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# ABELIAN HURWITZ-HODGE INTEGRALS

P. JOHNSON, R. PANDHARIPANDE, AND H.-H. TSENG

ABSTRACT. Hodge classes on the moduli space of admissible covers with monodromy group  $G$  are associated to irreducible representations of  $G$ . We evaluate all linear Hodge integrals over moduli spaces of admissible covers with abelian monodromy in terms of multiplication in an associated wreath group algebra. In case  $G$  is cyclic and the representation is faithful, the evaluation is in terms of double Hurwitz numbers. In case  $G$  is trivial, the formula specializes to the well-known result of Ekedahl-Lando-Shapiro-Vainshtein for linear Hodge integrals over the moduli space of curves in terms of single Hurwitz numbers.

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## 0. INTRODUCTION

0.1. **Moduli of covers.** Let  $\mathcal{M}_{g,n}$  be the moduli space of nonsingular, connected, genus  $g$  curves over  $\mathbb{C}$  with  $n$  distinct points. Let  $G$  be a finite group. Given an element  $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$ , we will consider principal  $G$ -bundles,

$$(1) \quad \begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & C \setminus \{p_1, \dots, p_n\} \end{array} ,$$

over the punctured curve. Denote the  $G$ -action on the fibers of  $\pi$  by

$$\tau : G \times P \rightarrow P.$$

The monodromy defined by a positively oriented loop around the  $i^{\text{th}}$  puncture determines a conjugacy class  $\gamma_i \in \text{Conj}(G)$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be the  $n$ -tuple of monodromies. The moduli space

of covers  $\mathcal{A}_{g,\gamma}(G)$  parameterizes  $G$ -bundles (1) with the prescribed monodromy conditions. There is a canonical morphism

$$\epsilon : \mathcal{A}_{g,\gamma}(G) \rightarrow \mathcal{M}_{g,n}$$

obtained from the base of the  $G$ -bundle. Both  $\mathcal{A}_{g,\gamma}(G)$  and  $\mathcal{M}_{g,n}$  are nonsingular Deligne-Mumford stacks.

A compactification  $\mathcal{A}_{g,\gamma}(G) \subset \overline{\mathcal{A}}_{g,\gamma}(G)$  by *admissible covers* was introduced by Harris and Mumford in [15]. An admissible cover

$$[\pi, \tau] \in \overline{\mathcal{A}}_{g,\gamma}(G)$$

is a degree  $|G|$  finite map of complete curves

$$\pi : D \rightarrow (C, p_1, \dots, p_n)$$

together with a  $G$ -action

$$\tau : G \times D \rightarrow D$$

on the fibers of  $\pi$  satisfying the following properties:

- (i)  $D$  is a possibly disconnected nodal curve,
- (ii)  $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$  is a stable curve,
- (iii)  $\pi$  maps the nonsingular points to nonsingular points and nodes to nodes,

$$\pi(D^{ns}) \subset C^{ns}, \quad \pi(D^{sing}) \subset C^{sing},$$

- (iv)  $[\pi, \tau]$  restricts to a principal  $G$ -bundle over the punctured nonsingular locus

$$\pi^{open} : D^{open} \rightarrow C^{ns} \setminus \{p_1, \dots, p_n\}$$

with monodromy  $\gamma$ ,

- (v) distinct branches of a node  $\eta \in D^{sing}$  map to distinct branches of  $\pi(\eta) \in C^{sing}$  with equal ramification orders over  $\pi(\eta)$ ,
- (vi) the monodromies of the  $G$ -bundle  $\pi^{open}$  determined by the two branches of  $C$  at  $\eta \in C^{sing}$  lie in opposite conjugacy classes.

Harris and Mumford originally considered only symmetric group  $\Sigma_d$  monodromy, but the natural setting for the construction is for all finite  $G$ .

An admissible cover may be alternatively viewed as a principal  $G$ -bundle over the stack quotient<sup>1</sup>  $[D/G]$  inducing a stable map to the classifying space

$$(2) \quad f : [D/G] \rightarrow \mathcal{B}G.$$

Then,  $\overline{\mathcal{A}}_{g,\gamma}(G)$  is simply a moduli space of stable maps [2, 5]<sup>2</sup>,

$$\overline{\mathcal{A}}_{g,\gamma}(G) \simeq \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G).$$

The deformation theory of stable maps endows  $\overline{\mathcal{A}}_{g,\gamma}(G)$  with a canonical nonsingular Deligne-Mumford stack structure. We take the stable maps perspective here.

There are two flavors of such stable map theories. If the base  $C$  is required to be connected as above, we write  $\overline{\mathcal{M}}_{g,\gamma}^\circ(\mathcal{B}G)$ . If disconnected bases  $C$  are allowed, we write  $\overline{\mathcal{M}}_{g,\gamma}^\bullet(\mathcal{B}G)$ . In the

<sup>1</sup> $[D/G]$  differs from  $C$  only by possible stack structure at the markings  $p_i$  and the nodes. In both cases, the order of the isotropy group is the order of the local monodromy in  $G$ .

<sup>2</sup>We do not trivialize the marked gerbes on the domain in the definition of  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ .

disconnected case, the genus  $g$  may be negative. If the superscript is omitted, the connected case is assumed.

Our results are restricted to abelian groups  $G$ . Here,  $\text{Conj}(G)$  is the set of elements of  $G$ . Of course, the cyclic groups  $\mathbb{Z}_a$  will play the most important role. In case  $G$  is trivial, there is no extra monodromy data, and the moduli space of maps  $\overline{\mathcal{M}}_{g,(0,\dots,0)}(\mathcal{B}\mathbb{Z}_1)$  specializes to  $\overline{\mathcal{M}}_{g,n}$ .

**0.2. Hodge integrals.** Let  $R$  be an irreducible  $\mathbb{C}$ -representation of  $G$ . If  $G$  is abelian,  $R$  is a character

$$\phi^R : G \rightarrow \mathbb{C}^*.$$

By associating to each map  $[f] \in \overline{\mathcal{M}}_{g,\gamma}(G)$  presented as (2) above the  $R$ -summand of the  $G$ -representation  $H^0(D, \omega_D)$ , we obtain a vector bundle

$$\mathbb{E}^R \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G).$$

The rank of  $\mathbb{E}^R$  is locally constant and determined by the orbifold Riemann-Roch formula discussed in Section 1. The *Hodge classes* on  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  are Chern classes of  $\mathbb{E}^R$ ,

$$\lambda_i^R = c_i(\mathbb{E}^R) \in H^{2i}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The  $i^{\text{th}}$  cotangent line bundle  $L_i$  on the moduli space of curves has fiber

$$L_i|_{(C,p_1,\dots,p_n)} = T_{p_i}^*(C).$$

Descendent classes on  $\overline{\mathcal{M}}_{g,n}$  are defined by

$$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Descendent classes  $\bar{\psi}_i$  on the space of stable maps are defined by pull-back via the morphism

$$\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,n}$$

to the moduli space of curves,

$$\bar{\psi}_i = \epsilon^*(\psi_i) \in H^2(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The *Hodge integrals* over  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  are the top intersection products of the classes  $\{\lambda_i^R\}_{R \in \text{Irr}(G)}$  and  $\{\bar{\psi}_j\}_{1 \leq j \leq n}$ . Linear Hodge integrals are of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \lambda_i^R \cdot \prod_{j=1}^n \bar{\psi}_j^{m_j}.$$

The term *Hurwitz-Hodge integral* was used in [3] to emphasize the role of the covering spaces.

**0.3. Hurwitz numbers.** Let  $g$  be a genus and let  $\nu$  and  $\mu$  be two (unordered) partitions of  $d \geq 1$ . Let  $\ell(\nu)$  and  $\ell(\mu)$  denote the lengths of the respective partitions. A Hurwitz cover of  $\mathbb{P}^1$  of genus  $g$  with ramifications  $\nu$  and  $\mu$  over  $0, \infty \in \mathbb{P}^1$  is a morphism

$$\pi : C \rightarrow \mathbb{P}^1$$

satisfying the following properties:

- (i)  $C$  is a nonsingular, connected, genus  $g$  curve,
- (ii) the divisors  $\pi^{-1}(0), \pi^{-1}(\infty) \subset C$  have profiles equal to the partitions  $\nu$  and  $\mu$  respectively,
- (iii) the map  $\pi$  is simply ramified over  $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$ .

By condition (ii), the degree of  $\pi$  must be  $d$ . Two covers

$$\pi : C \rightarrow \mathbb{P}^1, \pi' : C' \rightarrow \mathbb{P}^1$$

are isomorphic if there exists an isomorphism of curves  $\phi : C \rightarrow C'$  satisfying  $\pi' \circ \phi = \pi$ . Each cover  $\pi$  has an naturally associated automorphism group  $\text{Aut}(\pi)$ .

By the Riemann-Hurwitz formula, the number of simple ramification points of  $\pi$  over  $\mathbb{C}^*$  is

$$r_g(\nu, \mu) = 2g - 2 + \ell(\nu) + \ell(\mu).$$

Let  $U_r \subset \mathbb{C}^*$  be a fixed set of  $r_g(\nu, \mu)$  distinct points. The set of  $r_g(\nu, \mu)^{\text{th}}$  roots of unity is the standard choice. The *double Hurwitz number*  $H_g(\nu, \mu)$  is a weighted count of the distinct Hurwitz covers  $\pi$  of genus  $g$  with ramifications  $\nu$  and  $\mu$  over  $0, \infty \in \mathbb{P}^1$  and simple ramification over  $U_r$ . Each such cover is weighted by  $1/|\text{Aut}(\pi)|$ . The count  $H_g(\nu, \mu)$  does not depend upon the location of the points of  $U_r$ .

There are two flavors of Hurwitz numbers. The connected case defined above will be denoted  $H_g^\circ(\nu, \mu)$ . If  $C$  is allowed to be disconnected, the Hurwitz count is denoted  $H_g^\bullet(\nu, \mu)$ . Again, the absence of a superscript indicates the connected theory.

