Stabilizability of nonlinear infinite dimensional switched systems by measures of noncompactness in the space c_0

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Abstract

This article studies the problem of stabilizability of nonlinear infinite dimensional switched systems. The switching rule is arbitrary and takes place between a countably infinite number of subsystems, each of which is represented by a differential equation in some Banach space. Using a topological notion of a (locally finite) cover and the Hausdorff measure of noncompactness in the c_0 space, we show how the problem of approximate stabilizability of switched systems can be cast into a sequential framework and dealt with. Examples of application are given.

Keywords: switched systems, nonlinear infinite dimensional dynamical system, measure of noncompactness, sequence spaces

1. Introduction

Switched systems are dynamical systems that consist of a given number of subsystems, between which switching occurs. From the mathematical point of view, the subsystems (also called modes) are usually described by differential or difference equations, indexed according to a particular rule. The switching itself takes place according to a given method, which may be a function of time or state of the given mode. The switched systems are most frequently classified according to the dynamics of their modes. Hence, there are linear or nonlinear, continuous or discrete and finite or infinite dimensional switched systems. If among the modes of the system the representation of each type can be found, such switched system is usually referred to as a hybrid system. Hybrid systems frequently arise in modern modelling challenges in various disciplines. Among them are such diverse fields as biomechanics [2], molecular biology or oncology [16] or vibroacoustics [11].

From the theoretical point of view the analysis of switched systems strongly depends on the characteristics of their modes. It frequently requires, especially in the case of hybrid systems, a deep insight into the interactions between continuous and discrete counterparts of the system. Literature such as [20, 30, 21, 29] together with references therein shows a wide spectrum of the current state of knowledge of the analysis of dynamics of switched systems.

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In this article we focus on the problem of controllability of a specific type of dynamical system. In general, we call a dynamical system controllable if it is possible to steer it from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In mathematical control theory there are many different definitions of controllability. They strongly depend both on the class of dynamical systems and on the form of admissible controls [18]. The application of methods taken directly from differential geometry, topology, functional analysis, theory of ordinary and partial differential equations and theory of difference equations may be required in controllability analysis for different types of hybrid systems. As an example, a short survey of such applications can be found in [32].

The systems with controllability of which we deal in this article arise in e.g. analysis of partial differential equations (PDEs) [31] or neural networks [25]. Article [31] discusses the so called longitudinal method of lines for parabolic PDEs, which consists in replacing spatial derivatives by difference expressions. In [25] a sensitivity of randomly distributed neural cells in a neural net is modelled. Although in both cases the resulting dynamic systems have different origins, they both represent a countably infinite system of differential equations. Such systems were analysed from the perspective of a solution existence problem [24], but they are also worth to look at from the perspective of control theory. This becomes especially interesting when taking into account that, due to the origin of the equations forming an infinite system, one should expect a mutual influence between those equations. Incorporating such a phenomenon into the switched system framework opens up new perspectives.

When representing a distributed system dynamics in a standard switched system approach [20, 29], one should be aware that a state space of the system is infinitely dimensional to facilitate spatial changes of its counterparts. Among general control-oriented analyses of infinite dimensional switched systems the case of controller switching with a state equation (often a PDE) unchanged seems to be one of the main approaches [9]. Examples of it are in a design of boundary switching for semilinear hyperbolic equations [12] or construction of algorithms for abstract systems on Hilbert spaces with predefined switching times [17].

If the state equation is also switched the current state of knowledge is less developed. Preliminary considerations can be found in [28]. Switched systems based on hyperbolic semilinear PDEs with application to transport networks with switching control are addressed in [13] and [14]. In terms of adaptation of methods used for finite dimensional switched systems, especially in terms of stability analysis, a Lyapunov function approach is not very fruitful. Under the assumption of permutation of semigroup generators for abstract linear system on Hilbert space one result, providing the construction of common quadratic Lyapunov function, can be found in [27]. With less restrictive assumptions another result can be found in [15], where authors show the existence of Lyapunov (not necessarily quadratic) function for switched linear system in a Banach space. When a switching rule is subject to specific constraints, the authors of [22] present a set of algebraic conditions for switched nonlinear systems on Banach spaces with Lyapunov functions given for each mode. In [1] the authors consider an initial-boundary value problem governed by systems of linear hyperbolic PDEs and study conditions for exponential stability under switches between a finite set of modes. A similar problem for switched hyperbolic PDEs with the use of Lyapunov techniques is discussed by the authors of [26], where they give sufficient conditions of exponential stability under the assumption of a finite set of modes.

The above-mentioned literature analysis of infinite dimensional switched systems concerns mostly stability of such systems. According to the author's knowledge there are no previous results on controllability analysis in a setting proposed in this article. In this setting a state space of the system is assumed only to be a Banach space; the nonlinear infinite dimensional state equation is switched and a set of modes is countably infinite.

The main tool that carries the burden of the analysis presented here is the concept of a measure of noncompactness. It is the foundation on which the main supplementary theorem - Theorem 2.2 - is based. Specifically, it is the condensing property of a given operator which leads to the existence of a solution of differential equation, as was shown by Darbo [7]. The main theorem of this article - Theorem 3.2 - builds on the results obtained in [5], where the authors analyse the behaviour of abstract measures of noncompactness in various sequence spaces.

This article is organized as follows. The second section gives basic definitions. The third section shows the approach to stabilizability analysis in infinite dimensional switched systems. It also contains the main results of this article. The fourth section is devoted to discussion of results in the context of control theory and its applications. The fifth section gives examples of application of the results in three different settings. The article ends with conclusions and references.

