



Deposited via The University of Leeds.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/id/eprint/118991/>

Version: Accepted Version

---

**Proceedings Paper:**

Fairbairn, AI and Kelmanson, MA (2017) An exponentially convergent Volterra-Fredholm method for integro-differential equations. In: Chappell, DJ, (ed.) Proceedings of the Eleventh UK Conference on Boundary Integral Methods (UKBIM 11),. UKBIM 11, 10-11 Jul 2017, Nottingham, UK. Nottingham Trent University: Publications. ISBN: 9780993111297.

---

(c) 2017, The Authors. This is an author produced version of a paper published in Proceedings of the Eleventh UK Conference on Boundary Integral Methods (UKBIM 11) by Nottingham Trent University: Publications.

**Reuse**

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.

# Chapter 1

## An exponentially convergent Volterra-Fredholm method for integro-differential equations

A.I. Fairbairn and M.A. Kelmanson

Department of Applied Mathematics  
University of Leeds  
Leeds  
LS2 9JT

---

**Abstract.** *Extending the authors' recent work [15] on the explicit computation of error bounds for Nyström solvers applied to one-dimensional Fredholm integro-differential equations (FIDEs), presented herein is a study of the errors incurred by first transforming (as in, e.g., [21]) the FIDE into a hybrid Volterra-Fredholm integral equation (VFIE). The VFIE is solved via a novel approach that utilises  $N$ -node Gauss-Legendre interpolation and quadrature for its Volterra and Fredholm components respectively: this results in numerical solutions whose error converges to zero exponentially with  $N$ , the rate of convergence being confirmed via large- $N$  asymptotics. Not only is the exponential rate inherently far superior to the algebraic rate achieved in [21], but also it is demonstrated, via diverse test problems, to improve dramatically on even the exponential rate achieved in [15] via direct Nyström discretisation of the original FIDE; this improvement is confirmed theoretically.*

---

### 1.1 Introduction

Although there is a substantial body of literature devoted to the development of methods for approximating the numerical solution of one-dimensional Fredholm integro-differential equations (FIDEs), corresponding error analyses, an aspect considered to be a default element in the exposition of new numerical methods, are relatively scarce. For example, though the independent studies (in chronological order) [27, 4, 28, 20, 5, 11, 25, 29, 6, 22, 2, 1, 16, 26] present diverse FIDE-solution techniques of varying degrees of efficiency and (disparate) accuracy, only [20, 29, 22, 1] include a discussion of errors and, in even these cases, error analyses are limited (see summary in [15, §1]) to estimates of convergence rates: that is, the direct computation of theoretically predicted error bounds is absent.

The present work is therefore motivated on two fronts: to develop not only a novel numerical method that converges exponentially in the dimension  $N$  of the discrete numerical method, but also an explicit error analysis that is implementable and yields errors in terms of only the computed numerical solution. In [15], the authors develop a novel approach for achieving these

two goals, but the method developed therein—based on a combination of numerical quadrature and numerical differentiation—has a global error dictated by the latter process, which is considerably less accurate than the former. Accordingly, an approach independent of [15] is presently pursued in which the need for numerical differentiation is circumvented by first transforming the FIDE (as in, e.g., [21]) into a Volterra-Fredholm integral equation (VFIE); though the solution of this can be approximated in a number of ways (see, e.g., [18, 12, 10, 7, 24]), a novel approach is adopted herein.

The remainder of this paper is structured as follows. In §1.2 is presented an FIDE-to-VFIE conversion approach from [21], in which the VFIE is solved to (only; see below) *quadratic* order in the number  $N$  of Simpson’s-rule panels used. In §1.3 the VFIE is solved numerically to *spectral* order in  $N$ , the degree of the highest-order orthogonal polynomial used in the approximation of the VFIE solution. This approach obviates the need for the numerical differentiation matrices used in a related paper [15]. In §1.4 is presented a novel error analysis, for the VFIE numerical solution procedure, whose distinctive aspect is computation of the error in the numerical solution of the original FIDE *explicitly* in terms of the numerical approximation of the derivate that results from the VFIE reformulation. In §1.5 numerical results of test problems, some challenging, are presented that validate to spectral accuracy both the implementation outlined in §1.3 and the error analysis of §1.4. Brief conclusions are presented in §1.6.

## 1.2 Conversion from FIDE to VFIE

The canonical form on the normalised interval  $[-1, 1]$  of the first-order one-dimensional Fredholm integro-differential equation (FIDE) for the unknown function  $u(x)$  is

$$u(x) - \mu(x) \frac{du}{dx}(x) - \lambda \int_{-1}^1 K(x, y) u(y) dy = f(x), \quad x \in [-1, 1], \quad (1.1)$$

in which the source function  $f : [-1, 1] \rightarrow \mathbb{R}$ , the kernel  $K : [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}$  and coefficient function  $\mu : [-1, 1] \rightarrow \mathbb{R}$  are prescribed functions of  $x, y \in [-1, 1]$  and the parameter  $\lambda \in \mathbb{R}$  is a constant. By hypothesis, (1.1) is solvable and so  $u(x)$  exists. In symbolic form, (1.1) is

$$u - \mu \mathcal{D} u - \lambda \mathcal{K} u = f, \quad (1.2)$$

where  $u, f \in \mathcal{C} \equiv C[-1, 1]$ , the Banach space with supremum norm  $\|\cdot\|$  on which the action of the differential operator  $\mathcal{D}$  on  $u$  is defined by

