Book Section:
Yalonetzky, G (2017) The necessary requirement of median independence for relative bipolarisation measurement. In: Bandyopadhyay, S, (ed.) Research on Economic Inequality: Poverty, Inequality and Welfare. Research on Economic Inequality, 25. Emerald, Bingley, pp. 51-62. ISBN: 978-1-78714-522-1.

https://doi.org/10.1108/s1049-258520170000025002

The necessary requirement of median independence for relative bipolarisation measurement

Gaston YALONETZKY*

March 31, 2017

Abstract

The relative bipolarisation literature features examples of indices which depend on the median of the distribution, including the renowned Foster-Wolfson index. This note shows that the use of the median in the design and computation of relative bipolarisation indices is both unnecessary and problematic. It is unnecessary because we can rely on existing well-behaved, median-independent indices. It is problematic because, as the note shows, median-dependent indices violate the basic transfer axioms of bipolarisation (defining spread and clustering properties), except when the median is unaffected by the transfers. The convenience of discarding the median from index computations is further illustrated with a numerical example in which median-independent indices rank distributions according to the basic transfer axioms while median-dependent indices do not.

Keywords: Relative bipolarisation, Median.
JEL Classification: D30, D31.

*University of Leeds; Maurice Keyworth Building LS6 1AN, UK. Phone: (+44) 113 343 0199. E-mail: G.Yalonetzky@leeds.ac.uk.

1 Introduction

Bipolarisation indices are well-known for their departure from traditional inequality measurement in their treatment of progressive transfers. When these transfers involve one member from the bottom half of the population coupled with a member from the top half, then bipolarisation indices decrease, just as inequality indices do, thereby signalling a reduction in the spread between the two halves. Otherwise, if the transfer involves people on the same side of the median, then bipolarisation indices increase, signalling clustering away from the median. Meanwhile traditional inequality indices would decrease in the face of the same type of transfer.

Save for the above similarities in their treatment of progressive transfers, bipolarisation indices differ in numerous ways among themselves and can be classified accordingly. Depending on their sensitivity to changes in the variables' unit of measurement, we can construct relative (e.g. Foster and Wolfson, 2010, Wang and Tsui, 2000), absolute (e.g. Bossert and Schworm, 2008), intermediate (e.g. Chakravarty and D'Ambrosio, 2010), or simply unit-consistent indices (e.g. Lasso de la Vega, Urrutia, and Diez, 2010). In this note we focus on relative, scale-invariant bipolarisation indices, which feature the popular Foster-Wolfson index, but the problems we identify also crop up among non-relative alternatives.

Within this group of relative indices we can identify further sub-categories defined by how the indices are constructed and their satisfaction of desirable properties, or lack thereof. One main distinction relates to whether the index uses the median in its computation or not. Thus, we have relative *median-dependent* and relative *median-independent* indices. Examples of the former are the classes P_2^N and P_4^N of relative indices proposed by Wang and Tsui (2000), which include the famous Foster-Wolfson index (Foster and Wolfson, 2010) as a special case. Examples of median-independent indices include the P_1^N class proposed by Wang and Tsui (2000), the $P(v)$ class proposed by Rodriguez and Salas (2003), and the generalised-mean indices proposed by Kosny and Yalonetzky (2016).

The main purpose of this note is to show that the use of the median in the design and computation of relative bipolarisation indices is both unnecessary and problematic. It is unnecessary because we can rely on existing well-behaved, median-independent indices. It is problematic because, as the note shows, median-dependent indices violate the basic transfer axioms of bipolarisation (defining spread and clustering properties), except when the median is unaffected by the transfers. The convenience of discarding the median from index computations is further illustrated with a numerical example in which median-independent indices rank distributions according to the basic transfer axioms while median-dependent indices do not.

The rest of the note proceeds as follows. Section 2 provides the notation and the definition of the main relative bipolarisation axioms. Section 3 explains how and why median-dependent indices of relative bipolarisation are problematic due to their inability to satisfy the transfer axioms whenever the median is altered. Section 4 provides a simple numerical illustration of the problem posed by median-dependent indices. Section 5 offers some concluding remarks.