Disconnected Hurwitz numbers are easily expressed as products in the center  $\mathcal{Z}\Sigma_d$  of the group algebra of  $\Sigma_d$ ,

$$(3) \quad H_g^\bullet(\nu, \mu) = \frac{1}{d!} (C_\nu T^{r_g(\nu, \mu)} C_\mu)_{[\text{Id}]}.$$

Here,  $C_\nu$  and  $C_\mu$  are the sums in the group algebra of all elements of  $\Sigma_d$  with cycle types  $\nu$  and  $\mu$  respectively, and  $T$  is the sum of all transpositions. The subscript denotes the coefficient of the identity  $[\text{Id}]$ .

Multiplication in  $\mathcal{Z}\Sigma_d$  is diagonalized by the representation basis. Hurwitz numbers can be written as sums over characters of  $\Sigma_d$  and conveniently expressed as matrix elements in the infinite wedge representation. The latter formalism naturally connects Hurwitz numbers to integrable systems [20, 21, 24].

**0.4. Formula for  $\mathbb{Z}_a$ .** The formula for linear Hodge integrals is simplest in case the monodromy group is  $\mathbb{Z}_a$  and the representation  $U$  is given by

$$\phi^U : \mathbb{Z}_a \rightarrow \mathbb{C}^*, \quad \phi^U(1) = e^{\frac{2\pi i}{a}}.$$

Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a vector<sup>3</sup> of *nontrivial* elements of  $\mathbb{Z}_a$ ,

$$\gamma_i \in \{1, \dots, a-1\}.$$

Let  $\mu$  be a partition of  $d \geq 1$  with parts  $\mu_j$  and length  $\ell$ ,

$$\sum_{j=1}^{\ell} \mu_j = d.$$

Let  $\gamma - \mu$  denote the vector of elements of  $\mathbb{Z}_a$  defined by

$$\gamma - \mu = (\gamma_1, \dots, \gamma_n, -\mu_1, \dots, -\mu_\ell).$$

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<sup>3</sup> The length  $n$  may be taken to be 0 in which case  $\gamma = \emptyset$ .

While the parts of  $\mu$  are unordered, an ordering is chosen for  $\gamma - \mu$ . The vector  $\gamma - \mu$  may contain trivial parts. We will consider Hodge integrals over the moduli space  $\overline{\mathcal{M}}_{g, \gamma - \mu}(\mathcal{B}\mathbb{Z}_a)$ .

For nonemptiness, the parity condition

$$(4) \quad d - \sum_{i=1}^n \gamma_i = 0 \pmod{a}$$

is required. non-negativity,

$$d - \sum_{i=1}^n \gamma_i \geq 0,$$

and boundedness,

$$\forall i \neq j, \quad \gamma_i + \gamma_j \leq a$$

will also be imposed. If  $\gamma = \emptyset$ , non-negativity and boundedness are satisfied.

An automorphism of a partition is an element of the permutation group preserving equal parts. Let  $|\text{Aut}(\gamma)|$  and  $|\text{Aut}(\mu)|$  denote the orders of the automorphism groups.<sup>4</sup> Let  $\gamma_+$  be the partition of  $d$  determined by adjoining  $\frac{d - \sum_{i=1}^n \gamma_i}{a}$  parts of size  $a$ ,

$$\gamma_+ = (\gamma_1, \dots, \gamma_n, a, \dots, a).$$

A calculation shows

$$r_g(\gamma_+, \mu) = 2g - 2 + n + \ell + \frac{d}{a} - \sum_{i=1}^n \frac{\gamma_i}{a}.$$

Let the monodromy group  $\mathbb{Z}_a$  and representation  $\phi^U$  be specified as above. Our main result for linear  $\mathbb{Z}_a$ -Hodge integrals is the following formula.

**Theorem 1.** *Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be nontrivial monodromies in  $\mathbb{Z}_a$  satisfying the parity, non-negativity, and boundedness conditions with respect to the partition  $\mu$ . Then,*

$$H_g(\gamma_+, \mu) = \frac{r_g(\gamma_+, \mu)!}{|\text{Aut}(\gamma)| |\text{Aut}(\mu)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^{\ell} \frac{\mu_j \lfloor \frac{\mu_j}{a} \rfloor}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g, \gamma - \mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

The integer and fractional parts of a rational number are denoted in the above formula by

$$q = [q] + \langle q \rangle, \quad q \in \mathbb{Q}.$$

The cotangent lines in the denominator on the far right are associated to the stack points of the stable map domain corresponding to the parts of  $\mu$ .

Theorem 1 is proven by virtual localization on the moduli space of stable maps to the stack  $\mathbb{P}^1[a]$  with  $\mathbb{Z}_a$ -structure at 0 following the arguments of [9, 12]. The space of stable maps to  $\mathbb{P}^1[a]$  is discussed in Section 1, and the proof is given in Section 2. The formula is easily seen to determine all linear  $\mathbb{Z}_a$ -Hodge integrals with respect to  $U$  in terms of double Hurwitz numbers. In fact, the set of evaluations with  $\gamma = \emptyset$  is sufficient. Conversely, every double Hurwitz number is realized for  $a$  sufficiently large.

<sup>4</sup>Here,  $\gamma$  is considered as a partition by forgetting the ordering of the elements.

For the disconnected formula, we assume  $\gamma = \emptyset$  and the parity condition  $d = 0 \pmod{a}$ .<sup>5</sup> Then, Theorem 1 holds in exactly the same form,

$$(5) \quad H_g^\bullet(\emptyset_+, \mu) = \frac{r_g(\emptyset_+, \mu)!}{|\text{Aut}(\mu)|} a^{1-g+\sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\mathcal{M}_{g, -\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

The ELSV formula [6] for linear Hodge integrals on the moduli space of curves arises from the  $a = 1$  specialization of Theorem 1,

$$H_g(\mu) = \frac{(2g - 2 + d + \ell)!}{|\text{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{\mu_j^{\mu_j}}{\mu_j!} \int_{\mathcal{M}_{g, \ell}} \frac{\sum_{i=0}^g (-1)^i \lambda_i}{\prod_{j=1}^{\ell} (1 - \mu_j \psi_j)}.$$

For  $a = 1$ , we must have  $\gamma = \emptyset$ .

The conditions  $\gamma$  allow for greater freedom in the  $a > 1$  case. For example, the proof of Theorem 1 yields a remarkable vanishing property. The monodromy conditions  $\gamma$  satisfy negativity if

$$d - \sum_{i=1}^n \gamma_i < 0$$

and strong negativity if

$$d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} < 0.$$

Strong negativity is easily seen to imply negativity.

**Theorem 2.** *Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be nontrivial monodromies in  $\mathbb{Z}_a$  satisfying the parity condition with respect to the partition  $\mu$ . In addition, let  $\gamma$  satisfy at least one of the following two conditions:*

- (i) *negativity and boundedness, or*
- (ii) *strong negativity.*

*Then, a vanishing results for Hurwitz-Hodge integrals holds:*

$$\int_{\mathcal{M}_{g, \gamma - \mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)} = 0.$$

A few examples of Theorems 1 and 2 where alternative approaches to the integrals are available are presented in Section 3.

**0.5. Abelian  $G$ .** Since any faithful representation  $R$  of  $\mathbb{Z}_a$  differs from  $U$  by an automorphism of  $\mathbb{Z}_a$ , Theorem 1 determines linear Hodge integrals with respect to  $R$ . Representations of  $\mathbb{Z}_a$  with kernels require an additional analysis.

Let  $G$  be an abelian group with group law written additively. Consider an irreducible representation  $R$ ,

$$\phi^R : G \rightarrow \mathbb{C}^*,$$

<sup>5</sup>If  $\gamma \neq \emptyset$ , the non-negativity condition may be satisfied globally but be violated on connected components.

with associated exact sequence

$$(6) \quad 0 \rightarrow K \rightarrow G \xrightarrow{\phi^R} \text{Im}(\phi^R) \cong \mathbb{Z}_a \rightarrow 0.$$

The homomorphism  $\phi^R$  induces a canonical morphism

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a).$$

The morphism  $\rho$  satisfies

$$\rho^*(\lambda_i^U) = \lambda_i^R$$

and has the same degree over each component of  $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$ . Therefore, linear Hodge integrals with respect to  $R$  can be calculated by multiplying the formula of Theorem 1 by the degree of  $\rho$ .

In Section 4, the solution for arbitrary  $G$  and  $R$  is cast in a more appealing way. When

$$\phi^R(\gamma) = -\mu \in \mathbb{Z}_a,$$

Hodge integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^R}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}$$

are expressed in terms of Hurwitz numbers for  $K_d$ , the wreath product of  $K$  with the symmetric group  $\Sigma_d$ . Since the infinite wedge formalism for  $\Sigma_d$  extends to a Fock space formalism for the wreath product  $K_d$ , there is again a connection to integrable systems [25].

Conjugacy classes in  $K_d$  are indexed by  $\text{Conj}(K)$ -weighted partitions of  $d$ ,

$$\bar{\mu} = \{(\mu_1, \kappa_1), \dots, (\mu_{\ell(\mu)}, \kappa_{\ell(\mu)})\}.$$

Here,  $\mu$  is a partition of  $d$  with parts  $\mu_j$ , the weights  $\kappa_i \in \text{Conj}(K)$  are conjugacy classes in  $K$ , and  $\bar{\mu}$  is an unordered set of pairs. Let  $\text{Aut}(\bar{\mu})$  denote the automorphism group of  $\bar{\mu}$ . Let  $C_{\bar{\mu}} \in \mathcal{Z}K_d$  be the element of the group algebra associated to the conjugacy class  $\bar{\mu}$ . The transposition element  $T \in \mathcal{Z}K_d$  is associated to conjugacy class of  $K_d$  indexed by

$$\bar{\tau} = \{(2, 0), (1, 0), \dots, (1, 0)\}$$

where all the  $\text{Conj}(K)$ -weights are 0.