$$\mathcal{D} u = (\mathcal{D} u)(x) \equiv u'(x),$$

wherein a prime denotes differentiation with respect to  $x$ . The action in (1.2) of the compact integral operator  $\mathcal{K}$  on  $u$  is defined by

$$\mathcal{K} u = (\mathcal{K} u)(x) \equiv \int_{-1}^1 K(x, y) u(y) dy.$$

The FIDE (1.1) is augmented by the boundary condition (BC)

$$u(\xi) = \zeta, \quad \xi \in [-1, 1], \quad (1.3)$$

i.e.  $\xi$  is a prescribed real constant in the interval containing all Legendre nodes. When the BC is given for the end-points  $\xi = \pm 1$ , the FIDE can be converted into a Volterra-Fredholm integral equation (VFIE) following the approach in, e.g., [21]; the details of this conversion for  $\xi = -1$  are as follows. Define the function  $v(x)$  by

$$v(x) \equiv u'(x), \quad (1.4)$$

integration of which, upon using (1.3), yields

$$u(x) = \zeta + \int_{-1}^x v(y) dy, \quad (1.5)$$

whence the FIDE (1.1) becomes

$$\zeta + \int_{-1}^x v(y) dy - \mu(x) v(x) - \lambda \int_{-1}^1 K(x, y) \left( \zeta + \int_{-1}^y v(z) dz \right) dy = f(x). \quad (1.6)$$

By the existence of  $u(x)$  and (1.5),  $v(x)$  is integrable, hence the order of double integration in the final term on the right-hand side of (1.6) can be exchanged, thereby rendering (1.6) as the VFIE

$$v(x) = g(x) + \frac{1}{\mu(x)} \int_{-1}^x v(y) dy - \lambda \int_{-1}^1 k(x, y) v(y) dy, \quad (1.7)$$

in which the modified source function  $g(x)$  is given by

$$g(x) = \frac{1}{\mu(x)} \left( \zeta - \lambda \zeta \int_{-1}^1 K(x, y) dy - f(x) \right),$$

and the modified kernel  $k(x, y)$  by

$$k(x, y) = \frac{1}{\mu(x)} \int_y^1 K(x, z) dz. \quad (1.8)$$

By defining the action of the (Volterra) integral operator  $\mathcal{V}$  on  $v \in \mathcal{C}$  by

$$\mathcal{V}v = (\mathcal{V}v)(x) \equiv \int_{-1}^x v(y) dy, \quad (1.9)$$

and that of the (Fredholm) integral operator  $\mathcal{F}$  on  $v \in \mathcal{C}$  by

$$\mathcal{F}v = (\mathcal{F}v)(x) \equiv \int_{-1}^1 k(x, y) v(y) dy, \quad (1.10)$$

the symbolic form of the VFIE (1.7) corresponding to FIDE (1.2) is

$$v = g + \frac{\mathcal{V}v}{\mu} - \lambda \mathcal{F}v. \quad (1.11)$$

The FIDE-to-VFIE conversion for the case when the BC is at  $x = 1$  follows analogously by replacing integrals  $\int_{-1}^x$  with  $\int_x^1$  in (1.5), (1.6), (1.7) and (1.9) and replacing  $\int_y^1$  with  $\int_{-1}^y$  in (1.8).

The original FIDE (1.2) can now be solved via (1.5) and (1.7) without the need for numerical differentiation. The symbolic equation (1.11) will form the basis of the error analysis in section §1.4.

### 1.3 Numerical Solution of the VFIE

Let  $y_{j,N}$ ,  $j = 1(1)N$  be a set of  $N$  distinct nodes in  $[-1, 1]$  ordered so that  $-1 \leq y_{1,N} < y_{2,N} < \dots < y_{N-1,N} < y_{N,N} \leq 1$ , using which the action of the  $N$ -node Lagrange-interpolation operator  $\mathcal{L}_N$  on  $v \in \mathcal{C}$  is defined as

$$\mathcal{L}_N v = (\mathcal{L}_N v)(x) \equiv \sum_{j=1}^N L_{j,N}(x) v(y_{j,N}), \quad (1.12)$$

wherein the Lagrange basis functions are given by

$$L_{j,N}(x) = \prod_{\substack{l=1 \\ l \neq j}}^N \frac{x - y_{l,N}}{y_{j,N} - y_{l,N}}, \quad j = 1(1)N. \quad (1.13)$$

To approximate the Volterra term in (1.11), define the (Volterra-Lagrange) operator  $\mathcal{V}_N \equiv \mathcal{L}_N \mathcal{V}$ . Application of the operator  $\mathcal{V}$  to both sides of the approximate Lagrange interpolation  $v \approx \mathcal{L}_N v$  then yields

$$\mathcal{V} v \approx \mathcal{V}_N v = (\mathcal{V}_N v)(x) \equiv \sum_{j=1}^N \tau_{j,N}(x) v(y_{j,N}), \quad (1.14)$$

in which

$$\tau_{j,N}(x) = \mathcal{V} L_{j,N}(x), \quad j = 1(1)N.$$

To approximate the Fredholm term in (1.11), define the (Fredholm-Gauss) operator  $\mathcal{F}_N$  that approximates the action of  $\mathcal{F}$  by the Nyström quadrature