2 Preliminaries

2.1 Notation

Let $y_i \geqslant 0$ denote the income of individual i. Y is the income distribution with mean $\mu_Y > 0$, median $m_Y > 0$, and size $N \geqslant 4$.[1] If N is even, then we divide Y into two equally sized halves, each with a size $n = \frac{N}{2}$. Otherwise, if N is odd, we include and repeat the median observation on both equally sized halves, each with a size: $n = \frac{N}{2}+1$. Individuals are ranked in ascending order within each half so that, for example, y_1^L is the poorest individual in the lower-half set L and y_n^H is the richest individual in the higher-half set H. The means of the lower and higher half are μ_Y^L are μ_Y^H, respectively. The distributions of the lower and higher half are Y_L and Y_H, respectively.

We further define a bipolarisation index $I : Y \to \mathcal{R}_+$. We also require a definition of a rank-preserving Pigou-Dalton transfer, involving incomes $y_i < y_j$ and a positive amount $\delta > 0$ such that: $y_i + \delta \leqslant y_j - \delta$. And a definition of regressive transfer in the opposite direction, i.e. with $y_i - \delta$ and $y_j + \delta$.

We will be referring to the Gini coefficient of Y, $G(Y)$; and, following Lambert and Aronson (1993), define the between-group Gini coefficient, $G_B(Y)$, as well as the within-group Gini coefficient, $G_W(Y)$, for the situation in which the groups are the two non-overlapping equally sized halves:

$$G_B(Y) \equiv \frac{\mu_Y^H - \mu_Y^L}{4\mu_Y}; \tag{1}$$

and:

$$G_W(Y) \equiv \frac{1}{4}[\frac{\mu_Y^L}{\mu_Y}G(Y_L) + \frac{\mu_Y^H}{\mu_Y}G(Y_H)]. \tag{2}$$

Finally, in order to apply the general class of relative bipolarisation indices proposed by Rodriguez and Salas (2003) we will consider, first, a generalised Gini index:

$$G(Y;v) \equiv v(v-1)\frac{1}{N}\sum_{i=1}^{N}(1-\frac{i}{N})^{v-2}[\frac{i}{N} - L(Y;i)], \tag{3}$$

where $L(Y;i) \equiv \frac{1}{N\mu_Y}\sum_{j=1}^{i} y_j$ is the Lorenz curve of Y and the y_j are, naturally, ranked in ascending order. Secondly, we will consider a generalised between-group Gini index (Rodriguez and Salas, 2003):

$$G_B(Y;v) \equiv v(v-1)\frac{1}{N}\sum_{i=1}^{N}(1-\frac{i}{N})^{v-2}[\frac{i}{N} - L(\hat{Y};i)], \tag{4}$$

where $\hat{Y}$ is a smoothed distribution obtained from Y by replacing every y_i^L with μ_Y^L and every y_i^H with μ_Y^H.

[1] For the measurement of bipolarisation, ideally we would like to have at least two people on each half of the distribution.

2.2 Some desirable properties for a relative bipolarization index

Just like their inequality counterparts, bipolarisation indices are expected to satisfy axioms of symmetry and population replication. Some minimum normalisation is also expected, chiefly that the bipolarisation indices attain their minimum value (usually 0) only in the presence of perfectly egalitarian distributions. More narrowly, relative bipolarisation indices are also expected to fulfill an axiom of scale invariance imposing index insensitivity to any change in the unit of measurement through scalar multiplication.

Here we focus on defining the two key transfer axioms, whose violation among median-dependent indices is this note's main concern. Perhaps implicitly aware of this problem, Wang and Tsui (e.g. 2000, p. 356) proposed a stringent version of the transfer axioms requiring the median of the distribution undergoing the transfer to remain unchanged. This is an impractical restriction given that, in empirical applications, only by a fluke would we be comparing distributions of continuous variables with exactly the same median. Hence, here we follow Bossert and Schworm (2008) and do not impose such requirement of median invariance when defining the transfer axioms:

Axiom 1. *Spread-increasing transfer (SI): $I(X) > I(Y)$ if X is obtained from Y through a regressive transfer involving y_i^L and y_j^H.*

In other words, the transfer in SI involves pairs of incomes from different halves. The next axiom involves pairs of incomes from the same half:

