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Fractional Herglotz Variational Principles with Generalized Caputo Derivatives

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Abstract. We obtain Euler–Lagrange equations, transversality conditions and a Noether-like theorem for Herglotz-type variational problems with Lagrangians depending on generalized fractional derivatives. As an application, we consider a damped harmonic oscillator with time-dependent mass and elasticity, and arbitrary memory effects.

Mathematics Subject Classification (2010). 26A33; 49K05; 49S05.

Keywords. Fractional variational principles, Herglotz problem, Euler–Lagrange equations, generalized fractional operators.

1. Introduction

Fractional variational principles and their applications is a subject under strong current research [3, 19, 20]. For classical fields with fractional derivatives, by using the fractional Lagrangian formulation, we can refer to [7]. An Hamiltonian approach to fractional problems of the calculus of variations is given in [25], where the Hamilton equations of motion are obtained in a manner similar to the one found in classical mechanics. In addition, classical fields with fractional derivatives are investigated using the Hamiltonian formalism [25]. A method for finding fractional Euler–Lagrange equations with Caputo derivatives, by making use of a fractional generalization of the classical Faà di Bruno formula, can be found in [5]. There the fractional Euler–Lagrange and Hamilton equations are obtained within the so called $1 + 1$ field formalism [5]. For discrete versions of fractional derivatives with a nonsingular Mittag-Leffler function see [1], where the properties of such fractional differences are studied and discrete integration by parts formulas proved in order to obtain Euler–Lagrange equations for discrete variational problems [1]. The readers interested in the discrete fractional calculus of variations are referred to the pioneer work of Bastos et al. [8, 9]. Here we are interested in the generalized continuous calculus of variations introduced by Herglotz.

The generalized variational principle firstly proposed by Gustav Herglotz in 1930 [14] gives a variational principle description of non-conservative

systems even when the Lagrangian is autonomous [28, 30]. It is essentially based on the following problem: find the trajectories $x(t)$, satisfying given boundary conditions, that extremize (minimize or maximize) the terminal value $z(b)$ of the functional z that satisfies the differential equation

$$\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a, b],$$

subject to the initial condition $z(a) = \gamma$. Herglotz proved that the necessary condition for a trajectory to be an extremizer of the generalized variational problem is to satisfy the generalized Euler–Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial \dot{x}} = 0.$$

The main physical motivation for the development of generalized variational methods behind the classical calculus of variations is linked to the inverse variational problem of classical mechanics in the cases where dissipation is not negligible [18]. In [2], Almeida and Malinowska have considered a fractional variational Herglotz principle, where fractionality stands in the dependence of the Lagrangian by the Caputo fractional derivative of the generalized variables. See also the more recent papers [33, 34] on fractional Herglotz variational principles. In [33], new necessary conditions for higher-order generalized variational problems with time delay, which are semi-invariant under a group of transformations that depends on arbitrary functions, is obtained. Fractional variational problems of Herglotz type of variable order are investigated in [34], where necessary optimality conditions, described by fractional differential equations depending on a combined Caputo fractional derivative of variable order, are proved, both for one and several independent variables [34]. Using such results, it is possible to find, by using a variational approach, the equations of motion of a dissipative mechanical system with memory. This is useful, since in many classical cases, memory effects play a relevant role in systems with dissipation. A relevant example is given by the Basset memory force acting on a sphere rotating in a Stokes fluid. It is well-known that this force can be represented by means of Caputo derivatives of order 1/2 (see, e.g., [6] and references therein). However, a limit in this approach stands to the fact that the particular choice of the dependence of the Lagrangian by the Caputo fractional derivative of the generalized variable, implies that it describes equations of motion of systems with power-law memory kernels. On the other hand, in the framework of the fractional calculus of variations (a quite recent topic of research, started from the seminal investigations of Riewe [26] and then developed by many researchers, see for example the recent monographs [3, 4, 16, 19, 20]), Odziejewicz et al. [21, 22] discusses the case of the Lagrangian depending on generalized Caputo-type derivatives with arbitrary completely monotonic kernels [19]. Here we consider a generalized calculus of variations in the sense of Herglotz with Lagrangians depending on generalized Caputo-type operators treated in [12, 19, 21]. Our aim is to find a general and adequate variational approach to describe mechanical systems with arbitrary memory forces.

