*, outlined in this section, tells us that the coarse Baum-Connes conjecture, along with certain mild extra conditions, implies the analytic Novikov conjecture.

**Lemma 7.1** *We have a natural transformation  $\theta_*: K_n^P(X) \rightarrow K_n^{hP}(X)$  that is an isomorphism whenever  $X$  is a finite free  $P$ -CW-complex.*

**Proof:** Let  $U(X)$  and  $U_P(X)$  be the stable unitary groups of the  $C^*$ -algebras  $D^*(X)/C^*(X)$  and  $D_P^*(X)/C_P^*(X)$  respectively. Since the space  $EP$  is weakly contractible, we have a natural weak equivalence

$$\begin{aligned} j: U_P(X) &\simeq \text{Map}(EP, U_P(X)) \\ &= \text{Map}(EP, \text{Map}_P(P, U_P(X))) \\ &= \text{Map}_P(EP, \text{Map}(P, U_P(X))) \\ &= \text{Map}(P, U_P(X))^{hG} \end{aligned}$$

Let  $i: D_P^*(X)/C_P^*(X) \rightarrow D^*(X)/C^*(X)$  be defined by the inclusions  $C_P^*(X) \hookrightarrow C^*(X)$  and  $D_P^*(X) \hookrightarrow D^*(X)$ . Then we have a natural map  $k: \text{Map}(P, U_P(X)) \rightarrow U(X)$  defined by writing  $k(f) = i_*f(e)$ , where  $e$  is the identity element of the semigroup  $P$ .

Taking homotopy fixed point sets, we obtain a natural map  $k': \text{Map}(P, U_P(X))^{hP} \rightarrow U(X)^{hP}$ . Composing with the map  $j$ , we have a natural map

$$\theta = k' \circ j: U_P(X) \rightarrow U(X)^{hP}$$

and so a natural induced map  $\theta_*: K_n^P(X) \rightarrow K_n^{hP}(X)$ .

Let  $c: P \rightarrow +$  be the constant map onto the one point space. Then the composition  $c_* \circ i_* = i_* \circ c_*: U_P(P) \rightarrow U(+)$  is certainly a homotopy-equivalence, and the map  $k: \text{Map}(P, U_P(+)) \rightarrow U(+)$  is a homeomorphism, and so a weak equivalence. By corollary 5.2, the map  $k'$  is also a weak equivalence.

Thus the map  $\theta$  is a weak equivalence in this case, making the induced map  $\theta_*: K_n^P(P) \rightarrow K_n^{hP}(P)$  an isomorphism.

By theorem 4.7, the map  $\theta_*: K_n^P(X) \rightarrow K_n^{hP}(X)$  is therefore an isomorphism whenever  $X$  is a finite free  $P$ -CW-complex.  $\square$

Now, let  $X$  be a cocompact coarse  $P$ -space.

Now, we can define a map  $\eta_*: K_n(D_P^*(X)) \rightarrow K_n(D^*(X)^{hP})$  in much the same way as the map  $\theta_*$  in the above lemma, and so, whenever  $X$  is a cocompact coarse  $P$ -space, we have a commutative diagram

$$\begin{array}{ccccc} K_n(D_P^*(X)) & \xrightarrow{v_P} & K_n^P(X) & \xrightarrow{\beta} & K_n C_r^*(P) \\ \downarrow & & \downarrow & & \\ K_n(D^*(X)^{hP}) & \xrightarrow{v_{hP}} & K_n^{hP}(X) & & \end{array}$$

where  $\beta$  is the analytic assembly map.

**Theorem 7.2** *Let  $X$  be a free coarse  $P$ -space that is a free finite  $P$ -CW-complex as a topological space. Suppose the coarse Baum-Connes conjecture holds for  $X$ . Then the analytic assembly map  $\beta$  is split-injective for  $P$ .*

**Proof:** The coarse Baum-Connes conjecture for  $X$  implies that  $K_n D^*(X) = 0$  for all  $n$ . Hence, by proposition 5.1,  $K_n(D^*(X)^{hP}) = 0$  for all  $n$ .

Now, by the previous lemma, the map  $\theta_*: K_n^P(X) \rightarrow K_n^{hP}(X)$  in the above commutative diagram is an isomorphism.

It follows that the map  $v_p$  is zero, so the map  $\beta$  is split-injective as required.  $\square$

**Corollary 7.3** *Let  $P$  be a semigroup with a classifying space  $EP$  that is a coarse  $P$ -space and a finite  $P$ -CW-complex. Suppose the coarse Baum-Connes conjecture holds for the space  $EP$ .*

*Then the analytic Novikov conjecture holds for the semigroup  $P$ .*  $\square$

## 8 Ore Semigroups

**Definition 8.1** A semigroup  $P$  is a *left Ore semigroup* if:

- For all  $p, q, r \in P$ , if  $pq = pr$ , or  $qp = rp$ , then  $q = r$ .
- For all  $p, q \in P$ , we have  $pP \cap qP \neq \emptyset$ .