Axiom 2. *Clustering-increasing transfer (CI): $I(X) > I(Y)$ if X is obtained from Y by a Pigou-Dalton transfer, involving either the pair y_i^L and y_j^L, or the pair y_i^H and y_j^H.*

3 Existing *relative* bipolarisation indices and the problem of median-dependency

As mentioned above, relative bipolarisation indices can be classified into those which are functions of the median and those which are not. In this section we show that median-based indices violate axioms SI and CI. Therefore they are not really suitable for relative bipolarisation measurement. Median-based indices come in different functional shapes.[2] Therefore we will show their violation of the transfer axioms by looking into each specific class of existing median-dependent indices separately. We also provide a numerical illustration in the next section.

To the best of our knowledge, the existing median-dependent relative bipolarisation indices comprise, firstly, a class of rank-dependent indices (Wang and Tsui, 2000):

$$P_2^N(Y) \equiv \frac{\sum_{i=1}^n a_i y_i^L + \sum_{j=1}^n b_j y_j^H}{m_Y}, \tag{5}$$

with the restriction that $a_n < a_{n-1} < ... < a_1 < 0 < b_n < b_{n-1} < ... < b_1$ (Wang and Tsui, 2000, proposition 3, p. 356). A famous member of $P_2^N(Y)$ is the Foster-Wolfson index:

[2]By contrast, the median-independent indices are all functional forms consistent with Theorem 3 of Bossert and Schworm (2008).

$FW \equiv 2[G_B(Y) - G_W(Y)]\frac{\mu_Y}{m_Y}$. Secondly, there is the class of rank-independent indices also proposed by Wang and Tsui (2000):

$$P_4^N(Y) \equiv \frac{1}{N}\sum_{i=1}^{N}\psi(|\frac{y_i - m_Y}{m_Y}|), \tag{6}$$

with $\psi()$ being a continuous, "strictly increasing and strictly concave" (Wang and Tsui, 2000, p. 359) function mapping from the non-negative segment of the real line.

Now we present our result stating that the existing classes of median-dependent indices, i.e. $P_2^N(Y)$ and $P_4^N(Y)$, violate the key transfer axioms. Even though the result does not cover every conceivable median-dependent index, it does question the likelihood of ever finding suitable median-dependent indices of relative bipolarisation. Moreover, the result can be easily extended to show that the median-dependent relative bipolarisation curves, proposed by Foster and Wolfson (2010) in order to test for robust orderings, also violate the transfer axioms.

Proposition 1. *The existing median-dependent classes of relative bipolarisation indices, i.e. $P_2^N(Y)$ and $P_4^N(Y)$, violate the key transfer axioms SI and CI.*

Proof. See Appendix. ■

4 Numerical illustration

As shown by proposition 1, proposed median-dependent indices, including the popular Foster-Wolfson index, are actually unsuitable for relative bipolarisation measurement. The only way out of this problem is to use readily available median-independent indices of relative bipolarisation, e.g. the P_1^N class proposed by Wang and Tsui (2000), the $P(v)$ class proposed by Rodriguez and Salas (2003) (for $v \in [2, 3]$), or the generalised-mean indices proposed recently by Kosny and Yalonetzky (2016).

Table 1 provides a numerical illustration of the unsuitability of existing median-dependent indices alongside the good behaviour of three median-independent indices: a member from the $P_1^N(X)$ class (Wang and Tsui, 2000, p. 356), $P(v)$ (Rodriguez and Salas, 2003) for $v = 3$, and a multiple of $P(v)$ for $v = 2$ which is essentially a correction of the Foster-Wolfson index, as shown below. The table features three distributions (A, B, and C) each with $N = 10$. Distribution B was obtained from A through two Pigou-Dalton transfers involving the fifth richest person (with an initial income of 8) and two people from the bottom half with initial incomes of 3 and 1. After the transfer the three are left with incomes of 6, 4, and 2 respectively. Naturally the mean does not change, but the median decreases from 6 to 5. Since both transfers take place across the median then any index fulfilling axiom SI should yield a lower value for B vis-a-vis A. Meanwhile, distribution C was also obtained from A but this time using only one Pigou-Dalton transfer involving the fourth and the fifth richest people (with initial incomes of 33 and 8). After the transfer the two are left with incomes of 31 and 10, respectively. Again, the mean does not change, but the median rises from 6 to 7. Since the transfers take place on one side of the median then any index fulfilling axiom CI should yield a higher value for C vis-a-vis A.

