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The benchmark of maximum relative bipolarisation

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Abstract

Relative bipolarisation indices are usually constructed making sure that they achieve their minimum value of bipolarisation if and only if distributions are perfectly egalitarian. However, the literature has neglected discussing the existence of a benchmark of maximum relative bipolarisation. Consequently there is no discussion as to the implications of maximum bipolarisation for the optimal normalisation of relative bipolarisation indices either. In this note we characterize the situation of maximum relative bipolarisation as the only one consistent with the key axioms of relative bipolarisation. We illustrate the usefulness of incorporating the concept of maximum relative bipolarisation in the design of bipolarisation indices by identifying, among the family of rank-dependent Wang-Tsui indices, the only subclass fulfilling a normalisation axiom that takes into account both benchmarks of minimum and maximum relative bipolarisation.

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1 Introduction

Since the seminal contributions of Foster and Wolfson (Foster and Wolfson (2010); based on a 1992 working paper) and Wolfson (1994), proposals to measure bipolarisation have proliferated. All bipolarisation indices share in common that they take their minimum value when distributions are perfectly egalitarian; yet they all claim to depart from traditional inequality measurement in their treatment of progressive transfers. When these transfers involve one member from the bottom half of the population coupled with a member from the top half, then bipolarisation indices decrease, just as inequality indices do, thereby signalling a reduction in the spread between the two halves. Otherwise, if the transfer involves people on the same side of the median, then bipolarisation indices, unlike their inequality counterparts, increase, signalling clustering away from the median.

By contrast bipolarisation indices differ in numerous ways and can be classified accordingly. Akin to inequality indices, there are relative bipolarisation indices satisfying a property of scale invariance whereby their value remains unaltered when the unit of measurement of the continuous variable changes (e.g. pounds versus dollars). Examples of relative bipolarisation indices include those proposed by Foster and Wolfson (2010) and Wang and Tsui (2000). A less stringent property, that of unit consistency, requires only the relative ranking of distributions by a given index, not its value, to be insensitive to the unit of measurement. Lasso de la Vega, Urrutia, and Diez (2010) have proposed indices fulfilling this property. There are also absolute indices which are sensitive to the unit of measurement, but satisfy a property of translation invariance whereby their values remain unaffected if the same amount is added to all incomes. Examples of these indices include those by Wang and Tsui (2000), Deutsch, Silber, and Hanoka (2007), and the general class of Bossert and Schworm (2008). Finally, there are intermediate indices which are hybrids of relative and absolute measures. Examples include the family by Chakravarty and D'Ambrosio (2010).

In this paper we focus on relative, scale-invariant bipolarisation indices. Within this group there are further distinctions to be made in terms of how the indices are constructed and their satisfaction of desirable properties, or lack thereof. One distinction of interest is whether the indices are normalised or not. Most indices in the literature are constructed so that they achieve their minimum value of 0 if and only if distributions are perfectly egalitarian. However, the literature does not discuss the existence of a benchmark of maximum relative bipolarisation, with the exceptions of Yalonetzky (2014) and Duclos and Taptue (2014). But Yalonetzky (2014) does it only in the context of quasi-orderings. Moreover, while Yalonetzky (2014) correctly identifies the benchmark of maximum relative bipolarisation, he does not formally show that, indeed, relative bipolarisation cannot be increased any further from that benchmark distribution, or that it is the only situation consistent with the axioms of relative bipolarisation. Likewise, Duclos and Taptue (2014) correctly identify the benchmark of maximum relative bipolarisation in their discussion of the Foster-Wolfson bipolarisation index. However they do not formally show that relative bipolarisation cannot be increased any further from that benchmark distribution, or that it is the only situation consistent with the axioms of relative bipolarisation, either. Moreover, they fail to note that, unfortunately, the Foster-Wolfson index does not attain its maximum possible value when evaluated at the benchmark distribution.

