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On the integration of weakly geometric rough paths

Thomas Cass*, Bruce K. Driver†, Nengli Lim‡ and Christian Litterer§

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Abstract

We close a gap in the theory of integration for weakly geometric rough paths in the infinite-dimensional setting. We show that the integral of a weakly geometric rough path against a sufficiently regular one form is, once again, a weakly geometric rough path.

1 Introduction

The theory of rough paths was introduced by T. Lyons in [11], [12], [13] to study the evolution of highly oscillatory, non-differentiable systems modelled as differential equations driven by Banach-space-valued rough signals. Rough paths have found numerous successful applications in stochastic analysis, among them the study of the properties of stochastic differential equations driven by Gaussian processes, see e.g. [1], [3], [2] and the analysis of broad classes of stochastic differential equations, see e.g. [9], [8].

An important feature of rough path theory as developed by Lyons [13] is that it immediately applies in infinite dimensional settings. Two crucially important subsets of rough paths are the geometric and weakly geometric p- rough paths. These classes encompass many important examples, such as the (Stratonovich)

*Department of Mathematics, Imperial College London, 180 Queen's Gate, London, SW7 2AZ, United Kingdom (thomas.cass@imperial.ac.uk). The work of Thomas Cass was supported by EPSRC grant EP/M00516X/1.

†Department of Mathematics, University of California, San Diego, La Jolla, California 92093 USA (bdriver@math.ucsd.edu / 858-458-0544). The research of Bruce K. Driver was supported in part by NSF Grant DMS-0739164. The second author also gratefully acknowledges the generosity and hospitality of the Imperial College Mathematics Department, where part of this research was carried out, and was supported by EPSRC grant EP/M00516X/1 and a Nelder Visiting Fellowship.

‡Department of Mathematics, Imperial College London, 180 Queen's Gate, London, SW7 2AZ, United Kingdom (n.lim13@imperial.ac.uk)

§Centre de Mathématiques Appliquées, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France (christian.litterer@polytechnique.edu). Christian Litterer has received support from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 321111.

lifts of a wide range of stochastic processes, including Brownian motion or more generally semi-martingales. They also allow us immediately to define a theory of integration and differential equations for all $p \geq 1$ in a way that satisfies the usual rules of calculus. Geometric rough paths on the one hand are obtained by taking the closure of the smooth paths in a suitable rough path topology. Weakly geometric paths on the other hand are more algebraic in flavour and are characterised by taking values in the free nilpotent group of step $\lfloor p \rfloor$. In finite dimensions, it was demonstrated by Friz, Victoir [4] that the difference between geometric and weakly geometric rough paths is insignificant: every weakly geometric p -rough path is a geometric p' rough path for every $p' > p$. Their argument relies on the fact that, in finite dimensions, the sub-Riemannian topology induced by the Carnot-Carathéodory metric is equivalent to the sub-manifold topology that the free nilpotent group inherits from the tensor algebra. The extent to which this is true in general in infinite dimensions is at present unclear, though there have been recent attempts to show this, as well as to generalise Chow's connectivity theorem, a foundational result in the theory of sub-Riemannian geodesics, e.g. to Hilbert manifolds, c.f. [10].

The literature on rough paths is by now well established and the reader has a choice among a number of introductions to the subject, see e.g. [16], [5], [7]. However, somewhat surprisingly, there still appear to be rather basic gaps in the integration theory of weakly geometric rough paths in infinite dimensions. More specifically, we were unable to find a result, for example, that would demonstrate that the integral of a weakly geometric rough path remains in the free nilpotent group, i.e. that the rough integral is, as one might expect, a weakly geometric rough path in its own right (cf. also Remark 4.6). One way to resolve this issue would be to attempt to generalise the Friz-Victoir results relating geometric and weakly geometric rough paths to infinite dimensions. In this note we take a more direct approach: We provide a self-contained proof that allows us to close the gap, in keeping with the spirit of Lyons' original approach [13], [16] and without unduly increasing the technical complexity of his arguments. On the way, we also demonstrate that the rough path obtained by applying Lyons' fundamental extension theorem to a weakly geometric rough path remains in the group.

