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Inhabitants of interesting subsets of the Bousfield lattice

Andrew D. Brooke-Taylor^a, Benedikt Löwe^{b,c,d}, Birgit Richter^c

^a*School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom;
a.d.brooke-taylor@leeds.ac.uk*

^b*Institute for Logic, Language and Computation, Universiteit van Amsterdam, Postbus
94242, 1090 GE Amsterdam, The Netherlands; b.loewe@uva.nl*

^c*Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg,
Germany; birgit.richter@uni-hamburg.de*

^d*Churchill College & Faculty of Mathematics, University of Cambridge, Wilberforce Road,
Cambridge CB3 0WB, United Kingdom*

Abstract

In 1979, Bousfield defined an equivalence relation on the stable homotopy category. The set of Bousfield classes has some important subsets such as the distributive lattice, \mathbf{DL} , of all classes $\langle E \rangle$ which are smash idempotent and the complete Boolean algebra, \mathbf{cBA} , of closed classes. We provide examples of spectra that are in \mathbf{DL} , but not in \mathbf{cBA} ; in particular, for every prime p , the Bousfield class of the Eilenberg-MacLane spectrum $\langle H\mathbb{F}_p \rangle$ is in $\mathbf{DL} \setminus \mathbf{cBA}$.

1. Introduction

An important tool for understanding structural and computational phenomena in the stable homotopy category (*i.e.*, the homotopy category of spectra) is the Bousfield localization at a spectrum E , L_E [2]. In [1], Bousfield defines an equivalence relation on spectra such that the localization functor L_E depends only on the equivalence class of the spectrum E . These equivalence classes, called *Bousfield classes*, form a lattice.

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In the original paper [1] introducing the Bousfield lattice \mathbf{B} , Bousfield also introduces its subsets \mathbf{BA} and \mathbf{DL} and identifies the location of many explicit Bousfield classes. In [5, Definition 6.3], Hovey and Palmieri add a third interesting subset, denoted by \mathbf{cBA} . (We shall give definitions below.) It is easy to see that

$$\mathbf{BA} \subseteq \mathbf{cBA} \subseteq \mathbf{DL} \subseteq \mathbf{B}.$$

In this paper, we deal with the question of which and how many classes of spectra live in the various parts of \mathbf{B} defined by this chain of inclusions. We give lower bounds for the cardinality of $\mathbf{DL} \setminus \mathbf{cBA}$ and $\mathbf{cBA} \setminus \mathbf{BA}$ by identifying concrete examples of Bousfield classes in these complements. The cardinality results of this paper are graphically represented as in Figure 1 and concern the dark grey parts.

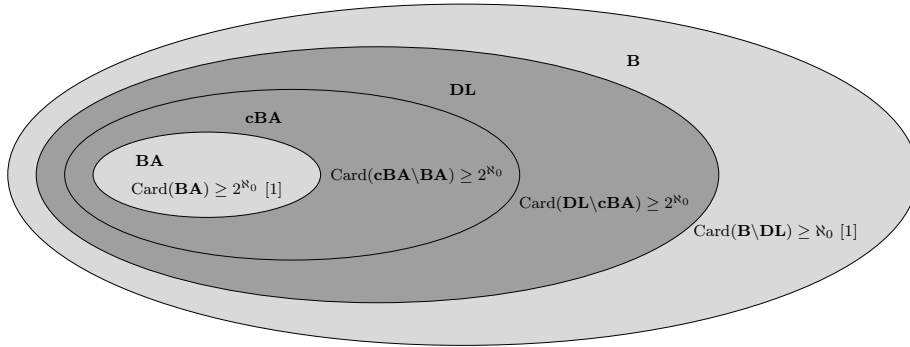


Figure 1: Lower bounds for the sizes of the four differences of subsets of \mathbf{B} .