By and large, there is no discussion in the literature as to the implications of maximum relative bipolarisation for the optimal normalisation of relative bipolarisation indices. Accordingly, the subsets of existing classes of indices that fulfill good normalisation properties have not yet been identified. Worse still, some otherwise appealing and popular indices, like the Foster-Wolfson (Foster and Wolfson, 2010) or the Deutsch-Silber-Hanoka (Deutsch et al., 2007), are not well normalised in the sense that they will not exclusively achieve their highest possible value in a situation of maximum relative bipolarisation. Therefore, performing relative bipolarisation comparisons with indices that are not properly normalised may be problematic if there is no connection between the maximum value they can attain and the benchmark of maximum relative bipolarisation. In this note we characterize the situation of maximum relative bipolarisation as the only one consistent with the axioms of relative bipolarisation, and we illustrate how to normalise existing (non-normalised) indices by identifying, among the family of rank-dependent Wang-Tsui indices ($P_1^N(X)$ from Wang and Tsui (2000)), the only subclass fulfilling a normalisation axiom that takes into account both benchmarks of minimum and maximum relative bipolarisation. A numerical example further elucidates the importance of adequately normalising indices of relative bipolarisation.

The rest of the note proceeds as follows. Section 2 provides the notation and the definition of the main relative bipolarisation axioms. Then section 3 compares the existing normalisation property which only takes into account the benchmark of minimum relative bipolarisation against an alternative normalisation property which accounts for both benchmarks of minimum and maximum relative bipolarisation. Section 4 provides the justification for preferring the latter normalisation axiom by characterizing the benchmarks of minimum and maximum relative bipolarisation. Then section 5 illustrates the usefulness of considering maximum relative bipolarisation in normalisation axioms, by identifying the only subclass from a family of rank-dependent Wang-Tsui indices that fulfills the normalisation axiom incorporating benchmarks of both minimum and maximum relative bipolarisation. Then section 6 provides a numerical illustration comparing the performance of well normalised indices against some poorly normalised available counterparts. Section 7 offers some concluding remarks.

2 Preliminaries

2.1 Notation

Let $y_i \geqslant 0$ denote the income of individual i. Y is the income distribution with mean $\mu_Y > 0$, median $m_Y > 0$, and size $N \geqslant 4$.[1] We divide Y into two equally sized halves, each with a size $n = \frac{N}{2}$.[2] Individuals are ranked in ascending order within each half so that, for example,

[1]For the measurement of bipolarisation, ideally we would like to have at least two people on each half of the distribution

[2]For the sake of simplicity we assume that N is even, but results can be easily adapted for the case of N being odd.

y_1^L is the poorest individual in the lower-half set L and y_n^H is the richest individual in the higher-half set H. The means of the lower and higher half are μ_Y^L are μ_Y^H, respectively.

We further define a bipolarisation index $I : Y \rightarrow \mathcal{R}_+$. It will also be useful to define a rank-preserving progressive transfer, involving incomes $y_i < y_j$ and a positive amount $\delta > 0$ such that: $y_i + \delta \leqslant y_j - \delta$. We also define a regressive transfer in the opposite direction, i.e. with $y_i - \delta$ and $y_j + \delta$.

Finally we should also define two sets of distributions which are necessary for the discussions of minimum and maximum relative bipolarisation. The first set, $\mathcal{E}$, is made of distributions exhibiting equal non-negative incomes. That is:

$$\mathcal{E} = \{Y \in \mathcal{R}_{++}^N : y_1^L = y_2^L = \ldots = y_n^L = y_1^H = \ldots = y_n^H = y > 0\}. \tag{1}$$

This is the set of all perfectly egalitarian distributions, which the literature also identifies with the benchmark of minimum relative bipolarisation. The second set, $\mathcal{B}$, is made of a bottom half of null incomes and a top half of egalitarian incomes. That is:

$$\mathcal{B} = \{Y \in \mathcal{R}_{+}^N : y_1^L = y_2^L = \ldots = y_n^L = 0 \wedge y_1^H = y_2^H = \ldots = y_n^H = y > 0\}. \tag{2}$$

This is the set that we will characterize below as the benchmark of maximum relative bipolarisation.