The paper is organised as follows. In Section 2 we carefully introduce the basic algebraic setup required to develop the theory of rough paths in infinite dimensions and recall the notion of Lip-γ functions in the sense of Stein. Section 3 introduces the concept of almost rough paths due to Lyons [13]. The Lyons construction of the rough integral proceeds in two steps. First, one defines an almost-rough path candidate motivated by calculations in the smooth category; the rough integral is then defined to be the unique rough path associated with the almost rough path. In Theorem 3.8 we prove that if an almost rough path is at least approximately taking values in the free nilpotent group, the uniquely associated rough path is also group-valued. As a corollary we see that Lyons' extension theorem takes weakly geometric to weakly geometric rough paths.

Section 4 is devoted to the proof of our main result, Theorem 4.5, which shows that, even in infinite dimensions, integration against weakly geometric rough paths exists and the result is again a weakly geometric rough path. We show in Proposition 4.8 that the almost rough path used to construct the rough integral for geometric rough paths in [16] still takes values in the free nilpotent group if we broaden the class of integrators to the weakly geometric rough paths. Finally, we apply Lemma 4.11 to prove that the candidate for almost rough paths indeed has the almost multiplicative property.

Shortly before submitting our paper we noticed a revised version of an independent work by Lyons and Yang [15]. Their work includes an important generalisation of the rough integral, but does not appear to address the question concerning our paper (the rough path integral in [15, Corollary 40] is still asserted to take values in the tensor algebra).

2 Preliminaries

Let $(E, ||\cdot||)$ be a real Banach space. For every $n \in \mathbb{N}$ we let

$$E^{\otimes_{\text{alg}} n} := \text{span}\{v_1 \otimes \cdots \otimes v_n : v_{1,\dots,}v_n \in E\}$$

denote the algebraic tensor product of E with itself n-times. In this paper we will drop the tensor product symbol and simply write $v_1 \dots v_n$ for $v_1 \otimes \cdots \otimes v_n$. If σ is a permutation of the set $\{1, .., n\}$ we let σ also denote the linear action on $E^{\otimes_{\text{alg}} n}$ which is uniquely determined by

$$\sigma : v_1 v_2 \dots v_n \mapsto v_{\sigma(1)} v_{\sigma(2)} \dots v_{\sigma(n)} \text{ for all } v_i \in E. \tag{1}$$

Let us now further assume that each space, $E^{\otimes_{\text{alg}} n}$, comes equipped with a norm, $\|\cdot\|_n$, which is invariant under permutations and satisfies;

$$||vw||_{n+m} \leq ||v||_n \, ||w||_m \qquad \text{for all } v \in E^{\otimes_{\text{alg}} n} \text{ and } w \in E^{\otimes_{\text{alg}} m}.$$

We let $E^{\otimes n}$ denote the completion of $E^{\otimes_{\text{alg}} n}$ in this given tensor norm, $||\cdot||_n$. In the sequel we will simply denote $\|\cdot\|_n$ by $\|\cdot\|$. It should be noted that under these assumptions the action σ on $E^{\otimes_{\text{alg}} n}$ determined by Eq. (1) extends by continuity uniquely to an action of $E^{\otimes n}$ which is still denote by σ.

Definition 2.1 (Truncated Tensor Algebras) *For $n \in \mathbb{N}$, let $T^{(n)}(E) := \oplus_{k=0}^{n} E^{\otimes k}$ with the convention that $E^{\otimes 0} := \mathbb{R}$. The vector space $T^{(n)}(E)$ is an algebra when equipped with the multiplication rule,*

$$gh = \sum_{k=0}^{n} \left(\sum_{l=0}^{k} g_l h_{k-l} \right) \text{ for all } g, h \in T^{(n)}(E), \tag{2}$$

where $g = \sum_{k=0}^{n} g_k$ and $h = \sum_{k=0}^{n} h_k$ with $g_k, h_k \in E^{\otimes k}$ for $0 \leq k \leq n$. We further define a norm on $T^{(n)}(E)$ by

$$\|g\| := \sum_{k=0}^{n} \|g_k\|.$$

Lemma 2.2 *$T^{(n)}(E)$ is a Banach algebra.*

Proof. It is clear that $\left(T^{(n)}(E), \|\cdot\|\right)$ is complete and hence a Banach space. It is a Banach algebra because the sub-multiplicative property of the tensor norms implies for $g, h \in T^{(n)}(E)$ that

$$\|gh\| = \sum_{k=0}^{n} \left\| \sum_{l=0}^{k} g_l h_{k-l} \right\| \leq \sum_{k=0}^{n} \sum_{l=0}^{k} \|g_l h_{k-l}\| \leq \sum_{k=0}^{n} \sum_{l=0}^{k} \|g_l\| \|h_{k-l}\| \leq \|g\| \|h\| . \tag{3}$$