Table 1 then shows the values of relative bipolarisation indices for the three distributions. First we show the Foster-Wolfson index (FW). It ranks B higher than A and C lower than A: exactly the opposite of what would be expected should the index satisfy both transfer axioms. Then we show the correction of the Foster-Wolfson index ($2P(2)$) proposed by Rodriguez and Salas (2003). Based on the notation from subsection 2.1, we define:

$$2P(2) \equiv 2[2G_B(Y;2) - G(Y;2)]. \tag{7}$$

$2P(Y;2)$ ranks the three distribution in accordance with the transfer axioms. We also compute another member from the $P(v)$ class: $P(3) \equiv 2G_B(Y;3) - G(Y;3)$. Likewise, as any member of $P(v)$ such that $v \in [2,3]$, $P(3)$ satisfies the transfer axioms and ranks the three distributions of Table 1 coherently.

Then we compute $I_{\theta,r}$ with $\theta = 1$ and $r = 0.5$, a member of the class $P_4^N(Y)$ (Wang and Tsui, 2000):

$$I_{\theta,r} \equiv \frac{\theta}{N}\sum_{i=1}^{N}|\frac{y_i - m_Y}{m_Y}|^r, \tag{8}$$

where θ is a positive constant. Again, being median-dependent, $I_{1,0.5}$ fails to rank the distributions consistently with the transfer axioms, as anticipated by proposition 1. Finally, we compute WT, a member of the median-independent class $P_2^N(Y)$ (Wang and Tsui, 2000):

$$WT \equiv \sum_{j=1}^{n}\frac{[n+1-j]}{\mu_Y n[n+1]}y_j^H - \sum_{j=1}^{n}\frac{[n+1-j]}{\mu_Y n[n+1]}y_{n+1-j}^L \tag{9}$$

WT also behaves consistently with the two transfer axioms and ranks the three distributions accordingly.

5 Conclusion

This note showed how the median is both unnecessary and problematic in the construction of sound indices of relative bipolarisation. This problem stems from the indices' reliance on percentile functions which, in turn, depend on a subset of the distribution. Hence the problem could also emerge in bipolarisation assessments based on uneven partitions of the distribution, i.e. relying on other quantiles, besides the median (see Bossert and Schworm, 2008, Kosny and Yalonetzky, 2016, for a discussion of these options).

Here we should also stress that, while certainly necessary, median independence is not a sufficient requirement for well-behaved indices. For example, the index proposed by Deutsch, Silber, and Hanoka (2007), $D \equiv \frac{G_B(Y) - G_W(Y)}{G(Y)}$, is median-independent but violates the axiom SI, unfortunately.

We note that most existing well-behaved, median-independent, relative bipolarisation indices tend to be rank-dependent. Hence, for the sake of easier computation, the effort to devise rank-independent, median-independent indices may be worthwhile. Kosny and Yalonetzky (2016) have already provided a promising route in that direction, using differences of generalised means.

Finally, the findings of this paper in relation to the unsuitability of the median can, and should, be applied to the construction of better non-relative bipolarisation indices.

6 Acknowledgments

I would like to thank Jacques Silber and an anonymous referee for very helpful comments.

Table 1: Suitability and unsuitability of some proposed relative bipolarisation indices: A numerical illustration

Distributions	A	B	C
	100	100	100
	50	50	50
	45	45	45
	33	33	31
	8	6	10
	4	4	4
	3	4	3
	1	2	1
	0	0	0
	0	0	0
Mean	24.4	24.4	24.4
Median	6	5	7
FW	1.193333	1.264	1.0285
$2P(2)$	0.586885	0.562295	0.590164
$P(3)$	0.562623	0.545164	0.56459
$I_{1,0.5}$	1.611164	1.667	1.529606
WT	0.666667	0.648148	0.669492

References

Bossert, W. and W. Schworm (2008). A class of two-group polarization measures. *Journal of public economic theory 10*(6), 1169–87.