2. Definitions

In order to fix notation, we give the relevant definitions, following closely the exposition in [5]. We consider the Bousfield equivalence of spectra [1]: two spectra X and Y are equivalent if for all spectra E , $X_*(E) = 0$ if and only if $Y_*(E) = 0$ (alternatively put: $X \wedge E \simeq *$ if and only if $Y \wedge E \simeq *$). For a spectrum X , we write $\langle X \rangle$ for the class of all spectra E with $X_*(E) = 0$. The class of all Bousfield classes is denoted by \mathbf{B} . By a theorem of Ohkawa [6, 3], it is known that \mathbf{B} is a set and

$$2^{\aleph_0} \leq \text{Card}(\mathbf{B}) \leq 2^{2^{\aleph_0}}.$$

This set is a poset with respect to reverse inclusion: $\langle X \rangle \leq \langle Y \rangle$ if and only if for all spectra Z , $Y_*Z = 0$ implies $X_*Z = 0$. The poset (\mathbf{B}, \leq) has a largest element $\mathbf{1} := \langle S \rangle$ where S is the sphere spectrum and we denote by $\mathbf{0}$ the minimal element which is the Bousfield class of the trivial spectrum. We work at a fixed but arbitrary prime p , *i.e.*, we consider p -local spectra.

For every prime p , $K(n)$ denotes the n th Morava K -theory spectrum with coefficients $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}]$ where the degree of v_n is $2p^n - 2$. We use the convention that $K(\infty)$ is the mod p Eilenberg-MacLane spectrum, $H\mathbb{F}_p$. For any subset $S \subseteq \mathbb{N} \cup \{\infty\}$, we denote by $K(S)$ the spectrum $\bigvee_{n \in S} K(n)$.

The topological operations \wedge and \vee of taking smash products and wedges, respectively, are well-defined on \mathbf{B} ; the class $\langle \bigvee_{i \in I} X_i \rangle$ is the least upper bound (“join”) in the structure (\mathbf{B}, \leq) of the classes $\langle X_i \rangle$ [1, (2.2)], but in general, \wedge does not produce the greatest lower bound. We can define the greatest lower bound (“meet”) by

$$\bigwedge \mathcal{X} := \bigvee \{ \langle Z \rangle ; \forall \langle X \rangle \in \mathcal{X} (\langle Z \rangle \leq \langle X \rangle) \},$$

and observe that \wedge and \bigwedge can differ quite a bit: the Brown-Comenetz dual I of the p -local sphere spectrum satisfies $\langle I \rangle \wedge \langle I \rangle = \mathbf{0} \neq \langle I \rangle = \langle I \rangle \bigwedge \langle I \rangle$ [1, Lemma 2.5].

The complete lattice $(\mathbf{B}, \bigwedge, \bigvee)$ is endowed with a pseudo-complementation function

$$aX := \bigvee \{ \langle Z \rangle ; Z \wedge X = \mathbf{0} \}$$

which is well-defined on Bousfield classes, *i.e.*, $a\langle X \rangle := \langle aX \rangle$ is independent of the choice of representative X of $\langle X \rangle$. The function a is not in general a complement. While $a^2 = \text{id}$ and $a\langle X \rangle \wedge \langle X \rangle = \mathbf{0}$, we may not have $a\langle X \rangle \vee \langle X \rangle = \mathbf{1}$ [1, Lemma 2.7]. Bousfield defined two subclasses of \mathbf{B} as follows:

$$\begin{aligned} \mathbf{BA} &:= \{ \langle X \rangle ; \langle X \rangle \vee a\langle X \rangle = \mathbf{1} \}, \text{ and} \\ \mathbf{DL} &:= \{ \langle X \rangle ; \langle X \rangle \wedge \langle X \rangle = \langle X \rangle \}. \end{aligned}$$

Many examples for classes in \mathbf{BA} or \mathbf{DL} are known. Bousfield showed in [1] that every Moore spectrum of an abelian group is in \mathbf{BA} and so are the periodic topological K -theory spectra $\langle KO \rangle = \langle KU \rangle$; furthermore, he shows that (arbitrary joins of) finite CW spectra also give classes in \mathbf{BA} . Every class of a ring spectrum is in \mathbf{DL} but not necessarily in \mathbf{BA} [1, § 2.6]; in particular, all Eilenberg-MacLane spectra of rings are in \mathbf{DL} , but, *e.g.*, the class of the Eilenberg-MacLane spectrum of the integers, $\langle H\mathbb{Z} \rangle$, is in $\mathbf{DL} \setminus \mathbf{BA}$ [1, Lemma 2.7]. However, the Brown-Comenetz duals of (p -local) spheres are not in \mathbf{DL} [1, Lemma 2.5].