■

Remark 2.3 *As with all associative algebras we may view $T^{(n)}(E)$ as Lie algebra with Lie bracket given by the **commutator;***

$$[A, B] = AB - BA, \text{ for } A, B \in T^{(n)}(E) . \tag{4}$$

*We will also let, for $A \in T^{(n)}(E)$, $ad_A : T^{(n)}(E) \rightarrow T^{(n)}(E)$ denotes the **adjoint operator** $ad_A B := [A, B]$.*

The following two closed subsets $T^{(n)}(E)$ will play an important role in what follows

$$T_0^{(n)} := \left\{ A \in T^{(n)}(E) : \pi_0(A) = 0 \right\} \text{ and}$$
$$T_1^{(n)} := \left\{ g \in T^{(n)}(E) : \pi_0(g) = 1 \right\} = 1 + T_0^{(n)},$$

where $\pi_0 : T^{(n)}(E) \rightarrow \mathbb{R}$ is the natural projection onto $E^{\otimes 0} = \mathbb{R}$. The set $T_0^{(n)}$ is a closed subalgebra of $T^{(n)}(E)$ and hence is also a Banach algebra while $T_1^{(n)}$ is a closed affine subspace of $T^{(n)}(E)$. More importantly $T_0^{(n)}$ is a Lie subalgebra of $T^{(n)}(E)$ and $T_1^{(n)}$ is a group under the algebra multiplication in $T^{(n)}(E)$ [For example, if $A \in T_0^{(n)}$ then $(1 + A)^{-1} = \sum_{k=0}^{n} (-1)^k A^k$.]

Theorem 2.4 *The map $\exp^{(n)} : T_0^{(n)} \rightarrow T_1^{(n)}$ is a diffeomorphism of affine spaces and the inverse is given by $\log^{(n)}$ where*

$$\exp^{(n)}(A) = \sum_{k=0}^{n} \frac{1}{k!} A^k \text{ and } \log^{(n)}(g) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} (g-1)^k .$$

Proof. The smoothness of $\exp^{(n)}$ and $\log^{(n)}$ follows from the smoothness of the polynomial functions in the Banach algebra. The fact that $\exp^{(n)}$ and $\log^{(n)}$ are mutually inverse is standard, see for example (3.13) of [17]. ■

Remark 2.5 *When no confusion is likely to arise we will simply write* $\exp$ *and* $\log$ *for* $\exp^{(n)}$ *and* $\log^{(n)}$ *respectively.*

Corollary 2.6 *If $\mathfrak{g} \subset T_0^{(n)}$ is a Lie sub-algebra of $T_0^{(n)}$, then $\bar{\mathfrak{g}} \subset T_0^{(n)}$ is a Lie algebra and $G_{\mathfrak{g}} := \exp(\bar{\mathfrak{g}}) = \overline{\exp(\mathfrak{g})}$ is a closed sub-group of $T_1^{(n)}$.*

Proof. Because the bracket operation in Eq. (4) is continuous one easily verifies that $\bar{\mathfrak{g}}$ is still a (necessarily nilpotent and closed) Lie subalgebra. Since $\exp : T_0^{(n)} \to T_1^{(n)}$ is a homeomorphism (Theorem 2.4), it follows that $\exp(\bar{\mathfrak{g}}) = \overline{\exp(\mathfrak{g})}$ which is a closed subset of $T_1^{(n)}$. To see that $G_{\mathfrak{g}}$ is a group we need to know that $C := \log\left(e^A e^B\right) \in \bar{\mathfrak{g}}$ whenever $A, B \in \bar{\mathfrak{g}}$. This however is a consequence of the Campbell-Baker-Hausdorff-Dynkin formula (see for example [18]).[1] ■

For our purposes we will most interested in the following Lie sub-algebra $T_0^{(n)}$.

