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POLYNOMIAL EXPONENTIAL EQUATIONS AND ZILBER'S CONJECTURE

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With an Appendix by V. Mantova and U. Zannier

Abstract. Assuming Schanuel's conjecture, we prove that any polynomial exponential equation in one variable must have a solution that is transcendental over a given finitely generated field. With the help of some recent results in Diophantine geometry, we obtain the result by proving (unconditionally) that certain polynomial exponential equations have only finitely many rational solutions.

This answers affirmatively a question of David Marker, who asked, and proved in the case of algebraic coefficients, whether at least the one-variable case of Zilber's strong exponential-algebraic closedness conjecture can be reduced to Schanuel's conjecture.

1. The problem

Based on model-theoretic arguments, Zilber conjectured in [Zil05] that the complex exponential function satisfies two strong properties about its algebraic behaviour. One is the long standing Schanuel's Conjecture, today considered out of reach, while the other, called "Strong Exponential-algebraic Closedness" or "Strong Exponential Closedness", states that all systems of polynomial-exponential equations compatible with Schanuel's conjecture have solutions of maximal transcendence degree. Zilber proved that this would imply that the complex exponential function has a good algebraic description in a very strong model-theoretic sense (most importantly, the structure would be axiomatizable with an uncountably categorical sentence).

While Schanuel's Conjecture is currently considered out of reach, except for some very special known instances, the second property is still relatively unexplored. Marker observed in [Mar06] that, at least in some cases, the second property would already follow from Schanuel's Conjecture. This suggests investigating whether the second property is actually a consequence of Schanuel's Conjecture, thereby implying that Schanuel's and Zilber's conjectures are equivalent.

In the case of systems in one variable the question of Zilber has a particularly simple shape. A system of equations in one variable compatible with Schanuel's Conjecture is just an equation

$$p(z, \exp(z)) = 0$$

where $p(x, y) \in \mathbb{C}[x, y]$ is an irreducible polynomial where both x and y appear, namely such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$.

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Zilber's Strong Exponential Closedness for one variable asserts the following:

Conjecture 1.1 ([Zil05]). *For any finitely generated field $k \subset \mathbb{C}$ and for any irreducible $p(x, y) \in k[x, y]$ such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$, there exists $z \in \mathbb{C}$ such that $p(z, \exp(z)) = 0$ and $\operatorname{tr.deg.}_k(z, \exp(z)) = 1$.*

It is well known and not too difficult to prove that the equation $p(z, \exp(z)) = 0$ has infinitely many solutions (see [Mar06]). Marker proved that Schanuel's Conjecture implies Conjecture 1.1 when $p \in \overline{\mathbb{Q}}[x, y]$, and he asked whether the same holds for any $p \in \mathbb{C}[x, y]$ [Mar06]. Günaydin suggested a different technique in [Gü12] and provided some steps towards this generalisation. Starting from [Gü12], we apply a Diophantine result about function fields and exponential equations from [Zan04] and ultimately give a positive answer.

Theorem 1.2. *If Schanuel's Conjecture holds, then Conjecture 1.1 is true.*

In fact, using Schanuel's conjecture it is easy to reduce Conjecture 1.1 to the problem of counting rational solutions of certain polynomial-exponential equations. Here we obtain the finiteness of such solutions.

Theorem 1.3. *Let $p(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial such that $\frac{\partial p}{\partial x} \neq 0$, $\frac{\partial p}{\partial y} \neq 0$, and let $\mathbf{b} \in \mathbb{C}^l$ be a vector of $\mathbb{Q}$-linearly independent complex numbers. Then the equation*

$$p(\mathbf{x} \cdot \mathbf{b}, \exp(\mathbf{x} \cdot \mathbf{b})) = 0$$

has only finitely many solutions $\mathbf{x} \in \mathbb{Q}^l$.

For simplicity, we shall first prove Theorem 1.3 assuming Schanuel's conjecture, as this will suffice to prove Theorem 1.2. We note that the proof uses only the fact that exp is a homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C}^*, \cdot)$ with cyclic kernel, and the Schanuel property in a few places. This implies that the conclusion of Theorem 1.2 holds in any exponential field having standard kernel, satisfying Schanuel's Conjecture and where $p(z, \exp(z)) = 0$ has infinitely many solutions. Similarly, the conclusion of Theorem 1.3 holds in any exponential field having standard kernel and satisfying the conclusion of Baker's theorem on logarithms. In particular, in the axiomatization of Zilber fields it is sufficient to state that $p(z, \exp(z)) = 0$ has infinitely many solutions, rather than requiring the solutions to have maximal transcendence degree.

In an appendix written with U. Zannier, we shall provide a different unconditional proof of Theorem 1.3 based on Baker's theorem on logarithms in place of Schanuel's conjecture.

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2. Proof

Let $p(x,y) \in \mathbb{C}[x,y]$ be an irreducible polynomial over $\mathbb{C}$ such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$ and let $k \subset \mathbb{C}$ be a finitely generated field. We shall prove that Schanuel's Conjecture implies that the equation

$$p(z, \exp(z)) = 0 \tag{2.1}$$

has only finitely many solutions z such that $z \in \overline{k}$, where $\overline{k}$ denotes the algebraic closure of k in $\mathbb{C}$. Since the equation has infinitely many solutions in $\mathbb{C}$, this easily implies the desired statement.

This is done in four steps:

(1) as explained in [Mar06], if Schanuel's conjecture is true, then there is a finite-dimensional $\mathbb{Q}$-vector space $L \subset \mathbb{C}$ containing all the $z \in \overline{k}$ such that $p(z, \exp(z)) = 0$; if $\mathbf{b}$ is a basis of L as a $\mathbb{Q}$-vector space, all such solutions can be written as a scalar product $\mathbf{q} \cdot \mathbf{b}$ with $\mathbf{q} \in \mathbb{Q}^n$ (§(2.1));
(2) a special case of the main result of [Gü12] says that there is an $N \in \mathbb{N}$ such that actually $\mathbf{q} \in \mathbb{Z}\left[\frac{1}{N}\right]$ for all the solutions; hence, our problem reduces to the one of counting the *integer* solutions $\mathbf{n} \in \mathbb{Z}^n$ of an equation of the form $p(\mathbf{x} \cdot \mathbf{b}/N, \exp(\mathbf{x} \cdot \mathbf{b}/N)) = 0$ (§(2.2));
(3) as suggested by Zannier, a function field version of a theorem of M. Laurent [Zan04] lets us reduce to the case where $\exp(\mathbf{n} \cdot \mathbf{b})$ is always algebraic for any solution $\mathbf{n}$ (§(2.3));
(4) finally, if $2\pi i$ is in L, we specialise it to 0; assuming Schanuel's Conjecture, some arithmetic on $\overline{\mathbb{Q}}$ is sufficient to prove finiteness (§(2.4)).