$$\mathcal{F} v \approx \mathcal{F}_N v = (\mathcal{F}_N v)(x) \equiv \sum_{j=1}^N w_{j,N} k(x, y_{j,N}) v(y_{j,N}), \quad (1.15)$$

in which  $w_{j,N}$  and  $y_{j,N}$  are respectively the weights and abscissae of the Gaussian integration rule. As the weight function in the integral (1.10) for  $\mathcal{F} v$  is unity, the nodes  $y_{j,N}$  can be chosen as Gauss-Legendre, Legendre-Gauss-Radau or Legendre-Gauss-Lobatto distributions. Via (1.14) and (1.15), the discrete approximation of VFIE (1.7) is obtained as

$$v_N(x) = g(x) + \sum_{j=1}^N \left\{ \frac{\tau_{j,N}(x)}{\mu(x)} - \lambda w_{j,N} k(x, y_{j,N}) \right\} v_N(y_{j,N}) \quad (1.16)$$

which, when collocated at nodes  $x = y_{i,N}$ ,  $i = 1(1)N$ , yields the  $N \times N$  linear system

$$(\mathbf{I}_N - \mathbf{M}_N) \mathbf{v}_N = \mathbf{g}_N. \quad (1.17)$$

The matrix and vector entries in (1.17) are given by, for  $i, j = 1(1)N$ ,

$$\{\mathbf{I}_N\}_{i,j} = \delta_{ij}, \quad \{\mathbf{M}_N\}_{i,j} = \frac{\tau_{j,N}(y_{i,N})}{\mu(y_{i,N})} - \lambda w_{j,N} k(y_{i,N}, y_{j,N}), \quad (1.18)$$

$$\{\mathbf{v}_N\}_i = v_N(y_{i,N}) \quad \text{and} \quad \{\mathbf{g}_N\}_i = g(y_{i,N}),$$

wherein  $\delta_{ij}$  is the Kronecker delta. Inversion of (1.17) yields the  $N$  nodal values  $v_N(y_{i,N})$  which, when substituted into the inversion formula (1.16), give the approximate solution  $v_N(x)$  of (1.7), which in symbolic form is

$$v_N = g + \frac{\mathcal{V}_N v_N}{\mu} - \lambda \mathcal{F}_N v_N. \quad (1.19)$$

Note that computing  $v_N(x)$  directly via the inversion formula (1.16) is more accurate [13] than using Lagrange interpolation (1.12). By (1.5), the exact solutions  $v$  and  $u$ , of the VFIE and FIDE respectively, satisfy the symbolic equation

$$u = \zeta + \mathcal{V} v, \quad (1.20)$$

to which application of  $\mathcal{D}$  to both sides yields  $\mathcal{D} u = \mathcal{D} \mathcal{V} v$ , i.e.  $v = \mathcal{D} \mathcal{V} v$ , so that  $(\mathcal{D})^{-1} = \mathcal{V}$ . Additionally, (1.20) implies that there are two cases to consider when recovering the numerical

solution  $u_N$  from its derivative  $v_N$  computed via (1.17)–(1.19). First, if  $v_N(x)$  is exactly integrable (case 1) then the approximate numerical solution  $u_N$  of (1.2) can be computed from  $v_N$  as

$$\tilde{u}_N = \zeta + \mathcal{V} v_N. \quad (1.21)$$

Second, if functions  $\mu(x)$ ,  $K(x, y)$  and  $f(x)$  in IDE (1.1) are such that (1.19) is not exactly integrable (case 2) then the approximate numerical solution  $u_N$  of (1.2) must in this case be computed from  $v_N$  as

$$\hat{u}_N = \zeta + \mathcal{V}_N v_N, \quad (1.22)$$

which yields  $\hat{u}_N(x)$  as a polynomial of degree  $N$  in  $x$ . Note that this method requires only (1.17)–(1.18), as  $v_N(x)$  does not need to be computed via (1.19) since only its nodal values, given by the solution vector  $\mathbf{v}_N$  of (1.17), are present in the last term in (1.22).

## 1.4 Error Analysis

A theoretical analysis of the error incurred in computing  $u_N$  is now presented. Though a basic consideration of errors appears in the VFIE approach in [21], it not only computes the Volterra component of the VFIE crudely using Simpson's rule, but also concerns only convergence rates of  $\|v - v_N\|$  (NB and not  $\|u - u_N\|$ ) using a known exact solution. By contrast, the present work computes both Volterra and Fredholm components of the VFIE to spectral accuracy and, moreover, determines explicit error bounds for  $\|u - u_N\|$  using only the approximate derivative  $v_N$  of the numerical solution  $u_N$ . The error analysis is now presented for cases 1 and 2 given in (1.21) and (1.22) respectively.