Definition 2.7 *For* $n \in \mathbb{N}$, *let* $\mathrm{Lie}_n(E)$ *denote the Lie subalgebra generated by* E *inside of* $T_0^{(n)}$.

It is well known and easy to check, using $ad_{[A,B]} = ad_A ad_B - ad_B ad_A$ for all $A, B \in T_0^{(n)}$, that $\mathrm{Lie}_n(E)$ may be described explicitly as

$$\mathrm{Lie}_n(E) = \oplus_{k=1}^{n} \mathrm{span}\left\{ad_{A_1} \dots ad_{A_{k-1}} A_k : A_1, ..., A_n \in E\right\}. \tag{5}$$

Remark 2.8 *For* $n \leq m$ *we may identify* $T^{(n)}(E)$ *as a subspace of* $T^{(m)}(E)$ *and therefore from (5) we may regard* $\mathrm{Lie}_n(E)$ *as a subspace of* $\mathrm{Lie}_m(E)$. *However, it should be noted that if* $n < m$, $\mathrm{Lie}_n(E)$ *is* ***not*** *a Lie subalgebra of* $\mathrm{Lie}_m(E)$.

Definition 2.9 *For* $n \in \mathbb{N}$ *let* $G_{alg}^{(n)}(E) := \exp(\mathrm{Lie}_n(E))$ *and*

$$G^{(n)}(E) := \overline{G_{alg}^{(n)}(E)} = \exp\left(\overline{\mathrm{Lie}_n(E)}\right) \subset T_1^{(n)}$$

be the closure of $G_{alg}^{(n)}(E)$ *in* $T^{(n)}(E)$. *[As* $\mathrm{Lie}_n(E)$ *is a Lie sub-algebra of* $T_0^{(n)}$, *Corollary 2.6 guarantees* $G^{(n)}(E)$ *is a closed sub-group of* $T_1^{(n)}$.*]*

Remark 2.10 *When* E *is finite dimensional* $G^{(n)}(E) = \exp(\mathrm{Lie}_n(E))$. *Moreover, in this case, it is well known (see e.g. Proposition 2.27 of [16]) that as a set* $G^{(n)}(E)$ *conincides with the range of the step−*n *signature map,* S_n, *defined on the set of continuous functions* $x : [0,1] \to E$ *of bounded variation by*

$$S_n(x) = 1 + \sum_{k=1}^{n} \int_{0<t_1<...<t_k<1} dx_{t_1} \otimes ... \otimes dx_{t_k}.$$

Under this characterisation if $g = S_n(x)$ *and* $h = S_n(y)$ *are elements of* $G^{(n)}(E)$ *then*

$$gh = S_n(x * y),$$

where $x * y$ *denotes the concatentation of the paths* x *and* y, *reparametrised to give a path over* $[0,1]$. *The inverse operation is given by* $g^{-1} = S_n(\overleftarrow{x})$, *where* $\overleftarrow{x}_t := x_{1-t}$ *denotes* x *run backwards in time.*

[1] The results in [18] are valid in our infinite dimensional setting. However this infinite dimensional analysis may be avoided because inside the nilpotent Lie algebra, T_0, the Lie algebra generated by $A, B \in \bar{\mathfrak{g}}$ is finite dimensional.

Finally, we recall the notion of Lip-γ functions in the sense of Stein, see e.g. [16, Definition 1.21].

Definition 2.11 *Let $k \geq 1$ an integer and $\gamma \in (k, k+1]$. Let E, W be Banach spaces, $F \subset E$, and $\alpha : F \to W$ a function. Furthermore for $j = 1, \ldots k$ let*

$$\alpha^j : F \to L\left(E^{\otimes j}, W\right)$$

be symmetric $k-$ linear mappings from E to W . We say that the collection $\left(\alpha^0 = \alpha, \alpha^1, \ldots, \alpha^k\right)$ is in Lip(γ, F) if there exists a constant M such that for each $j = 0, \ldots, k$

$$\sup_{x \in F} \left|\alpha^j(x)\right| \leq M$$

and there exist functions $R_j : E \times E \to L\left(E^{\otimes j}, W\right)$ such that for all $x, y \in F$ and each $v \in E$

$$\alpha^j(y)(v) = \sum_{l=0}^{k-j} \frac{1}{l!} \alpha^{j+l}(x)\left(v \otimes (y-x)^{\otimes l}\right) + R_j(x, y)(v)$$

and

$$\left|R_j(x, y)\right| \leq M |x-y|^{\gamma - j}.$$

The smallest constant M for which these inequalities hold defines a norm on Lip(γ, F).