Step (4) can be replaced with a more complicated, but unconditional, argument where Baker's theorem on logarithms is used in place of Schanuel's Conjecture; its details are given in the appendix. Steps (2) and (3) are obtained unconditionally.

2.1. Reduction to linear spaces.

Let us recall Schanuel's Conjecture:

Conjecture 2.1 (Schanuel)**.** *For all $z_1, \dots, z_n \in \mathbb{C}$,*

$$\operatorname{tr.deg.}(z_1, \dots, z_n, \exp(z_1), \dots, \exp(z_n)) \geq \operatorname{lin.d.}_{\mathbb{Q}}(z_1, \dots, z_n).$$

We shall denote by $\mathbf{z}$ a finite tuple of elements of $\mathbb{C}$, and by $\exp(\mathbf{z})$ the tuple of their exponentials, so that Schanuel's Conjecture can be rewritten as

$$\operatorname{tr.deg.}(\mathbf{z}, \exp(\mathbf{z})) \geq \operatorname{lin.d.}_{\mathbb{Q}}(\mathbf{z}).$$

Proposition 2.2. *Let $p(x,y) \in \mathbb{C}[x,y]$ be an irreducible polynomial such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$ and k be a finitely generated subfield of $\mathbb{C}$.*

If Schanuel's conjecture holds, then all the solutions of (2.1) in $\overline{k}$ are contained in some finite-dimensional $\mathbb{Q}$-linear space $L \subset \overline{k}$.

Proof. Let k' be the field generated by k and the coefficients of p. If Schanuel's conjecture holds, for any $\mathbf{z} \subset \mathbb{C}$ we have

$$\operatorname{tr.deg.}(\mathbf{z}, \exp(\mathbf{z})) \geq \operatorname{lin.d.}_{\mathbb{Q}}(\mathbf{z}).$$

However, if each entry z of $\mathbf{z}$ is in $\overline{k}$ and $p(z, \exp(z)) = 0$, we also have $\exp(z) \in \overline{k}'$, so that in particular

$$\operatorname{tr.deg.}(\overline{k}') = \operatorname{tr.deg.}(k') \geq \operatorname{tr.deg.}(\mathbf{z}, \exp(\mathbf{z})) \geq \operatorname{lin.d.}_{\mathbb{Q}}(\mathbf{z}).$$

This implies that the solutions of (2.1) in $\overline{k}$ live in a $\mathbb{Q}$-linear subspace $L \subset \overline{k}$ of dimension at most $\operatorname{tr.deg.}(\overline{k}')$. Since k' is finitely generated, L is finite-dimensional, as desired. □

In particular, if $\mathbf{b} \subset L$ is a $\mathbb{Q}$-linear basis of L, then all the solutions of Equation (2.1) in $\overline{k}$ are of the form $\mathbf{q} \cdot \mathbf{b}$ for some vector $\mathbf{q}$ with rational coefficients.

Corollary 2.3. *Let $p(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$ and k be a finitely generated subfield of $\mathbb{C}$.*

If Schanuel's Conjecture holds, then there exist $l \in \mathbb{N}$ and $\mathbf{b} \in \mathbb{C}^l$ with $\mathbb{Q}$-linearly independent entries such that (2.1) has only finitely many solutions in $\overline{k}$ if and only if

$$p(\mathbf{x} \cdot \mathbf{b}, \exp(\mathbf{x} \cdot \mathbf{b})) = 0 \tag{2.2}$$

has only finitely many solutions in $\mathbb{Q}^l$.

2.2. **Reduction to integer solutions.** In [Gü12] it is shown that the rational solutions of polynomial equations like (2.2) have bounded denominators, so that our problem becomes one of counting *integer* solutions. We recall the original statement in its full form.

Consider some $\mathbf{b}_i \in \mathbb{C}^t$, where $t \in \mathbb{N}_{>0}$ and i ranges in $\{1, \dots, s\}$ for some integer $s > 1$. We study the rational solutions of the equation

$$\sum_{i=1}^{s} q_i(\mathbf{x}) \exp(\mathbf{x} \cdot \mathbf{b}_i) = 0, \tag{2.3}$$

where the $q_i(\mathbf{x})$'s are polynomials in $\mathbb{C}[\mathbf{x}]$. This includes (2.2) as a special case.

First of all, we exclude the degenerate solutions. A solution $\mathbf{q}$ of (2.3) is *degenerate* if there is a finite proper subset $B \subset \{1, \dots, s\}$ such that

$$\sum_{i \in B} q_i(\mathbf{q}) \exp(\mathbf{q} \cdot \mathbf{b}_i) = 0.$$

A solution is *non-degenerate* otherwise. Moreover, we project away the 'trivial part' of the solutions given by the $\mathbf{q}$'s such that $\exp(\mathbf{q} \cdot \mathbf{b}_i) = \exp(\mathbf{q} \cdot \mathbf{b}_j)$ for all i, j. Let V be the subspace of $\mathbb{Q}^t$ of such $\mathbb{Q}$-linear relations, i.e.,

$$V := \{\mathbf{q} \in \mathbb{Q}^t : \mathbf{q} \cdot \mathbf{b}_i = \mathbf{q} \cdot \mathbf{b}_j \text{ for all } i, j = 1, \dots, s\},$$

and let $\pi' : \mathbb{Q}^t \to V'$ be the projection onto some complement V' of V in $\mathbb{Q}^t$.

With this notation, we have the following.

Theorem 2.4 ([Gü12, Thm. 1.1])**.** *Given $q_1, \dots, q_s \in \mathbb{C}[\mathbf{x}]$ and $\mathbf{b}_1, \dots, \mathbf{b}_s \in \mathbb{C}^t$, there is $N \in \mathbb{N}_{>0}$ such that if $\mathbf{q} \in \mathbb{Q}^t$ is a non-degenerate solution of*

$$\sum_{i=1}^{s} q_i(\mathbf{x}) \exp(\mathbf{x} \cdot \mathbf{b}_i) = 0,$$

then $\pi'(\mathbf{q}) \in \frac{1}{N}\mathbb{Z}^t$.