2. Preliminaries

This section gives the basic definitions and background material. It also defines the notation. If for lemmas or theorems given without reference to a particular source the proof is short and simple, they are immediately followed by the $\square$ sign.

Following [3], we will introduce the axiomatic definition of the measure of noncompactness (MNC).

Definition 2.1 (Axiomatic measure of noncompactness)**.** Let E be a Banach space, $\mathcal{M}_E$ be a family of all non-empty and bounded subsets of E and $\mathcal{N}_E$ be the family of all non-empty and relatively compact subsets of E. A function $\mu : \mathcal{M}_E \to [0, +\infty)$ is called the measure of noncompactness if it satisfies all the following conditions:

(M1) The family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is non-empty and $\ker \mu \subset \mathcal{N}_E$,

(M2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

(M3) $\mu(\mathrm{cl}_E X) = \mu(\mathrm{conv}\, X) = \mu(X)$, where cl_E stands for closure in E,

(M4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for all $\lambda \in [0, 1]$,

(M5) If (X_n) is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is non-empty,

Remark 1. From axiom (M5) we infer that $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, \ldots,$ what implies that $\mu(X_\infty) = 0$ and that X_∞ is a member of the kernel of μ.

Definition 2.2. Let μ be a measure of noncompactness in the Banach space E. We call the measure μ to be

(M6) *homogeneous* if $\mu(\lambda X) = |\lambda|\mu(X)$ for all $\lambda \in \mathbb{R}$,

(M7) *subadditive* if $\mu(X+Y) \leq \mu(X) + \mu(Y)$,

(M8) *sublinear* if μ is both, homogeneous and subadditive,

(M9) with the *maximum property* if $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$,

(M10) *regular* if μ is sublinear, has the maximum property and $\ker \mu = \mathcal{N}_E$.

In the theory of differential equations in Banach spaces [4], the kernel set of a measure of noncompactness plays an important role. It is given by the following

Definition 2.3 (The kernel set). The kernel set of a measure of noncompactness $\mu : \mathcal{M}_E \to [0, \infty)$ is given by

$$E_\mu := \big\{x \in E : \{x\} \in \ker \mu\big\}.$$

Remark 2. For X being a member of the family $\ker \mu$ all singletons belonging to X are elements of the kernel set E_μ. Notice also that if μ is a regular MNC in E, then $E_\mu = E$. But such an equality is not always true. For example, if we take the measure μ in a Banach space E defined as $\mu(X) = \|X\|$ for $X \in \mathcal{M}_E$, then $E_\mu = \{0_E\}$. On the other hand for the measure $\mu(X) = \operatorname{diam} X$, we have $E_\mu = E$ [5].

In this article, we will make use of the specific measure of noncompactness, namely the Hausdorff MNC. It is given by the following

Definition 2.4 (Hausdorff measure of noncompactness). For a bounded subset $A \in \mathcal{M}_E$ of a metric space Ξ we call

$$\chi(A) := \inf\{\epsilon \geq 0 : A \subseteq \bigcup_{i=1}^{n} B(x_i, r_i);\ x_i \in \Xi,\ r_i \leq \epsilon,\ i = 1, \dots, n;\ n \in \mathbb{N}\}$$

the *Hausdorff measure of noncompactness*, where the set $B(x_i, r_i) \subset \Xi$ is a ball centred at x_i with a radius r_i.

Particularly, in the case of the sequence space c_0, the following theorem gives the possibility to effectively calculate the Hausdorff MNC.

Theorem 2.1 (Hausdorff MNC in c_0 space [5]). *Let A be a bounded subset of a Banach space $E = c_0$. Then*

$$\chi(A) = \lim_{n \to \infty} \left\{ \sup_{x \in A} \left(\max_{k \geq n} |x_k| \right) \right\}.$$

Consider a standard Cauchy problem of the form

$$\begin{cases} \frac{d}{dt} x(t) = f(t, x(t)) \\ x(0) = x_0, \end{cases} \tag{1}$$

where $J := [0, T]$ is a given time interval, $x : J \to E$, $f : J \times E \to E$, x_0 is the initial condition and E is a Banach space.

Among the existence theorems for the Cauchy problem (1) used in this paper, the following one plays an important role [5]:

Theorem 2.2. *Assume that $f : [0,T] \times E \to E$ is such that*

(i) for any $x \in E$, with P and Q being a non-negative constants there is

$$\|f(t,x)\| \leq P + Q\|x\|,$$

(ii) f is uniformly continuous on the set $[0,T'] \times B(x_0, r)$, where $QT' < 1$ and $r = (PT' + Q'\|x_0\|)/(1 - QT')$,

(iii) f satisfies a condition

$$\mu(f(t,X)) \leq p(t)\mu(X)$$

with a sublinear measure of noncompactness μ such that $x_0 \in E_\mu$ and p is a Lebesgue integrable function on the interval $[0,T]$.

Then the problem (1) *has a solution x such that $x(t) \in E_\mu$ for every $t \in [0,T']$.*

Remark 3. In the case when $\mu = \chi$, that is the Hausdorff MNC, the assumption of the uniform continuity of f can be replaced by only its continuity. This is also true if μ is a regular MNC equivalent to the Hausdorff MNC [23].

We will also use the following definitions of controllability and stabilizability.

Definition 2.5 (approximate controllability)**.** The control process is said to be *approximately controllable* in time $T_1 \leq T$ when for any given admissible initial and target state and any $\varepsilon > 0$ there exists an admissible control function such that the state of system at time T_1 falls within the ε-neighbourhood of the target state.

Definition 2.6 (approximate stabilizability)**.** The control process is said to be *approximately stabilizable* if the target state from the definition of approximate controllability is zero.