We have that $\mathbf{BA} \subseteq \mathbf{DL}$; on \mathbf{DL} , \wedge and \bigwedge coincide, and $(\mathbf{DL}, \wedge, \bigvee)$ is a distributive lattice. Furthermore, on \mathbf{BA} , a is a true complement, so $(\mathbf{BA}, \wedge, \bigvee, \mathbf{0}, \mathbf{1}, a)$ is a Boolean algebra, but not complete.

There is a retraction from \mathbf{B} to \mathbf{DL} defined by

$$r\langle X \rangle := \bigvee \{ \langle Z \rangle ; \langle Z \rangle \in \mathbf{DL} \text{ and } \langle Z \rangle \leq \langle X \rangle \}.$$

The pseudo-complementation function a may not respect \mathbf{DL} , *i.e.*, it could be that $\langle X \rangle \in \mathbf{DL}$, but $a\langle X \rangle \notin \mathbf{DL}$. On \mathbf{DL} , we therefore define a new pseudo-complement by

$$A\langle X \rangle := ra\langle X \rangle.$$

While $A^3 = A$ and $\langle X \rangle \leq A^2\langle X \rangle$, it is not in general the case that $A^2 = \text{id}$. It is known [5, Lemma 6.2(d)] that A converts joins to meets, *i.e.*,

$$A(\bigvee \mathcal{X}) = \bigwedge \{A\langle X \rangle; \langle X \rangle \in \mathcal{X}\}.$$

Following [5, Definition 6.3], we define

$$\mathbf{cBA} := \{\langle X \rangle \in \mathbf{DL}; A^2\langle X \rangle = \langle X \rangle\}.$$

If $\langle X \rangle \in \mathbf{BA}$, then $A^2\langle X \rangle = a^2\langle X \rangle = \langle X \rangle$, thus $\mathbf{BA} \subseteq \mathbf{cBA}$. The set \mathbf{cBA} carries a complete Boolean algebra structure [5, Theorem 6.4]; however, it is not $(\mathbf{cBA}, \wedge, \vee, \mathbf{0}, \mathbf{1}, A)$, but instead $(\mathbf{cBA}, \wedge, \Upsilon, \mathbf{0}, \mathbf{1}, A)$ with Υ defined by

$$\Upsilon \mathcal{X} := A^2\bigvee \mathcal{X}.$$

Note that $\Upsilon \mathcal{X}$ is perfectly well-defined for subsets \mathcal{X} of \mathbf{DL} or even \mathbf{B} , it just will not in general produce the least upper bound in these contexts.

3. Results

We start with an observation on joins of elements in \mathbf{BA} and use this to derive lower bounds for the size of $\mathbf{DL} \setminus \mathbf{cBA}$ and $\mathbf{cBA} \setminus \mathbf{BA}$.

Lemma 1. *If $\mathcal{X} \subseteq \mathbf{BA}$, then $\Upsilon \mathcal{X} = \bigvee \mathcal{X}$. In particular, $\bigvee \mathcal{X} \in \mathbf{cBA}$.*

Proof. We have that

$$\Upsilon \mathcal{X} = A^2\bigvee \mathcal{X} = \text{rara} \bigvee \mathcal{X},$$

and as a converts joins to meets, the latter is equal to

$$\text{rar} \bigwedge \{a\langle X \rangle; \langle X \rangle \in \mathcal{X}\}.$$

Since every $a\langle X \rangle$ is in \mathbf{BA} , it is also in \mathbf{DL} , and as \mathbf{DL} is complete,

$$\Xi := \bigwedge \{a\langle X \rangle; \langle X \rangle \in \mathcal{X}\} \in \mathbf{DL}$$

and hence $r\Xi = \Xi$. Therefore, as a sends meets to joins,

$$\begin{aligned} \text{rar}\Xi &= r a \Xi \\ &= r \bigvee \{a^2\langle X \rangle; \langle X \rangle \in \mathcal{X}\} \\ &= r \bigvee \{\langle X \rangle; \langle X \rangle \in \mathcal{X}\} \\ &= \bigvee \mathcal{X}. \end{aligned}$$

q.e.d.