In the case of (2.2), we can easily deduce that there must be an $N > 0$ such that *all* its rational solutions are in $\frac{1}{N}\mathbb{Z}^t$.

Proposition 2.5. *Let $p(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$ and $\mathbf{b} \in \mathbb{C}^l$ be a vector with $\mathbb{Q}$-linearly independent entries. Then there exists an integer $N > 0$ such that the rational solutions of (2.2) are contained in $\frac{1}{N}\mathbb{Z}^l$.*

In particular, there exists a $\mathbf{b}' \in \mathbb{C}^l$ with $\mathbb{Q}$-linearly independent entries such that (2.2) has only finitely many rational solutions if and only if

$$p(\mathbf{x} \cdot \mathbf{b}', \exp(\mathbf{x} \cdot \mathbf{b}')) = 0 \tag{2.4}$$

has only finitely many integer solutions.

Proof. We can rewrite (2.2) as

$$p(\mathbf{x} \cdot \mathbf{b}, \exp(\mathbf{x} \cdot \mathbf{b})) = \sum_{i=0}^{d} q_i(\mathbf{x} \cdot \mathbf{b}) \cdot \exp(i\mathbf{x} \cdot \mathbf{b}) = 0.$$

There are at most finitely many solutions such that $q_i(\mathbf{x} \cdot \mathbf{b}) = 0$ for all $i = 0, \ldots, d$; indeed, for such solutions the value $\mathbf{x} \cdot \mathbf{b}$ ranges in a finite set, and since the entries of $\mathbf{b}$ are $\mathbb{Q}$-linearly independent, each value of $\mathbf{x} \cdot \mathbf{b}$ determines at most one value of $\mathbf{x}$. Therefore, there is a positive integer N_0 such that these solutions are contained in $\frac{1}{N_0}\mathbb{Z}^l$.

Consider now the solutions such that $q_i(\mathbf{x} \cdot \mathbf{b}) \neq 0$ for at least one i. For each such solution $\mathbf{q}$, there must be a subset $B \subset \{0, \ldots, d\}$ with $|B| \geq 2$ such that $\mathbf{q}$ is a non-degenerate solution of

$$\sum_{i \in B} q_i(\mathbf{x} \cdot \mathbf{b}) \cdot \exp(i\mathbf{x} \cdot \mathbf{b}) = 0.$$

We apply Theorem 2.4 to the equation given by such a B. The corresponding

$$V_B = \{\mathbf{q} \in \mathbb{Q}^l \,:\, \mathbf{q} \cdot (i\mathbf{b}) = \mathbf{q} \cdot (j\mathbf{b}) \text{ for all } i, j \in B\}$$

is null, since the entries of $\mathbf{b}$ are $\mathbb{Q}$-linearly independent and $|B| \geq 2$; therefore, the non-degenerate solutions lie in $\frac{1}{N_B}\mathbb{Z}^l$ for some positive integer N_B.

We now define N as the least common multiple of the various N_B and of N_0, so that all the rational solutions are contained in $\frac{1}{N}\mathbb{Z}^l$. In particular, the rational solutions of (2.2) are in bijection with the integer solutions of the equation

$$p(\mathbf{x} \cdot \mathbf{b}/N, \exp(\mathbf{x} \cdot \mathbf{b}/N)) = p(\mathbf{x} \cdot \mathbf{b}', \exp(\mathbf{x} \cdot \mathbf{b}')) = 0,$$

where $\mathbf{b}' := \mathbf{b}/N$, proving the desired conclusion. □

2.3. **Reduction to algebraic exponentials.** Using the main result of [Zan04], we can reduce the problem of finding integer solutions of (2.4) to the case where $\exp(\mathbf{b}') \subset \overline{\mathbb{Q}}^*$. In the following, if $\mathbf{A}$ is a vector $(a_1, \ldots, a_n)$ in $\mathbb{C}^n$ and $\mathbf{m}$ is a vector $(m_1, \ldots, m_n)$ in $\mathbb{N}^n$, we write $\mathbf{A}^{\mathbf{m}}$ to denote the product $a_1^{m_1} \cdot \cdots \cdot a_n^{m_n}$.

As in [Zan04], we start from the equation

$$\sum_{i=1}^{s} q_i(\mathbf{x})\mathbf{A}_i^{\mathbf{x}} = 0 \tag{2.5}$$

where $\mathbf{A}_i \in \mathbb{C}^t$. In [Lau89], M. Laurent showed that in a precise sense the non-degenerate solutions are not far away from the submodule

$$H := \{\mathbf{n} \in \mathbb{Z}^t \,:\, \mathbf{A}_i^{\mathbf{n}} = \mathbf{A}_j^{\mathbf{n}} \text{ for all } i, j = 1, \ldots, s\}.$$

Note that H is indeed defined similarly to V. If the polynomials q_i are constant, then it turns out that the solutions actually lie in a finite union of translates of H. However, this is not a finiteness result, and as mentioned in [Gü12], even when combined with Theorem 2.4 it is not sufficient to prove Theorem 1.2.

On the other hand, we may control the solutions using a "function field" version of Laurent's theorem as found in [Zan04]. In this version, consider two finitely generated fields $K \subset F$ such that the polynomials and the $\mathbf{A}_i$'s are defined over F and $\operatorname{tr.deg.}_K(F) \geq 1$. Rather than looking for zeroes, the theorem makes a statement regarding the set

$$S := \{\mathbf{n} \in \mathbb{Z}^t : \text{the } q_i(\mathbf{n})\mathbf{A}_i^{\mathbf{n}} \text{ are } \overline{K}\text{-linearly dependent}\}$$

which contains at least the solutions of (2.5).

If we know that $\left(\mathbf{A}_i\mathbf{A}_j^{-1}\right)^{\mathbf{n}} \in \overline{K}^*$ is true for all i, j, then we can rewrite $q_i(\mathbf{n})\mathbf{A}_i^{\mathbf{n}} = q_i(\mathbf{n})\mathbf{A}_1^{\mathbf{n}}\left(\mathbf{A}_i\mathbf{A}_1^{-1}\right)^{\mathbf{n}}$, and it follows that the $q_i(\mathbf{n})\mathbf{A}_i^{\mathbf{n}}$'s are $\overline{K}$-linearly dependent if and only if the $q_i(\mathbf{n})$'s are. Moreover, if there is some $B \subseteq \{1, \ldots, s\}$ such that the $q_i(\mathbf{n})\mathbf{A}_i^{\mathbf{n}}$ for $i \in B$ are $\overline{K}$-linearly dependent, we may deduce the analogous conclusion if $\left(\mathbf{A}_i\mathbf{A}_j^{-1}\right)^{\mathbf{n}} \in \overline{K}^*$ is true just for i, j varying in B. We group the elements of S accordingly.