The wreath product  $K_d$  has a forgetful map to  $\Sigma_d$  which sends elements of cycle type  $\bar{\mu}$  to elements of type  $\mu$ . The  $K_d$ -Hurwitz number  $H_{g,K}(\bar{\nu}, \bar{\mu})$  counts the degree  $d|K|$ -fold covers of  $\mathbb{P}^1$  with monodromy in  $K_d$  given by  $\bar{\nu}$  and  $\bar{\mu}$  at  $0, \infty \in \mathbb{P}^1$  and  $\bar{\tau}$  at all the points of

$$U_{r_g(\nu, \mu)} \subset \mathbb{P}^1.$$

Since  $K \subset K_d$  is contained in the center, any such cover has a canonical  $K$ -action which defines a  $K$ -bundle over a punctured Hurwitz cover counted by  $H_g(\nu, \mu)$ . The connectivity requirement we place on covers counted by  $H_{g,K}(\bar{\nu}, \bar{\mu})$  is *not* that the  $d|K|$ -fold cover is connected, but only that the associated Hurwitz  $d$ -fold cover is connected. Similarly,  $g$  is the genus of the  $d$ -fold cover.

The natural extension of formula (3) for disconnected Hurwitz covers for the wreath product  $K_d$  is

$$H_{g,K}^{\bullet}(\bar{\nu}, \bar{\mu}) = \frac{1}{|K_d|} (C_{\bar{\nu}} T^{r_g(\nu, \mu)} C_{\bar{\mu}})_{[\text{Id}]},$$

where the product on the right takes place in the group algebra of  $K_d$ .

Select an element  $x \in G$  with  $\phi^R(x) = 1$ . Let  $k = ax \in K$ . Denote by  $-\bar{\mu}$  the  $\ell(\mu)$ -tuple of elements of  $G$  defined by:

$$-\bar{\mu} = (\kappa_1 - \mu_1 x, \kappa_2 - \mu_2 x, \dots, \kappa_{\ell(\mu)} - \mu_{\ell(\mu)} x).$$

Although the parts of  $\bar{\mu}$  are unordered, an ordering is chosen for  $-\bar{\mu}$ . The parity condition is now

$$\sum_{j=1}^{\ell} \kappa_j - \mu_j x = 0 \in G.$$

Denote by  $\emptyset_+(k)$  the conjugacy class given by

$$\emptyset_+(k) = \underbrace{\{(a, -k), \dots, (a, -k)\}}_{d/a \text{ times}}.$$

**Theorem 3.** *For weighted-partitions  $\bar{\mu}$  satisfying the parity condition,*

$$H_{g,K}(\emptyset_+(k), \bar{\mu}) = \frac{r_g(\emptyset_+, \mu)!}{|\text{Aut}(\bar{\mu})|} a^{1-g+\sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g, -\bar{\mu}}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^R}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

Theorem 3 determines all linear Hurwitz-Hodge integrals for  $G$  and holds in exactly the same form for the disconnected theories  $H_{g,K}^{\bullet}(\emptyset_+(k), \bar{\mu})$  and  $\overline{\mathcal{M}}_{g, -\bar{\mu}}^{\bullet}(\mathcal{B}G)$ .

**0.6. Future directions.** The ELSV formula has two immediate applications in Gromov-Witten theory. The first is the determination of descendent integrals over  $\overline{\mathcal{M}}_{g,n}$  via asymptotics to remove the Hodge classes [18, 21]. The second is the exact evaluation of the vertex integrals in the localization formula for  $\mathbb{P}^1$  in [22, 23]. The latter requires the Hodge classes.

Since  $\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,n}$  is a finite map, a geometric approach to the descendent integrals is not strictly necessary [16]. However, for the calculation of the Gromov-Witten theory of target curves with orbifold structure [17], Theorem 3 is essential. The results may be viewed as a first step for orbifolds along the successful line of exact Hodge integral formulas which have culminated in the topological and equivariant vertices in ordinary Gromov-Witten theory.

Hurwitz-Hodge integrals can be viewed as pairings of tautological classes

$$\epsilon_*(\lambda_i^R) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

against the descendents  $\psi_i$ . Given an action

$$\alpha : G \times \{1, \dots, k\} \rightarrow \{1, \dots, k\}$$

on a set with  $k$  elements, there is a second map to the moduli space of curves. Let

$$\mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \quad \mathcal{D} \rightarrow \mathcal{C}$$

be the universal domain curve and the universal  $G$ -bundle respectively. A second universal curve

$$\mathcal{D}^{\alpha} = \mathcal{D} \times_G \{1, \dots, k\} \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$$

is obtained by the mixing construction. We obtain

$$\epsilon^{\alpha} : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g^{\alpha}, n^{\alpha}},$$

where  $g^\alpha$  and  $n^\alpha$  are the genus and the number of distinguished sections<sup>6</sup> of the universal curve  $\mathcal{D}^\alpha$ . Two questions immediately arise:

- (i) Do the classes  $\epsilon_*^\alpha(\lambda_i^R)$  lie in the tautological ring of  $\overline{\mathcal{M}}_{g^\alpha, n^\alpha}$ ?
- (ii) Do the pairings of  $\epsilon_*^\alpha(\lambda_i^R)$  against the descendents of  $\overline{\mathcal{M}}_{g^\alpha, n^\alpha}$  admit simple evaluations?

The answer to (i) is known [11] to be false for  $g = 1$ , but may be true for  $g = 0$ . See [8] for positive results related to (i) for the standard action of the symmetric group  $\Sigma_k$  in the  $g = 0$  case.

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## 1. STABLE RELATIVE MAPS

**1.1. Definitions.** For  $a \geq 1$ , let  $\mathbb{P}^1[a]$  be the projective line with a single stack point of order  $a$  at 0. Let

$$\langle \zeta_a \rangle \subset \mathbb{C}^*, \quad \zeta_a = e^{\frac{2\pi i}{a}}$$

be the group of  $a^{\text{th}}$ -roots of unity. Locally at 0,  $\mathbb{P}^1[a]$  is the quotient stack  $\mathbb{C}/\langle \zeta_a \rangle$ . Alternatively,  $\mathbb{P}^1[a]$  is the  $a^{\text{th}}$ -root stack of  $\mathbb{P}^1$  along the divisor 0.

Let  $\overline{\mathcal{M}}_{g, \gamma}(\mathbb{P}^1[a], \mu)$  be the stack of stable relative maps to  $(\mathbb{P}^1[a], \infty)$  where  $\gamma = (\gamma_1, \dots, \gamma_n)$  is a vector of nontrivial elements

$$1 \leq \gamma_i \leq a - 1, \quad \gamma_i \in \mathbb{Z}_a,$$

and  $\mu$  is a partition of  $d \geq 1$  with parts  $\mu_j$  and length  $\ell$ . The moduli space parametrizes maps

$$[f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_{g, \gamma}(\mathbb{P}^1[a], \mu)$$

for which

- (i) the domain  $C$  is a nodal curve of genus  $g$  with stack structure at  $p_i$  determined by  $\gamma_i$ ,
- (ii) relative conditions over  $\infty \in \mathbb{P}^1[a]$  are given by the partition  $\mu$ .

The isotropy group of  $p_i \in C$  is the subgroup of  $\mathbb{Z}_a$  generated by  $\gamma_i$ . Let  $a_i$  denote the order of  $\gamma_i$ . The domain  $C$ , called a *twisted curve*, may have additional stack structure at the nodes, see [2].

We recall the Riemann-Roch formula for twisted curves.<sup>7</sup> Let  $C$  be a twisted curve whose non-singular stack points are  $p_1, \dots, p_n$  with cyclic isotropy groups  $I_1, \dots, I_n$ . The group  $I_i$  is identified with the  $a_i^{\text{th}}$ -roots of unity via the action on  $T_{p_i}C$ ,

$$I_i \xrightarrow{\sim} \langle \zeta_{a_i} \rangle \subset \mathbb{C}^*, \quad \zeta_{a_i} = e^{\frac{2\pi i}{a_i}}.$$

<sup>6</sup>We suppress the ordering issues here.

<sup>7</sup>See Theorem 7.2.1 of [1] for precisely our situation.

Let  $E$  be a locally free sheaf over the stack  $C$ . Then,  $I_i$  acts on the restriction  $E|_{p_i}$ . Let

$$E|_{p_i} = \bigoplus_{0 \leq s \leq a_i - 1} V_s^{\oplus e_s}$$

be the direct sum decomposition, where  $V_s$  is the irreducible representation of  $\mathbb{Z}_{a_i}$  associated to the character

$$\phi^s : I_i \rightarrow \mathbb{C}^*, \quad \phi^s(\zeta_{a_i}) = \zeta_{a_i}^s.$$

The *age* of  $E$  at  $p_i$  is defined by

$$\text{age}_{p_i}(E) = \sum_{0 \leq s \leq a_i - 1} e_s \frac{s}{a_i}.$$

The Riemann-Roch formula for twisted curves is given as follows:

$$(7) \quad \chi(C, E) = \text{rk}(E)(1 - g) + \text{deg}(E) - \sum_{i=1}^n \text{age}_{p_i}(E).$$

The virtual dimension of  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  is calculated by the Riemann-Roch formula (7). Let

$$[f : (C, p_1, \dots, p_n) \rightarrow \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu).$$

Certainly,  $\text{deg}(f^*T_{\mathbb{P}^1[a]}(-\infty)) = d/a$ . By the quotient presentation of  $\mathbb{P}^1[a]$ , the character of  $f^*T_{0,\mathbb{P}^1[a]}$  at  $p_i$  is

$$\zeta_{a_i} \mapsto \zeta_{a_i}^{\frac{\gamma_i a_i}{a}} = \zeta_a^{\gamma_i}.$$

Therefore,  $\text{age}_{p_i}(f^*T_{\mathbb{P}^1[a]}(-\infty)) = \frac{\gamma_i}{a}$  and

$$\begin{aligned} \text{vdim } \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) &= 3g - 3 + n + \ell + \chi(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) \\ &= 3g - 3 + n + \ell + 1 - g + \frac{d}{a} - \sum_{i=1}^n \frac{\gamma_i}{a} \\ &= 2g - 2 + n + \ell + \frac{d}{a} - \sum_{i=1}^n \frac{\gamma_i}{a}. \end{aligned}$$

To simplify notation, let  $r$  denote the above virtual dimension. Since  $r$  must be an integer,  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  is empty unless the parity condition  $d = \sum_{i=1}^n \gamma_i \pmod{a}$  holds.