### Case 1

Defining the linear operators  $\mathcal{S}$  and  $\mathcal{S}_N$  as

$$\mathcal{S} \equiv \frac{\mathcal{V}}{\mu} - \lambda \mathcal{F} \quad \text{and} \quad \mathcal{S}_N \equiv \frac{\mathcal{V}_N}{\mu} - \lambda \mathcal{F}_N, \quad (1.23)$$

the exact solution (1.11) of VFIE (1.7) can be written as

$$v = g + \mathcal{S} v \quad (1.24)$$

and the numerical solution (1.19) of (1.11) can be written as

$$v_N = g + \mathcal{S}_N v_N. \quad (1.25)$$

Subtraction of (1.25) from (1.24) yields

$$v - v_N = \mathcal{S} v - \mathcal{S}_N v_N = \mathcal{S} (v - v_N) + (\mathcal{S} - \mathcal{S}_N) v_N. \quad (1.26)$$

Since  $v = \mathcal{D} u$  and  $v_N = \mathcal{D} \tilde{u}_N$ , (1.26) can be rearranged to yield

$$(\mathcal{J} - \mathcal{S}) \mathcal{D} (u - \tilde{u}_N) = (\mathcal{S} - \mathcal{S}_N) v_N,$$

giving an explicit error formula for the exact solution  $u$  of the FIDE (1.1) as

$$u - \tilde{u}_N = (\mathcal{D} - \mathcal{S} \mathcal{D})^{-1} ((\mathcal{S} - \mathcal{S}_N) v_N),$$

yielding the error bound

$$\|u - \tilde{u}_N\| \leq C \sigma_N, \quad (1.27)$$

where

$$C = \|(\mathcal{D} - \mathcal{S}\mathcal{D})^{-1}\| \quad \text{and} \quad \sigma_N = \|(\mathcal{S} - \mathcal{S}_N)v_N\|. \quad (1.28)$$

The term  $\sigma_N$  can be expressed via (1.25) as

$$\sigma_N = \|\mathcal{S}v_N - v_N + g\|, \quad (1.29)$$

which demonstrates that the error is proportional to the residual obtained when the numerical solution  $v_N(x)$  is inserted into the exact VFIE (1.7). Alternatively, via (1.23), a bound on  $\sigma_N$  can be obtained as

$$\sigma_N \leq \frac{\|(\mathcal{V} - \mathcal{V}_N)v_N\|}{\|\mu\|} + |\lambda| \|(\mathcal{F} - \mathcal{F}_N)v_N\|, \quad (1.30)$$

in which  $\|(\mathcal{V} - \mathcal{V}_N)v_N\|$  is obtained from the definition of  $\mathcal{V}_N$ , which gives

$$(\mathcal{V} - \mathcal{V}_N)v_N(x) = \mathcal{V}(\mathcal{J} - \mathcal{L}_N)v_N(x) = \frac{\mathcal{V}p_N(x)}{N!}v_N^{(N)}(\eta), \quad \eta \in (-1, 1), \quad (1.31)$$

wherein  $p_N(x)$  is the monic polynomial whose roots are the  $N$  nodes  $y_{i,N}$ , i.e.

$$p_N(x) = \prod_{i=1}^N (x - y_{i,N}). \quad (1.32)$$

Therefore, in (1.31), there results

$$\|(\mathcal{V} - \mathcal{V}_N)v_N\| \leq Q_N \|v_N^{(N)}\| = Q_N \|\tilde{u}_N^{(N+1)}\|, \quad (1.33)$$

in which

$$Q_N \equiv \frac{\|\mathcal{V}p_N(x)\|}{N!}; \quad (1.34)$$

moreover, by standard Gaussian quadrature results [19],

$$\|(\mathcal{F} - \mathcal{F}_N)v_N\| \leq \psi_N^{(\nu)} \mathbb{F}_{2N-\nu}, \quad (1.35)$$

in which [15]

$$\psi_N^{(\nu)} \sim \frac{2^{2\nu-1} \sqrt{\pi}}{N^{(1-2\nu)/2}} \left(\frac{e}{4N}\right)^{2N}, \quad N \rightarrow \infty \quad \text{and} \quad \mathbb{F}_M = \max_{x,y \in [-1,1]} \left| \frac{\partial^M}{\partial y^M} (k(x,y)v_N(y)) \right|, \quad (1.36)$$

in which  $\nu$  corresponds to the number of endpoints included in the distribution, i.e.  $\nu = 0, 1$  and  $2$  for Legendre, Radau and Lobatto nodes respectively. Combining (1.30), (1.33) and (1.35) yields

$$\sigma_N \leq \frac{Q_N \|v_N^{(N)}\|}{\|\mu\|} + |\lambda| \psi_N^{(\nu)} \mathbb{F}_{2N-\nu}. \quad (1.37)$$

With  $\sigma_N$  in (1.27) bounded by (1.37), the constant  $C$  given by (1.28) can be bounded via

$$C = \|((\mathcal{J} - \mathcal{S})\mathcal{D})^{-1}\| = \|\mathcal{D}^{-1}(\mathcal{J} - \mathcal{S})^{-1}\| = \|\mathcal{V}(\mathcal{J} - \mathcal{S})^{-1}\| \leq \|\mathcal{V}\| \|(\mathcal{J} - \mathcal{S})^{-1}\|, \quad (1.38)$$

in which, adopting the approach of Atkinson [3, Eqns. (4.1.13)–(4.1.17)],  $\|\mathcal{V}\|$  is computed as

$$\|\mathcal{V}\| = \|\mathcal{V}1\| = \max_{x \in [-1,1]} |x+1| = 2.$$

By (1.23), operators  $\mathcal{S}$  and  $\mathcal{S}_N$  are linear combinations of  $\mathcal{V}$ ,  $\mathcal{F}$ ,  $\mathcal{V}_N$  and  $\mathcal{F}_N$ , for which, by the definitions of Lagrangian interpolation and Gaussian quadrature respectively,  $(\mathcal{V} - \mathcal{V}_N)v(x) \rightarrow 0$  and  $(\mathcal{F} - \mathcal{F}_N)v(x) \rightarrow 0$  as  $N \rightarrow \infty$  for all  $v \in \mathcal{C}$  and  $x \in [-1, 1]$ . That is,  $\mathcal{S}_N v$  is pointwise