The paper is organized as follows. In Section 2, we recall the necessary definitions and results from the generalized fractional variational calculus. Our results are then given in Sections 3, 4 and 5: we prove in Section 3 necessary optimality conditions of Euler–Lagrange type (Theorem 3.1) and transversality conditions (Theorem 3.3) to the generalized fractional variational problem of Herglotz; we show in Section 4 how our approach can deal, in an elegant way, with dissipative dynamical systems with memory effects and time-varying mass and elasticity; and we obtain a generalized fractional Herglotz Noether theorem (Theorem 5.2) in Section 5. We end with Section 6 of conclusions and some directions of future work.

2. Preliminaries

In this section, we recall the main definitions of the generalized Riemann–Liouville and Caputo-like operators and their properties, according to the analysis of generalized fractional variational principles developed in [19, 21]. For a general introduction to fractional differential operators and equations we refer to the classical encyclopedic book [27]. See also [24]. For an introduction to the fractional variational methods, and in particular integration by parts formulas for fractional integrals and derivatives, we refer to the monographs [4, 20]. For computational and numerical aspects see [3, 11, 15].

Definition 2.1. The operator K_P^α is given by

$$\begin{aligned} K_P^\alpha[f](x) &= K_P^\alpha[t \rightarrow f(t)](x) \\ &= p \int_a^x k_\alpha(x, t)f(t)dt + q \int_x^b k_\alpha(t, x)f(t)dt, \end{aligned}$$

where $P = \langle a, x, b, p, q \rangle$ is the parameter set, $x \in [a, b]$, $p, q \in \mathbb{R}$, and $k_\alpha(x, t)$ is a completely monotonic kernel.

For the sake of completeness, we should remark that similar generalizations of the Riemann–Liouville integrals have been considered in the framework of the fractional action-like variational approach (FALVA) [13, 17]. On the other hand, a similar generalization is considered in a probabilistic framework in [35].

Theorem 2.2 (See [21, Theorem 3]). *Let $k_\alpha \in L_1([a, b])$ and*

$$k_\alpha(x, t) = k_\alpha(x - t).$$

Then, the operator $K_P^\alpha : L_1([a, b]) \rightarrow L_1([a, b])$ is a well-defined bounded and linear operator.

Definition 2.3. Let P be a given parameter set and $\alpha \in (0, 1)$. The operator $A^\alpha = D \circ K_P^{1-\alpha}$ is the generalized Riemann–Liouville derivative, where D stands for the conventional integer order derivative operator.

The corresponding generalized Caputo derivative is defined as $B_P^\alpha = K_P^{1-\alpha} \circ D$. A key-role in the following analysis is played by the following theorem that provides the integration by parts formula for the generalized operators defined before.

Theorem 2.4 (See [21, Theorem 11]). *Let $\alpha \in (0, 1)$ and $P = \langle a, x, b, p, q \rangle$. If $f, K_P^{1-\alpha}g \in AC([a, b])$, where $P^* = \langle a, x, b, q, p \rangle$ and $k_\alpha(\cdot)$ is a square-integrable function on $\Delta = [a, b] \times [a, b]$, then*

$$\int_a^b g(x)B_P^\alpha[f](x)dx = f(x)K_{P^*}^{1-\alpha}[g](x)\Big|_b^a - \int_a^b f(x)A_{P^*}^\alpha[g](x)dx.$$