It is shown in [2] that a semigroup  $P$  can be embedded into a group  $G$  such that  $G = P^{-1}P = \{q^{-1}p \mid p, q \in P\}$  if and only if  $P$  is a left Ore semigroup.

One of the main results of [4] is a computation of the  $K$ -theory of the reduced  $C^*$ -algebra of an Ore semigroup. It is more precise than corollary 7.3, but covers what appears to be a smaller class of semigroups. To state it, we need some more terminology.

**Definition 8.2** Let  $P$  be a semigroup. A *right ideal* of  $P$  is a subset  $X \subseteq P$  such that for all  $x \in X$  and  $p \in P$ , we have  $xp \in X$ .

Let us write

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{q \in P \mid pq \in X\}.$$

Let  $\mathcal{J}$  be the smallest family of right ideals of  $P$  such that:

- $\emptyset, P \in \mathcal{J}$ .
- If  $X \in \mathcal{J}$  and  $p \in P$ , then  $pX, p^{-1}X \in \mathcal{J}$ .
- If  $X, Y \in \mathcal{J}$ , then  $X \cap Y \in \mathcal{J}$ .

The following comes from [7].

**Definition 8.3** We call elements of  $\mathcal{J}$  the *constructible right ideals* of  $P$ . We say that the constructible right ideals of  $P$  are *independent* if for all right ideals  $X_1, \dots, X_n \in \mathcal{J}$ , if the union

$$X = \bigcup_{j=1}^n X_j$$

is a right ideal, then  $X = X_j$  for some  $j$ .



We refer to [7] for examples and further analysis. Now, let  $P$  be an Ore semigroup, and let  $G = P^{-1}P$ . For a right ideal  $X$ , form the group

$$G_X = \{g \in G \mid gX = X\}.$$

Let  $E_X$  be the orthogonal projection from the Hilbert space  $l^2(P)$  to the Hilbert space  $l^2(X)$ . Let  $D_r^\infty(P)$  be the smallest  $C^*$ -subalgebra of the space of bounded linear operators  $\mathcal{L}(l^2(P))$  that contains the set of projections  $\{E_{p^{-1}X} \mid p \in P, X \in \mathcal{J} \setminus \{\emptyset\}\}$ .

For an index set  $I$ , let us write  $c_0(I) = \bigoplus_{i \in I} \mathbb{C}$ . Let  $\chi \subseteq \mathcal{J} \setminus \{\emptyset\}$  be a set containing precisely one representative for each set in the collection  $G \setminus (P^{-1}(\mathcal{J} \setminus \{\emptyset\}))$ .

Theorem 7.3 of [4] states the following.

**Theorem 8.4** *Let  $P$  be a left Ore semigroup whose constructible right ideals are independent. Suppose that the group  $G = P^{-1}P$  satisfies the Baum-Connes conjecture with coefficients in the  $G$ - $C^*$ -algebras  $c_0(P^{-1}(\mathcal{J} \setminus \{\emptyset\}))$  and  $D_r^\infty(P)$ .*

*Then the groups  $K_n(C_r^*(P))$  and  $\bigoplus_{X \in \chi} K_n(C_r^*(G_X))$  are isomorphic.  $\square$*

See [1] for more details on the Baum-Connes conjecture.

## 9 Examples

We conclude this article by looking at some simple examples where corollary 7.3 applies. We can use Yu's theorem from [17] that the coarse Baum-Connes conjecture holds for any bounded geometry coarse space which can be uniformly embedded in Hilbert space. Of course, Yu's theorem is a heavy piece of machinery for this example; the author's proof of the coarse Baum-Connes conjecture for coarse  $CW$ -complexes in [9, 10] also suffices.

The explicit examples all satisfy the conditions of theorem 8.4, so the result applies to them. In particular, by split-injectivity, we know that the  $K$ -theory group  $K_*^P(EP)$  embeds as a subgroup of the  $K$ -theory group  $K_*(C_r^*P)$ , which the techniques from [4] can be used to compute.

### The semigroup $\mathbb{N}$

The semigroup  $\mathbb{N}$  acts freely on  $\mathbb{R}^+$  by writing  $(n, x) \mapsto n + x$ , where  $n \in \mathbb{N}$ , and  $x \in \mathbb{R}^+$ . With the coarse structure defined by the metric, the space  $\mathbb{R}^+$  is certainly uniformly embeddable in Hilbert space, so the coarse Baum-Connes conjecture holds.