uniformly convergent to  $\mathcal{S}v$  as  $N \rightarrow \infty$  for all  $v \in \mathcal{C}$  and  $x \in [-1, 1]$ , and hence, by [3, Thm 4.1.2] and [17, Eq. (4.7.17b)],  $(\mathcal{J} - \mathcal{S})^{-1}$  in (1.38) exists and is uniformly bounded by

$$\|(\mathcal{J} - \mathcal{S})^{-1}\| \leq \frac{1 + \|(\mathcal{J} - \mathcal{S}_N)^{-1}\| \|\mathcal{S}\|}{1 - \|(\mathcal{J} - \mathcal{S}_N)^{-1}\| \|(\mathcal{S} - \mathcal{S}_N)\mathcal{S}\|}, \quad (1.39)$$

the denominator of which is positive by construction. The sub-elements on the right-hand side of (1.39) are computed using the approach in Atkinson [3, Eqns. (4.1.13)–(4.1.17)], which gives  $\|\mathcal{S}\|$  as

$$\|\mathcal{S}\| = \|\mathcal{S}\mathbf{1}\| \equiv \|s\|,$$

say, in which  $s(x)$  is given by (1.9), (1.10) and (1.23) as

$$s(x) = \frac{x+1}{\mu(x)} - \lambda \int_{-1}^1 k(x, y) dy. \quad (1.40)$$

Similarly,  $\|(\mathcal{S} - \mathcal{S}_N)\mathcal{S}\|$  in (1.39) is computed as

$$\|(\mathcal{S} - \mathcal{S}_N)\mathcal{S}\| = \|(\mathcal{S} - \mathcal{S}_N)\mathcal{S}\mathbf{1}\| = \|(\mathcal{S} - \mathcal{S}_N)s\|$$

and  $\|(\mathcal{J} - \mathcal{S}_N)^{-1}\|$  as

$$\|(\mathcal{J} - \mathcal{S}_N)^{-1}\| = \|(\mathcal{J} - \mathcal{S}_N)^{-1}\mathbf{1}\| \equiv \|r_N\|,$$

say, in which  $r_N(x)$  is the solution of

$$r_N - \mathcal{S}_N r_N = \mathbf{1},$$

whose left-hand side is of the same form as VFIE (1.25). Consequently, nodal values of  $r_N(x)$  are found by solving a linear system with the same matrix as in (1.17), i.e.

$$(\mathbf{I}_N - \mathbf{M}_N) \mathbf{r}_N = \mathbf{1}, \quad (1.41)$$

in which  $\mathbf{I}_N$  and  $\mathbf{M}_N$  are as given in (1.18) and the entries of the vectors  $\mathbf{r}_N$  and  $\mathbf{1}$  are given by

$$\{\mathbf{r}_N\}_i = r_N(y_{i,N}) \quad \text{and} \quad \{\mathbf{1}\}_i = 1, \quad i = 1(1)N.$$

It is noted that, for the purposes of efficiency, (1.17) and (1.41) can be solved in the partitioned form

$$(\mathbf{I}_N - \mathbf{M}_N) (\mathbf{v}_N | \mathbf{r}_N) = (\mathbf{g}_N | \mathbf{1}).$$

Solving (1.41) gives the nodal vector  $\mathbf{r}_N$ , the elements of which are used in the Nyström inversion formula

$$r_N(x) = 1 + \sum_{j=1}^N \left\{ \frac{\tau_{j,N}(x)}{\mu(x)} - \lambda w_{j,N} k(x, y_{j,N}) \right\} r_N(y_{j,N}),$$

from which  $\|r_N\|$  can be computed directly; similarly,  $\|s\|$  can be computed directly from (1.40). Finally, (1.27), (1.29) and (1.38) give the case-1 theoretical bound

$$\|u - \tilde{u}_N\| \leq \frac{2(1 + \|r_N\| \|s\|)}{1 - \|r_N\| \|(\mathcal{S} - \mathcal{S}_N)s\|} \|\mathcal{S}v_N - v_N + g\| \quad (1.42)$$

on the (case-1) error  $u - \tilde{u}_N$  that is explicitly computable in terms of only the derivative  $v_N$  of the case-1 numerical solution  $\tilde{u}_N$ .