3. Generalized fractional Herglotz variational principles

One of the main aims of this work is to prove generalized Euler–Lagrange equations related to the generalized fractional variational principle of Herglotz. In particular, the generalization is based on the fact that the Lagrangian depends on the generalized Caputo derivative B_P^α . As explained before, by using this approach, we will be able to find a variational approach to mechanical systems involving an arbitrary (suitable) memory kernel $k_\alpha(t)$. Therefore, let us consider the differential equation

$$\dot{z}(t) = L(t, x(t), B_P^\alpha[x](t), z(t)), \quad t \in [a, b], \quad (3.1)$$

with the initial condition $z(a) = z_a$. We moreover assume that

- $x(a) = x_a, x(b) = x_b, x_a, x_b \in \mathbb{R}^n$,
- $\alpha \in (0, 1)$,
- $x \in C^1([a, b], \mathbb{R}^n), B_P^\alpha[x] \in C^1([a, b], \mathbb{R}^n)$,
- the Lagrangian $L : [a, b] \times \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ is of class C^1 and the maps $t \rightarrow \lambda(t) \frac{\partial L}{\partial B_P^\alpha x_j}[x, z](t)$ exist and are continuous on $[a, b]$, where we use the notations

$$[x, z](t) := (t, x(t), B_P^\alpha[x](t), z(t)),$$

$$\lambda(t) = \exp\left(-\int_a^t \frac{\partial L}{\partial z}[x, z](\tau)d\tau\right).$$

The generalized fractional Herglotz variational principle is formulated as follows:

Let functional $z(t) = z[x; t]$ be given by the differential equation (3.1) and $\eta \in C^1([a, b], \mathbb{R})$ be an arbitrary function such that $\eta(a) = \eta(b) = 0$ and $B_P^\alpha[\eta] \in C^1([a, b], \mathbb{R})$. Then, the value of the functional z has an extremum for the function x if and only if

$$\frac{d}{d\epsilon} z[x + \epsilon\eta; b] \Big|_{\epsilon=0} = 0.$$

We are now able to state and prove a generalized fractional necessary optimality condition of Euler–Lagrange type that, together with the fractional Herglotz Noether theorem (see Theorem 5.2), constitute the central results of the paper.

Theorem 3.1 (Generalized fractional Herglotz Euler–Lagrange equations).

Let function x be such that $z[x; b]$ in (3.1) attains an extremum. Then $x(t)$, $t \in [a, b]$, is a solution to the generalized Euler–Lagrange equations

$$\lambda(t) \frac{\partial L}{\partial x_j} [x, z](t) + A_{P^*}^\alpha \left(\lambda(t) \frac{\partial L}{\partial B_{P^*}^\alpha x_j} [x, z](t) \right) = 0, \quad (3.2)$$

$j = 1, \dots, n$.

Proof. The proof uses the generalized integration by parts formula for Caputo-like operators B_P^α given by Theorem 2.4. Let x be such that the functional $z[x; b]$ attains an extremum. The rate of change of z in the direction η is given by

$$\theta(t) = \left. \frac{d}{d\epsilon} z[x + \epsilon\eta; t] \right|_{\epsilon=0}.$$

The variation $\epsilon\eta$ of the argument in equation (3.1) is given by

$$\frac{d}{dt} z[x + \epsilon\eta; t] = L(t, x(t) + \epsilon\eta(t), B_P^\alpha[x](t) + \epsilon B_P^\alpha[\eta](t), z[x + \epsilon\eta; t]).$$

