# Solving Vlasov-Maxwell Equations by Using Hamiltonian Splitting


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**Abstract.** In this paper, we reformulate the Vlasov-Maxwell equations based on the Morrison-Marsden-Weinstein Poisson bracket. In order to get the numerical solutions preserving the Poisson bracket, we split the Hamiltonian of the Vlasov-Maxwell equations into five parts. We construct the numerical methods for the time direction via composing the exact solutions of subsystems. By combining an appropriate spatial discretization, we can prove that the resulting numerical discretization preserves the discrete Poisson bracket. We present numerical simulations for the problems of Landau damping and two-stream stability.


**Keywords:** Hamiltonian splitting, Poisson bracket, Vlasov-Maxwell equations
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## Vlasov-Maxwell equations

The Vlasov-Maxwell(VM) system of equations describes the collective motion of particles interacting with self-consistent electromagnetic fields. The system of dimensionless Vlasov-Maxwell equations regardless of the relativistic effects reads

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{1}$$

$$\nabla \times \mathbf{B} = \int f \mathbf{v} d\mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \cdot \mathbf{E} = \int f d\mathbf{v} - 1, \quad \nabla \cdot \mathbf{B} = 0, \tag{4}$$

where $f(\mathbf{x}, \mathbf{v}, t)$ is the electron distribution function, $\mathbf{x} \in U \subset \mathbb{R}^3$ denotes the position, $\mathbf{v} \in \mathbb{R}^3$ denotes the velocity, and $(\mathbf{E}, \mathbf{B}) \in \mathbb{R}^3 \times \mathbb{R}^3$ are the electromagnetic fields. Denote $\mathcal{M} = \{(f, \mathbf{E}, \mathbf{B}) | \nabla \cdot \mathbf{B} = 0\}$. The Vlasov-Maxwell system is an infinite dimensional Hamiltonian system defined on $\mathcal{M}$ . In particular, VM equations (1-3) can be written in the form,

$$\frac{\partial \mathcal{Z}}{\partial t} = \{\mathcal{Z}, \mathcal{H}\}, \tag{5}$$

where $\mathcal{Z} \in \mathcal{M}$. The bracket $\{\cdot, \cdot\}$is the Morrison-Marsden-Weinstein (MMW) Poisson bracket presented in [1, 2],

$$\{\mathcal{F}, \mathcal{G}\} = \int f \left[\frac{\delta \mathcal{F}}{\delta f}, \frac{\delta \mathcal{G}}{\delta f}\right]_{\mathbf{xv}} d\mathbf{x} d\mathbf{v} + \int \left(\frac{\delta \mathcal{F}}{\delta \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{\delta \mathcal{G}}{\delta f} - \frac{\delta \mathcal{G}}{\delta \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{\delta \mathcal{F}}{\delta f}\right) d\mathbf{x} d\mathbf{v} \tag{6}$$

$$+ \int \left(\frac{\delta \mathcal{F}}{\delta \mathbf{E}} \cdot \left(\nabla \times \frac{\delta \mathcal{G}}{\delta \mathbf{B}}\right) - \frac{\delta \mathcal{G}}{\delta \mathbf{E}} \cdot \left(\nabla \times \frac{\delta \mathcal{F}}{\delta \mathbf{B}}\right)\right) d\mathbf{x} \tag{7}$$

$$+ \int f \mathbf{B} \cdot \left(\frac{\partial}{\partial \mathbf{v}} \frac{\delta \mathcal{F}}{\delta f} \times \frac{\partial}{\partial \mathbf{v}} \frac{\delta \mathcal{G}}{\delta f}\right) d\mathbf{x} d\mathbf{v}, \tag{8}$$


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where $\mathcal{F}$ and $\mathcal{G}$ are any two functionals defined on $\mathcal{M}$. Here $[\cdot,\cdot]_{\mathbf{xv}}$ denotes the canonical Poisson bracket. The Hamiltonian functional $\mathcal{H}$ is

$$\mathcal{H}(f,\mathbf{E},\mathbf{B}) = \frac{1}{2}\int \mathbf{v}^2 f d\mathbf{x}d\mathbf{v} + \frac{1}{2}\int \left(\mathbf{E}^2+\mathbf{B}^2\right)d\mathbf{x}. \tag{9}$$

It is clear that the energy of the system $\mathcal{H}$ remains constant along the solution of the system.