Now,  $\mathbb{R}^+$  is a finite free  $\mathbb{N}$ - $CW$ -complex, with a single 0-cell,  $\mathbb{N}$ , and 1-cell  $\mathbb{N} \times [0, 1]$ , with attaching map  $f: \mathbb{N} \times \{0, 1\} \rightarrow \mathbb{N}$  defined by the formula  $f(n, k) = n + k$ .

Now  $\mathbb{R}_+$  is weakly contractible, so we can take  $E\mathbb{N} = \mathbb{R}_+$ , and, by corollary 7.3, the analytic Novikov conjecture holds for  $\mathbb{N}$ .

Similarly, let  $\mathbb{N}^\times$  be the group of non-zero natural numbers with group operation defined by multiplication. Then  $E\mathbb{N} = [1, \infty)$ , with free  $\mathbb{N}$ -action defined by writing  $(n, x) \mapsto nx$ . As above, the analytic Novikov conjecture holds for  $\mathbb{N}^\times$ .

Actually, the first example shows us why we do not have  $K_*^P(EP) \cong K_*(EP/P)$  in general. To see this, observe, first of all, that the  $C^*$ -algebra  $C_r^*(\mathbb{N})$  is simply the Toeplitz algebra. It is well-known from any course in  $K$ -theory (see for example [14, 16]) that we therefore have  $K_1 C_r^*(\mathbb{N}) = 0$ . It follows from the analytic Novikov conjecture for  $\mathbb{N}$  that  $K_1^{\mathbb{N}}(E\mathbb{N}) = 0$ .

On the other hand,  $\mathbb{R}^+/\mathbb{N} = S^1$ , and  $K_1(S^1) = \mathbb{Z}$ , so  $K_*^{\mathbb{N}}(\mathbb{R}^+) \not\cong K_*(\mathbb{R}^+/\mathbb{N})$ .

## Free semigroups

The *free semigroup on  $n$  generators*,  $V_n$ , is the set of words in an alphabet with  $n$  letters, say  $e_1, \dots, e_n$ . Let  $S = \{e_1, \dots, e_n\}$ . Then  $V_n$  certainly acts freely on the Cayley graph  $\text{Cay}(V_n; S)$ , which is weakly contractible.

So we can take  $EV_n = \text{Cay}(V_n; S)$ . The space  $EV_n$  is a finite  $V_n$ - $CW$ -complex, with a single 0-cell, the set of vertices, and 1-cells  $P \times \{e_1\} \times [0, 1], \dots, P \times \{e_n\} \times [0, 1]$ . The attaching map  $f: P \times \{e_i\} \times \{0, 1\} \rightarrow P$  is defined by writing  $f(p, e_i, 0) = p$  and  $f(p, e_i, 1) = pe_i$ .

The space  $EV_n$  is certainly uniformly embeddable in an infinite-dimensional Hilbert space, so by corollary 7.3, the analytic Novikov conjecture holds for the free semigroup  $V_n$ .

## Products

Let  $P$  and  $Q$  be semigroups such that  $EP$  and  $EQ$  are finite free  $P$ - and  $Q$ - $CW$ -complexes respectively, and  $EP$  and  $EQ$  have compatible coarse structures where  $P$  and  $Q$  act respectively by coarse continuous maps. Suppose  $EP$  and  $EQ$  are uniformly embeddable in Hilbert spaces  $H_P$  and  $H_Q$  respectively.

Then we can take  $E(P \times Q) = EP \times EQ$ . The space  $EP \times EQ$  is a free finite  $P \times Q$ - $CW$ -complex, which is a coarse  $P \times Q$ -space uniformly embeddable in  $H_P \times H_Q$ . Thus the analytic Novikov conjecture holds for  $P \times Q$ .

In particular, by the above, the analytic Novikov conjecture holds for the semigroup  $\mathbb{N}^m \times (\mathbb{N}^\times)^n$  for all  $m$  and  $n$ .

## The $ax + b$ semigroup over $\mathbb{N}$

The  $ax + b$  semigroup over  $\mathbb{N}$  is defined in [3] as the set

$$P_{\mathbb{N}} = \left\{ \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \mid n \in \mathbb{N}^\times, k \in \mathbb{N} \right\}$$

with group operation defined by matrix multiplication.

The group  $P_{\mathbb{N}}$  acts freely and cocompactly on the space  $[0, \infty) \times [1, \infty)$  by the formula

$$\begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ky \\ ny \end{pmatrix}.$$

As above, the space  $[0, \infty) \times [1, \infty)$  has the structure of a finite  $P_{\mathbb{N}}$ -CW-complex. As a coarse space, it is uniformly embeddable in Hilbert space. So the analytic Novikov conjecture holds for  $P_{\mathbb{N}}$ .

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