Observing that, from equation (3.1), we have

$$\frac{d}{dt} \theta(t) = \left. \frac{d}{d\epsilon} L \left(t, x(t) + \epsilon\eta(t), B_P^\alpha[x](t) + \epsilon B_P^\alpha[\eta](t), z[x + \epsilon\eta; t] \right) \right|_{\epsilon=0},$$

this gives a differential equation of the form

$$\frac{d\theta(t)}{dt} - \frac{\partial L}{\partial z} [x, z](t) \theta(t) = \sum_{j=1}^n \left(\frac{\partial L}{\partial x_j} [x, z](t) \eta_j(t) + \frac{\partial L}{\partial B_{P^*}^\alpha x_j} B_{P^*}^\alpha [\eta_j](t) \right),$$

whose solution is

$$\int_a^t \sum_{j=1}^n \left(\frac{\partial L}{\partial x_j} [x, z](\tau) \eta_j(\tau) + \frac{\partial L}{\partial B_{P^*}^\alpha x_j} B_{P^*}^\alpha [\eta_j](\tau) \right) \lambda(\tau) d\tau = \theta(t) \lambda(t) - \theta(a),$$

where $\lambda(t) = \exp \left(- \int_a^t \frac{\partial L}{\partial z} [x, z](\tau) d\tau \right)$ and $\theta(a) = 0$. For $t = b$, we get

$$\int_a^b \sum_{j=1}^n \left(\frac{\partial L}{\partial x_j} [x, z](t) \eta_j(t) + \frac{\partial L}{\partial B_{P^*}^\alpha x_j} B_{P^*}^\alpha [\eta_j](t) \right) \lambda(t) dt = \theta(b) \lambda(b).$$

Since $\theta(b)$ is the variation of $z[x; b]$, if x gives a maximum, also $\theta(b) = 0$, and therefore we get

$$\int_a^b \sum_{j=1}^n \left(\frac{\partial L}{\partial x_j} [x, z](t) \eta_j(t) + \frac{\partial L}{\partial B_{P^*}^\alpha x_j} B_{P^*}^\alpha [\eta_j](t) \right) \lambda(t) dt = 0.$$

Then, by using the integration by parts formula (see Theorem 2.4), and the fact that $\eta(a) = \eta(b) = 0$, we obtain the claimed result by the fundamental lemma of the calculus of variations. \square

Remark 3.2. Let $k_\alpha(x, t) = \frac{1}{\Gamma(1-\alpha)}(x-t)^{-\alpha}$, $\alpha \in (0, 1)$, and the parameter set be given by $P = \langle a, x, b, 1, 0 \rangle$. In this particular case, the operator B_P^α coincides with the standard Caputo fractional derivative ${}_a^C D_x^\alpha$, and our Theorem 3.1 gives the Euler–Lagrange equation of [2].

Observe that, in order to determine uniquely the unknown function that satisfies equation (3.1), the following system of differential equations

$$\begin{cases} \dot{z}(t) = L[x, z](t), \\ \lambda(t) \frac{\partial L}{\partial x_j}[x, z](t) + A_{P^*}^\alpha \left(\lambda(t) \frac{\partial L}{\partial B_{P^*}^\alpha x_j}[x, z](t) \right) = 0, \end{cases} \quad (3.3)$$

$j = 1, \dots, n$, should be studied with the given boundary conditions. In the case $x_j(b)$ is not fixed, similar arguments as those used in the proof of Theorem 3.1 allow us to obtain the generalized fractional integral transversality conditions (3.4).

Theorem 3.3 (Generalized fractional Herglotz transversality conditions). *Let x be such that $z(b) = z[x; b]$ in equation (3.1) attains an extremum. Then x is a solution to the system (3.3). Moreover, if $x_j(b)$ is not fixed, $j \in \{1, \dots, n\}$, then the integral transversality condition*

$$K_{P^*}^{1-\alpha} \left[t \rightarrow \lambda(t) \frac{\partial L}{\partial B_{P^*}^\alpha x_j}[x, z](t) \right] (b) = 0 \quad (3.4)$$

holds.