For the purpose of constructing numerical methods which can preserve the Poisson bracket, we split the Hamiltonian (9) as five parts

$$\mathcal{H} = \mathcal{H}_E + \mathcal{H}_B + \mathcal{H}_{1f} + \mathcal{H}_{2f} + \mathcal{H}_{3f} \tag{10}$$

with $\mathcal{H}_E = \frac{1}{2}\int \mathbf{E}^2 d\mathbf{x}$, $\mathcal{H}_B = \frac{1}{2}\int \mathbf{B}^2 d\mathbf{x}$, and $\mathcal{H}_{if} = \frac{1}{2}\int v_i^2 f d\mathbf{x}d\mathbf{v}$ for $i = 1,2,3$. Substituting (10) into (5) provides five solvable subsystems which are

$$\dot{\mathcal{Z}} = \{\mathcal{Z},\mathcal{H}_E\},\quad \dot{\mathcal{Z}} = \{\mathcal{Z},\mathcal{H}_B\},\quad \dot{\mathcal{Z}} = \left\{\mathcal{Z},\mathcal{H}_{1f}\right\},\quad \dot{\mathcal{Z}} = \left\{\mathcal{Z},\mathcal{H}_{2f}\right\},\quad \dot{\mathcal{Z}} = \left\{\mathcal{Z},\mathcal{H}_{3f}\right\}. \tag{11}$$

For example, the $i$-th subsystem corresponding to Hamiltonian $\mathcal{H}_{if}$ is,

$$\frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} + \sum_{j=1}^{3}\hat{B}_{ji}v_i\frac{\partial f}{\partial v_j} = 0, \tag{12}$$

$$\frac{\partial E_i}{\partial t} = -\int v_i f d\mathbf{v}, \tag{13}$$

$$\frac{\partial E_j}{\partial t} = 0,\quad j \neq i, \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial t} = 0. \tag{15}$$

Here $\hat{B}_{ij}$ is the element of the matrix $\hat{\mathbf{B}} = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix}$, where $\mathbf{B} = (B_1,B_2,B_3)$. By the method of characteristics, with the initial condition $(f_0,\mathbf{E}_0,\mathbf{B}_0) \in \mathcal{M}$, the exact solution to the $i$-th subsystem (12–15) is

$$f(\mathbf{x},\mathbf{v},t) = f_0\left(\mathbf{x} - tv_i\mathbf{e}_i,\quad \mathbf{v} - \sum_{l=1}^{3}\mathbf{e}_l F_l^{(i)}\right), \tag{16}$$

$$E_i(\mathbf{x},t) = E_i(\mathbf{x},0) - \int_0^t\int v_i f(\mathbf{x},\mathbf{v},\tau)d\mathbf{v}d\tau,\quad E_j(\mathbf{x},t) = E_{0j}(\mathbf{x}),\quad j\neq i, \tag{17}$$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0(\mathbf{x}), \tag{18}$$

where $F_l^{(i)} = \int_{x_i - tv_i}^{x_i}\hat{B}_{li}(\mathbf{x},0)dx_i$, $\mathbf{e}_i$ is the unit vector in the $i$-th Cartesian direction.

Temporal discretizations can be constructed by composing the solutions of the subsystems, first order methods can be constructed by the Lie splitting method , and second order methods can be constructed by the Strang splitting method. To get high-order methods, the composition approaches developed in [4] can be employed.