In the whole remaining part of this article, unless explicitly stated otherwise, X is a Banach space of system trajectories called *trajectory space*, an element $x \in X$ is a given system trajectory, J is a time interval in which system operates, E is a Banach space of trajectory values called *state space*, $x(t) \in E$ is a state of the system at the time instant $t \in J$.

3. Switched systems in sequence spaces

Following a canonical procedure we introduce a switched system by

$$\begin{cases} \frac{d}{dt}x(t) = f_{\sigma(t)}\big(t, x(t)\big) \\ x(0) = x^0, \end{cases} \tag{2}$$

where $\sigma : J \to \mathbb{N}$ is a piecewise constant map called *the switching signal*, with an image $Im(\sigma) := \{\sigma(t) : t \in J\} \subset \mathbb{N}$. Before introducing a formal definition of the trajectory of the switched system (2), for every $i \in Im(\sigma)$ consider a separate Cauchy problem

$$\begin{cases} \frac{d}{dt}x_i(t) = f_i\big(t, x_i(t)\big) \\ x_i(0) = x_i^0, \end{cases} \tag{3}$$

where $t \in J$, $f_i : J \times E \to E$ is an appropriate function and $x_i(0)$ is the i-th mode initial condition. The family of solutions of (3) forms a system of integral equations

$$x_i(t) = x_i(0) + \int_0^t f_i\big(\tau, x_j(\tau)\big) d\tau, \quad i \in Im(\sigma), \tag{4}$$

where $x_i : J \to E$ is called *the trajectory of the i-th mode* of the switched system (2). To proceed further we introduce the following general topology definition [10], namely

Definition 3.1. For any topological space X

(i) a family $\{A_s\}_{s\in S}$ of subsets of X such that $\bigcup_{s\in S} A_s = X$ is called *a cover* of X,

(ii) for any two covers $\mathscr{A} = \{A_s\}_{s\in S}$ and $\mathscr{B} = \{B_t\}_{t\in T}$ of X we say that $\mathscr{A}$ is *embedded* in $\mathscr{B}$ if for every $s \in S$ there exists such $t \in T$ that $A_s \subset B_t$,

(iii) a cover $\mathscr{A}' = \{A'_s\}_{s\in S'}$ is called *a subcover* of a cover $\mathscr{A} = \{A_s\}_{s\in S}$ of X if $S' \subset S$ and for every $s \in S'$ we have $A'_s = A_s$.

(iv) a cover $\mathscr{A} = \{A_s\}_{s\in S}$ of X is called *locally finite* if every $x \in X$ has a neighbourhood U such that the set $\{s \in S : A_s \cap U \neq \emptyset\}$ is finite.

To take a full advantage of the abstract setting stated above, we will explicitly relate the switching signal σ to a cover of J. To do so, we firstly introduce the following

Definition 3.2 (Realisable cover)**.** Consider $J = [0, T]$ with a standard Euclidean topology. A cover $\{J_s\}_{s\in S}$ of J is called a *realisable cover* if for every $s \in S$ the set J_s is connected and for every $s_1, s_2 \in S$, $s_1 \neq s_2$ there is $J_{s_1} \cap J_{s_2} = \emptyset$.

Denote by $\mathcal{M} \subset \mathbb{N}$ the set of indices of modes of the switched system (2), such that each mode is represented by only one element of $\mathcal{M}$ and let the number of modes be $|\mathcal{M}| = \mathfrak{m}$.

In general approach fix $S \subset \mathbb{N}$, $|S| \geq \mathfrak{m}$ and a corresponding realisable cover $\{J_s\}_{s\in S}$ of J with a surjective function $m : S \to \mathcal{M}$ called a translation function. Consider an equivalence relation R_m in S defined by a decomposition of S into layers $\{m^{-1}(i)\}_{i\in\mathcal{M}}$. The switching signal σ is now defined as a quotient map $\sigma : J \to S/R_m$, $\sigma(t) := [s]$ for every $t \in J_s$, $s \in S$, where $[s]$ indicates a class of abstraction of the relation R_m containing s. It is obvious that $|Im(\sigma)| = |\mathcal{M}| = \mathfrak{m}$.

Remark 4. Of course the switching signal σ results from both the selected cover $\{J_s\}_{s\in S}$ and the translation function m. In case of a countably infinite number of modes $|\mathcal{M}| = \aleph_0 = |S|$, the function m becomes a bijection and $\sigma : J \to \mathbb{N}$, $\sigma(t) := i$ for every $t \in J_i$, $i \in \mathbb{N}$.

We may now formally define the trajectory of the switched system (2), namely

Definition 3.3. Let $\{J_s\}_{s\in S}$ be a realisable cover of J, $\{x_s\}_{s\in S}$ be a family of functions corresponding to the cover $\{J_s\}_{s\in S}$, that is $x_s : J_s \to E$. By taking $x(t) := x_s(t)$ for $t \in J_s$ we define a mapping $x : J \to E$ called *a combination* of functions $\{x_s\}_{s\in S}$ and denote it by $\triangledown_{s\in S}\, x_s$.

Definition 3.4 (Trajectory of a switched system)**.** Let $\{J_s\}_{s\in S}$ be a realisable cover of J with a translation function m and a resulting switching signal σ. Let $\{x_i\}_{i\in Im(\sigma)}$ be a family of integral solutions (4) . The *trajectory of the switched system* (2) is a combination $x : J \to E$, $x := \bigtriangledown_{s\in S} x_s$, where $x_s(t) := x_{\sigma(t)}(t)$ for $t \in J_s$.