4. An application: generalized damped harmonic oscillator with memory effects

As already discussed in the literature, generalized variational methods can be useful to treat the inverse problem for dissipative systems where memory or damping effects are not negligible. For example, in [21], the authors discussed an application of generalized variational problems to the Caldirola–Kanai approach to quantum dissipative systems; while in [6] an application to the inverse problem for the Basset system was discussed. One can argue that the starting point to research on generalized variational problems was given by the Lemma of Bauer [10], stating that the equations of motion of a classical dissipative system with constant coefficients cannot be derived from a classical variational approach. Here we show the peculiarity of the approach considered in this paper, to treat dissipative dynamical systems with memory effects and time-varying mass and elasticity. Let us consider the mechanical system described by the following autonomous Lagrangian:

$$L(x(t), B_P^\alpha[x](t), z(t)) = \frac{1}{2}m(B_P^\alpha[x](t))^2 - \frac{1}{2}kx^2(t) + \lambda_0 z(t), \quad (4.1)$$

where we take $a = 0$, $b > 0$, $p = 1$ and $q = 0$. By using the generalized Euler–Lagrange equations discussed in Theorem 3.1, we have that the equation of

motion in this case is given by

$$mA_{\mathcal{P}^*}^\alpha \left(e^{-\lambda_0 t} (B_{\mathcal{P}}^\alpha[x](t)) \right) - ke^{-\lambda_0 t} x(t) = 0. \quad (4.2)$$

Equation (4.2) describes a dynamical oscillatory system with exponentially time-decaying mass and elasticity coefficient and arbitrary memory. The generalized *velocity* $B_{\mathcal{P}}^\alpha[x](t)$ can be physically interpreted as a time-weighted velocity, where memory effects are induced by viscosity (and clearly depends by the relaxation kernel k_α in the definition of the generalized operators $B_{\mathcal{P}}^\alpha$ and $A_{\mathcal{P}^*}^\alpha$). Therefore, the generalized Herglotz approach here considered provides a variational method to treat the inverse problem of a damped harmonic oscillator with time-depending mass and elasticity (as a consequence of the Herglotz approach) and with arbitrary memory effect implying a velocity delay (due to the dependence of the Lagrangian by a generalized fractional integral). Clearly, in the case in which the memory is neglected, that is, $\alpha \rightarrow 1$, the Lagrangian (4.1) takes the form

$$L(x(t), \dot{x}(t), z(t)) = \frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}kx^2(t) + \lambda_0 z(t) \quad (4.3)$$

and we recover from (4.2) the equation $m\frac{d}{dt}(e^{-\lambda_0 t}\dot{x}(t)) + ke^{-\lambda_0 t}x(t) = 0$ of an harmonic oscillator with dissipation. Moreover, if $\lambda_0 = 0$, then we are in the case in which the Lagrangian (4.3) does not depend on $z(t)$ and we have the classical equation of the harmonic oscillator: if $\alpha \rightarrow 1$ and $\lambda_0 = 0$, then the Euler-Lagrange equation (4.2) reduces to $m\ddot{x}(t) + kx(t) = 0$.

5. Noether theorem for generalized fractional Herglotz variational problems

The analysis of possible generalizations of Noether-type theorems to fractional and Herglotz variational principles, has been subject of recent research: we refer, for example, to [30, 31] and the references therein. Noether-like theorems play indeed a key-role in mathematical-physics by giving the relation between the invariance of the action with respect to some parametric transformation and the existence of conserved quantities [36, 37]. Here we consider a one-parameter family of transformations

$$\bar{x}_j = h_j(t, x, s), \quad j = 1, \dots, n, \quad (5.1)$$

depending on the parameter $s \in (-\epsilon, +\epsilon)$, with $h_j \in C^2$ such that

$$h_j(t, x, 0) = x_j \text{ for all } (t, x) \in [a, b] \times \mathbb{R}^n.$$

The Taylor expansion is given by

$$\begin{aligned} h_j(t, x, \delta) &= h_j(t, x, 0) + s\xi_j(t, x) + o(s) \\ &= x_j + s\xi_j(t, x) + o(s), \end{aligned}$$

where $\xi_j(t, x) = \partial_s h_j(t, x, s) \Big|_{s=0}$. In this case the linear approximation of the transformation (5.1) is simply given by

$$\bar{x}_j(t) = h_j(t, x(t), s), \quad j = 1, \dots, n. \quad (5.2)$$

Let

$$\theta(t) = \frac{d}{ds} \bar{z}[x + s\xi, t] \Big|_{s=0}$$

be the total variation produced by the transformation (5.1).