Remark 2.12 *Intuitively the notion of Lip-γ captures that a function is well approximated by polynomials of degree up to k. In the case where F is an open set, the α^j are uniquely identified as the j^{th} derivative α (which have to exist in that case). We usually say $\alpha \in$ Lip(γ, F), without referring to $\left(\alpha^0 = \alpha, \alpha^1, \ldots, \alpha^k\right)$.*

3 Almost rough paths and Lyons' extension theorem

Define the simplex

$$\Delta := \left\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\right\}.$$

We recall that for $p > 1$ a (ω-controlled) $p-$**rough path** is a multiplicative functional $\mathbf{X} := 1 + \sum_{i=1}^{\lfloor p \rfloor} \mathbf{X}^i : \Delta \to T^{(\lfloor p \rfloor)}(E)$ for which there exists a control[2] ω, such that:

$$\left\|\mathbf{X}_{s,t}^i\right\| \leq \omega(s, t)^{i/p}, \forall i = 1, ..., \lfloor p \rfloor \text{ and } \forall (s, t) \in \Delta. \quad (6)$$

[2] A control $\omega : \Delta \to [0, \infty)$ is a continuous, superadditive function which vanishes on the diagonal.

[The multiplicative property of $\mathbf{X}$ means that $\mathbf{X}_{st}\mathbf{X}_{tu} = \mathbf{X}_{su}$ for all $0 \leq s \leq t \leq u < \infty$.] If such an $\mathbf{X}$ in addition satisfies $\mathbf{X}_{s,t} \in G^{\lfloor p \rfloor}(E) \ \forall \ (s,t) \in \Delta$ we say that $\mathbf{X}$ is **weakly geometric**. We denote the set of p-rough paths on E by $\Omega_p(E)$ and write $W\Omega_p(E)$ for the weakly geometric rough paths.

In applications it is often easy to find an approximation to a rough path which has the necessary regularity but which fails to be multiplicative. In these circumstances one may nonetheless be able to identify a rough path which is close to this approximation. To do this in a unique way we must demand – in a quantitative way – that the approximation was not all that far from being multiplicative. This motivates the following definition which is again due to Lyons [13].

Definition 3.1 *Let $\theta > 1$ and $p > 1$. A θ –**almost** p**-rough path** is a function $\mathbf{X} := 1 + \sum_{i=1}^{\lfloor p \rfloor} \mathbf{X}^i : \Delta \to T^{(\lfloor p \rfloor)}(E)$ for which there exists a control ω such that*

1. $\left\| \mathbf{X}_{s,t}^i \right\| \leq \omega(s,t)^{i/p}$, $\forall i = 1, ..., \lfloor p \rfloor$, $\forall \ (s,t) \in \Delta$, *and*
2. $\left\| \mathbf{X}_{s,t}^i - [\mathbf{X}_{s,u}\mathbf{X}_{u,t}]^i \right\| \leq \omega(s,t)^{\theta}$, $\forall i = 1, ..., \lfloor p \rfloor$, $\forall 0 \leq s \leq u \leq t \leq T$.

*If we wish to emphasise ω as well as θ and p we will call such an $\mathbf{X}$ an (ω, p, θ) –**almost rough path**. If there exists some $\theta > 1$ for which $\mathbf{X}$ is a θ –almost p rough path then we say $\mathbf{X}$ is an **almost** p **rough path***

We will frequently make use of the following notation.