To show our main results we will also make use of the following

Lemma 3.1. *Let $f : X \times Y \to E$, where (X, d_X), (Y, d_Y), (E, d_E) are metric spaces, $a \in X$ and $b \in Y$ are accumulation points, respectively. If both conditions, namely*

(i) there exists finite or infinite double limit

$$A = \lim_{\substack{x\to a\\ y\to b}} f(x, y),$$

(ii) for every $y \in Y$ there exists finite limit

$$\varphi(y) = \lim_{x\to a} f(x, y),$$

are satisfied, then there also exists the iterated limit

$$\lim_{y\to b} \varphi(y) = \lim_{y\to b} \lim_{x\to a} f(x, y)$$

and it is equal to the double limit.

Proof. Although the proof is not complicated we include it for the reader's convenience. We show the proof in following steps:

1. From (i), for a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_E(f(x, y), A) < \varepsilon$$

   if only $d_X(x, a) < \delta$ and $d_Y(y, b) < \delta$, where $x \in X$ and $y \in Y$.

2. Now, let us fix such y that the inequality $d_Y(y, b) < \delta$ holds, and go to the limit in 1 with $x \to a$. Because of (ii) the value $f(x, y)$ goes to the limit $\varphi(y)$. As a result, we obtain

$$d_E(\varphi(y), A) \le \varepsilon.$$

3. Because y is an arbitrary chosen element of Y satisfying only the condition $d_Y(y, b) < \delta$, we obtain

$$A = \lim_{y\to b} \varphi(y) = \lim_{y\to b} \lim_{x\to a} f(x, y).$$

□

In the remaining part, to keep a sufficient level of generality and yet obtain specific results, we will restrict our considerations to a particular case of the trajectory space X. Specifically, we will investigate cases when trajectory $x \in X$ is a combination of functions which themselves form sequence spaces. The direct motivation behind such approach is given by the examples of switched systems, where although each mode represents a stable system, the specific switching strategy makes the overall switched system unstable [20, 21, 30].

3.1. Systems with solutions in the space c_0

Consider a realisable cover $\{J_s\}_{s\in S}$ of J where $S = \mathbb{N}$, with a given translation function m. For this setting let a switching signal be defined as previously, namely $\sigma : J \to S/R_m$, $\sigma(t) := [s]$ for every $t \in J_s,\ s \in S$. As explained in the introduction, we will analyse a switched system model where, due to the origin of the model itself, modes are allowed to influence each other.

Let a switched system be given by

$$\begin{cases} \frac{d}{dt}x(t) = f_{\sigma(t)}\big(t, (x_j(t))_{j\in Im(\sigma)}\big) \\ x(0) = x^0. \end{cases} \tag{5}$$

An infinite system of Cauchy problems corresponding to the above system is given by

$$\begin{cases} \frac{d}{dt}x_s(t) = f_{[s]}\big(t, (x_j(t))_{j\in \mathbb{N}}\big) \\ x_s(0) = x^0_{[s]}, \end{cases} \tag{6}$$

where $\sigma(t) = [s]$ for every $t \in J_s,\ s \in S$. The family $\{x_s\}_{s\in\mathbb{N}}$ of integral solutions to (6) is then

$$x_s(t) = x_{[s]}(0) + \int_0^t f_{[s]}\big(\tau, (x_j(\tau))_{j\in\mathbb{N}}\big)d\tau, \quad s \in \mathbb{S}, \tag{7}$$

where $x_s \in \mathcal{C}(J, E)$ with a standard supremum norm $\|x_s\|_{\mathcal{C}} := \sup_{t\in J} \|x_s(t)\|_E$.

Let $\tilde{X}$ be a sequential space consisting of all sequences $\tilde{x} = (x_i)_{i\in\mathbb{N}}$ of elements $x_i \in \mathcal{C}(J, E)$ for $i \in \mathbb{N}$. Considering only sequences converging to zero with the norm

$$\|\tilde{x}\|_{\tilde{X}} = \|(x_i)_{i\in\mathbb{N}}\|_{\tilde{X}} := \max\{\|x_i\|_{\mathcal{C}} : i \in \mathbb{N}\},$$

space $\tilde{X}$ becomes a Banach c_0 space.

Fix a realisable cover $\{J_s\}_{s\in S}$ of J and a family $\{x_s\}_{s\in S}$, $x_s \in C(J, E)$ such that $x_s := x_i$ for every $s \in m^{-1}(i)$, $i \in Im(\sigma)$. We may associate two functional sequences with a trajectory $x \in X$, $x := \bigtriangledown_{s\in S}\, x_s$ of the switched system (5), namely $(x_s)_{s\in S}$ and $(\tilde{x}_s)_{s\in S}$, with the latter defined as

$$\tilde{x}_s(t) := \begin{cases} \lim_{\tau\downarrow t_{s-1}} x_s(\tau), & t \in \bigcup_{k=1}^{s-1} J_k \\ x_s(t), & t \in J_s \\ \lim_{\tau\uparrow t_s} x_s(\tau), & t \in \bigcup_{k=s+1}^{\infty} J_k, \end{cases}$$

where $t_{s-1} := \inf\{t : t \in J_s\}$, $t_s := \sup\{t : t \in J_s\}$ and $t_0 = 0$. It is clear that if $(x_s)_{s\in S} \in \tilde{X}$ than $(\tilde{x}_s)_{s\in S} \in \tilde{X}$ and $\|(x_s)_{s\in S}\|_{\tilde{X}} \geq \|(\tilde{x}_s)_{s\in S}\|_{\tilde{X}}$. In what follows, it is possible to analyse the switched system trajectory x by its sequence representation $(\tilde{x}_s)_{s\in S}$ or $(x_s)_{s\in S}$. The following theorem, being the main result of this article, shows that under certain assumptions infinite switching leads to approximate stabilizability.