## Discretization of the $1+\frac{1}{2}$ dimensional Vlasov-Maxwell system.

We consider a $1+\frac{1}{2}$ dimensional VM equations introduced in [5]. Assume that the distribution $f$ depends only on $(x_1,v_1,v_2)$, $\mathbf{B}$ and $\mathbf{E}$ depend only on $x_1$, $B_1 = B_2 = E_3 = 0$, and $f$, $\mathbf{B}$ and $\mathbf{E}$ are periodic in the $x_1$-direction. Then the Vlasov-Maxwell equations (1)–(4),

$$\frac{\partial f}{\partial t} + v_1\frac{\partial f}{\partial x_1} + (E_1 + v_2B_3)\frac{\partial f}{\partial v_1} + (E_2 - v_1B_3)\frac{\partial f}{\partial v_2} = 0, \tag{19}$$

$$\frac{\partial E_1}{\partial t} = -\int v_1 f\, dv_1\, dv_2,\quad \frac{\partial E_2}{\partial t} = -\frac{\partial B_3}{\partial x_1} - \int v_2 f\, dv_1\, dv_2, \tag{20}$$

$$\frac{\partial B_3}{\partial t} = -\frac{\partial E_2}{\partial x_1}. \tag{21}$$

We apply the Hamiltonian splitting method above to discretize this system in time. In this reduced case, the exact solution of subsystem $H_{1f}$ is:

$$f(x_1, v_1, v_2; t) = f\left(x_1 - tv_1, v_1, v_2 + \int_{x_1-tv_1}^{x_1} B_3(\xi)\, d\xi; 0\right), \tag{22}$$

$$E_1(x_1; t) = E_1(x_1; 0) - \iint v_1 \int_{x_1-tv_1}^{x_1} f(\xi, v_1, v_2; 0)\, d\xi\, dv_1\, dv_2, \tag{23}$$

$$E_2(t) = E_2(0), \quad B_3(t) = B_3(0). \tag{24}$$

Next, we discretize (22,23) in space. As the solution is periodic w.r.t $x_1$, we use the Fourier spectral method in the $x_1$ direction. For the $v_1$ and $v_2$ directions, we use the finite volume method. It can be investigated from (22) that we need evaluate the value of $f$ off the grid point. Here, we use the Parabolic Spline Method (PSM) introduced in [6] to reconstruct a continuous function. For (23), we use Fourier coefficients of $f$ to compute the Fourier coefficients of $E_1$. We can handle with the other subsystems like this subsystem.

## Numerical experiments

**Landau damping.**

We simulate the problem of Landau damping. We use the above developed numerical discretizations to solve the Vlasov-Poisson equations. For this case, we do not concern the magnetic effect. For the computation, the initial values are taken as

$$f(x_1, v_1, v_2) = \frac{1}{2\pi} e^{-\frac{1}{2}|v|^2} (1 + \alpha \cos(kx_1)), \tag{25}$$

$$E_1(x_1) = \frac{\alpha}{k} \sin(x_1), \quad E_2(x_1) = 0, \quad B_3(x_1) = 0, \tag{26}$$

where k=0.4, $x_1 \in [0, 2\pi/k]$, $v \in \mathbb{R}^2$. Figure.1,2 display the evolution of the numerical electric and total energy. The Figure.1 displays the result of linear Landau damping. We can see an exponential decrease of electric energy, which recovers the analytic decay rate. The Figure.2 displays the results of nonlinear Landau damping. We can see from the left figure a short time exponential decrease of electric energy and oscillation in later times. In the right figure, the error of total energy is bounded by $10^{-2}$.

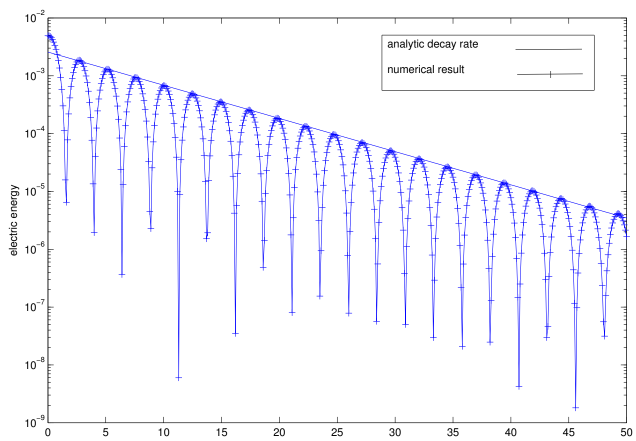


**FIGURE 1.** Time evolution of the electric energy for the linear Landau damping test using the Strang splitting with $32 \times 64 \times 64$ grid points and a time step of $\Delta t = 0.1$, $\alpha = 0.01$.