**Case 2**

Subtraction of (1.22) from (1.21) and addition of  $u - u = 0$  to the resulting left-hand side gives a bound on the case-2 error as

$$\tilde{u}_N - u + u - \hat{u}_N = (\mathcal{V} - \mathcal{V}_N)v_N \Rightarrow \|u - \hat{u}_N\| \leq \|u - \tilde{u}_N\| + \|(\mathcal{V} - \mathcal{V}_N)v_N\|$$

which, by (1.33) and (1.42), yields

$$\|u - \hat{u}_N\| \leq \frac{2(1 + \|r_N\| \|s\|)}{1 - \|r_N\| \|(\mathcal{S} - \mathcal{S}_N)s\|} \|\mathcal{S}v_N - v_N + g\| + Q_N \|v_N^{(N)}\|. \quad (1.43)$$

As the case-2 solution arises when  $v_N(x)$  is not integrable, the bound (1.43) is not computable as the operator  $\mathcal{S}$  contains the Volterra operator  $\mathcal{V}$  via (1.23). Therefore, the term  $\|\mathcal{S}v_N - v_N + g\|$  in (1.43)—defined as  $\sigma_N$  in (1.29)—must be bounded using (1.37). Similarly, as  $\mathcal{S}s$  will in general be uncomputable, a bound (analogous to (1.37)) on  $\|(\mathcal{S} - \mathcal{S}_N)s\|$  can be found as

$$\|(\mathcal{S} - \mathcal{S}_N)s\| \leq \frac{Q_N \|s^{(N)}\|}{\|\mu\|} + |\lambda| \psi_N^{(\nu)} \mathbb{S}_{2N-\nu},$$

in which

$$\mathbb{S}_M = \max_{x,y \in [-1,1]} \left| \frac{\partial^M}{\partial y^M} (k(x,y) s(y)) \right|.$$

Collecting results, the computable case-2 error bound is given by

$$\|u - \hat{u}_N\| \leq \frac{2(1 + \|r_N\| \|s\|) (Q_N \|v_N^{(N)}\| + |\lambda| \|\mu\| \psi_N^{(\nu)} \mathbb{F}_{2N-\nu})}{\|\mu\| - \|r_N\| (Q_N \|s^{(N)}\| + |\lambda| \|\mu\| \psi_N^{(\nu)} \mathbb{S}_{2N-\nu})} + Q_N \|v_N^{(N)}\|. \quad (1.44)$$

Computable error bounds (1.42) and (1.44) have now been derived for the general FIDE (1.2).

Asymptotic large- $N$  approximations for  $Q_N$  were obtained in terms of  $N$  and  $\nu$ ; the details are cumbersome and omitted for reasons of space. These approximations explicitly reveal that the present VFIE approach is of order  $O(N^{(\nu^2 - \nu - 7)/2})$  times more accurate than the direct FIDE approach, henceforth denoted as “case 0”, used in [15].

**1.5 Numerical Results**

Using the algebraic manipulator `Maple`, the methods and bounds derived above were respectively implemented and validated on four test problems, each with known solutions, chosen to demonstrate the accuracy of the theory on potentially challenging problems. The components of each test problem are shown in Table 1.1. As the results were qualitatively similar for each nodal distribution, only the results for the Legendre distribution, for which  $\nu = 0$ , are presented.

Problem	Type	Solution $u(x)$	$\mu(x)$	Kernel $K(x, y)$	$\lambda$
1	Smooth	$\sin x + x^2$	$\sec x$	$(x^3 - 1) y \cos y$	$\frac{1}{3}$
2	Runge	$\frac{1}{1+25x^2}$	$\frac{1}{x-2}$	$(x+1)(y^2 - 5)$	$-\frac{1}{2}$
3	Steep	$e^{15x}$	$e^x$	$e^{x+y}$	1
4	Oscillatory	$\cos 12x$	$\frac{1}{x^5 - 3x + 1}$	$\sin x y^3$	2

Table 1.1: Test problems with solutions of four qualitatively distinct forms. The Runge phenomenon [8, 9], extreme gradient and high-frequency oscillations, in the solutions of problems 2, 3 and 4 respectively, offer well-documented challenges to approximation methods.

Figure 1.1 shows that, for each test problem, the case-1 errors are lower than the case-2 errors and so, as expected, it is more accurate to integrate the numerical VFIE solution exactly to obtain the FIDE solution rather than to integrate its Lagrange interpolant. Additionally, as predicted at the end of Section 1.4, the new case-1 and case-2 errors are smaller in magnitude than the case-0 errors incurred in [15], confirming that bypassing the need for numerical differentiation by converting from FIDE to VFIE form yields a more accurate numerical solution.

Figure 1.1 also reveals that the case-1 error bound is more accurate (by comparison with the actual computed errors) than the case-2 error bound, particularly for problem 2 in which the case-2 error bound diverges whilst the true errors converge with increasing  $N$ : this divergence, and the large discrepancy between true case-2 errors and error bounds for the other problems, is due to the terms  $\|v_N^{(N)}\|$  and  $\mathbb{F}_{2N-\nu}$  in the error bound (1.44). Via the mean-value theorem used to derive (1.31), the truncation parameter  $\eta \in (-1, 1)$  that yields the true error  $(\mathcal{V} - \mathcal{V}_N)v_N$  is unknown, so  $v_N^{(N)}(\eta)$  must be replaced by  $\|v_N^{(N)}\|$ , the latter of which may be much greater than the former. The same argument applies to the Gaussian-quadrature error term (1.35), which includes the unknown values of  $x$  and  $y$  in (1.36); as these are unknown,  $\mathbb{F}_{2N-\nu}$  must be computed by maximising over  $x, y \in [-1, 1]$ , and so the quadrature error may also be over-estimated.

## 1.6 Conclusions

A novel method for the accurate numerical solution of one-dimensional, first-order Fredholm integro-differential equations has been developed by first converting the problem into a Volterra-Fredholm integral equation. The technique has been validated on diverse and challenging test problems. A novel error analysis has been conducted and validated to yield explicitly computable (using only the numerical solution) error bounds that predict true computational errors to spectral accuracy. Two independent sub-approaches have been analysed depending upon whether or not intermediate stages of the novel process admit exact integration. For both cases, errors are shown theoretically and numerically to be smaller in magnitude than the errors incurred by a previous approach [15].