Notation 3.2 *Given a function, $\mathbf{X} : \Delta \to T^{(\lfloor p \rfloor)}(E)$ and a partition,*

$$\Pi = \{s = t_0 < t_1 < \cdots < t_r = t\},$$

of $[s,t]$, let

$$\mathbf{X}(\Pi) := \prod_{\tau \in \Pi} \mathbf{X}_{\tau_-,\tau} := \mathbf{X}_{t_0,t_1}\mathbf{X}_{t_1,t_2} \ldots \mathbf{X}_{t_{r-1},t_r}. \tag{7}$$

Furthermore, given a partition, Π, of $[0,T]$ and $(s,t) \in \Delta$ let $\Pi_{[s,t]}$ denote the partition of $[s,t]$ given by

$$\Pi_{[s,t]} = \{s,t\} \cup (\Pi \cap [s,t])$$

and then define

$$\mathbf{X}(\Pi)_{st} := \mathbf{X}(\Pi_{[s,t]}) = \prod_{\tau \in \Pi_{[s,t]}} \mathbf{X}_{\tau_-,\tau}. \tag{8}$$

A key result in Lyons' development of rough integration theory is the one which, as mentioned above, identifies a unique rough path which is close to an almost rough path.

Theorem 3.3 (Lyons [13, Theorem 3.3.1]) *If* $\mathbf{X} : \Delta \to T^{(\lfloor p \rfloor)}(E)$ *is a an* (ω, p, θ) *– almost rough path then there exists a unique* $\tilde{\mathbf{X}} \in \Omega_p(E)$ *such that*

$$\max_{i=1,\ldots,\lfloor p \rfloor} \sup_{(s,t)\in\Delta} \frac{\left\| \mathbf{X}_{s,t}^i - \tilde{\mathbf{X}}_{s,t}^i \right\|}{\omega(s,t)^\theta} < \infty, \tag{9}$$

where by convention, in such formulas, we define $0/0$ *to be* 0.

Proof. The full proof of this result may be found in [13, Theorem 3.3.1], [14, Theorem 3.2.1] or [16, Theorem 4.3]. We recall and outline the construction of $\tilde{\mathbf{X}}$ given in these references.

First one shows that the following limit exists,

$$1 + \tilde{\mathbf{X}}_{s,t}^1 := \lim_{|\Pi|\to 0} \left(1 + \mathbf{X}^1\right)\left(\Pi_{[s,t]}\right),$$

$\left\{1 + \tilde{\mathbf{X}}_{s,t}^1\right\}_{(s,t)\in\Delta_T}$ is multiplicative, and that Eq. (9) holds for $i = 1$. The proof is then finished by induction. In more detail, one assumes that for some $1 \le j < \lfloor p \rfloor$ we have constructed $\left\{\tilde{\mathbf{X}}_{s,t}^i\right\}_{i=1}^j$ so that

$$\tilde{\mathbf{X}}^{\le j} := \left\{\tilde{\mathbf{X}}_{s,t}^{\le j} = 1 + \tilde{\mathbf{X}}_{s,t}^1 + \cdots + \tilde{\mathbf{X}}_{s,t}^j\right\}_{(s,t)\in\Delta_T} \tag{10}$$

is multiplicative and Eq. (9) holds for $1 \le i \le j$. It is then shown that $\tilde{\mathbf{X}}_{s,t}^{j+1}$ may be constructed by the following limit,

$$\tilde{\mathbf{X}}_{s,t}^{\le j} + \tilde{\mathbf{X}}_{s,t}^{j+1} = \lim_{|\Pi|\to 0} \left[\left(\tilde{\mathbf{X}}_{s,t}^{\le j} + \mathbf{X}_{s,t}^{j+1}\right)\left(\Pi_{[s,t]}\right)\right]^{\le j+1}.$$

■

For our purposes, we would like to strengthen this result to demonstrate that if the almost rough path takes values in the group $G^{(\lfloor p \rfloor)}$ the uniquely associated rough path remains in the group as well. To this end, it will be useful to see that the inductive construction of $\tilde{\mathbf{X}}$ in Theorem 3.3 may be replaced by a limiting construction which constructs all components of $\tilde{\mathbf{X}}$ in one step. This will be the content of Proposition 3.5 below. In preparation we require the following simple lemma which is easily proved by induction.

Lemma 3.4 *Suppose* $a_1, \ldots, a_r$ *and* $b_1, \ldots, b_r$ *are elements of an associative algebra,* $\mathcal{A}$, *then*

$$a_1 \ldots a_r - b_1 \ldots b_r = \sum_{i=1}^r b_1 \ldots b_{i-1}(a_i - b_i)\, a_{i+1} \ldots a_r \tag{11}$$

where $b_1 \ldots b_{i-1} := 1$ *when* $i = 1$ *and* $a_{i+1} \ldots a_r = 1$ *when* $i = r$.