Chakravarty, S. and C. D'Ambrosio (2010). Polarization orderings of income distributions. *Review of Income and Wealth 56*(1), 47–64.

Deutsch, J., J. Silber, and M. Hanoka (2007). On the link between the concepts of kurtosis and bipolarization. *Economics Bulletin 4*(36), 1–5.

Foster, J. and M. Wolfson (2010). Polarization and the decline of the middle class: Canada and the US. *Journal of Economic Inequality 8*(2), 247–73.

Kosny, M. and G. Yalonetzky (2016). Relative p-bipolarisation measurement with generalised means and hybrid lorenz curves. ECINEQ WP 2016 - 404.

Lambert, P. and R. Aronson (1993). Inequality decomposition analysis and the Gini coefficient revisited. *The Economic Journal 103*(420), 1221–7.

Lasso de la Vega, C., A. Urrutia, and H. Diez (2010). Unit consistency and bipolarization of income distributions. *Review of Income and Wealth 56*(1), 65–83.

Rodriguez, J. G. and R. Salas (2003). Extended bi-polarization and inequality measures. *Research on Economic Inequality 9*, 69–83.

Wang, Y.-Q. and K.-Y. Tsui (2000). Polarization orderings and new classes of polarization indices. *Journal of Public Economic Theory 2*(3), 349–63.

7 Appendix: Proof of proposition 1

Proof. Violation of axiom SI by $P_2^N(Y)$: Consider a Pigou-Dalton transfer of an amount $\delta > 0$ involving y_1^L and y_1^H which does not change any rank in the distribution. Then we have the following new value of $P_2^N(Y) : \frac{1}{m_Y - \frac{\delta}{2}}[\sum_{i=2}^n a_i y_i^L + a_1(y_1^L + \delta) + \sum_{i=2}^n b_i y_i^H + b_1(y_1^H - \delta)]$. The denominator is also affected because N is even, therefore the median depends on y_1^H. The change due to the transfer is equal to:

$$\frac{\partial P_2^N(Y)}{\partial \delta} = \frac{1}{m_Y}[a_1 - b_1 + \frac{P_2^N(Y)}{2}] \tag{10}$$

From Wang and Tsui (2000, proposition 3), we know that $a_1 - b_1$ is negative, but $P_2^N(Y)$ is positive. Hence the sign of $\frac{\partial P_2^N(Y)}{\partial \delta}$ is a priori ambiguous. In fact, it is easy to find distributions (e.g. with relatively high values of y_1^H and low values of y_1^L) such that: $\frac{\partial P_2^N(Y)}{\partial \delta} > 0$, i.e. in contradiction with (SI).

Violation of axiom CI by $P_2^N(Y)$: Consider a Pigou-Dalton transfer of an amount $\delta > 0$ involving y_1^H and y_2^H which does not change any rank. Then we have the following new value of $P_2^N(Y) : \frac{1}{m_Y + \frac{\delta}{2}}[\sum_{i=1}^n a_i y_i^L + \sum_{i>2}^n b_i y_i^H + b_1(y_1^H + \delta) + b_2(y_2^H - \delta)]$. Again, the denominator is also affected. The change due to the transfer is equal to:

$$\frac{\partial P_2^N(Y)}{\partial \delta} = \frac{1}{m_Y}[b_1 - b_2 - \frac{P_2^N(Y)}{2}] \tag{11}$$

From Wang and Tsui (2000, proposition 3), we know that $b_1 - b_2$ is positive, but $-P_2^N(Y)$ is negative. Hence the sign of $\frac{\partial P_2^N(Y)}{\partial \delta}$ is a priori ambiguous. In fact, it is easy to find distributions (e.g. with relatively high values of y_1^H) such that: $\frac{\partial P_2^N(Y)}{\partial \delta} < 0$, i.e. in contradiction with (CI).