Theorem 3.2 (Approximate stabilizability in c_0). *Let $\{J_s\}_{s\in\mathbb{N}}$ be a realisable cover of J, $y := (y_j)_{j\in\mathbb{N}}$ be a sequence of elements of the Banach space E such that $(\|y_j\|_E)_{j\in\mathbb{N}} \in c_0$. Assume also that*

(i) initial values of functions in (7) are such that $(\|x_s(0)\|_E)_{s\in\mathbb{N}} \in c_0$,

(ii) *the functions* $f_{[s]} : J \times E^\infty \to E$ *are such that* $\big(\|f_s(t,y)\|_E\big)_{s\in\mathbb{N}} \in c_0$ *for* $t \in J$ *and the mapping* $f : J \times E^\infty \to E^\infty$, $f\Big(t, \big(f_s(t,y)\big)_{s\in\mathbb{N}}\Big) := \big(f_s(t,y)\big)_{s\in\mathbb{N}}$ *is continuous,*

(iii) *there exists an increasing sequence* (k_n) *of natural numbers such that for any* $t \in J$ *the following equality holds*

$$\|f_n(t,(y_j)_{j\in\mathbb{N}})\|_E \le p_n(t) + q_n(t)\sup\{\|y_j\|_E : \ j \ge k_n\},$$

where $n = 1, 2, \dots$*;* $p_n : J \to \mathbb{R}$ *is continuous and the sequence* $(p_n)_{n\in\mathbb{N}}$ *converges uniformly on* J *to the function vanishing identically;* $q_n : J \to \mathbb{R}$ *is continuous and the sequence* $(q_n)_{n\in\mathbb{N}}$ *is equibounded on* J*,*

(iv) *denote also*

$$\begin{aligned} q(t) &:= \sup\{q_n(t) : \ n = 1,2,\dots\},\\ Q &:= \sup\{q(t) : \ t \in J\},\\ p(t) &:= \sup\{p_n(t) : \ n = 1,2,\dots\},\\ p(n) &:= \sup\{p_n(t) : t \in J\},\\ P &:= \sup\{p(t) : \ t \in J\}. \end{aligned}$$

Then the switched system (5) *is approximately stabilizable on the interval* $J' := [0, T']$ *where* $T' < T$ *and* $QT' < 1$*.*

Proof. 1. Take an arbitrary element $y = (y_j)_{j\in\mathbb{N}} \in E^\infty$ such that $(\|y_j\|_E)_{j\in\mathbb{N}} \in c_0$. From assumption *(iii)* for any $t \in J$ and for a fixed $n \in \mathbb{N}$ we have

$$\begin{aligned} \|f_n(t,y)\|_E = \|f_n(t,(y_j)_{j\in\mathbb{N}})\|_E &\le p_n(t) + q_n(t)\sup_{j\ge k_n}\{\|y_j\|_E\}\\ &\le P + Q\sup_{j\ge k_n}\{\|y_j\|_E\} \le P + Q\|y\|_{\tilde{X}}. \end{aligned} \tag{8}$$

As a result we get

$$\|f\|_{\tilde{X}} \le P + Q\|y\|_{\tilde{X}}. \tag{9}$$

2. Take the ball $B\big((\|x_s(0)\|_E)_{s\in\mathbb{N}}, r\big) \subset c_0$, where r is chosen according to theorem 2.2. Then for arbitrarily fixed non-empty subset B_0 of the ball $B\big((\|x_s(0)\|_E)_{s\in\mathbb{N}}, r\big)$ and for $t \in J'$ the following estimate holds

$$\begin{aligned} \chi(f(t,B_0)) &= \lim_{n\to\infty}\{\sup_{x\in B_0}\sup_{s\ge n}\|f_s(t,x)\|_E\}\\ &= \lim_{n\to\infty}\{\sup_{(x_s)\in B_0}\sup_{s\ge n}\|f_s\big(t,(x_s(t))_{s\in\mathbb{N}}\big)\|_E\}\\ &\le \lim_{n\to\infty}\Big\{\sup_{(x_s)\in B_0}\big\{\sup_{s\ge n}\{p_s(t) + q_s(t)\sup_{j\ge k_s}\|x_j(t)\|_E\}\big\}\Big\}\\ &\le \lim_{n\to\infty}\{\sup_{s\ge n}p_s(t)\} + q(t)\lim_{n\to\infty}\{\sup_{(x_s)\in B_0}\sup_{s\ge n}\sup_{j\ge k_s}\|x_j(t)\|_E\}\\ &\le q(t)\chi(B_0). \end{aligned} \tag{10}$$

3. From assumptions, results of points 1 and 2 and from theorem 2.2 with remark 3 it follows that there exists a solution $x = (x_s)_{s\in\mathbb{N}}$ of the initial value problem

$$\begin{cases} \frac{d}{dt}x_s(t) = f_s\big(t, (x_j(t))_{j\in\mathbb{N}}\big) \\ x_s(0) = x_s^0, \end{cases} \tag{11}$$

and is such that $(\|x_s(t)\|_E)_{s\in\mathbb{N}} \in c_0$ for any $t \in J'$.