Proposition 3.5 *Suppose* $\mathbf{X} : \Delta \to T^{(\lfloor p \rfloor)}(E)$ *is a* θ *– almost* p *– rough path and* $\tilde{\mathbf{X}} \in \Omega_p(E)$ *is the unique* p *-rough path (as in Theorem 3.3) such that*

$$\max_{i=1,\ldots,\lfloor p \rfloor} \sup_{(s,t)\in\Delta} \frac{\left\| \mathbf{X}^i_{s,t} - \tilde{\mathbf{X}}^i_{s,t} \right\|}{\omega(s,t)^\theta} < \infty$$

Then $\tilde{\mathbf{X}}_{st}$ *may be computed as* $\tilde{\mathbf{X}}_{st} = \lim_{|\Pi|\to 0} \mathbf{X}(\Pi)_{st}$.

Proof. From the proof of Theorem 3.3 we know that $\tilde{\mathbf{X}}^{\leq 1}_{st} = \lim_{|\Pi|\to 0} \mathbf{X}(\Pi)^{\leq 1}_{st}$. We now show by induction on j that

$$\tilde{\mathbf{X}}^{\leq j}_{st} = \lim_{|\Pi|\to 0} \mathbf{X}(\Pi)^{\leq j}_{st} \text{ for } 1 \leq j \leq \lfloor p \rfloor,$$

where $\tilde{\mathbf{X}}^{\leq j}$ and $\mathbf{X}^{\leq j}$ are defined as in (10). To this end, suppose that for some $1 \leq i < \lfloor p \rfloor$ we have already shown,

$$\tilde{\mathbf{X}}^{\leq i}_{st} = \lim_{|\Pi|\to 0} \mathbf{X}(\Pi)^{\leq i}_{st}. \tag{12}$$

If we set $\mathbf{Z}_{st} = \tilde{\mathbf{X}}^{\leq i}_{st} + \mathbf{X}^{i+1}_{st}$, the construction used in Theorem 3.3 then shows

$$\tilde{\mathbf{X}}^{\leq i+1}_{st} = \lim_{|\Pi|\to 0} \mathbf{Z}(\Pi)^{\leq i+1}_{st}. \tag{13}$$

Letting $\Pi = \{s = t_0 < t_1 < \cdots < t_r = t\}$ be a partition of $[s,t]$, we then have by Lemma 3.4 that

$$\begin{aligned}
&\mathbf{Z}(\Pi)^{\leq i+1} - \mathbf{X}(\Pi)^{\leq i+1} \\
&= \left[\mathbf{Z}_{t_0,t_1}\mathbf{Z}_{t_1,t_2}\ldots\mathbf{Z}_{t_{r-1},t_r} - \mathbf{X}_{t_0,t_1}\mathbf{X}_{t_1,t_2}\ldots\mathbf{X}_{t_{r-1},t_r}\right]^{\leq i+1} \\
&= \sum_{j=1}^{r} \left[\mathbf{Z}_{t_0t_1}\ldots\mathbf{Z}_{t_{j-2}t_{j-1}}\left(\mathbf{X}_{t_{j-1}t_j} - \mathbf{Z}_{t_{j-1}t_j}\right)\mathbf{X}_{t_jt_{j+1}}\ldots\mathbf{X}_{t_{r-1}t_r}\right]^{\leq i+1} \\
&= \sum_{j=1}^{r} \left[\mathbf{Z}_{t_0t_1}\ldots\mathbf{Z}_{t_{j-2}t_{j-1}}\left(\mathbf{X}^{\leq i}_{t_{j-1}t_j} - \mathbf{Z}^{\leq i}_{t_{j-1}t_j}\right)\mathbf{X}_{t_jt_{j+1}}\ldots\mathbf{X}_{t_{r-1}t_r}\right]^{\leq i+1}
\end{aligned}$$

wherein we work relative to the truncated tensor algebra, $T^{(i+1)}(E)$. Therefore it follows that