Proposition 2. *If $S \subseteq \mathbb{N}$ is infinite, then $\langle K(S) \rangle = \bigvee_{i \in S} \langle K(i) \rangle \in \mathbf{cBA} \setminus \mathbf{BA}$ and $\langle K(S) \rangle \geq \langle I \rangle$.*

Proof. Hovey and Palmieri [5, §5] proved that for each n , $\langle K(n) \rangle$ is in \mathbf{BA} , so by Lemma 1, $\langle K(S) \rangle$ is in \mathbf{cBA} . Hovey showed [4, Proof of Theorem 3.6] that the mod- p Moore spectrum, $M(p)$ is $K(S)$ -local, so in particular $K(S)$ has a finite local and [5, Proposition 7.2] gives that $\langle K(S) \rangle \geq \langle I \rangle$. If $K(S)$ were in \mathbf{BA} , having a finite local implies [5, Lemma 7.9] that $\langle K(S) \wedge I \rangle \neq \mathbf{0}$. But we know that $\langle K(n) \wedge I \rangle = \mathbf{0}$ and hence using distributivity we get that $\langle K(S) \wedge I \rangle = \mathbf{0}$. q.e.d.

Corollary 3. *We have a proper inclusion $\mathbf{BA} \subsetneq \mathbf{cBA}$; in fact, the set $\mathbf{cBA} \setminus \mathbf{BA}$ has at least 2^{\aleph_0} elements.*

Proof. As noted above, $\mathbf{BA} \subseteq \mathbf{cBA}$. For the non-equality, if $S \neq S'$ are infinite subsets of \mathbb{N} , then Dwyer and Palmieri showed that $\langle K(S) \rangle \neq \langle K(S') \rangle$ [3, Lemma 3.4], so there are continuum many elements in the complement. q.e.d.

To sum up, we have

$$\mathbf{BA} \subsetneq \mathbf{cBA} \subseteq \mathbf{DL} \subsetneq \mathbf{B}.$$

Hovey and Palmieri argue that the middle inclusion is also proper:

This argument also implies that A^2 is not the identity—indeed, if A^2 were the identity, one can check that A would have to convert meets to joins. However, we do not know a specific spectrum X in \mathbf{DL} for which $A^2 \langle X \rangle \neq \langle X \rangle$. [5, p. 185]

We analyse the argument sketched in the above quote:

Lemma 4. *Let $\mathcal{X} \subseteq \mathbf{DL}$ be any set such that A^2 is the identity for each $\langle X \rangle \in \mathcal{X}$ and for $\bigvee \{A \langle X \rangle; \langle X \rangle \in \mathcal{X}\}$. Then*

$$A(\bigwedge \mathcal{X}) = \bigvee \{A \langle X \rangle; \langle X \rangle \in \mathcal{X}\}.$$

Proof. Since A converts joins to meets, under the assumption of the lemma, we have

$$\begin{aligned} A(\bigwedge \mathcal{X}) &= A \bigwedge \{A^2 \langle X \rangle; \langle X \rangle \in \mathcal{X}\} \\ &= A^2 \bigvee \{A \langle X \rangle; \langle X \rangle \in \mathcal{X}\} \\ &= \bigvee \{A \langle X \rangle; \langle X \rangle \in \mathcal{X}\}. \end{aligned}$$

q.e.d.