4. Consider now the double limit

$$\begin{aligned} &\lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(t)\|_E = \lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(0) + \int_0^t f_s\big(\tau, (x_j(\tau))_{j\in\mathbb{N}}\big)d\tau\|_E \\ &\le \lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(0)\|_E + \lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t \|f_s\big(\tau, (x_j(\tau))_{j\in\mathbb{N}}\big)\|_E d\tau \\ &\le \lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(0)\|_E + \lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t \Big((p_s(\tau) + q_s(\tau) \sup_{j\ge k_s}\{\|x_j(\tau)\|_E\}\Big)d\tau \\ &= \lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(0)\|_E + \lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t p_s(\tau)d\tau \\ &\quad + \lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t q_s(\tau) \sup_{j\ge k_s}\{\|x_j(\tau)\|_E\}d\tau = 0. \end{aligned} \tag{12}$$

Really, consider further evaluation of the elements of the above estimation

4.1 From assumption (i) we obtain

$$\lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(0)\|_E = \lim_{s\to\infty} \|x_s(0)\|_E = 0.$$

4.2 From assumption (iii) we deduce

$$\begin{aligned} &\lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t p_s(\tau)d\tau \le \lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t p(s)d\tau \\ &\le \lim_{s\to\infty} \int_0^{T'} |p(s)|d\tau = \lim_{s\to\infty} \frac{1}{T'}|p(s)| = 0. \end{aligned}$$

4.3 From assumption (iii) and point 3 we get

$$\begin{aligned} &\lim_{\substack{s\to\infty \\ t\to T'^-}} \int_0^t q_s(\tau) \sup_{j\ge k_s}\{\|x_j(\tau)\|_E\}d\tau \le \lim_{\substack{s\to\infty \\ t\to T'^-}} Q \int_0^t \sup_{j\ge k_s}\{\|x_j(\tau)\|_E\}d\tau \\ &\le \lim_{s\to\infty} Q \int_0^{T'} \sup_{j\ge k_s}\{\|x_j(\tau)\|_E\}d\tau \end{aligned}$$

4.4 The result of point 3 states that $(\|x_s(t)\|_E)_{s\in\mathbb{N}} \in c_0$ for any $t \in J'$, what means that the sequence of mappings $(x_s)_{s\in\mathbb{N}}$ is pointwise convergent to the zero of

$\mathcal{C}(J', E)$ space. On the other side, for each $s \in \mathbb{N}$ there is $\|x_s(t)\|_E \leq \|x\|_{\tilde{X}}$ for every $t \in J'$. Now the Lebesgue dominated convergence theorem gives

$$\lim_{s\to\infty} Q \int_0^{T'} \sup_{j\geq k_s} \{\|x_j(\tau)\|_E\} d\tau = Q \int_0^{T'} \lim_{s\to\infty} \sup_{j\geq k_s} \{\|x_j(\tau)\|_E\} d\tau = 0.$$

5. From the results of points 3 and 4 and from lemma 3.1, it follows that

$$\lim_{s\to\infty} \|x_s(T')\|_E = 0$$

and

$$\lim_{t\to T'^-} \lim_{s\to\infty} \|x_s(t)\|_E = \lim_{\substack{s\to\infty \\ t\to T'^-}} \|x_s(t)\|_E = 0,$$

what means that regardless of the switching strategy, if only assumptions are met, the switched system (5) is approximately stabilizable.

□

4. Discussion

It is worthwhile to discuss some aspects of the above considerations in more detail, especially from the control theory and its applications point of view.

4.1. Modes of the system

The switched system (5) is usually a result of a process of mathematical modelling - an attempt to describe some physical phenomenon in mathematical terms. In such a case the number of modes present in the model and their possible order of execution results from physical properties of the phenomenon being described or the method of description itself.

In this work we are concerned only with the number of modes, that is with the cardinality $\mathfrak{m}$ of the set $\mathcal{M}$. If of interest, the possibility of their physical consecutive execution must be incorporated into the translation function m.

4.2. Switching strategy

In the theory of switched systems a switching strategy is an important aspect of analysis. It is frequently formulated as a problem of determining conditions which must by satisfied by coefficients of (5) that guarantee that the system performs accurately e.g. is stable, for arbitrary switching signal. The examples of approach to this problem in case of linear systems include such techniques as common Lyapunov functions [29, 20, 15, 27], theory of Lie algebras [19], generalised spectral radius [6] etc.

In this work such problem may be formulated in terms of a selection of a given pair of a realisable cover $\{J_s\}_{s\in S}$ of J and a translation function m.

4.3. Zeno executions

An execution of a hybrid system is called *Zeno* if it takes infinitely many discrete transitions in a finite time - see [33] and references therein. Typically, non-Zenoness is hypothesized in the analysis of switched systems, either explicitly e.g [1] or implicitly e.g. [29]. As physical systems are not Zeno but their hybrid models may be so due to modelling abstraction, it is important to identify Zeno phenomenon and manage it accordingly. This becomes of utmost interest when developing computational tools for hybrid systems or when designing a regulator for a physical system modelled as Zeno system.

In this article in terms of Zeno analysis it is worth to note that we are interested only in approximate stabilizability. The required "accuracy" from Definition 2.5 is expressed by $\varepsilon > 0$. In the proof of Theorem 3.2, point 3 says that for every $t \in J'$ there is $\lim_{s\to\infty} \|x_s(t)\|_E = 0$, what means that for every fixed time instant $t \in J'$ by increasing the index s - what corresponds to switching - it would be possible to approach zero with an arbitrary accuracy. This by itself would be a Zeno execution with countably many switches in a time interval degenerated to one point t. For this reason there are double limit considerations which show that is possible to approach zero not only along coordinate axes of the product space $S \times J'$, but along any "route" leading to the boundary point $(\infty, T') \in S \times J'$, cf. Lemma 3.1. What now point 5. of the same proof says is that with t approaching T' there will always be a sufficiently high index number s such that the given accuracy will be attained.