Definition 2.6. Let B be a nonempty subset of $\{1, \ldots, s\}$. A set $S' \subset \mathbb{Z}^t$ is a *class relative to B* if

(1) for each $\mathbf{n} \in S'$, the elements $q_i(\mathbf{n})\mathbf{A}_i^{\mathbf{n}}$ for $i \in B$ are $\overline{K}$-linearly dependent;
(2) there is an $\mathbf{n}_0 \in S'$ such that for all $\mathbf{n} \in \mathbb{Z}^t$, the vector $\mathbf{n}$ is in S' if and only if it satisfies (1) and for all $i, j \in B$ we have $(\mathbf{A}_i\mathbf{A}_j^{-1})^{\mathbf{n}-\mathbf{n}_0} \in \overline{K}^*$.

Theorem 2.7 ([Zan04, Thm. 1])**.** *The set S is a union of finitely many classes.*

Remark 2.8. In the original article, the theorem is only stated when F has transcendence degree 1 over K; it is however noted that the arguments would actually carry on to larger transcendence degrees (a summary of the few required changes is given in [Zan04, Rmk. 3]). The restricted version with $\operatorname{tr.deg.}(F/K) = 1$ would also work for our purposes, as it lets us reduce the transcendence degree of the exponentials by one; a careful argument with specialisations would then let us reduce the transcendence degree of the coefficients, so that the theorem may be applied again, ultimately leading to algebraic exponentials.

The original theorem also puts explicit bounds to the number of classes in terms of the degrees of the polynomials q_i and on their number.

With this theorem we can reduce our problem regarding (2.4) to the special case in which the exponentials are contained in $\overline{\mathbb{Q}}$.

Proposition 2.9. *Let $p(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$ and $\mathbf{b}' \in \mathbb{C}^l$ be a vector with $\mathbb{Q}$-linearly independent entries. Then there are finitely many irreducible polynomials $r_m(x, y) \in \mathbb{C}[x, y]$ and a vector $\mathbf{c}$ with $\mathbb{Q}$-linearly independent entries such that $\frac{\partial r_m}{\partial x}, \frac{\partial r_m}{\partial y} \neq 0$, $\exp(\mathbf{c}) \subset \overline{\mathbb{Q}}^*$, and such that (2.4) has only finitely many integer solutions if and only if each equation*

$$r_m(\mathbf{x} \cdot \mathbf{c}, \exp(\mathbf{x} \cdot \mathbf{c})) = 0 \tag{2.6}$$

has only finitely many integer solutions.

Proof. Let us write

$$p(\mathbf{x}\cdot\mathbf{b}',\exp(\mathbf{x}\cdot\mathbf{b}')) = \sum_{i=0}^{d} p_i(\mathbf{x}\cdot\mathbf{b}')\exp(i\mathbf{x}\cdot\mathbf{b}') = \sum_{i=0}^{d} q_i(\mathbf{x})\exp(i\mathbf{x}\cdot\mathbf{b}') = 0.$$

This is a special case of (2.5), so we can apply Theorem 2.7. We choose as F a field of definition of p_i, $\mathbf{b}'$ and $\exp(\mathbf{b}')$, and we pick $K=\mathbb{Q}$. We may assume that $\operatorname{tr.deg.}_{\mathbb{Q}}(F)\geq 1$, otherwise the conclusion is trivial as we would have $\exp(\mathbf{b}')\subset\overline{\mathbb{Q}}$. Note that $\operatorname{tr.deg.}_{\mathbb{Q}}(F)\geq 1$ is actually always true, since at least one between $\mathbf{b}'$ and $\exp(\mathbf{b}')$ must contain a transcendental element by the Hermite-Lindemann-Weierstrass theorem [Her73, vL82, Wei85].

Let us drop the terms such that $q_i\equiv 0$, as they do not contribute to the sum. As before, there are at most finitely many solutions $\mathbf{n}$ such that $q_i(\mathbf{n}) = p_i(\mathbf{n}\cdot\mathbf{b}') = 0$ for some non-zero q_i, because the entries of $\mathbf{b}'$ are $\mathbb{Q}$-linearly independent. The remaining solutions are such that the corresponding terms in the sum are all non-zero, but $\overline{\mathbb{Q}}$-linearly dependent, since their sum is 0. By Theorem 2.7, these solutions are contained in finitely many classes, which means that there exist $\mathbf{n}_1,\ldots,\mathbf{n}_k$ and $B_1,\ldots,B_k$ such that for every such solution $\mathbf{n}$ there is some m satisfying

$$\exp((i-j)(\mathbf{n}-\mathbf{n}_m)\cdot\mathbf{b}')\in\overline{\mathbb{Q}}^*$$

for all i, j in B_m. Since no term vanishes, we must have $|B_m|\geq 2$, and the latter condition becomes equivalent to

$$\exp((\mathbf{n}-\mathbf{n}_m)\cdot\mathbf{b}')\in\overline{\mathbb{Q}}^*.$$

Let $\mathbf{c}$ be a $\mathbb{Z}$-linear basis of $\log(\overline{\mathbb{Q}}^*)\cap\operatorname{span}_{\mathbb{Z}}(\mathbf{b}')$, so that for all of the above vectors $\mathbf{n}-\mathbf{n}_m$ we have $(\mathbf{n}-\mathbf{n}_m)\cdot\mathbf{b}' = \mathbf{n}'\cdot\mathbf{c}\in\operatorname{span}_{\mathbb{Z}}(\mathbf{c})$ for some $\mathbf{n}'\in\mathbb{Z}^{l'}$. Each $\mathbf{n}'$ is an integer solution of

$$r_m(\mathbf{x}\cdot\mathbf{c},\exp(\mathbf{x}\cdot\mathbf{c})) := p(\mathbf{n}_m\cdot\mathbf{b}'+\mathbf{x}\cdot\mathbf{c},\exp(\mathbf{n}_m\cdot\mathbf{b}'+\mathbf{x}\cdot\mathbf{c})) = 0$$

for some m; moreover, the map $\mathbf{n}'\mapsto(\mathbf{n}-\mathbf{n}_m)\mapsto\mathbf{n}$ is clearly injective and, as m varies, it covers all the integer solutions of (2.4), except at most finitely many ones. Therefore, (2.4) has only finitely many integer solutions if and only if each of the above equations have only finitely many integer solutions, as desired. □

2.4. **Finiteness.** We can finally prove that the integer solutions of (2.6) are only finitely many. In order to prove that, we use some classical results about the arithmetic of $\overline{\mathbb{Q}}$, and more specifically the properties of the logarithmic Weil height $h:\overline{\mathbb{Q}}^*\to\mathbb{R}_{\geq 0}$.