**1.2. Hurwitz numbers.** We now impose the non-negativity condition,

$$d - \sum_{i=1}^n \gamma_i \geq 0.$$

Let  $H_{g,a}(\gamma, \mu)$  denote the weighted count of degree  $d$  representable maps from nonsingular, connected, genus  $g$  twisted curves with stack points of type  $\gamma$  to  $\mathbb{P}^1[a]$  with profile  $\mu$  over  $\infty$  and simple ramification over  $r$  fixed points in  $\mathbb{P}^1[a] \setminus \{0, \infty\}$ .

**Lemma 1.**  $H_{g,a}(\gamma, \mu)$  is well-defined and equal to  $|\text{Aut}(\gamma)| \cdot H_g(\gamma_+, \mu)$ .

Given a stack map  $[f : C \rightarrow \mathbb{P}^1[a]] \in \mathcal{M}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  satisfying the simple ramification condition over the  $r$  points, the associated coarse map

$$f^c : C^c \rightarrow \mathbb{P}^1$$

is a usual Hurwitz covering counted by  $H_g(\gamma_+, \mu)$ . The representability condition implies the point  $p_i$  has ramification profile  $\gamma_i$  over 0 for the coarse map. Conversely, we have the following result.

**Lemma 2.** *Let  $C^c$  be a nonsingular curve and let  $f^c : C^c \rightarrow \mathbb{P}^1$  be a nonconstant map. Then, there is a unique (up to isomorphism) twisted curve  $(C, p_1, \dots, p_m)$  and a representable morphism  $f : C \rightarrow \mathbb{P}^1[a]$  whose induced map between coarse curves is  $f^c$ .*

*Proof.* Since the natural map  $\mathbb{P}^1[a] \rightarrow \mathbb{P}^1$  is an isomorphism over  $\mathbb{P}^1[a] \setminus [0/\mathbb{Z}_a]$ , we may consider the composite

$$C^c \setminus (f^c)^{-1}(0) \xrightarrow{f^c} \mathbb{P}^1 \setminus \{0\} \xrightarrow{\sim} \mathbb{P}^1[a] \setminus \{[0/\mathbb{Z}_a]\} \subset \mathbb{P}^1[a].$$

The Lemma follows by applying Lemma 7.2.6 of [2].  $\square$

To proceed, we need to identify the ramification profile of  $f^c$  over 0. Since  $\mathbb{P}^1[a]$  is a root stack, we may use classification results on maps to root stacks proven in [4]. According to Theorem 3.3.6 of [4], maps considered in our stack Hurwitz problem are in bijective correspondence with maps  $f^c : C^c \rightarrow \mathbb{P}^1$  from a coarse curve  $C^c$  satisfying

$$(8) \quad (f^c)^*[0] = \sum_{i=1}^n \gamma_i[\bar{p}_i] + aD,$$

where  $\bar{p}_1, \dots, \bar{p}_n \in C^c$  are distinct points and  $D \subset C^c$  is a divisor consisting of  $\frac{d - \sum_{i=1}^n \gamma_i}{a}$  additional distinct points.

The proof of Lemma 1 is complete. The factor  $|\text{Aut}(\gamma)|$  occurs since the stack points of  $C$  are labelled while the corresponding ramification points on the Hurwitz covers enumerated by  $H_g(\gamma_+, \mu)$  are not.  $\square$

**1.3. Branch maps.** There exists a basic branch morphism for stable maps,

$$\text{br} : \overline{\mathcal{M}}_g(\mathbb{P}^1, \mu) \rightarrow \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1),$$

constructed in [9]. By composing with the coarsening map, we obtain

$$\text{br} : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \rightarrow \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

To proceed, we impose the boundedness condition,

$$\forall i \neq j, \quad \gamma_i + \gamma_j \leq a.$$

**Lemma 3.** *If the parity, non-negativity, and boundedness conditions are satisfied,*

$$\text{Im}(\text{br}) \subset \left( d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} \right) [0] + \text{Sym}^r(\mathbb{P}^1) \subset \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

*Proof.* Let  $f : C \rightarrow \mathbb{P}^1[a]$  be a Hurwitz cover counted by  $H_{g,a}(\gamma, \mu)$ . The expression

$$E = d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a}$$

is the order of  $[0]$  in  $\text{br}([f])$ . The claim of the Lemma is simply that the minimum order of  $[0]$  in  $\text{br}(f)$  is achieved at such Hurwitz covers  $f$ .

The proof requires checking all possible degenerations of  $f$  over  $0$ . If the stack points  $p_1, \dots, p_n$  do not bubble off the domain, the claim follows easily as in the coarse case. We leave the details to the reader.

A more interesting calculus is encountered if a subset of stack points  $p_1, \dots, p_l$  bubbles off the domain together over  $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$ . We do the analysis for a single bubble. We can assume the bubble is of genus 0 since higher genus increases the branching order. The multi-bubble calculation is identical.

The genus 0 bubble is attached to the rest of the curve in  $m$  stack points of type

$$\delta_1, \dots, \delta_m \in \mathbb{Z}_a, \quad 1 \leq \delta_j \leq a$$

on the noncollapsed side. The parity condition

$$(9) \quad \sum_{i=1}^l \gamma_i - \sum_{j=1}^m \delta_j = ka$$

must be satisfied with  $k \in \mathbb{Z}$ .

The branch contribution over  $0$  of the bubbled map is at least

$$E' = \sum_{i=l+1}^n (\gamma_i - 1) + \sum_{j=1}^m (\delta_j - 1) + 2m - 2 + \frac{d - \sum_{i=l+1}^n \gamma_i - \sum_{j=1}^m \delta_j}{a} (a - 1).$$

All the terms on the right are obtained from the ramifications on the noncollapsed side except for the  $2m$  from the  $m$  nodes of the bubble and the  $-2$  from the bubble itself, see [9]. Rewriting using the parity condition (9), we find

$$E' = E + l + m - 2 - k.$$

By connectedness and bubble stability, we have

$$m \geq 1, \quad l + m \geq 3.$$

If  $k \leq 0$ , we conclude  $E' > E$ . If  $k \geq 0$ , then  $k \leq l - 2$  by the boundedness condition and the positivity of  $\delta_1$ . Again,  $E' > E$ .  $\square$

By Lemma 3, we may view the branch map with restricted image,

$$\text{br}_0 : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \rightarrow \text{Sym}^r(\mathbb{P}^1).$$

The proof of Lemma 3 shows the maps  $f : C \rightarrow \mathbb{P}^1[a]$  satisfying  $[0] \notin \text{br}_0(f)$  have no contraction over  $0$  and coarse profile exactly  $\gamma_+$ . The usual nonsingularity and Bertini arguments [9] then imply the following result.

**Lemma 4.** *If the parity, non-negativity, and boundedness conditions are satisfied,*

$$H_{g,a}(\gamma, \mu) = \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{vir}} \text{br}_0^*(H^r),$$

where  $H \in H^2(\text{Sym}^r(\mathbb{P}^1), \mathbb{Q})$  is the hyperplane class.

## 2. LOCALIZATION

**2.1. Fixed loci.** The standard  $\mathbb{C}^*$ -action on  $\mathbb{P}^1$ , defined by  $\xi \cdot [z_0, z_1] = [z_0, \xi z_1]$ , lifts canonically to  $\mathbb{C}^*$ -actions on  $\mathbb{P}^1[a]$  and  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ . We will evaluate the integral

$$(10) \quad \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{vir}} \mathbf{br}_0^*(H^r)$$

by virtual localization for relative maps [10, 13] following [9, 12]. We assume the parity, non-negativity, and boundedness conditions.

The first step is to define a lift of the  $\mathbb{C}^*$ -action to the integrand. Certainly the  $\mathbb{C}^*$ -action lifts canonically to  $\mathrm{Sym}^r(\mathbb{P}^1)$ . A lift of  $H^r$  can be defined by choosing the  $\mathbb{C}^*$ -fixed point  $r[0] \in \mathrm{Sym}^r(\mathbb{P}^1)$ . The tangent weights at  $[0/\mathbb{Z}_a], \infty \in \mathbb{P}^1[a]$  are  $\frac{t}{a}$  and  $-t$  respectively. The equivariant Euler class of the normal bundle to  $r[0]$  in  $\mathrm{Sym}^r(\mathbb{P}^1)$  has weight  $r!t^r$ .

The second step is to identify the  $\mathbb{C}^*$ -fixed locus  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)^{\mathbb{C}^*} \subset \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ . The components of the  $\mathbb{C}^*$ -fixed locus lie over the  $r+1$  points of  $\mathrm{Sym}^r(\mathbb{P}^1)^{\mathbb{C}^*}$ . By our lifting of  $H^r$ , we need only consider

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)^{\mathbb{C}^*} \cap \mathbf{br}_0^{-1}(r[0]).$$

Because of the strong restriction on the branching, the maps

$$[f : C \rightarrow \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_0^{\mathbb{C}^*}$$

have a very simple structure:

- (i)  $C = C_0 \cup \coprod_{j=1}^{\ell} C_j$ ,
- (ii)  $f|_{C_0}$  is a constant map from a genus  $g$  curve to  $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$ ,
- (iii) the coarse map  $f^c|_{C_j} : C_j^c \rightarrow \mathbb{P}^1$  is a  $\mathbb{C}^*$ -fixed Galois cover of degree  $\mu_j$  for  $j > 0$ ,
- (iv)  $C_0$  meets  $C_j$  at a node  $q_j$ .