Corollary 5 (Hovey-Palmieri). *The operation A^2 is not the identity on \mathbf{DL} ; i.e., $\mathbf{cBA} \subsetneq \mathbf{DL}$.*

Proof. Let $X := K(\mathbb{N})$, $Y := H\mathbb{F}_p = K(\infty)$, and $\mathcal{X} := \{\langle X \rangle, \langle Y \rangle\} \subseteq \mathbf{DL}$. We assume towards a contradiction that A^2 is the identity on \mathbf{DL} , so in particular, the assumptions of Lemma 4 are satisfied for \mathcal{X} . But $\langle X \rangle \wedge \langle Y \rangle = \langle X \rangle \wedge \langle Y \rangle = \mathbf{0}$, hence $A(\langle X \rangle \wedge \langle Y \rangle) = \mathbf{1}$. On the other hand, $A\langle X \rangle \vee A\langle Y \rangle \leq a\langle I \rangle < \mathbf{1}$, in contradiction to Lemma 4. q.e.d.

The proof of Corollary 5 due to Hovey and Palmieri yields a trichotomy result: at least one of $\langle K(\mathbb{N}) \rangle$, $\langle H\mathbb{F}_p \rangle$, and $A\langle K(\mathbb{N}) \rangle \vee A\langle H\mathbb{F}_p \rangle$ is not in \mathbf{cBA} . We improve this in our Dichotomy Lemma 7 to a dichotomy which will allow us to identify concrete elements in $\mathbf{DL} \setminus \mathbf{cBA}$, including in particular $\langle H\mathbb{F}_p \rangle$ (Corollary 10).

Lemma 6. *For any spectrum, the condition $A\langle E \rangle < \mathbf{1}$ is equivalent to $\langle E \rangle \neq \mathbf{0}$.*

Proof. If $\langle E \rangle = \mathbf{0}$, then clearly $A\langle E \rangle = \mathbf{1}$. Conversely, if $A\langle E \rangle = \mathbf{1}$, then $a\langle E \rangle \geq A\langle E \rangle = \mathbf{1}$, and so

$$\langle E \rangle = \mathbf{1} \wedge \langle E \rangle = a\langle E \rangle \wedge \langle E \rangle = \mathbf{0}.$$

q.e.d.

Lemma 7 (Dichotomy Lemma). *Let X and Y be spectra, and let E be a spectrum such that $\langle E \rangle \neq \mathbf{0}$. Suppose that the following conditions hold:*

1. $\langle X \rangle \in \mathbf{DL}$,
2. $\langle Y \rangle \in \mathbf{DL}$,
3. $\langle X \rangle \wedge \langle Y \rangle = \mathbf{0}$,
4. $\langle E \rangle \leq \langle X \rangle$, and
5. $\langle E \rangle \leq \langle Y \rangle$.

Then $\langle X \rangle$ or $\langle Y \rangle$ is not in \mathbf{cBA} .

Note that conditions 4 and 5 are equivalent to saying that $\langle X \rangle \wedge \langle Y \rangle \neq \mathbf{0}$, and thus the Dichotomy Lemma extracts the failure of $A^2 = \text{id}$ from the discrepancy between \wedge and \wedge in \mathbf{B} .

Proof. Assume that $A^2\langle X \rangle = \langle X \rangle$ and $A^2\langle Y \rangle = \langle Y \rangle$. Since A converts joins to meets, we get by our assumption on X and Y

$$\mathbf{1} = A\mathbf{0} = A(\langle X \rangle \wedge \langle Y \rangle) = A(A^2\langle X \rangle \wedge A^2\langle Y \rangle) = A^2(A\langle X \rangle \vee A\langle Y \rangle)$$

and the latter is $A\langle X \rangle \gamma A\langle Y \rangle$ by definition of γ . As A is order-reversing we get $A\langle X \rangle \leq A\langle E \rangle$ and $A\langle Y \rangle \leq A\langle E \rangle$ and hence (using Lemma 6)

$$\mathbf{1} = A^2(A\langle X \rangle \vee A\langle Y \rangle) = A\langle X \rangle \gamma A\langle Y \rangle \leq A\langle E \rangle \gamma A\langle E \rangle = A\langle E \rangle < \mathbf{1},$$

a contradiction, showing that our assumption that both $\langle X \rangle$ and $\langle Y \rangle$ are in **cBA** cannot hold. q.e.d.

As usual, we call a set $S \subset \mathbb{N} \cup \{\infty\}$ *coinfinite* if its complement $(\mathbb{N} \cup \{\infty\}) \setminus S$ is infinite.