We just recall the following facts about the function h. Let $\boldsymbol{\gamma}\in(\overline{\mathbb{Q}}^*)^t$ and $\mathbf{m}\in\mathbb{Z}^t$. We denote by $|\mathbf{m}|_1$ the 1-norm of $\mathbf{m}$. There are positive numbers $a_1=a_1(\boldsymbol{\gamma})$, $a_2=a_2(\boldsymbol{\gamma})$, $a_3=a_3(\boldsymbol{\gamma})$ depending on $\boldsymbol{\gamma}$ only such that:

(1) $h(\mathbf{m}\cdot\boldsymbol{\gamma})\leq a_1(\boldsymbol{\gamma})\log|\mathbf{m}|_1$;
(2) $h(\boldsymbol{\gamma}^{\mathbf{m}})\leq a_2(\boldsymbol{\gamma})|\mathbf{m}|_1$;
(3) if the entries of $\boldsymbol{\gamma}$ are multiplicatively independent, then $h(\boldsymbol{\gamma}^{\mathbf{m}})\geq a_3(\boldsymbol{\gamma})|\mathbf{m}|_1$.

Moreover, if $f\in\overline{\mathbb{Q}}[x,y]$ is such that $\frac{\partial f}{\partial x}\neq 0$, there is $a_4=a_4(f)$ depending on f only such that if $\alpha,\beta\in\overline{\mathbb{Q}}^*$ and $f(\alpha,\beta)=0$, then $h(\alpha)\leq c_4(f)h(\beta)$. We also recall that if $n\in\mathbb{Z}^*$, then $h(n)=\log|n|$.

Proposition 2.10. *Let $r_m(x,y) \in \mathbb{C}[x,y]$ be an irreducible polynomial such that $\frac{\partial r_m}{\partial x}, \frac{\partial r_m}{\partial y} \neq 0$ and $\mathbf{c} \in \mathbb{C}^l$ be a vector with $\mathbb{Q}$-linearly independent entries such that $\exp(\mathbf{c}) \subset \overline{\mathbb{Q}}^*$.*

If Schanuel's Conjecture holds, then (2.6) has only finitely many integer solutions.

Proof. We distinguish two cases. Let K be a finitely generated field containing $\mathbf{c}$, $\exp(\mathbf{c})$ and the coefficients of r_m.

If $\mathrm{span}_{\mathbb{Q}}(\mathbf{c})$ does not contain $2\pi i$, then $\exp(\mathbf{c})$ is a multiplicatively independent set. We pick a specialisation $\sigma : K \to \overline{\mathbb{Q}} \cup \{\infty\}$ such that the coefficients of r_m and of $\mathbf{c}$ become non-zero elements of $\overline{\mathbb{Q}}$. Let $r_m^\sigma \in \overline{\mathbb{Q}}[x,y]$ be the specialisation of the polynomial r_m. Note that $\frac{\partial r_m^\sigma}{\partial x}, \frac{\partial r_m^\sigma}{\partial y} \neq 0$. For any $\mathbf{n}$, if α is a root of $r_m^\sigma(x, \exp(\mathbf{n}\cdot\mathbf{c}))$, then $h(\alpha) \leq a_4(r_m^\sigma)h(\exp(\mathbf{n}\cdot\mathbf{c}))$. Symmetrically, $\exp(\mathbf{n}\cdot\mathbf{c})$ must be a root of $r_m^\sigma(\alpha, y) \in \overline{\mathbb{Q}}[y] \setminus \{0\}$, hence $h(\exp(\mathbf{n}\cdot\mathbf{c})) \leq a_4(r_m^\sigma)h(\alpha)$ as well.

Now note that the specialisation σ must leave $\exp(\mathbf{n}\cdot\mathbf{c}) \in \overline{\mathbb{Q}}^*$ fixed, so that if $\mathbf{n}$ is a solution of (2.6), then

$$r_m^\sigma(\sigma(\mathbf{n}\cdot\mathbf{c}), \exp(\mathbf{n}\cdot\mathbf{c})) = 0,$$

hence

$$h(\exp(\mathbf{n}\cdot\mathbf{c})) \leq a_4(r_m^\sigma)h(\sigma(\mathbf{n}\cdot\mathbf{c})) \leq a_4(r_m^\sigma)a_1(\mathbf{c})\log|\mathbf{n}|_1.$$

Since $\exp(\mathbf{c})$ is multiplicatively independent, this implies that

$$a_3(\exp(\mathbf{c}))|\mathbf{n}|_1 \leq h(\exp(\mathbf{n}\cdot\mathbf{c})) \leq a_4(r_m^\sigma)a_1(\mathbf{c})\log|\mathbf{n}|_1,$$

and therefore that there are only finitely many such $\mathbf{n}$, as desired.

If $\mathrm{span}_{\mathbb{Q}}(\mathbf{c})$ contains $2\pi i$, we may assume, without loss of generality, that the first coordinate of $\mathbf{c}$ is $2\pi i/N$ for some integer N; after splitting (2.6) into N different equations, we may directly assume that the first coordinate of $\mathbf{c}$ is $2\pi i$ itself. Write $\mathbf{c} = (2\pi i)^\frown \mathbf{c}'$, where $\frown$ indicates vector concatenation, so that $\mathbf{c}'$ is the vector containing all the entries of $\mathbf{c}$ except for the first one.