The stack structure at  $q_j \in C_j$  is easily determined using the relationship between stack Hurwitz covers of  $\mathbb{P}^1[a]$  and ordinary Hurwitz covers of  $\mathbb{P}^1$  discussed in Section 1.2. The stack structure at  $q_j \in C_j$  is of type  $\mu_j \in \mathbb{Z}_a$ . The stack structure at  $q_j \in C_0$  where  $C_j$  is attached is of the *opposite* type  $-\mu_j \in \mathbb{Z}_a$ . The map

$$f|_{C_0} : (C, p_1, \dots, p_n, q_1, \dots, q_\ell) \rightarrow [0/\mathbb{Z}_a]$$

is an element of  $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ .

The  $\mathbb{C}^*$ -fixed locus may be identified with a quotient of a fibered product,

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} \cong \left( \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times_{(\overline{I}\mathcal{B}\mathbb{Z}_a^\ell)} P_1 \times \dots \times P_\ell \right) /_{\mathrm{Aut}(\mu)},$$

where  $\overline{I}\mathcal{B}\mathbb{Z}_a$  is the rigidified inertia stack of  $\mathcal{B}\mathbb{Z}_a$  and  $P_j$  is the moduli stack of  $\mathbb{C}^*$ -fixed Galois covers of degree  $\mu_j$ . By the standard multiplicity obtained from gluing stack  $\mathbb{Z}_a$ -bundles, the projection

$$(11) \quad \overline{\mathcal{M}}_0^{\mathbb{C}^*} \rightarrow \left( \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times \dots \times P_\ell \right) /_{\mathrm{Aut}(\mu)}$$

has degree  $\prod_{j=1}^{\ell} \frac{a}{b_j}$  where  $b_j$  is the order of  $\mu_j \in \mathbb{Z}_a$ .

Fortunately, the residue integral over  $\overline{\mathcal{M}}_0^{\mathbb{C}^*}$  in the virtual localization formula for (10) is pulled-back via (11). Instead of integrating over  $\overline{\mathcal{M}}_0^{\mathbb{C}^*}$ , we will integrate over

$$\widetilde{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times \dots \times P_{\ell}$$

and multiply by

$$\frac{1}{|\text{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{a}{b_j}.$$

**2.2. Virtual normal bundle.** The virtual localization formula for (10) with our choice of equivariant lifts takes the following form:

$$(12) \quad \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \text{br}_0^*(H^r) = \frac{1}{|\text{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{a}{b_j} \int_{\widetilde{\mathcal{M}}_0^{\mathbb{C}^*}} \frac{r! t^r}{e(\text{Norm}^{vir})}.$$

The equivariant Euler class of the virtual normal bundle is

$$(13) \quad \frac{1}{e(\text{Norm}^{vir})} = \frac{e(H^1(C, f^*T_{\mathbb{P}^1[a]}(-\infty)))}{e(H^0(C, f^*T_{\mathbb{P}^1[a]}(-\infty)))} \frac{1}{\prod_{j=1}^{\ell} e(N_j)},$$

see [10]. The last product is over the nodes of  $C$ , and  $N_j$  is the equivariant line bundle associated to the smoothing of  $q_j$ . The terms in (13) are computed via the normalization sequence of the domain  $C$ . The various contributions over the components  $C_0, C_1, \dots, C_{\ell}$  are computed separately.

First consider the collapsed component  $C_0$ . The space  $H^0(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$  is identified with the subspace of  $T_{\mathbb{P}^1[a]}(-\infty)|_{[0/\mathbb{Z}_a]}$  consisting of vectors invariant under the action of the image of the monodromy representation  $\pi_1^{orb}(C_0) \rightarrow \mathbb{Z}_a$ . Therefore,  $H^0$  vanishes unless the monodromy representation is trivial, in which case  $H^0$  is 1-dimensional with weight  $\frac{t}{a}$ .

The trivial monodromy representation  $\pi_1^{orb}(C_0) \rightarrow \mathbb{Z}_a$  is possible only if

$$\gamma = \emptyset \quad \text{and} \quad \forall j, \mu_j = 0 \pmod{a}.$$

Even then, the locus with trivial monodromy is just a component<sup>8</sup> of  $\overline{\mathcal{M}}_{g,(0,\dots,0)}(\mathcal{B}\mathbb{Z}_a)$ . The trivial monodromy representation locus will play a slightly special role throughout the calculation. But, in the final formula, no different treatment is required.

The space  $H^1(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$  yields the vector bundle

$$\mathbb{B} = (\mathbb{E}^U)^{\vee}$$

over  $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$  whose rank may be calculated by the orbifold Riemann-Roch formula. Over the component of the fixed locus where the monodromy representation  $\pi_1^{orb}(C_0) \rightarrow \mathbb{Z}_a$  is trivial, the rank of  $\mathbb{B}$  is  $g$ . Otherwise, the rank is

$$(14) \quad r_{\mathbb{B}} = g - 1 + \sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 \pmod{a}} \left(1 - \left\langle \frac{\mu_j}{a} \right\rangle\right).$$

<sup>8</sup>If  $g > 0$ , there will typically be other components as well.

The  $H^1 - H^0$  contribution from the collapsed component to the localization formula is

$$(15) \quad \sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} c_i(\mathbb{B}) = \sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U.$$

For the component where the monodromy representation is trivial, an additional factor of  $\frac{a}{t}$  must be inserted in (15).

Next consider the  $H^1 - H^0$  contribution from the  $\mathbb{C}^*$ -fixed Galois covers. Since

$$\deg(f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = \frac{\mu_j}{a},$$

we have

$$H^k(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = H^k\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right).$$

The  $H^0$  weights are

$$\frac{t}{\mu_j}, 2\frac{t}{\mu_j}, \dots, \left\lfloor \frac{\mu_j}{a} \right\rfloor \frac{t}{\mu_j},$$

where the weight 0 is omitted.<sup>9</sup> The group  $H^1$  vanishes. The  $H^1 - H^0$  contribution is

$$t^{-\left\lfloor \frac{\mu_j}{a} \right\rfloor} \frac{\mu_j^{\left\lfloor \frac{\mu_j}{a} \right\rfloor}}{\left\lfloor \frac{\mu_j}{a} \right\rfloor!}.$$

Finally, consider the  $H^1 - H^0$  contribution from the nodal point  $q_j$ . If  $\mu_j \not\equiv 0 \pmod{a}$ , then  $q_j$  is a stack point and

$$H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j}) = 0$$

as there is no invariant section. If  $\mu_j \equiv 0 \pmod{a}$  then  $H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})$  is 1-dimensional and contributes a factor  $\frac{t}{a}$ . Certainly,  $H^1$  vanishes here for dimension reasons.

The contribution from smoothing the node  $q_j$  is the tensor product of the tangent lines of the two branches incident to  $q_j$ ,

$$e(N_j) = \frac{1}{b_j} \left( -\bar{\psi}_j + \frac{t}{\mu_j} \right).$$

After putting the component calculations together in (13), we obtain the following expression for  $1/e(\text{Norm}^{vir})$ :

$$\left( \sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U \right) \cdot \prod_{j=1}^{\ell} \left( t^{-\left\lfloor \frac{\mu_j}{a} \right\rfloor} \frac{\mu_j^{\left\lfloor \frac{\mu_j}{a} \right\rfloor}}{\left\lfloor \frac{\mu_j}{a} \right\rfloor!} \frac{1}{\frac{1}{b_j} \left( -\bar{\psi}_j + \frac{t}{\mu_j} \right)} \right) \cdot \prod_{j=1}^{\ell} \left(\frac{t}{a}\right)^{\delta_{0, \langle \frac{\mu_j}{a} \rangle}}.$$

Regrouping of terms yields

$$(16) \quad \frac{\prod_{j=1}^{\ell} b_j \mu_j}{a^{r_{\mathbb{B}} + \sum_{j=1}^{\ell} \delta_{0, \langle \frac{\mu_j}{a} \rangle}}} \left( \prod_{j=1}^{\ell} \frac{\mu_j^{\left\lfloor \frac{\mu_j}{a} \right\rfloor}}{\left\lfloor \frac{\mu_j}{a} \right\rfloor!} \right) \left( \sum_{i=0}^{r_{\mathbb{B}}} t^{r_{\mathbb{B}}-i} (-a)^i \lambda_i^U \right) \cdot t^{-\sum_{j=1}^{\ell} \left\lfloor \frac{\mu_j}{a} \right\rfloor} \prod_{j=1}^{\ell} \frac{t^{\delta_{0, \langle \frac{\mu_j}{a} \rangle}}}{(t - \mu_j \bar{\psi}_j)}.$$

For the component with trivial monodromy representation, a factor of  $\frac{a}{t}$  must be inserted in the formulas for  $1/e(\text{Norm}^{vir})$ .

<sup>9</sup>The 0 weight is from reparameterization of the domain  $C_j$  and is not in the virtual normal bundle.

**2.3. Proof of Theorem 1.** Putting the calculations of Section 2.2 together and passing to the non-equivariant limit, we obtain the following evaluation

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathbf{br}_0^*(H^r) = \frac{r!}{|\mathbf{Aut}(\mu)|} \frac{a^\ell}{a^{r_{\mathbb{B}} + \sum_{j=1}^{\ell} \delta_{0, \langle \frac{\mu_j}{a} \rangle}}} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{BZ}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

On the right side, we have included the fundamental class factors

$$\prod_{j=1}^{\ell} \frac{1}{\mu_j}$$

of the moduli spaces  $P_j$ . For the component with trivial monodromy representation, a factor of  $a$  must be inserted in the formula.