2.2 Desirable properties for a relative bipolarization index

Now we define the desirable axioms that a measure of relative bipolarisation should fulfil:

Axiom 1. *Symmetry (SY): $I(X) = I(Y)$ if $X = BY$ where B is an $N \times N$ permutation matrix.*

Axiom 2. *Population principle (PP): $I(X) = I(Y)$ if $X \in \mathcal{R}_+^{\lambda N}$ is obtained from $Y \in \mathcal{R}_+^N$ through an equal replication of each individual income, λ times.*

Axiom 3. *Scale invariance (SC): $I(X) = I(Y)$ if $X = \lambda Y$, with $\lambda > 0$.*

SC is, of course, the key axiom defining the relative approach to measuring bipolarisation. Now we mention the two key transfer axioms common, in one form or another, to all measurement proposals. Some versions require the medians of the distributions undergoing the transfer to remain unchanged (e.g. Wang and Tsui, 2000, p. 356). Here we follow Bossert and Schworm (2008) and do not impose such requirement:

Axiom 4. *Spread-increasing transfer (SI): $I(X) > I(Y)$ if X is obtained from Y through a regressive transfer involving y_i^L and y_j^H.*

In other words, the transfer in SI involves pairs of incomes from different halves. The next axiom involves pairs of income from the same half:

Axiom 5. *Clustering-increasing transfer (CI): $I(X) > I(Y)$ if X is obtained from Y by a progressive transfer, involving either the pair y_i^L and y_j^L, or the pair y_i^H and y_j^H.*

3 Normalisation properties

Many indices in distributional analysis are also commonly expected to fulfill some kind of normalisation axioms, whereby the indices adopt a particular value if (or only if) the distribution represents some extreme situation (e.g. a perfectly egalitarian income distribution, an income distribution without any people below a poverty line, etc.).

As discussed by Yalonetzky (2014), the set of perfectly egalitarian distributions, i.e. $\mathcal{E}$, is widely deemed the benchmark of minimum relative bipolarisation in the literature. Accordingly, all relative bipolarisation indices proposed to date take their minimum value (usually 0) when, or only when, the distribution is perfectly egalitarian. By contrast, except for some intuitive comments by Yalonetzky (2014), there is no formal discussion of a benchmark of maximum relative bipolarisation in the literature. Hence, unsurprisingly, not all indices proposed in the literature are explicitly, or readily, normalised so that they take a maximum value (e.g. 1) when, or only when, the distribution matches the benchmark of maximum relative bipolarisation. In the next section we prove that the benchmark of maximum relative bipolarisation is, essentially, the set $\mathcal{B}$.

The universally accepted axiom of normalisation in the relative bipolarisation literature only requires that the index takes its minimum value (usually 0) only in the presence of perfectly egalitarian distributions (e.g. Chakravarty, 2009, p. 108). We call it minimum normalisation (MN) and express it formally as part (a) of axiom definition (6) below. However, the next section shows that there exists a benchmark of maximum relative bipolarisation which is the only one consistent with the axioms of the previous section. Therefore, for the sake of improved comparability, we claim that relative bipolarisation indices should fulfill the following complete normalisation axiom, which we call normalisation:

Axiom 6. *Complete normalisation (CN): (a)* $I(Y) > I(X) = 0$ *if and only if* $X \in \mathcal{E}$ *and* $Y \notin \mathcal{E}$*; and (b):* $I(Y) < I(X) = 1$ *if and only if* $X \in \mathcal{B}$ *and* $Y \notin \mathcal{B}$.

The next section justifies the preferability of fulfilling axiom CN over just MN. Then the subsequent section illustrates this point with an application to the class of rank-depedent Wang-Tsui indices.