Assuming Schanuel's Conjecture, we deduce that the entries of $\mathbf{c}$ are algebraically independent. Let F be the field generated by $\mathbf{c}$, $\exp(\mathbf{c})$ and the coefficients of r_m. Let K' be the field generated by $\mathbf{c}'$ and $\exp(\mathbf{c}')$ only. Since the entries of $\mathbf{c}$ are algebraically independent, $2\pi i \notin K'$. We can then easily find some $K \supseteq K'$ such that F/K is a finitely generated geometric extension (i.e., K is relatively algebraically closed in F) of transcendence degree one, while $2\pi i \notin K$.

Let $\sigma : F \to K \cup \{\infty\}$ be a specialisation of F which leaves K fixed and such that $\sigma(2\pi i) = 0$. If we multiply r_m by a suitable element of F, we may assume that the specialisation r_m^σ is a non-zero polynomial in $K[x,y]$. Let $\mathbf{n}$ is an integer solution of (2.6) written as $\mathbf{n} = (n_0)^\frown \mathbf{n}'$, so that $\mathbf{n}\cdot\mathbf{c} = 2\pi i n_0 + \mathbf{n}'\cdot\mathbf{c}'$. Then

$$\sigma(r_m(\mathbf{n}\cdot\mathbf{c}, \exp(\mathbf{n}\cdot\mathbf{c}))) = r_m^\sigma(\mathbf{n}'\cdot\mathbf{c}', \exp(\mathbf{n}'\cdot\mathbf{c}')) = 0.$$

Again, the specialisation leaves $\exp(\mathbf{n}'\cdot\mathbf{c}) \in \overline{\mathbb{Q}}^*$ fixed, while any mention of n_0 disappears since $\sigma(n_0 \cdot 2\pi i) = 0$ and $\exp(n_0 \cdot 2\pi i) = 1$. In this way, we have reduced to the case where $\mathrm{span}_{\mathbb{Q}}(\mathbf{c}')$ does not contain $2\pi i$.

If the polynomial r_m^σ satisfies $\frac{\partial r_m^\sigma}{\partial x}, \frac{\partial r_m^\sigma}{\partial y} \neq 0$, we may reapply the previous argument with the heights and deduce that there are at most finitely many vectors $\mathbf{n}'$. Otherwise, if r_m^σ actually lives in $\mathbb{C}[x]$ or $\mathbb{C}[y]$, then it has finitely many solutions because $\mathbf{c}'$ and $\exp(\mathbf{c}')$ are respectively $\mathbb{Q}$-linearly independent and multiplicatively independent and $r_m^\sigma \neq 0$. In any case, the vectors $\mathbf{n}'$ are at most finitely many.

It remains to check n_0. It must be an integer solution of

$$r_m(2\pi i x + \mathbf{n}' \cdot \mathbf{c}', \exp(2\pi i x + \mathbf{n}' \cdot \mathbf{c}')) = r_m(2\pi i x + \mathbf{n}' \cdot \mathbf{c}', \exp(\mathbf{n}' \cdot \mathbf{c}')) = 0.$$

Since r_m is irreducible in $\mathbb{C}[x, y]$ and $\frac{\partial r_m}{\partial x} \neq 0$, the above equation is never trivial for any choice of $\mathbf{n}'$, and therefore it has only finitely many solutions for each possible $\mathbf{n}'$. Therefore, there are at most finitely many integer solutions $\mathbf{n}$, as desired. □

Chaining together all of the above statements, we obtain that if Schanuel's conjecture holds, then (2.2) has only finitely many rational solutions. This is a conditional version of Theorem 1.3; an unconditional proof will be given in the appendix.

Proposition 2.11. *Let $p(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial such that $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \neq 0$ and $\mathbf{b} \in \mathbb{C}^l$ be a vector with $\mathbb{Q}$-linearly independent entries.*

If Schanuel's Conjecture holds, then (2.2) has only finitely many rational solutions.

Proof. It suffices to apply Proposition 2.5, Proposition 2.9, and Proposition 2.10 in sequence. □

This is enough to prove the main theorem.

Proof of Theorem 1.2. By Corollary 2.3, Schanuel's Conjecture implies that, given a finitely generated field k, the solutions of (2.1) in $\overline{k}$ are in bijection with the rational solutions of (2.2), which are at most finitely many by Proposition 2.11, again assuming Schanuel's conjecture. However, (2.1) has infinitely many solutions in $\mathbb{C}$, and therefore at least one is not in $\overline{k}$, as desired. □

3. Appendix

By V. Mantova and U. Zannier

The purpose of this appendix is to give a proof of the conclusion of Proposition 2.10 without assuming Schanuel's Conjecture. Following all the previous implications, this gives an unconditional proof of Theorem 1.3. It is not surprising that we still use a deep theorem of transcendental number theory due to Alan Baker, which is a special true case of the conjecture.

We shall prove the following.

Proposition 3.1. *Let $r_m(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial such that $\frac{\partial r_m}{\partial x}, \frac{\partial r_m}{\partial y} \neq 0$ and $\mathbf{c} \in \mathbb{C}^l$ be a vector with $\mathbb{Q}$-linearly independent entries such that $\exp(\mathbf{c}) \subset \overline{\mathbb{Q}}^*$. Then (2.6) has only finitely many integer solutions.*

Informally, the idea is to consider (2.6) as an equation between algebraic functions

$$\mathbf{x} \cdot \mathbf{c} = \psi(\exp(\mathbf{x} \cdot \mathbf{c})),$$

where ψ is an element of $\overline{\mathbb{C}(y)}$ such that $r_m(\psi, y) = 0$. We then exploit the algebraic properties of $\mathbf{c}$ to show that the entries of $\mathbf{x}$ are actually themselves algebraic functions of $\exp(\mathbf{x} \cdot \mathbf{c})$. Once we know this, we can look at the heights of the functions to deduce the finiteness similarly to the proof of Proposition 2.10.

In place of Schanuel's Conjecture, we use the following deep theorem of transcendental number theory by Alan Baker.

Theorem 3.2 (A. Baker [Bak66, Bak67a, Bak67b]). *If $\alpha_1, \dots, \alpha_n \in \log(\overline{\mathbb{Q}}^*)$ are $\mathbb{Q}$-linearly independent, then they are also $\overline{\mathbb{Q}}$-linearly independent.*

It is easy to see that this is one of the consequences of Schanuel's Conjecture. In our case, it shows that our vector $\mathbf{c}$ is actually $\overline{\mathbb{Q}}$-linearly independent.

Before going on with the proof of Proposition 3.1, we recall a couple of classical results about function fields.