We can simplify the integral evaluation by using the calculation (14) of  $r_{\mathbb{B}}$ ,

$$\begin{aligned} & r_{\mathbb{B}} + \sum_{i=1}^{\ell} \delta_{0, \langle \frac{\mu_i}{a} \rangle} - \ell \\ &= g - 1 + \sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 \pmod{a}} \left(1 - \left\langle \frac{\mu_j}{a} \right\rangle\right) + \left( \sum_{\mu_j = 0 \pmod{a}} 1 \right) - \ell \\ &= g - 1 + \sum_{i=1}^n \frac{\gamma_i}{a} - \sum_{j=1}^{\ell} \left\langle \frac{\mu_j}{a} \right\rangle. \end{aligned}$$

The above calculation is not valid for the component with trivial monodromy since  $r_{\mathbb{B}} = g$  not  $g-1$ . The discrepancy is exactly fixed by the extra factor  $a$  required for the trivial monodromy case. We conclude

$$(17) \quad \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathbf{br}_0^*(H^r) = \frac{r!}{|\mathbf{Aut}(\mu)|} a^{1-g - \sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{BZ}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)}.$$

holds uniformly. Theorem 1 is then obtained from Lemmas 1 and 4.  $\square$

In degenerate cases, unstable integrals may appear on the right side of the formula in Theorem 1. The unstable integrals come in two forms and are defined by the localization contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{BZ}_a)} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^U}{(1 - x \bar{\psi}_1)} = \frac{1}{a} \cdot \frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{BZ}_a)} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^U}{(1 - x \bar{\psi}_1)(1 - y \bar{\psi}_2)} = \frac{1}{a} \cdot \frac{1}{x + y}.$$

With the above definitions, Theorem 1 holds in all cases.

The disconnected formula (5) follows easily from the connected case by the usual combinatorics of distributing ramification points to the components of Hurwitz covers.

**2.4. Proof of Theorem 2.** Suppose  $\gamma$  satisfies the parity and strong negativity condition with respect to  $\mu$ . Since

$$\delta = d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} < 0,$$

the virtual dimension  $r$  of  $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$  is greater than  $2g - 2 + d + \ell$ . As a consequence, we immediately obtain the vanishing

$$(18) \quad \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{vir}} \mathbf{br}^*(H^r) = 0$$

since  $H^r = 0 \in H^*(\mathrm{Sym}^{2g-2+d+\ell}(\mathbb{P}^1), \mathbb{Q})$ .

We may nevertheless calculate (18) by localization with the lift

$$H^r = (2g - 2 + d + \ell)[0] \cdot t^{-\delta}$$

which does *not* vanish equivariantly. The analysis is identical to the calculations of Sections 2.1-2.3. We find the integral (18) is proportional (with nonzero factor) to

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)},$$

and therefore conclude the vanishing.

Assume now strong negativity does not hold, but  $\gamma$  satisfies the parity, negativity, and boundedness condition. By the proof of Lemma 3, using the boundedness condition, the maps

$$f : C \rightarrow \mathbb{P}^1[a]$$

which satisfy  $[0] \notin \mathrm{br}_0(f)$  have no contraction over 0 and coarse profile determined by  $\gamma$ . By the negativity condition, no such maps exists. Hence,  $[0]$  is always in  $\mathrm{br}_0(f)$ . Therefore,

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)]^{vir}} \mathbf{br}_0^*(H^r) = 0$$

and we conclude as above.  $\square$

### 3. EXAMPLES

**3.1.  $\mathbb{Z}_2$  example.** The Hodge bundle  $\mathbb{E}^U$  has a very simple interpretation in the  $\mathbb{Z}_2$  case. Let

$$\mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2), \quad \mathcal{D} \rightarrow \mathcal{C}$$

be the universal domain curve and the universal  $\mathbb{Z}_2$ -bundle. Let

$$\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \rightarrow \overline{\mathcal{M}}_g, \quad \tilde{\epsilon} : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \rightarrow \overline{\mathcal{M}}_{g-1+\frac{n}{2}}$$

be the maps to moduli obtained from  $\mathcal{C}$  and  $\mathcal{D}$  respectively. The exact sequence

$$0 \rightarrow \epsilon^*(\mathbb{E}_g) \rightarrow \tilde{\epsilon}^*(\mathbb{E}_{g-1+\frac{n}{2}}) \rightarrow \mathbb{E}^U \rightarrow 0.$$

exhibits  $\mathbb{E}^U$  as the  $K$ -theoretic difference of the pulled-back Hodge bundles. If  $g = 0$ , then the situation<sup>10</sup> is even simpler,

$$(19) \quad \mathbb{E}^U \cong \tilde{\epsilon}^*(\mathbb{E}_{g-1+\frac{n}{2}}).$$

<sup>10</sup>The map  $\epsilon$  is not well-defined here for stability reasons.

Consider the case of Theorem 1 where  $g = 0$ ,  $\gamma = (1, 1)$ , and  $\mu = (1, 1)$ . The statement is

$$H_0((1, 1), (1, 1)) = \frac{2}{2!2!} 2^1 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1 - 2\lambda_1^U}{(1 - \bar{\psi}_1)(1 - \bar{\psi}_2)}.$$

The double Hurwitz number on the left is  $\frac{1}{2}$ . Expansion of the right side yields:

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1 - 2\lambda_1^U}{(1 - \bar{\psi}_1)(1 - \bar{\psi}_2)} &= \frac{1}{2} \int_{\overline{\mathcal{M}}_{0,4}} \frac{1}{(1 - \psi_1)(1 - \psi_2)} - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U \\ &= 1 - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U. \end{aligned}$$

To evaluate the last integral, we note the map

$$\tilde{\epsilon} : \overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2) \rightarrow \overline{\mathcal{M}}_{1,1},$$

where the first branch point is selected for the marking on the elliptic curve, is of degree 6. Moreover,  $\lambda_1^U$  is the pull-back of  $\lambda_1$  under  $\tilde{\epsilon}$  by (19). Hence,

$$1 - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U = 1 - 2 \cdot 6 \cdot \frac{1}{24} = \frac{1}{2}.$$

**3.2. Vanishing example.** The simplest example of the vanishing of Theorem 2 occurs for  $\mathbb{Z}_2$ . Let  $g = 0$ ,

$$\gamma = \underbrace{(1, \dots, 1)}_n$$

and  $\mu = (1)$ . By the parity condition,  $n$  must be odd. Boundedness holds. For the negativity condition, we require  $n \geq 2$ . By Theorem 2 (i),

$$\int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i \geq 0} (-2)^i \lambda_i^U}{1 - \bar{\psi}_1}$$

vanishes for all odd  $n \geq 3$ .

We now use the identification of  $\lambda_i^U$  with the Chern classes of the Hodge bundle  $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$  whose fiber over

$$f : [D/\mathbb{Z}_2] \rightarrow \mathcal{B}\mathbb{Z}_2$$

is simply given by the space of differential forms on the genus  $\frac{n-1}{2}$  curve  $D$ . The Chern roots of  $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$  can be identified by the vanishing sequence at a Weierstrass point of  $D$ . The Weierstrass point can be chosen to lie above the marking corresponding to the single part of  $\mu$ . The Chern roots of  $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$  are then  $L, 3L, \dots, (n-2)L$  where  $L$  is the Chern class of the cotangent line of the Weierstrass point. The class  $L$  on  $\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)$  is  $\frac{1}{2}\bar{\psi}_1$ . Expanding the Chern roots, we find

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i \geq 0} (-2)^i \lambda_i^U}{1 - \bar{\psi}_1} &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\prod_{i=1}^{\frac{n-1}{2}} (1 - (2i-1)\bar{\psi}_1)}{(1 - \bar{\psi}_1)} \\ &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \prod_{i=2}^{\frac{n-1}{2}} (1 - (2i-1)\bar{\psi}_1) \\ &= 0, \end{aligned}$$

where the last integral vanishes for dimension reasons.

**3.3.  $\mathbb{Z}_\infty$  example.** An interesting feature of Theorem 1 is the possibility of studying the behavior for large  $a$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  determine a partition of  $d$ ,

$$d = \sum_{i=1}^n \gamma_i.$$

Let  $\mu = (d)$  consist of a single part. For  $a > d$ , the rank of the Hodge bundle

$$\mathbb{E}^U \rightarrow \overline{\mathcal{M}}_{0, \gamma - \mu}(\mathcal{B}\mathbb{Z}_a)$$

is 0 by (14). Since the parity, non-negativity, and boundedness conditions hold for  $a > d$ , we may apply Theorem 1 to conclude

$$\begin{aligned} H_0(\gamma, (d)) &= \frac{(n-1)!}{|\text{Aut}(\gamma)|} a \int_{\overline{\mathcal{M}}_{0, \gamma - \mu}(\mathcal{B}\mathbb{Z}_a)} \frac{1}{1 - d\bar{\psi}_1} \\ &= \frac{(n-1)!}{|\text{Aut}(\gamma)|} d^{n-2}, \end{aligned}$$

which is a well-known formula for genus 0 double Hurwitz numbers.

**3.4. 1-point series.** If  $\mu = (d)$  consists of a single part, the entire generating series for double Hurwitz numbers has been computed <sup>11</sup> in [14]:

$$(20) \quad \sum_{g \geq 0} t^{2g} (-1)^g H_g(\nu, (d)) = \frac{r! d^{r-1}}{|\text{Aut}(\nu)|} \prod_{k \geq 1} \left( \frac{\sin(kt/2)}{kt/2} \right)^{m_k(\nu) - \delta_{k,1}},$$

where  $r = r_g(\nu, (d))$  and  $m_k(\nu)$  is the number of times  $k$  appears as a part of  $\nu$ . Single part double Hurwitz numbers are considerably simpler because such covers are automatically connected and the only characters with nonzero evaluation on the  $d$ -cycle are exterior powers of the standard  $(d-1)$ -dimensional representation.