4 Characterization of the benchmarks of extreme relative bipolarisation

The literature has usually emphasized part (a) of axiom CN (i.e. axiom MN), but not part (b), with the exception of Yalonetzky (2014) in the context of pre-orders for relative bipolarisation measurement. Now we highlight the importance of adopting axiom CN by characterizing these two relative bipolarisation benchmarks (minimum and maximum) as the only ones consistent with the other desirable axioms, in particular SC, SI, and CI (but also POP and SYM). We do this with the following two propositions:

Proposition 1. *Let* $\underline{v}$ *be the minimum value that* I *can take. Then, if* I *satisfies SC, SI, CI, POP and SYM,* $I(X) = \underline{v}$ *if and only if* $X \in \mathcal{E}$.

Proof. Consider the relative bipolarisation Lorenz (RBL) curve introduced by Yalonetzky (2014) (here in its non-continuous version). The RBL curve is a handy representation of the relative bipolarisation quasi-ordering satisfying the key transfer axioms originally proposed by Foster and Wolfson (2010) and recently endorsed by Bossert and Schworm (2008) and Yalonetzky (2014):

$$\psi_Y(k) \equiv \frac{\sum_{i=1}^{k}[y_i^H - y_{n-i+1}^L]}{N_Y \mu_Y};\ k = 1, 2, ..., n. \tag{3}$$

We know from Yalonetzky (2014, Theorem 1) that $I(X) > I(Y)$ for all I satisfying SC, SI, CI, POP and SYM, if and only if $\psi_X(k) \geqslant \psi_Y(k)$ for all $k = 1, 2, ..., n$, with at least one strict inequality. Now, inspecting definition (3) it is easy to show that $\psi_Y(k) = 0\ \forall k = 1, 2, ..., n$ if and only if $Y \in \mathcal{E}$. Hence, it must be the case that $I(Y) = \underline{v}$ if and only if $Y \in \mathcal{E}$. ■

Proposition 2. *Let $\overline{v}$ be the maximum value that I can take. Then, if I satisfies SC, SI, CI, POP and SYM, $I(X) = \overline{v}$ if and only if $X \in \mathcal{B}$.*

Proof. Consider distributions $X \in \mathcal{B}$ and $Y \notin \mathcal{B}$, each respectively with means μ_X and μ_Y, and population sizes N_X and N_Y. We need to prove that any distribution $X \in \mathcal{B}$ can be obtained from any other distribution $Y \notin \mathcal{B}$ through a sequence of operations such that the relative bipolarisation index I either increases or yields the same value through every intermediate step of the transformation.

Firstly, note that, as a corollary to Muirhead's theorem (Marshall, Olkin, and Arnold, 2011, p. 7-8), we can obtain any perfectly egalitarian distribution with μ_Y and N_Y from a non-egalitarian distribution with the same mean and size through a sequence of progressive transfers. Hence we perform such sequence of transfers, independently and in parallel, on each half of distribution Y. Call this ensuing perfectly bimodal distribution T_1. Then clearly $I(T_1) > I(Y)$ for any I satisfying CI.

Then generate distribution T_2 from T_1 by performing a sequence of n regressive transfers across the median, each involving a pair of incomes $y_i^L = \underline{\mu}$ and $y_j^H = \overline{\mu}$; where $\underline{\mu}$ and $\overline{\mu}$ are, respectively, the mean of the bottom half and the mean of the top half. In each case a transfer of $\underline{\mu}$ is performed, thereby rendering $y_i^L = 0\ \forall i$ and $y_j^H = \overline{\mu} + \underline{\mu}\ \forall j$. Then clearly $I(T_2) > I(T_1)$ for any I satisfying SI.

Then obtain distribution T_3 from T_2, firstly by multiplying every income by $\frac{\mu_X}{\mu_Y}$, and then by replicating every income $\lambda = \frac{N_X}{N_Y}$ times. Thus, $I(T_3) = I(T_2)$ for any I satisfying SC and POP. Finally, obtain distribution X by multiplying $T3$ by a permutation matrix. Then: $I(X) = I(T3)$ for any I satisfying SYM. We conclude that $I(X) > I(Y)$ for any pair of distributions $X \in \mathcal{B}$ and $Y \notin \mathcal{B}$. Since I is supposed to fulfill SC, SYM, and POP, then it should also be the case that: $I(X) = I(Y)$ for any pair of distributions $\{X, Y\} \in \mathcal{B}$. Hence if I satisfies SC, SI, CI, POP and SYM, $I(X) = \overline{v}$ if and only if $X \in \mathcal{B}$.