To guarantee a non-Zeno execution additional conditions must be imposed on the realisable cover $\{J_s\}_{s\in S}$ of J. We express this in the following

Corollary 4.1. *Suppose that all assumptions of the theorem 3.2 are met and let* $\{J_s\}_{s\in\mathbb{N}}$ *be a selected realisable cover of* J *such that for every* $T' < T$ *there exists its subcover* $\{J'_s\}_{s\in\mathbb{S}'}$ *such that* $\{J'_s\}_{s\in\mathbb{S}'}$ *is a locally finite cover of* $[0, T'']$*, where* $T' < T'' < T$*. Then the switched system* (5) *is approximately stabilizable on the interval* J' *and does not exhibit Zeno executions.* □

Corollary 4.1 shows how to formally avoid Zeno executions, but one has to bear in mind that the number of switches, although finite, depends on the accuracy $\varepsilon > 0$. As a general rule as ε becomes smaller the number of switches becomes bigger, what means that is the cover $\{J'_s\}_{s\in\mathbb{S}'}$ gets "denser" near T'. From the perspective of developing a computational tool the remaining problem now is whether the numerical algorithm will be able to handle such a situation.

5. Examples

To analyse the examples below it is important to note that we do not assume any control action other than the switching itself. As a consequence, once a given mode is selected, the behaviour of the switched system entirely depends on this mode dynamics.

Let the set of modes be $|\mathcal{M}| = \mathfrak{m} \leq \aleph_0$, $S = \mathbb{N}$, $s_j := j$ and $\{J_s\}_{s\in\mathbb{N}}$ be a realisable cover of J.

5.1. Finite number of modes

In this trivial example $|\mathcal{M}| = \mathfrak{m} < \aleph_0$. To be able to apply Theorem 3.2 among the modes $i \in \mathcal{M}$, there has to be a null mode, say i_0, for which $x_{i_0}(0) = 0$ and $f_{i_0}(t, y) = 0$ for all $t \in J$. The translation function $m : S \to \mathcal{M}$ has to be such that $m^{-1}(i_0) = \bigcup_{j=\mathfrak{m}}^{\infty} s_j \subset \mathbb{N}$ and the order of indices $\{s_j : j < \mathfrak{m}\}$ does not play a role. This is the only method to make $(\|x_s(0)\|_E)_{s\in\mathbb{N}} \in c_0$ and $\big(\|f_s(t, y)\|_E\big)_{s\in\mathbb{N}} \in c_0$ for every $t \in J$, that is to satisfy assumptions (i) and (ii) of Theorem 3.2.

5.2. Finite dimensional system with infinitely many modes

Let $|\mathcal{M}| = \mathfrak{m} = \aleph_0$ and $E = \mathbb{R}^2$ with a standard Euclidean topology. The translation function $m : S \twoheadrightarrow \mathcal{M}$ becomes a bijection and $\sigma : J \to \mathbb{N}$, $\sigma(t) := s$ for every $t \in J_s,\ s \in \mathbb{N}$.

Remark 5. Note that in this case a realisable cover $\{J_s\}_{s\in\mathbb{N}}$ of J is not locally finite - cf. corollary 4.1.

Let each mode $x_s : J \to \mathbb{R}^2$ be given by

$$x_s(t) := \begin{pmatrix} x_s^1(t) \\ x_s^2(t) \end{pmatrix} = \begin{pmatrix} a_s \sin(\frac{b_s}{a_s} t - \phi_s) \\ b_s \cos(\frac{b_s}{a_s} t - \phi_s) \end{pmatrix}, \tag{13}$$

where $t \in J := [0, 4\pi]$, $\phi_s \in \mathbb{R}$, $a_s \geq b_s$ for every $s \in \mathbb{N}$ and $\{a_s : s \in \mathbb{N}\}$ and $\{b_s : s \in \mathbb{N}\}$ are countable subsets of $[0, 1]$.

Every member x_s of an infinitely countable family $\{x_s\}_{s\in\mathbb{N}}$ of ellipses in $\mathbb{R}^2$ centred at $(0, 0)$ is a classical control theory example of a conservative system a trajectory of which depends only on the initial condition $x_s(0)$. It is not difficult to modify slightly a known example (see e.g. [20], p.19) of switching scheme between stable systems that produces an unstable one, to obtain a switched system that will not be stabilizable for sufficiently small ε if not allowing Zeno executions. What is more, this property is preserved even with a continuous trajectory.

The family $\{x_s\}_{s\in\mathbb{N}}$ is a family of integral solutions to the system of Cauchy problems

$$\begin{cases} \frac{d}{dt} x_s(t) = A_s x_s(t) \\ x_s(0) = x_s^0, \end{cases} \tag{14}$$

where $s \in \mathbb{N}$ and

$$A_s = \begin{pmatrix} 0 & 1 \\ \frac{b_s^2}{a_s^2} & 0 \end{pmatrix}.$$

Let us fix $s \in \mathbb{N}$. Now $f_s : J \times \mathbb{R}^2 \to \mathbb{R}^2$, $f_s(t, x_s(t)) := A_s x_s(t)$ where we can regard $A_s : \mathbb{R}^2 \to \mathbb{R}^2$ as a linear bounded operator with a standard norm $\|A_s\| := \sup_{y\in\mathcal{B}(0,1)} \|A_s y\|$ with $\mathcal{B}(0, 1)$ being a closed unit ball in $\mathbb{R}^2$ centred at $(0, 0)$. What follows, for every $t \in J$ there is

$$\|A_s x_s(t)\| \leq \|A_s\|\|x_s(t)\| \leq \|A_s\| a_s.$$

As above is true for every $s \in \mathbb{N}$, theorem 3.2 with corollary 4.1 say that the sufficient condition for the switched system

$$\begin{cases} \frac{d}{dt} x(t) = A_s x_s(t) \\ x(0) = x^0 \end{cases} \tag{15}$$

to be stabilizable without Zeno executions is that $lim_{s\to\infty} a_s = 0$. This is an intuitive result which can be interpreted as a requirement of proper order of mode selection by the switching signal σ.