Violation of axiom SI by $P_4^N(Y)$: Consider a Pigou-Dalton transfer of an amount $\delta > 0$ involving y_j^L and y_1^H which does not change any rank. Then we have the following new value of $NP_4^N(Y) : \sum_{i \neq j}^n \psi(|\frac{y_i^L}{m_Y - \frac{\delta}{2}} - 1|) + \sum_{i>1}^n \psi(|\frac{y_i^H}{m_Y - \frac{\delta}{2}} - 1|) + \psi(|\frac{y_j^L + \delta}{m_Y - \frac{\delta}{2}} - 1|) + \psi(|\frac{y_1^H - \delta}{m_Y - \frac{\delta}{2}} - 1|)$. The denominator is also affected since the median depends on y_1^H. The change due to the transfer is equal to:

$$\begin{aligned} \frac{\partial NP_4^N(Y)}{\partial \delta} &= -\frac{1}{2}\sum_{i \neq j}^n \psi'(|\frac{y_i^L}{m_Y} - 1|)\frac{y_i^L}{(m_Y)^2} \\ &\quad + \frac{1}{2}\sum_{i>1}^n \psi'(|\frac{y_i^H}{m_Y} - 1|)\frac{y_i^H}{(m_Y)^2} \\ -\psi'(|\frac{y_j^L}{m_Y} - 1|)\frac{m_Y + \frac{1}{2}y_j^L}{(m_Y)^2} &+ \psi'(|\frac{y_1^H}{m_Y} - 1|)\frac{\frac{1}{2}y_1^H - m_Y}{(m_Y)^2} \end{aligned} \tag{12}$$

where $\psi' \equiv \frac{\partial \psi(x)}{\partial x} > 0$, and $\psi'' \equiv \frac{\partial^2 \psi(x)}{(\partial x)^2} < 0$.

Now note the signs of the right-hand side elements: The first one is negative, the second one is positive, the third one is negative, and finally the fourth one is negative because $m_Y = \frac{1}{2}y_1^H + \frac{1}{2}y_n^L$. Clearly we can render $\frac{\partial NP_4^N(Y)}{\partial \delta} > 0$ by "choosing" a distribution

Y with relatively high values for y_i^H $\forall i = 2, 3, ..., n$, in order to enhance the second, positive element. Then the class $P_4^N(Y)$ violates (SI).

Violation of axiom CI by $P_4^N(Y)$: Consider again a Pigou-Dalton transfer of an amount $\delta > 0$ involving y_1^H and y_2^H which does not change any rank. Then we have the following new value of $NP_4^N(Y) : \sum_{i=1}^n \psi(|\frac{y_i^L}{m_Y+\frac{\delta}{2}} - 1|) + \sum_{i>2}^n \psi(|\frac{y_i^H}{m_Y+\frac{\delta}{2}} - 1|) + \psi(|\frac{y_1^H+\delta}{m_Y+\frac{\delta}{2}} - 1|) + \psi(|\frac{y_2^H-\delta}{m_Y+\frac{\delta}{2}} - 1|)$. The denominator is also affected since the median depends on y_1^H. The change due to the transfer is equal to:

$$\begin{aligned} \frac{\partial NP_4^N(Y)}{\partial \delta} &= \frac{1}{2}\sum_{i=1}^n \psi'(|\frac{y_i^L}{m_Y} - 1|)\frac{y_i^L}{(m_Y)^2} \\ &-\frac{1}{2}\sum_{i>2}^n \psi'(|\frac{y_i^H}{m_Y} - 1|)\frac{y_i^H}{(m_Y)^2} \\ +\psi'(|\frac{y_1^H}{m_Y} - 1|)\frac{m_Y - \frac{1}{2}y_1^H}{(m_Y)^2} - \psi'(|\frac{y_2^H}{m_Y} - 1|)&\frac{\frac{1}{2}y_2^H + m_Y}{(m_Y)^2} \end{aligned} \tag{13}$$

Now note again the signs of the right-hand side elements: The first one is positive, the second one is negative, the third one is positive because $m_Y = \frac{1}{2}y_1^H + \frac{1}{2}y_n^L$, and finally the fourth one is negative. Clearly we can render $\frac{\partial NP_4^N(Y)}{\partial \delta} < 0$ by "choosing" a distribution Y with relatively high values for y_i^H $\forall i = 2, 3, ..., n$, in order to enhance the second and fourth, negative elements. Then the class $P_4^N(Y)$ violates CI. ∎