■

5 Illustration: Identifying the subclass of normalised, rank-dependent Wang-Tsui indices

Now we illustrate the relevance of taking into account both benchmarks of extreme relative bipolarisation, by identifying, among the broadest class of rank-dependent Wang-Tsui indices, i.e. $P_1^N(Y)$ from Wang and Tsui (2000, p. 356), the only subclass fulfilling the normalisation axiom CN, in addition to scale invariance SC. We focus on the median-independent, rank-dependent, $P_1^N(Y)$ class, since median-dependent indices are not guaranteed to satisfy the transfer axioms when the median is altered. Precisely because we use median-independent indices, we aim to identify a subclass of $P_1^N(X)$ indices which fulfill both CN and SC.

In our notation, the class $P_1^N(Y)$ is defined by:

$$P_1^N(Y) \equiv \sum_{i=1}^{n} a_i y_i^L + \sum_{j=1}^{n} b_j y_j^H \tag{4}$$

Following Wang and Tsui (2000, Proposition 3) it is easy to show that the only subclass of $P_1^N(Y)$ satisfying SI and CI is defined by the following parametric constraint:[3]

$$a_n < a_{n-1} < ... < a_1 < 0 < b_n < b_{n-1} < ... < b_1 \tag{5}$$

Now proposition 3 identifies the subclass from $P_1^N(Y)$ fulfilling both the normalisation axiom N and scale invariance SC. We call it the subclass $\mathcal{WT}$:

Proposition 3. *$\mathcal{WT}$ fulfills axioms CN and SC if and only if: (a) $\sum_{i=1}^n a_i + \sum_{j=1}^n b_j = 0$; (b) $\sum_{j=1}^n b_j = \frac{1}{2\mu_Y}$; (c) the coefficients b_j and a_i are ratios featuring in the denominator an homogeneous function which is linear on an embedded linear function of a subset of Y.*

Proof. Consider $I \in \mathcal{WT}$. CN requires that $I = 0$ if and only if $\forall i, j : y_i^L = y_j^H = y$, which can only be ensured if $\sum_{i=1}^n a_i + \sum_{j=1}^n b_j = 0$. On the other extreme, CN requires that $I = 1$ if and only if $\forall i : y_i^L = 0 \wedge \forall j : y_j^H = y > 0$, which can only be ensured if $\sum_{j=1}^n b_j = \frac{1}{2\mu_Y}$.

As for fulfilment of SC, the axiom requires that if we multiply every income by $\lambda > 0$, then we should get: $\sum_{i=1}^n a_i y_i^L + \sum_{j=1}^n b_j y_j^H = \sum_{i=1}^n a_i \lambda y_i^L + \sum_{j=1}^n b_j \lambda y_j^H$, i.e. no change whatsoever. That can only take place if all the b_j and a_i are in the form of ratios and their respective denominators are equally multiplied by λ. For that to happen, the denominators must be a linear function of an embedded linear function of a subset of Y, and so they will grow by the same factor λ in order to compensate (for instance, they could be a linear function of the mean). ∎

Based on proposition 3, the constraints characterizing $I \in \mathcal{WT}$ can be met with different choices for b_j and a_i. Perhaps the most straightforward choice is coefficients which are linear functions of the rank, like the following:

[3]Wang and Tsui (2000) consider a restricted version of SI and CI in which the medians do not change across distributions and derive their Proposition 3 for a median-dependent class $P_2^N(Y)$. However it is straightforward to show that their Proposition 3 also applies to $P_1^N(Y)$ when we work with the more general axioms SI and CI.