5.3. Infinite dimensional system with infinitely many modes

This example, adapted from [5], tackles the problem of switched systems with solutions forming a c_0 space in the sense given in section 3.1. The example, in a simplified form, was originally considered in [8] and later applied to the numerical analysis of line method approximations to the Cauchy problem for nonlinear parabolic differential equations [31] modelling a heat conduction in a one-dimensional rod.

Consider a rectangular plate of size l_1 by l_2. Let $|\mathcal{M}| = \mathfrak{m} = \aleph_0$ and $E = \mathcal{C}([0, l_2], \mathbb{R})$ with a standard norm-induced topology, the translation function $m : S \to \mathcal{M}$ is again a bijection and $\sigma : J \to \mathbb{N}$, $\sigma(t) := s$ for every $t \in J_s,\ s \in \mathbb{N}$.

Let Q be a countable subset of $[0, l_1] \subset \mathbb{R}$ and $\rho : \mathbb{N} \to Q$ be a bijection. For every $s \in \mathbb{N}$ and $t \in J$ let $x_{\rho(s)}(t) : [0, l_2] \to \mathbb{R}$ be a temperature profile at time t along a cross section $\rho(s) \in [0, l_1]$ of the plate. To simplify the notation we will write simply x_s instead of $x_{\rho(s)}$.

Let $(k_s)_{s\in\mathbb{N}}$ be an increasing sequence of natural numbers. Consider an infinite system of Cauchy problems of the form

$$\begin{cases} \frac{d}{dt}x_s(t) = f_s\big(t, x_1(t), x_2(t), \ldots, x_{k_s}(t)\big) + \Sigma_{j=k_s+1}^{\infty} a_{sj}(t)x_j(t) \\ x_s(0) = x_s^0, \end{cases} \tag{16}$$

where $t \in J := [0, T]$. Assume also that

(i) initial values of system are such that $(\|x_s(0)\|_E)_{s\in\mathbb{N}} \in c_0$,

(ii) functions $f_s : J \times E^{k_s} \to E$ are uniformly continuous for $s \in \mathbb{N}$ and there exists a function sequence $(p_s)_{s\in\mathbb{N}}$ such that $p_s : J \to \mathbb{R}$ is continuous on J for $s \in \mathbb{N}$ and $(p_s)_{s\in\mathbb{N}}$ converges uniformly on J to the function vanishing identically. Also the following inequality holds

$$\|f_s(t, x_1(t), x_2(t), \ldots, x_{k_s}(t))\|_E \le p_s(t),$$

for every $t \in J$, $(x_1(t), x_2(t), \ldots, x_{k_s}(t)) \in E^{k_s}$ and $s \in \mathbb{N}$.

(iii) functions $a_{sj} : J \to \mathbb{R}$ are continuous and function series $\Sigma_{j=k_s+1}^{\infty} a_{sj}$ converges absolutely and uniformly on J to the function $a_s : J \to \mathbb{R}$, for every $s \in \mathbb{N}$,

(iv) the sequence $(a_s)_{s\in\mathbb{N}}$ is equibounded on J,

(v) $QT < 1$, where $Q = \sup\{a_s(t) : s \in \mathbb{N},\ t \in J\}$.

Then a switched system

$$\begin{cases} \frac{d}{dt}x(t) = f_s\big(t, x_1(t), x_2(t), \ldots, x_{k_s}(t)\big) + \Sigma_{j=k_s+1}^{\infty} a_{sj}(t)x_j(t) \\ x(0) = x^0, \end{cases} \tag{17}$$

satisfies all assumptions of Theorem 3.2. The switched system approach in this case is an example of longitudinal method known from numerical analysis of PDEs where, as described in the introduction, spatial derivatives are replaced by difference equations. In this case the distribution and order of consecutive cross sections on which the behaviour of the plate is analysed is dependent on translation function m and function ρ. The former is to be selected in such a way that the assumptions of Theorem 3.2 hold, while the latter is to be selected according to agiven property of the plate which is of particular interest. Note also that in the setting presented here there is no mention about the so called growth condition, which in numerical analysis is frequently imposed on the initial condition [31].

Another aspect of this case is that the Theorem 3.2 does not make use of any particular norm on the state space E. The above assumption about continuous temperature profile may be changed to better suit any particular analysis. This is the case when the plate is not homogeneous and the state space may be changed to, say, $\mathcal{L}^2([0,l_2],\mathbb{R}])$ to account for some specific artefacts or properties inside the plate.

6. Conclusions

This article showed the analysis of stabilizability of an infinite dimensional switched dynamical system by means of the concept of a measure of noncompactness. After presentation and fitting the problem into the realm of the sequence space c_0, appropriate tools were used to deliver the results. The application of the results was also shown to illustrate the technique and show the usefulness of the approach.

The future work consists of performing similar analysis, but in the case of a semilinear infinite dimensional dynamical switched system with a particular set of admissible controls. The author's expectation is that imposing the requirements of the form of equations forming the switched system will allow to obtain stronger results.

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