Let  $\gamma = (\gamma_1, \dots, \gamma_n)$  be a vector of nontrivial elements of  $\mathbb{Z}_a$  satisfying the boundedness condition. We will consider degrees  $d$  for which the parity and non-negativity conditions are satisfied. Then,

$$d - \sum_{i=1}^n \gamma_i = ab$$

for an integer  $b \geq 0$ . Consider the generating series

$$F_\gamma(t, z) = \sum_{g=0}^{\infty} \sum_{l=-\infty}^g t^{2g} z^l \int_{\overline{\mathcal{M}}_{g, \gamma - (d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{2g-2+\ell(\gamma)+l} \lambda_{g-1}^U$$

where  $\bar{\psi}_0$  is the class corresponding to the point with monodromy  $-d$ .

<sup>11</sup>We write Theorem 3.1 of [14] in terms of  $\sin$  instead of  $\sinh$  and divide by  $|\text{Aut}(\nu)|$  since we do not mark ramifications in our definition of Hurwitz numbers.

The double Hurwitz number formula of Theorem 1 is

$$\begin{aligned} H_g(\gamma_+, (d)) &= \frac{r!}{|\text{Aut}(\gamma)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a} + \lfloor \frac{d}{a} \rfloor} \frac{d^{\lfloor \frac{d}{a} \rfloor}}{\lfloor \frac{d}{a} \rfloor!} \sum_{l=-\infty}^g d^{r-b-1+l} (-a)^{g-l} \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l} \lambda_{g-l}^U \\ &= (-1)^g \frac{ad^{r-1}r! \left(\frac{d}{a}\right)^{\lfloor \frac{\sum \gamma_i}{a} \rfloor}}{|\text{Aut}(\gamma)| \left(b + \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!} \sum_{l=-\infty}^g \left(\frac{-d}{a}\right)^l \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l} \lambda_{g-l}^U \end{aligned}$$

or, equivalently,

$$\sum_{g \geq 0} (-1)^g t^{2g} H_g(\gamma_+, (d)) = \frac{ad^{r-1}r!}{|\text{Aut}(\gamma)| \left(b + \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!} \left(\frac{d}{a}\right)^{\lfloor \frac{\sum \gamma_i}{a} \rfloor} F_\gamma(t, -d/a)$$

where  $r = r_g(\gamma_+, (d))$ . After combining with (20), we obtain

$$(21) \quad F_\gamma(t, -d/a) = \frac{1}{a} \frac{\left(b + \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!}{b!} \left(\frac{a}{d}\right)^{\lfloor \frac{\sum \gamma_i}{a} \rfloor} \prod_{k \geq 1} \left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma_+) - \delta_{k,1}}.$$

for  $b \geq 0$ .

**Theorem 4.**  $F_\gamma(t, z)$  equals

$$\frac{1}{a} \frac{\left(-z - \sum \frac{\gamma_i}{a} + \sum \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!}{\left(-z - \sum \frac{\gamma_i}{a}\right)!} (-z)^{-\lfloor \frac{\sum \gamma_i}{a} \rfloor} \left(\frac{\sin(at/2)}{at/2}\right)^{-z - \sum \frac{\gamma_i}{a}} \prod_{k \geq 1} \left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma) - \delta_{k,1}}.$$

*Proof.* Using the standard polynomial expansion

$$\frac{\left(-z - \sum \frac{\gamma_i}{a} + \sum \lfloor \frac{\sum \gamma_i}{a} \rfloor\right)!}{\left(-z - \sum \frac{\gamma_i}{a}\right)!} = \left(-z - \sum \frac{\gamma_i}{a} + \sum \lfloor \frac{\sum \gamma_i}{a} \rfloor\right) \cdots \left(-z - \sum \frac{\gamma_i}{a} + 1\right),$$

we see the  $t^{2g}$  coefficients of both sides of Theorem 4 are Laurent polynomials in  $z$ . Equation (21) shows Theorem 4 holds for all evaluations of the form  $z = -d/a$  where

$$d - \sum_{i=1}^n \gamma_i = ab$$

and  $b$  is a non-negative integer. Since there are infinitely many such evaluations, the coefficient Laurent polynomials in  $z$  must be equal for all  $t^{2g}$ .  $\square$

If we specialize Theorem 4 to the case where  $\gamma = \emptyset$ , we obtain

$$(22) \quad \frac{1}{a} + \sum_{g>0} \sum_{l=0}^g t^{2g} z^l \int_{\overline{\mathcal{M}}_{g,1}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_1^{2g-2+l} \lambda_{g-l}^U = \frac{1}{a} \left(\frac{at/2}{\sin(at/2)}\right)^z \frac{t/2}{\sin(t/2)}$$

If  $\gamma = \emptyset$  and  $a = 1$  we recover

$$(23) \quad 1 + \sum_{g>0} \sum_{l=0}^g t^{2g} z^l \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2+l} \lambda_{g-l} = \left(\frac{t/2}{\sin(t/2)}\right)^{z+1}$$

first calculated in [7].

In (22), the term  $\lambda_g^U$  vanishes for dimensional reasons except over the trivial monodromy component, where it agrees with the usual  $\lambda_g$ . Indeed, setting  $z = 0$  in (22) yields

$$\frac{1}{a} + \sum_{g>0} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}(\mathcal{B}\mathbb{Z}_a)} \psi_1^{2g-2} \lambda_g^U = \frac{1}{a} \frac{t/2}{\sin(t/2)}$$

which is the expected contribution from (23) with a factor of  $1/a$  coming from the automorphisms.

#### 4. ABELIAN GROUPS

**4.1. Pull-back.** For an abelian group  $G$  and irreducible representation  $R$ , recall the sequence (6),

$$0 \rightarrow K \rightarrow G \xrightarrow{\phi^R} \text{Im}(\phi^R) \cong \mathbb{Z}_a \rightarrow 0.$$

By construction  $R \cong \phi^{R*}(U)$ . The homomorphism  $\phi^R$  induces a canonical map

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$$

by sending a principal  $G$ -bundle to its quotient by  $K$ .

**Lemma 5.**  $\mathbb{E}^R \cong \rho^*(\mathbb{E}^U)$ .

*Proof.* Recall  $\mathbb{E} \rightarrow \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$  is the bundle whose fiber over

$$[f] : [D/H] \rightarrow \mathcal{B}H \in \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$$

is  $H^0(D, \omega_D)$ . The latter can be understood as the space of 1-forms  $\alpha$  on the normalization  $\tilde{D}$  of  $D$  with possible simple poles with opposite residues at the two preimages of each node  $q_i$ .

Let  $\tilde{\rho}$  be the map between the universal principal  $G$ - and  $\mathbb{Z}_a$ -curves that induces  $\rho$ . We obtain

$$d\tilde{\rho} : \rho^*(\mathbb{E}) \rightarrow \mathbb{E}$$

by pulling-back differential forms. An easy verification shows  $\tilde{\rho}$  is well-defined even at points in the moduli space  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  for which the  $G$ -curve is nodal.

The map  $d\tilde{\rho}$  is injective on each fiber since the pull-back of a nonzero differential form by a finite surjective map is nonzero. Certainly  $d\tilde{\rho}$  carries the subbundle  $\rho^*(\mathbb{E}^U)$  to the subbundle  $\mathbb{E}^R$ . These bundles have the same dimension by the Riemann-Roch formula for twisted curves. Hence,  $d\tilde{\rho}$  is an isomorphism.  $\square$

The map  $\rho$  does not preserve the isotropy groups at the marked points. However, since the classes  $\bar{\psi}_i$  are pulled-back from  $\overline{\mathcal{M}}_{g,n}$ ,

$$\rho^*(\bar{\psi}) = \bar{\psi}.$$

By Lemma 5, we concluded the integrand in Theorem 3 is exactly the integrand of Theorem 1 pulled-back via  $\rho$ .

**4.2. Degree.** The degree of  $\rho$  is determined by the following result.

**Lemma 6.** *We have*

$$\deg(\rho) = \begin{cases} 0 & \sum_i \gamma_i \neq 0 \\ |K|^{2g-1} & \sum_i \gamma_i = 0 \end{cases}.$$

*Proof.* Consider a nonsingular curve  $[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n}$ . Let

$$\Gamma = \pi_1(C \setminus \{p_1, \dots, p_n\}) = \left\langle \Gamma_i, A_j, B_j \mid \prod_{i=1}^n \Gamma_i \prod_{j=1}^g [A_j, B_j] \right\rangle,$$

where  $\Gamma_i$  is a loop around  $p_i$  and the loops  $A_j, B_j$  are the standard generators of  $\pi_1(C)$ .

The elements of  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  lying above  $[C, p_1, \dots, p_n]$  are in bijective correspondence with the homomorphisms<sup>12</sup>  $\varphi : \Gamma \rightarrow G$  with

$$(24) \quad \varphi(\Gamma_i) = \gamma_i.$$

Since  $G$  is abelian,  $\varphi([A_j, B_j]) = 0$ . Hence, the parity condition

$$(25) \quad \sum_{i=1}^n \gamma_i = 0$$

must be satisfied for  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$  to be nonempty.