$$b_j = \frac{4[n+1-j]}{2n[n+1]\mu_Y} = -a_{n+1-j} \tag{6}$$

6 Numerical illustration

We consider three relative bipolarisation indices. Firstly, the Foster-Wolfson index, which is defined by:

$$FW \equiv 2(G_Y - GB_Y)\frac{\mu_Y}{m_Y}, \tag{7}$$

where G_Y is the Gini coefficient and GB_Y is the between-group Gini coefficient computed after smoothing distribution Y by replacing every income in the bottom half with μ_Y^L and every income in the top half with μ_Y^H. Secondly, we consider WT_1, defined as a member of the class $\mathcal{WT}$ satisfying the parameter restriction in equation 6. Finally, we also consider WT_2 which is a member of the class $P_1^N(Y)$ (in equation (4)) satisfying the following parameter restriction: $b_j = \frac{[n+1-j]}{\mu_Y} = -a_{n+1-j}$.

Table 1 shows the values of the three indices for distributions A and B. Distribution B is characterized by maximum relative bipolarisation, whereas A exhibits some dispersion among the top half. Note three results: (1) WT_1 and WT_2 agree in ranking B as more relatively bipolarised than A; (2) FW and WT_1 agree in yielding a value of 1 when evaluated at B; (3) FW does not reach its maximum value in the presence of maximum relative bipolarisation (it ranks A above B).

Table 1: Normalisation properties of relative bipolarisation indices: A numerical illustration

Distributions	A	B
	17	12
	15	12
	12	12
	10	12
	6	12
	0	0
	0	0
	0	0
	0	0
	0	0
Mean	6	6
Median	3	6
FW	1.64	1
WT_1	0.85	1
WT_2	2.55	3

What is going on? WT_1 behaves well as it fulfils all the key desirable axioms including CN. By contrast, FW takes a value of 1 in situations of maximum relative bipolarisation. In fact, it is easy to show that FW is equal to 1 for all $Y \in \mathcal{B}$. However, FW can attain higher values when evaluated at some $Y \notin \mathcal{B}$. Therefore, with the Foster-Wolfson index it is not possible to establish a connection between the benchmark of maximum relative bipolarisation and the maximum value that the index can attain. This is clearly problematic since we have established above that this benchmark is the only one consistent with the key axioms of relative bipolarisation. Meanwhile WT_2 will never rank any $A \notin \mathcal{B}$ above any $B \in \mathcal{B}$. However, it violates the axiom CN in a matter of convenience: its maximum value is 3 not 1 (as required by the axiom). In other words, it does not attain any intuitively recognizable value representing the benchmark of maximum relative bipolarisation. Therefore, we can perform neater comparisons of relative bipolarisation with indices from the class $P_1^N(Y)$ satisfying restriction (5) by considering only members of its subclass $\mathcal{WT}$.

7 Conclusion

This note demonstrated that the relative approach to bipolarisation measurement features not only a set of distributions characterized by minimum bipolarisation, but also a set of distributions characterized by maximum bipolarisation. The latter set comprises all situations of perfect bimodality with a bottom half of zero incomes and a top half of equal positive incomes. Moreover, the note showed the logical consistency of maximum relative bipolarisation with the key axioms defining the relative approach.

In the face of these benchmarks, it is only natural to advocate that indices of relative bipolarisation should fulfill a complete normalisation axiom whereby they attain their minimum and maximum values only in the presence of minimum and maximum relative bipolarisation, respectively. Such complete normalisation axiom effectively improves the usefulness of relative bipolarisation indices for distributional comparisons. In practice, some classes of indices, like the class proposed by Kosny and Yalonetzky (2016), already fulfill this complete normalisation axiom. By contrast, as shown in the illustration with the median-independent Wang-Tsui indices, some classes of indices require further parametric constraints in order to comply with the complete normalisation axiom.

Whether maximum bipolarisation benchmarks likewise exist in alternative bipolarisation measurement approaches, and whether corresponding normalisation axioms may be warranted in turn, is left for future inquiry.

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