Theorem 8. *For any coinfinite set $S \subseteq \mathbb{N} \cup \{\infty\}$ with $\infty \in S$, we have that $\langle K(S) \rangle$ is not in **cBA**.*

Proof. In Lemma 7, choose E to be the Brown-Comenetz dual of the p -local sphere spectrum, I . We know by [5, Lemma 7.1(c)] that $\langle H\mathbb{F}_p \rangle \geq \langle I \rangle$, and hence $\langle K(S) \rangle \geq \langle I \rangle$. As the complement $\bar{S} := (\mathbb{N} \cup \{\infty\}) \setminus S$ is infinite, we get by Proposition 2 that $\langle K(\bar{S}) \rangle \geq \langle I \rangle$. Both $\langle K(S) \rangle$ and $\langle K(\bar{S}) \rangle$ are in **DL** and $\langle K(S) \rangle \wedge \langle K(\bar{S}) \rangle = \mathbf{0}$. Thus all conditions of the Dichotomy Lemma are satisfied, and we get that one of $\langle K(S) \rangle$ and $\langle K(\bar{S}) \rangle$ is not in **cBA**. However, by Proposition 2, $\langle K(\bar{S}) \rangle \in \mathbf{cBA}$, so $\langle K(S) \rangle \in \mathbf{DL} \setminus \mathbf{cBA}$. q.e.d.

Corollary 9. *There are at least 2^{\aleph_0} Bousfield classes in $\mathbf{DL} \setminus \mathbf{cBA}$.*

Proof. This follows directly from Theorem 8 and [3, Lemma 3.4], as there are 2^{\aleph_0} many coinfinite subsets of $\mathbb{N} \cup \{\infty\}$. q.e.d.

By Corollaries 3 and 9, we get 2^{\aleph_0} as a lower bound for the cardinality for three of the four areas depicted in Figure 1; for $\mathbf{B} \setminus \mathbf{DL}$ we only get \aleph_0 as a lower bound. A natural project for future research would be to improve this to 2^{\aleph_0} by finding concrete inhabitants of that set. Getting even larger lower bounds than 2^{\aleph_0} is connected to the famous open question about the cardinality of **B**; as a consequence, we believe that this needs entirely novel ideas.

4. Applications

Several conjectures made by Hovey and Palmieri in [5] suggest that $\langle H\mathbb{F}_p \rangle$ is not in **cBA** [5, Proposition 6.14]. This follows directly from our Theorem 8:

Corollary 10. *For every prime p , we have that $\langle H\mathbb{F}_p \rangle \in \mathbf{DL} \setminus \mathbf{cBA}$.*

Proof. This is clear from Theorem 8, as $\langle H\mathbb{F}_p \rangle = \langle K(\infty) \rangle = \langle K(\{\infty\}) \rangle$ where $\{\infty\}$ is coinfinite in $\mathbb{N} \cup \{\infty\}$. q.e.d.

Our method also identifies several other explicit Bousfield classes in $\mathbf{DL} \setminus \mathbf{cBA}$. The following examples exploit the fact that for any self-map of a spectrum X , $f: \Sigma^{[f]} X \rightarrow X$ one gets by [7, Lemma 1.34] that

$$\langle X \rangle = \langle C_f \rangle \vee \langle X[f^{-1}] \rangle.$$

Here, C_f denotes the cofiber of f and $X[f^{-1}]$ is the telescope. Then the Bousfield class of the Eilenberg-MacLane spectrum of the p -local integers, $H\mathbb{Z}_{(p)}$, is

$\langle K(\{0, \infty\}) \rangle$. This is a special case of a truncated Brown-Peterson spectrum $BP\langle n \rangle$ with $\pi_*(BP\langle n \rangle) = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ ($|v_i| = 2p^i - 2$). Multiplication by v_n is a self-map on $BP\langle n \rangle$ with cofiber $BP\langle n-1 \rangle$ and $BP\langle n \rangle[v_n^{-1}] = E(n)$. An iteration then gives (cf. [7, Theorem 2.1]) $\langle BP\langle n \rangle \rangle = \langle E(n) \rangle \vee \langle H\mathbb{F}_p \rangle$. As the Bousfield class of $E(n)$ is $\langle K(0) \rangle \vee \dots \vee \langle K(n) \rangle$ we obtain $\langle BP\langle n \rangle \rangle = \langle K(\{0, \dots, n, \infty\}) \rangle$.