Notation 3.3. Let $\mathbf{x}, y$ be independent variables over $\overline{\mathbb{Q}}$. Let K be a finite extension of $\overline{\mathbb{Q}}(\mathbf{x})$ and E be a finite extension of $\overline{\mathbb{Q}}(y)$. We look at E as the function field of a normal, projective and irreducible curve $\mathcal{D}$ over $\overline{\mathbb{Q}}$. We call KE their compositum inside an algebraic closure of $\mathbb{Q}(\mathbf{x}, y)$.

Recall that KE can be seen as the quotient field of the ring of the finite sums $a_1b_1+\cdots+a_mb_m$, with $a_i \in K$ and $b_i \in E$; the ring itself, both as an E-vector space and as a K-vector space, is isomorphic to $E \otimes_{\overline{\mathbb{Q}}} K$ because K and E are linearly disjoint over $\overline{\mathbb{Q}}$. Moreover, each specialisation from $E/\overline{\mathbb{Q}}$ to $\overline{\mathbb{Q}}$ extends (uniquely) to a specialisation from KE to K that leaves the elements of K fixed.

Let $f(z) \in KE[z]$, where z is a further independent variable. For all $P \in \mathcal{D}(\overline{\mathbb{Q}})$ except for at most finitely many points, if $\mathcal{O}_P$ is the local ring of $\mathcal{D}(\overline{\mathbb{Q}})$ at P, and $\mathcal{O}_P^K$ the local ring of $\mathcal{D}(K)$ at P, we have that $f \in \mathcal{O}_P^K[z]$; it suffices to avoid the (finitely many) poles of the coefficients of f. When $f \in \mathcal{O}_P^K[z]$, we shall call $f_P \in K[z]$ the specialisation of f at P.

Proposition 3.4. *Let $f \in KE[z]$ be a monic polynomial irreducible in $K\overline{E}[z]$. Then the polynomial f_P is well-defined and irreducible in $K[z]$ for all $P \in \mathcal{D}(\overline{\mathbb{Q}})$ except at most finitely many points.*

Remark 3.5. In the case when $K = \overline{\mathbb{Q}}(x)$, the result is a special case of a well known theorem usually denominated "Bertini-Noether theorem" [FJ08, Prop. 9.4.3]. We have not been able to locate in the literature this slightly more general version, so we provide a proof by reduction to the Bertini-Noether theorem.

Proof. The polynomial f_P is well-defined as long as we choose $P \in \mathcal{D}(\overline{\mathbb{Q}})$ outside the poles of the coefficients of f, which are finitely many. Therefore, we may assume that f_P is well-defined.

Let F be the field extension of KE generated by a root α of f. Choose a primitive element β of $F/E(\mathbf{x})$ and let $g(\mathbf{x}, z) \in E(\mathbf{x})[z]$ be its minimal polynomial over $E(\mathbf{x})$. Without loss of generality, we may assume that g is actually a monic irreducible polynomial in $E[\mathbf{x}, z]$.

Since K is linearly disjoint from E over $\overline{\mathbb{Q}}$, we have $[K : \overline{\mathbb{Q}}(\mathbf{x})] = [KE : E(\mathbf{x})] = [K\overline{E} : \overline{E}(\mathbf{x})]$; moreover, since f is irreducible over $K\overline{E}$, we have $[F : KE] = [F\overline{E} : K\overline{E}]$. In particular, we have

$$[F : E(\mathbf{x})] = [F : KE] \cdot [KE : E(\mathbf{x})] = [F\overline{E} : K\overline{E}] \cdot [K\overline{E} : \overline{E}(\mathbf{x})] = [F\overline{E} : \overline{E}(\mathbf{x})].$$

This implies that the polynomial g is absolutely irreducible as a polynomial in several variables over E. By the Bertini-Noether theorem, in the form [FJ08, Prop. 9.4.3], for all the points $P \in \mathcal{D}(\overline{\mathbb{Q}})$ except at most finitely many ones, g_P is well-defined (i.e., $g \in \mathcal{O}_P[\mathbf{x}, z]$) and absolutely irreducible.

Now take a P such that g_P is absolutely irreducible, let $\mathcal{O}_P^F$ be an extension of $\mathcal{O}_P^K$ to a valuation subring of F and let $\sigma : \mathcal{O}_P^F \to \overline{K}$ be the corresponding specialisation which extends the specialisation at P. We note that both α and β are in $\mathcal{O}_P^F$, and except for at most finitely many choices of P, $\overline{\mathbb{Q}}(\mathbf{x}, \sigma(\beta)) = K(\sigma(\alpha))$. Since $g_P(\sigma(\beta)) = 0$, we have that $[K(\sigma(\alpha)) : \overline{\mathbb{Q}}(\mathbf{x})] = [F : E(\mathbf{x})]$.

Therefore, the degrees of the subextensions are preserved as well. In particular, $[K(\sigma(\alpha)) : K] = [F : KE]$. Since $f_P(\sigma(\alpha)) = 0$, this shows that f_P must be irreducible for all but finitely many $P \in \mathcal{D}(\overline{\mathbb{Q}})$, as desired. □

Now, let $\psi_1, \ldots, \psi_m$ be functions in KE. As before, we may define their specialisations ψ_j^P for all $P \in \mathcal{D}(\overline{\mathbb{Q}})$ except at most finitely many points.

Proposition 3.6. *If $\psi_1, \ldots, \psi_m \in KE$ are E-linearly independent, then for all $P \in \mathcal{D}(\overline{\mathbb{Q}})$ except at most finitely many points, the specialisations $\psi_1^P, \ldots, \psi_n^P$ are well-defined and $\overline{\mathbb{Q}}$-linearly independent.*

Proof. Let $\psi_1, \ldots, \psi_m$ be functions as in the hypothesis. Since KE is the quotient field of the finite sums $a_1 b_1 + \cdots + a_k b_k$ with $a_i \in K$ and $b_i \in E$, after multiplying by a common denominator we may assume that each function ψ_j is of the form

$$\psi_j = a_1 b_1 + \cdots + a_k b_k$$

with $a_i \in K$ and $b_i \in E$. In particular, we may find a $\overline{\mathbb{Q}}$-linearly independent set $d_1, \ldots, d_l \in K$ and coefficients $b_{jk} \in E$ such that

$$\psi_j = \sum_{k=1}^{l} d_k b_{jk}.$$

Since $d_1, \ldots, d_l$ are $\overline{\mathbb{Q}}$-linearly independent, they are also E-linearly independent, as E and K are linearly disjoint. The fact that the functions ψ_j are E-linearly independent translates to the fact that the matrix $(b_{jk})_{j,k}$ must have rank m. But then the matrix $(b_{jk}^P)_{j,k}$ is well-defined and has rank m for all but finitely many points, in which case the functions ψ_j^P are $\overline{\mathbb{Q}}$-linearly independent, as desired. □

Now we are able to prove Proposition 3.1.