**Two-stream instability.**

We simulate a very common instability in plasma physics, the two-stream instability. For this test, we take the initial particle density defined on $[0, 2\pi] \times [0.4, 0.4] \times [0.4, 0.4]$ as

$$f(x_1, v_1, v_2) = \frac{1}{2\pi k} e^{-\frac{v_2^2}{k}} (e^{-\frac{(v_1-0.2)^2}{k}} + e^{-\frac{(v_1+0.2)^2}{k}}),$$

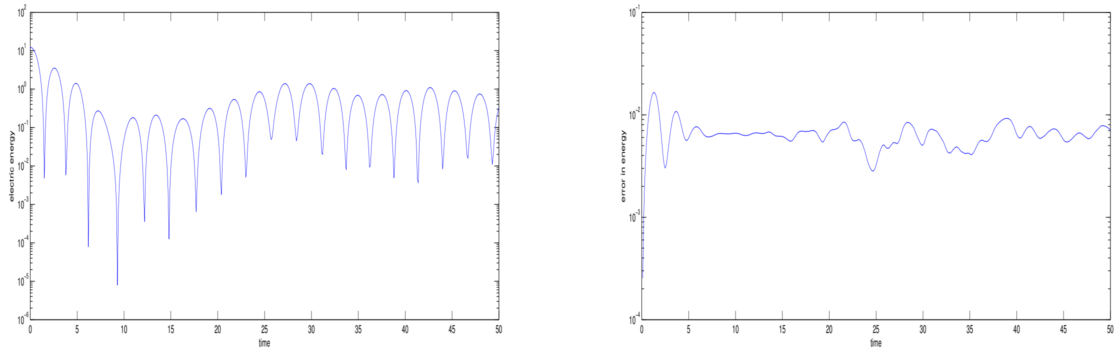


**FIGURE 2.** Time evolution of the electric energy and energy error for the nonlinear Landau damping test using the Strang splitting with $32 \times 64 \times 64$ grid points and a time step of $\Delta t = 0.1$, $\alpha = 0.5$.

where $k = 0.002$. We take the initial electromagnetic fields as

$$B_3(x_1) = 0.001 \sin(x_1), \quad E_1(x_1) = 0, \quad E_2(x_1) = 0.$$

The numerical result is shown in Fig.3. In theory, there is a exponential increase of the electric energy. And in the left figure, the exponential increase and a saturation in the electric field can be investigated. The right figure indicates the evolution behavior of electric energy has effect on the total energy error.

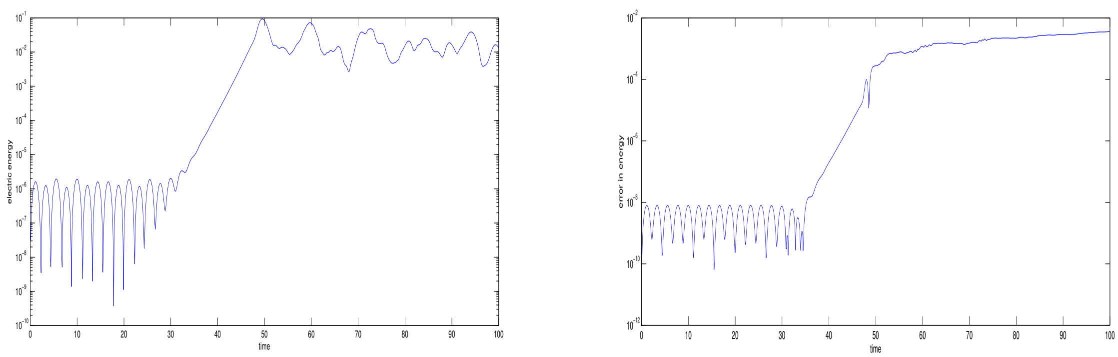


**FIGURE 3.** Time evolution of the electric energy and energy error for the two-stream instability. The Strang splitting scheme with $32 \times 64 \times 64$ grid points and time step $\Delta t = 0.1$ are used.

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