## References

- [1] O. A. ARQUB, M. AL-SMADI AND N. SHAWAGFEH, *Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method*, Appl. Math. Comput., 219 (2013), 8938–8948.
- [2] E. ARUCHUNAN AND J. SULAIMAN, *Quarter-sweep Gauss-Seidel method for solving first order linear Fredholm integro-differential equations*, Matematika, 27 (2011), 199–208.
- [3] K. E. ATKINSON, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, UK, 1997.
- [4] E. BABOLIAN AND L. M. DELVES, *A fast Galerkin scheme for linear integro-differential equations*, IMA J. Numer. Anal., 1 (1981), 193–213.
- [5] E. BABOLIAN, M. T. KAJANI AND M. GHASEMI, *Numerical solution of linear integro-differential equation by using sine-cosine wavelets*, Appl. Math. Comput., 180 (2006), 569–574.
- [6] N. BILDIK, A. KONURALP AND S. YALÇINBAŞ, *Comparison of Legendre polynomial approximation and variational iteration method for the solutions of general linear Fredholm integro-differential equations*, Comput. Math. Appl., 59 (2010), 1909–1917.

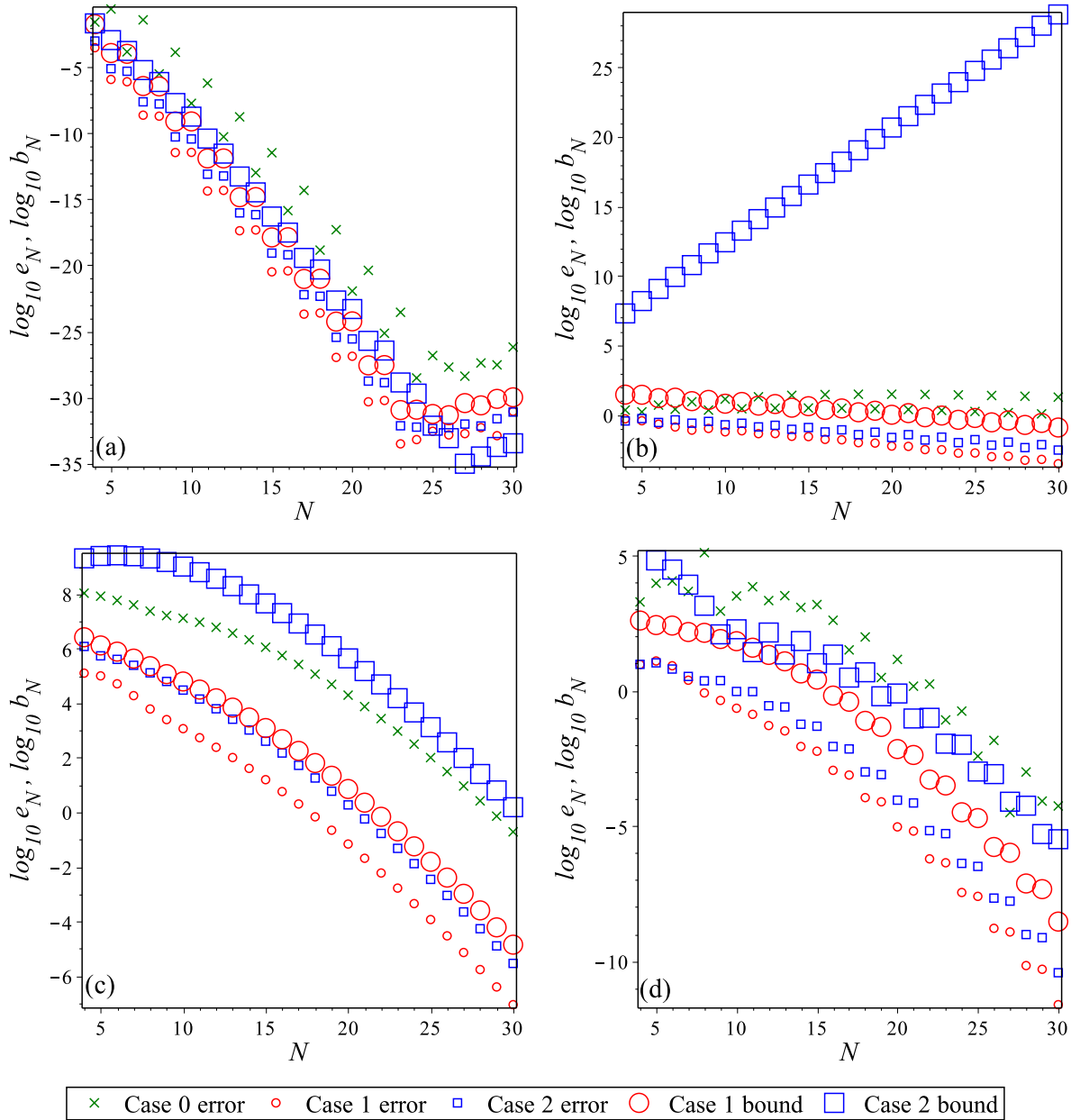


Figure 1.1: Logarithmic plots showing convergence or divergence with  $N$  of error  $e_N = \|u - u_N\|$  and bound  $b_N$  given by (1.42) and (1.44), for cases 0, 1 and 2 for each of problems (a) 1 “smooth”, (b) 2 “Runge”, (c) 3 “steep” and (d) 4 “oscillatory”. All computations are conducted on Legendre nodes, i.e.  $\nu = 0$ . Note the divergence of the case-2 bound for the Runge problem, as discussed in the text. In all four problems, the new case-1 and case-2 errors are smaller than those incurred in [15].