If the parity condition holds, then the images of  $A_j$  and  $B_j$  are completely unconstrained. There are  $|G|^{2g}$  homomorphisms  $\phi$  satisfying (24). Stated in terms of homomorphisms, the map  $\rho$  corresponds to composing  $\varphi : \Gamma \rightarrow G$  with  $\phi^R : G \rightarrow \mathbb{Z}_a$ . Since there are  $|K|$  elements of  $G$  in the preimage of any element of  $\mathbb{Z}_a$ , there are  $|K|^{2g}$  elements in a generic fiber of  $\rho$ . Since  $G$  is abelian, a cover in  $\mathcal{M}_{g,\gamma}(\mathcal{B}G)$  has automorphism group  $G$ . A cover in the image of  $\rho$  only has automorphism group  $\mathbb{Z}_a$ . Thus, the degree of  $\rho$  is  $|K|^{2g-1}$ .  $\square$

Although  $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$  may have several components, Lemma 6 implies the degree of  $\rho$  is the same over each component. In the nonabelian case, the situation is much more complicated. For example, let  $\eta$  be the conjugacy class of a 3-cycle in  $\Sigma_3$ , let

$$s : \Sigma_3 \rightarrow \mathbb{Z}_2$$

be the sign representation, and let

$$\rho : \overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3) \rightarrow \overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$$

be the map induced by  $s$ . The space  $\overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$  consists of two components: one with trivial monodromy, and one with nontrivial monodromy. There are covers in  $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$  lying above the nontrivial monodromy component. If  $t_1 \neq t_2 \in \Sigma_3$  are two transpositions, then  $[t_1, t_2]$  is a 3-cycle. On the other hand, there are no elements of  $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$  lying above the trivial monodromy component. All the monodromy in such a cover would lie in the abelian group  $\mathbb{Z}_3 = \ker(s)$ , and there are no such covers with nontrivial monodromy about the one marked point by (25). As the formula in Theorem 1 considers all components of  $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$  at once, a more nuanced approach would be required to understand Hurwitz-Hodge integrals for nonabelian groups, even for 1-dimensional representations.

<sup>12</sup> Composition in  $\Gamma$  is written multiplicatively while composition in  $G$  is additive.

In the disconnected case  $\rho : \overline{\mathcal{M}}_{g,\gamma}^\bullet(\mathcal{B}G) \rightarrow \overline{\mathcal{M}}_{g,\phi^R(\gamma)}^\bullet(\mathcal{B}\mathbb{Z}_a)$ , Lemma 6 has a few minor complications:

- (i) The monodromy condition  $\sum_i \gamma_i = 0 \in G$  cannot be checked globally, but must be verified separately on each domain component.
- (ii) The number of components matters. For disconnected curves with  $h$  components, each of which satisfies the monodromy requirements, the degree of  $\rho$  is  $|K|^{2g-2+h}$ .

When  $\rho$  is nonzero, the degree  $|K|^{2g-2+h}$  is independent of  $G$  and the monodromy conditions (25). The only role these conditions play is to determine when the degree is nonzero.

**4.3. Wreath Hurwitz numbers.** The wreath product  $K_d$  is defined by

$$K_d = \{(k, \sigma) \mid k = (k_1, \dots, k_d) \in K^d, \sigma \in \Sigma_d\},$$

$$(k, \sigma)(k', \sigma') = (k + \sigma(k'), \sigma\sigma').$$

Conjugacy classes of  $K_d$  are determined by their cycle types [19]. Since  $K$  is abelian, for each  $m$ -cycle  $(i_1 i_2 \dots i_m)$  of  $\sigma$ , the element  $k_{i_m} + k_{i_{m-1}} + \dots + k_{i_1}$  is well-defined. The resulting  $\text{Conj}(K)$ -weighted partition of  $d$  is called the *cycle type* of  $(k, \sigma)$ . Two elements of  $K_d$  are conjugate exactly when they have the same cycle type.

We index the conjugacy classes of  $K_d$  by  $\text{Conj}(K)$ -weighted partitions of  $d$ . Let

$$\overline{\nu} = \{(\nu_1, \iota_1), \dots, (\nu_{\ell(\nu)}, \iota_{\ell(\nu)})\},$$

$$\overline{\mu} = \{(\mu_1, \kappa_1), \dots, (\mu_{\ell(\mu)}, \kappa_{\ell(\mu)})\}$$

be two such partitions. Let  $\nu^*$  be the partition with parts  $\nu_j$  with a partial labelling given by  $\iota_j$ . Then

$$\text{Aut}(\nu^*) = \text{Aut}(\overline{\nu}).$$

The Hurwitz number  $H_g(\nu^*, \mu^*)$  counts covers with the additional labelling data,

$$H_g(\nu^*, \mu^*) = \frac{|\text{Aut}(\nu)|}{|\text{Aut}(\nu^*)|} \frac{|\text{Aut}(\mu)|}{|\text{Aut}(\mu^*)|} H_g(\nu, \mu).$$

**Lemma 7.**  $H_{g,K}(\overline{\nu}, \overline{\mu})$  is the count of the covers  $\pi : C \rightarrow \mathbb{P}^1$  enumerated by  $H_g(\nu^*, \mu^*)$  with multiplicity  $m_\pi$ . The multiplicity  $m_\pi$  is the automorphism-weighted count of principal  $K$ -bundles on  $C \setminus \pi^{-1}(\{0, \infty\})$  with monodromy  $\iota_i$  at  $p_i \in \pi^{-1}(0)$  and  $\kappa_j$  at  $q_j \in \pi^{-1}(\infty)$ .

*Proof.* Let  $\pi' : D \rightarrow \mathbb{P}^1$  be a cover counted by  $H_{g,K}(\overline{\nu}, \overline{\mu})$ . By definition,  $\pi'$  is a  $d|K|$ -fold cover of  $\mathbb{P}^1$  with monodromies  $\overline{\nu}, \overline{\mu}$  and  $\overline{\tau}$  over  $0, \infty$  and the points of  $U_r$  respectively.

Each such cover has an associated cover  $\pi : C \rightarrow \mathbb{P}^1$  counted by  $H_g(\nu^*, \mu^*)$ . Algebraically, the cover is obtained by the forgetful map from  $K_d \rightarrow \Sigma_d$ . Geometrically, the cover is obtained by taking the quotient of  $D$  by the diagonal subgroup  $K \subset K_d$ . There is a natural map  $f : D \rightarrow C$ . Away from the preimages of  $0, \infty$  and  $U_r$ , the map  $f$  is a principal  $K$ -bundle.

Consider the point  $p_i \in \pi^{-1}(0)$  corresponding to a cycle  $\nu_i$  which is labelled with  $\iota_i \in K$ . A small loop winding once around  $p_i$  on  $C$  has an image that winds  $\nu_i$  times around  $0$ . But we know that the monodromy for  $\pi' : D \rightarrow \mathbb{P}^1$  around  $0$  is given by  $\overline{\nu}$ . By the definition of the cycle type, the monodromy of  $f$  around  $p_i$  is  $\iota_i$ . An identical argument shows the monodromy at  $q_i$  over  $\infty$  is  $\kappa_i$  and the monodromy around all preimages of a point in  $U_r$  is zero.

The above process is reversible. We start with a  $d$ -fold cover  $\pi' : C \rightarrow \mathbb{P}^1$  counted by  $H_g(\nu^*, \mu^*)$  and a principal  $K$ -bundle  $f : D \rightarrow C$  with monodromy  $\nu_i$  around  $p_i$  and  $\kappa_i$  around  $q_i$ . Then, the composition  $\pi = \pi' \circ f$  is a cover counted by  $H_{g,K}(\bar{\nu}, \bar{\mu})$ .  $\square$

In other words, if  $\rho' : \overline{\mathcal{M}}_{g,\ell\kappa}(\mathcal{BK}) \rightarrow \overline{\mathcal{M}}_{g,\ell(\lambda)+\ell(\mu)}$  is the natural map, then

$$H_{g,K}(\bar{\nu}, \bar{\mu}) = \deg(\rho') H_g(\nu^*, \mu^*).$$

**4.4. Proof of Theorem 3.** By Lemma 5, we can compute the integral in Theorem 3 by computing the analogous Hurwitz-Hodge integral (appearing in Theorem 1) over  $\overline{\mathcal{M}}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a)$  and multiplying by the degree of

$$\rho : \overline{\mathcal{M}}_{g,-\bar{\mu}}(\mathcal{BG}) \rightarrow \overline{\mathcal{M}}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a).$$

On the other hand, by Lemma 7, we can calculate  $H_{g,K}(\emptyset_+(k), \bar{\mu})$  by computing  $H_g(\emptyset_+, \mu)$ , multiplying by the degree of

$$\rho' : \overline{\mathcal{M}}_{g,(-k)^{d/a}\cup\kappa}(\mathcal{BK}) \rightarrow \overline{\mathcal{M}}_{g,d/a+\ell(\mu)},$$

and correcting for the difference in the sizes of the automorphism groups  $\text{Aut}(\mu)$  and

$$\text{Aut}(\bar{\mu}) = \text{Aut}(\mu^*).$$

Thus, to deduce Theorem 3 from Theorem 1, we need only check that the degrees of  $\rho$  and  $\rho'$  agree. By Lemma 6, the degrees agree when nonzero. The last step is to check the parity condition (25) is the same for  $\rho$  and  $\rho'$ . For  $\rho$ , the parity condition is

$$0 = \sum_{j=1}^{\ell} (-\bar{\mu})_j = \sum_{j=1}^{\ell} (\kappa_j - \mu_j x) = \sum_{j=1}^{\ell} \kappa_j - dx.$$

For  $\rho'$ , the parity condition is

$$0 = -\frac{d}{a}k + \sum_{j=1}^{\ell} \kappa_j.$$

Since  $ax = k$ , the conditions are equivalent.  $\square$

As in the faithful case, unstable integrals may appear on the right side of the formula in Theorem 3. These unstable terms are defined in a completely analogous manner, and extend Theorem 3 to all contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{BG})} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^R}{(1 - x\bar{\psi}_1)} = \frac{1}{|G|} \cdot \frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{BG})} \frac{\sum_{i \geq 0} (-a)^i \lambda_i^R}{(1 - x\bar{\psi}_1)(1 - y\bar{\psi}_2)} = \frac{1}{|G|} \cdot \frac{1}{x + y}.$$

Alternatively, using a theory of stable maps relative to a stack divisor<sup>13</sup> at  $\infty$ , Theorem 3 could be proven in a manner closely parallel to the proof of Theorem 1.

<sup>13</sup>We avoid the foundational discussion of this theory.

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