Proof of Proposition 3.1. If $2\pi i \notin \mathrm{span}_{\mathbb{Q}}(\mathbf{c})$, the argument in the proof of Proposition 2.10 already shows that the desired result holds unconditionally, so we may assume that $2\pi i \in \mathrm{span}_{\mathbb{Q}}(\mathbf{c})$. As in the previous proof, up to replacing r_m with finitely many polynomials of the same shape, we may assume that the first coordinate of $\mathbf{c}$ is $2\pi i$. We write again $\mathbf{c} = (2\pi i)^\frown \mathbf{c}'$.

Suppose first that the solutions $\mathbf{n} = (n_0)^\frown \mathbf{n}'$ are such that the vectors $\mathbf{n}'$ are only finitely many. As in the proof of Proposition 2.10, for each such given $\mathbf{n}'$, (2.6) becomes a polynomial in n_0, so that the solutions are only finitely many, as desired.

Now assume by contradiction that the vectors $\mathbf{n}'$ are infinitely many. We also have that $\exp(\mathbf{n} \cdot \mathbf{c}) = \exp(\mathbf{n}' \cdot \mathbf{c}')$ takes infinitely many different values on the solutions, since the entries of $\exp(\mathbf{c}')$ are multiplicatively independent.

Let K be the field generated by $\overline{\mathbb{Q}}$, $\mathbf{c}$ and the coefficients of r_m. By Baker's Theorem 3.2, the entries of $\mathbf{c}$ are $\overline{\mathbb{Q}}$-linearly independent. Let z and y be new elements algebraically independent over K, and look at $r_m(z, y)$ as a polynomial in z with coefficients in $K(y)$. After taking a finite extension $E/\overline{\mathbb{Q}}(y)$, we may split $r_m(z, y)$ into finitely many factors that are irreducible in $K\overline{E}[z]$. Let $\mathcal{D}$ a normal, irreducible and projective curve over $\overline{\mathbb{Q}}$ whose function field is E.

If $\mathbf{n}$ is a solution of (2.6), there is a point $P \in \mathcal{D}(\overline{\mathbb{Q}})$ such that $r_m(z, y_P)$ has a linear factor of the form $(z - \mathbf{n}\cdot\mathbf{c})$ and $y_P = \exp(\mathbf{n}\cdot\mathbf{c})$. Since we are assuming that $\exp(\mathbf{n}\cdot\mathbf{c}) = \exp(\mathbf{n}'\cdot\mathbf{c})$ takes infinitely many values as $\mathbf{n}$ varies on the solutions, there are infinitely many points such that r_m has a linear factor. By Proposition 3.4, except for at most finitely many of these points, each such linear factor must be the specialisation of a linear factor of $r_m(z, y)$ over KE.

In particular, there must be a function $\phi \in KE$ such that $(z - \phi)$ is a linear factor of $r_m(z, y)$, and such that for infinitely many points $P \in \mathcal{D}(\overline{\mathbb{Q}})$ there is a solution $\mathbf{n}_P$ of (2.6) such that

$$y_P = \exp(\mathbf{n}_P \cdot \mathbf{c}), \quad \phi_P = \mathbf{n}_P \cdot \mathbf{c}.$$

As before, write $\mathbf{n}_P = (n_{0,P})^\frown \mathbf{n}'_P$. Let $\mathcal{P}$ be the (infinite) set of such points P; without loss of generality, we may further assume that if $P \neq Q \in \mathcal{P}$, then $\mathbf{n}'_P \neq \mathbf{n}'_Q$. Since K and E are linearly disjoint over $\overline{\mathbb{Q}}$, $\mathbf{c}$ is E-linearly independent. Since ϕ_P is $\overline{\mathbb{Q}}$-linearly dependent on $\mathbf{c}$ for all $P \in \mathcal{P}$, Proposition 3.6 implies that ϕ is E-linearly dependent on $\mathbf{c}$, so that

$$\phi = \boldsymbol{\psi} \cdot \mathbf{c}$$

for some vector $\boldsymbol{\psi}$ of algebraic functions in E. Moreover, when $\phi_P = \mathbf{n}_P \cdot \mathbf{c}$ we must have $\boldsymbol{\psi}_P = \mathbf{n}_P$. Let us write $\boldsymbol{\psi} = (\psi_0)^\frown \boldsymbol{\psi}'$.

We now use the logarithmic Weil height again, as in §(2.4). Let ψ be an entry of $\boldsymbol{\psi}'$. If ψ is non-constant, we have that y is algebraic over $\overline{\mathbb{Q}}(\psi)$, hence $f(y, \psi) = 0$ for some $f \in \overline{\mathbb{Q}}[z, w]$ such that $\frac{\partial f}{\partial z} \neq 0$. In particular, $f(y_P, \psi_P) = 0$, hence $h(y_P) \leq a_4(f)h(\psi_P)$. Therefore, for each $P \in \mathcal{P}$ we have

$$h(y_P) = h(\exp(\mathbf{n}_P \cdot \mathbf{c})) = h(\exp(\mathbf{n}'_P \cdot \mathbf{c}')) \geq a_3(\exp(\mathbf{c}'))|\mathbf{n}'_P|_1,$$

while we also have

$$h(y_P) \leq a_4(f)h(\psi_P) = a_4(f) \log |\psi_P| \leq a_4(f) \log |\mathbf{n}'_P|_1.$$

This implies that the range of ψ as P varies in $\mathcal{P}$ is finite. Since $\mathcal{P}$ is infinite, this implies that ψ is actually constant. Since this must be true for any entry of $\boldsymbol{\psi}'$, the vector $\boldsymbol{\psi}'$ itself is constant, which implies that the vectors $\mathbf{n}'_P$ are only finitely many, and in particular that $\mathcal{P}$ is finite, a contradiction. □

The above proposition now yields the unconditional proof of Theorem 1.3.

Proof of Theorem 1.3. It suffices to apply Proposition 2.5, Proposition 2.9, and Proposition 3.1 in sequence. □

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