- [7] A. H. BORZABADI AND M. HEIDARI, *A successive numerical scheme for some classes of Volterra-Fredholm integral equations*, Iranian Journal of Mathematical Sciences and Informatics, 10 (2015), 1–10.
- [8] J. P. BOYD, *Defeating the Runge Phenomenon for equispaced polynomial interpolation via Tikhonov regularization*, Appl. Math. Lett., 5 (1992), 57–59.
- [9] J. P. BOYD, *Exponentially accurate Runge-free approximation of non-periodic functions from sam-*

- ples on an evenly spaced grid, *Appl. Math. Lett.*, 20 (2007), 971–975.
- [10] Z. CHEN AND W. JIANG, *An approximate solution for a mixed linear Volterra-Fredholm integral equation*, *Appl. Math. Lett.*, 25 (2012), 1131–1134.
- [11] P. DARANIA AND A. EBADIAN, *A method for the numerical solution of the integro-differential equations*, *Appl. Math. Comput.*, 188 (2007), 657–668.
- [12] H. LAELI DASTJERDI AND F. M. MAALEK GHAINI, *Numerical solution of Volterra–Fredholm integral equations by moving least square method and Chebyshev polynomials*, *Appl. Math. Model.*, 36 (2012), 3283–3288.
- [13] A. I. FAIRBAIRN AND M. A. KELMANSON, *Computable theoretical error bounds for Nyström methods for Fredholm integral equations of the second kind*. In *Proc. 10th UK Conf. on Boundary Integral Methods*, 85–94, 2015.
- [14] A. I. FAIRBAIRN AND M. A. KELMANSON, *Spectrally accurate Nyström-solver error bounds for 1-D Fredholm integral equations of the second kind*, submitted to *Appl. Math. Comput.*, June 29 2016, 20pp.
- [15] A. I. FAIRBAIRN AND M. A. KELMANSON, *Computable a priori error bounds for Nyström methods applied to 1-D integro-differential Fredholm equations*, submitted to *IMA J. Num. Anal.*, February 15 2017, 19pp.
- [16] M. FATHY, M. EL-GAMEL AND M. S. EL-AZAB, *Legendre-Galerkin method for the linear Fredholm integro-differential equations*, *Appl. Math. Comput.*, 243 (2014), 789–800.
- [17] W. HACKBUSCH, *Integral Equations: Theory and Numerical Treatment*, Birkhäuser Verlag, Basel, Switzerland, 1995.
- [18] F. A. HENDI AND A. M. ALBUGAMI, *Numerical solution for Fredholm–Volterra integral equation of the second kind by using collocation and Galerkin methods*, *Journal of King Saud University - Science*, 22 (2010), 37–40.
- [19] F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1974.
- [20] S. M. HOSSEINI AND S. SHAHMORAD, *Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases*, *Appl. Math. Model.*, 27 (2003), 145–154.
- [21] P. LINZ, *A method for the approximate solution of linear integro-differential equations*, *SIAM J. Numer. Anal.*, 11 (1974), 137–144.
- [22] K. MALEKNEJAD AND M. ATTARY, *An efficient numerical approximation for the linear class of Fredholm integro-differential equations based on Cattani’s method*, *Commun. Nonlinear Sci. Numer. Simulat.*, 16 (2011), 2672–2679.
- [23] M. M. MUSTAFA AND I. N. GHANIM, *Numerical solution of linear Volterra-Fredholm integral equations using Lagrange polynomials*, *Mathematical Theory and Modeling*, 4 (2014), 137–146.
- [24] S. NEMATI, *Numerical solution of Volterra–Fredholm integral equations using Legendre collocation method*, *J. Comput. Appl. Math.*, 278 (2015), 29–36.
- [25] J. RASHIDINIA AND M. ZAREBNIA, *The numerical solution of integro-differential equation by means of the Sinc method*, *Appl. Math. Comput.*, 118 (2007), 1124–1130.
- [26] P. K. SAHU AND S. RAY, *Legendre spectral collocation method for Fredholm integro-differential equation with variable coefficients and mixed conditions*, *Appl. Math. Comput.*, 268 (2015), 575–580.
- [27] M. A. WOLFE, *The numerical solution of non-singular integral and integro-differential equations by iteration with Chebyshev series*, *Comput. J.*, 12 (1969), 193–196.
- [28] S. YALÇINBAŞ AND M. SEZER, *The approximate solution of high-order linear volterra-fredholm integro-differential equations in terms of taylor polynomials*, *Appl. Math. Comput.*, 112 (2000), 291–308.
- [29] S. YALÇINBAŞ, M. SEZER AND H. H. SORKUN, *Legendre polynomial solutions of high-order linear Fredholm integro-differential equations*, *Appl. Math. Comput.*, 210 (2009), 334–349.