Corollary 11. *For every prime p and every natural number n , we have that $\langle H\mathbb{Z}_{(p)} \rangle$ and $\langle BP\langle n \rangle \rangle$ are in $\mathbf{DL} \setminus \mathbf{cBA}$.*

Proof. The subsets $\{0, \infty\}$ and $\{0, \dots, n, \infty\}$ are coinfinite in $\mathbb{N} \cup \{\infty\}$. **q.e.d.**

For the connective Morava K -theory $k(n)$ (with $\pi_*k(n) = \mathbb{F}_p[v_n]$) we get $\langle k(n) \rangle = \langle K(n) \rangle \vee \langle H\mathbb{F}_p \rangle = \langle K(\{n, \infty\}) \rangle$.

Corollary 12. *For every natural number n , $\langle k(n) \rangle \in \mathbf{DL} \setminus \mathbf{cBA}$.*

Proof. This follows from Theorem 8, as $\{n, \infty\}$ is coinfinite in $\mathbb{N} \cup \{\infty\}$. **q.e.d.**

Similar to the Morava K -theory spectra $K(n)$ we can consider the telescopes $T(n)$ of v_n -maps. (Cf. [5, §5] for details.) It is known that

$$\langle T(n) \rangle = \langle K(n) \rangle \vee \langle A(n) \rangle$$

where $A(n)$ is the spectrum describing the failure of the telescope conjecture. We set $\langle T(\infty) \rangle = \langle H\mathbb{F}_p \rangle$. The classes $\langle T(n) \rangle$ and $\langle A(n) \rangle$ are in \mathbf{BA} but $\bigvee_{\mathbb{N}} \langle T(n) \rangle \notin \mathbf{BA}$ by [5, Corollary 7.10]. By Lemma 1, we know that for any $S \subseteq \mathbb{N}$, we have that $\bigvee_{n \in S} \langle T(n) \rangle \in \mathbf{cBA}$. An argument similar to the proof of Proposition 2 yields Proposition 13.

Proposition 13. *If $S \subseteq \mathbb{N}$ is infinite, then $\langle T(S) \rangle = \bigvee_{i \in S} \langle T(i) \rangle \in \mathbf{cBA} \setminus \mathbf{BA}$ and $\langle T(S) \rangle \geq \langle I \rangle$.*

Theorem 14. *Let $S \subseteq \mathbb{N} \cup \{\infty\}$ be a coinfinite subset with $\infty \in S$. Then $\langle T(S) \rangle$ is not in \mathbf{cBA} .*

Proof. Again, we use the Brown-Comenetz dual of the p -local sphere as E in the Dichotomy Lemma. Let \bar{S} be the complement of S . As $\langle T(n) \rangle \geq \langle K(n) \rangle$ and as $\infty \in S$ we have that

$$\bigvee_{n \in S} \langle T(n) \rangle \geq \bigvee_{n \in S} \langle K(n) \rangle \geq \langle I \rangle$$

and $\bigvee_{n \in \bar{S}} \langle T(n) \rangle \geq \langle I \rangle$. The telescopes satisfy $\langle T(n) \rangle \wedge \langle T(m) \rangle = \mathbf{0}$ for $m \neq n$: cf. [5, §5] for the cases $n \neq \infty \neq m$ and the proof of [5, Proposition 6.14] for $\langle H\mathbb{F}_p \rangle \wedge \bigvee_{\mathbb{N}} \langle T(n) \rangle = \mathbf{0}$. Therefore we obtain that one of $\bigvee_{n \in S} \langle T(n) \rangle$ or $\bigvee_{n \in \bar{S}} \langle T(n) \rangle$ cannot be an element of \mathbf{cBA} , but $\bigvee_{n \in \bar{S}} \langle T(n) \rangle$ is in \mathbf{cBA} by Proposition 13. **q.e.d.**

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