Definition 5.1. The transformation \bar{x} given by (5.1) leaves the functional z (3.1) invariant if $\theta(t) \equiv 0$.

By using the generalized fractional Euler–Lagrange equation (3.2) of Theorem 3.1, we are now able to state the following Noether-type result.

Theorem 5.2 (Generalized fractional Herglotz Noether theorem). *If the functional z in (3.1) is invariant in the sense of Definition 5.1, then*

$$\sum_{j=1}^n \hat{O}^\alpha \left[\lambda(t) \frac{\partial L}{\partial B_{P^*}^\alpha x_j} [x, z](t), \xi_j(t, x(t)) \right] = 0$$

holds along the solutions of the generalized Euler–Lagrange equation (3.2), where

$$\lambda(t) = \exp \left(- \int_0^t \frac{\partial L}{\partial z} [x, z](\tau) d\tau \right)$$

and

$$\hat{O}^\alpha [f, g] := f B_{P^*}^\alpha g - g A_{P^*}^\alpha f.$$

Proof. By using the transformation (5.2) into equation (3.1), we get

$$\frac{d}{dt} \bar{z}(t) = L(t, \bar{x}(t), B_{P^*}^\alpha \bar{x}(t), \bar{z}(t)).$$

Differentiating with respect to s and setting $s = 0$, we obtain

$$\dot{\theta}(t) - \frac{\partial L}{\partial z} \theta(t) = \sum_{j=1}^n \left(\frac{\partial L}{\partial x_j} \xi_j + \frac{\partial L}{\partial B_{P^*}^\alpha x_j} B_{P^*}^\alpha \xi_j \right), \quad (5.3)$$

where we omit, in order to simplify the notation, that the partial derivatives of L are evaluated at $[x, z](t)$ and ξ_j at $(t, x(t))$. The solution of (5.3) is given by

$$\theta(t)\lambda(t) - \theta(a) = \int_a^t \sum_{j=1}^n \left(\frac{\partial L}{\partial x_j} \xi_j + \frac{\partial L}{\partial B_{P^*}^\alpha x_j} B_{P^*}^\alpha \xi_j \right) \lambda(\tau) d\tau. \quad (5.4)$$

Since, along the solutions of the generalized Euler–Lagrange equation (3.2), we have that

$$\lambda(t) \frac{\partial L}{\partial x_j} = -A_{P^*}^\alpha \left(\lambda(t) \frac{\partial L}{\partial B_{P^*}^\alpha x_j} \right), \quad (5.5)$$

we obtain the claimed result by substituting (5.5) into (5.4). \square

6. Conclusions

We introduced the study of fractional variational problems of Herglotz type that depend on generalized fractional operators in the sense of [19, 23]. As a particular case, one gets a generalization of the Herglotz variational principle for non-conservative systems with Caputo derivatives. Main results give a necessary optimality condition of Euler–Lagrange type (Theorem 3.1), integral transversality conditions (Theorem 3.3), and a Noether-type theorem (Theorem 5.2). Our motivation comes from physics, where such variational principles can be used to describe mechanical systems with memory of arbitrary form. As an application, a fractional mechanical system is analyzed with a fractionally generalized velocity that reproduces, for $\alpha = 1$, the standard Lagrangian of a harmonic oscillator with exponential damping, which also contains the non-damped conservative oscillator.

The research here initiated can now be enriched in different directions, by trying to bring to the fractional setting the recent results [28, 29, 30, 31, 32] of Santos et al. on Herglotz variational problems.

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