$$\begin{aligned}
&\left\|\mathbf{Z}\left(\Pi\right)^{\leq i+1}-\mathbf{X}\left(\Pi\right)^{\leq i+1}\right\| \\
&\leq \sum_{j=1}^{r}\left\|\left[\mathbf{Z}_{t_0t_1}\dots\mathbf{Z}_{t_{j-2}t_{j-1}}\left(\mathbf{X}^{\leq i}_{t_{j-1}t_j}-\mathbf{Z}^{\leq i}_{t_{j-1}t_j}\right)\mathbf{X}_{t_jt_{j+1}}\dots\mathbf{X}_{t_{r-1}t_r}\right]^{\leq i+1}\right\| \\
&= \sum_{j=1}^{r}\left\|\left[\mathbf{Z}_{t_0t_{j-1}}\left(\mathbf{X}^{\leq i}_{t_{j-1}t_j}-\mathbf{Z}^{\leq i}_{t_{j-1}t_j}\right)\mathbf{X}_{t_jt_r}\right]^{\leq i+1}\right\| \\
&\leq \sum_{j=1}^{r}\left\|\mathbf{Z}_{t_0t_{j-1}}\right\|\left\|\mathbf{X}^{\leq i}_{t_{j-1}t_j}-\mathbf{Z}^{\leq i}_{t_{j-1}t_j}\right\|\left\|\mathbf{X}^{\leq i+1}_{t_jt_r}\right\| \\
&\leq C\sum_{j=1}^{r}\omega\left(t_{j-1},t_j\right)^{\theta} \qquad (14)
\end{aligned}$$

wherein we have used $\mathbf{Z}$ and $\mathbf{X}$ are bounded and Theorem 3.3 to conclude,

$$\left\|\mathbf{X}^{\leq i}_{t_{j-1}t_j}-\mathbf{Z}^{\leq i}_{t_{j-1}t_j}\right\|=\left\|\mathbf{X}^{\leq i}_{t_{j-1}t_j}-\tilde{\mathbf{X}}^{\leq i}_{t_{j-1}t_j}\right\|\leq C\omega\left(t_{j-1}t_j\right)^{\theta}$$

in the last inequality. Since,

$$\begin{aligned}
\sum_{j=1}^{r}\omega\left(t_{j-1},t_j\right)^{\theta} &\leq \max_j \omega\left(t_{j-1},t_j\right)^{\theta-1}\sum_{j=1}^{r}\omega\left(t_{j-1},t_j\right) \\
&\leq \max_j \omega\left(t_{j-1},t_j\right)^{\theta-1}\omega\left(s,t\right)\to 0 \text{ as } |\Pi|\to 0
\end{aligned}$$

we may now conclude from Eq. (14) that

$$\lim_{|\Pi|\to 0}\left\|\mathbf{Z}\left(\Pi\right)^{\leq i+1}-\mathbf{X}\left(\Pi\right)^{\leq i+1}\right\|=0$$

which combined with Eq. (13) shows

$$\tilde{\mathbf{X}}^{\leq i+1}_{st}=\lim_{|\Pi|\to 0}\mathbf{Z}\left(\Pi\right)^{\leq i+1}_{st}=\lim_{|\Pi|\to 0}\mathbf{X}\left(\Pi\right)^{\leq i+1}_{st}.$$

This completes the induction step and hence the proof of the theorem. ■

As the group $G^{(\lfloor p\rfloor)}(E)$ is closed the following corollary is an immediate consequence of the preceding proposition.

Corollary 3.6 *With the notation and assumptions of Proposition 3.5 suppose that* $\mathbf{X}$ *takes values in* $G^{(\lfloor p\rfloor)}(E)$*. Then we have* $\tilde{\mathbf{X}}\in W\Omega_p(E)$*.*

In fact, it turns out that if the almost $p-$rough path we start with happens to be only approximately in $G^{(\lfloor p\rfloor)}(E)$, the associated rough path is still in $W\Omega_p(E)$. To make this precise we first need to define what we mean by a path being approximately in the group. In the following let $n:=\lfloor p\rfloor$.

Definition 3.7 *Given a control* ω*, we say a path* $\mathbf{X} := 1 + \sum_{i=1}^{n} \mathbf{X}^i : \Delta \rightarrow T^{(n)}(E)$ *is* $\theta-$ ***approximately in*** $G^{(n)}(E)$ *if there exists* $\mathbf{Y} : \Delta \rightarrow \overline{\mathrm{Lie}_n(E)}$ *such that*

$$\max_{i=1,\ldots,n} \sup_{(s,t)\in\Delta} \frac{\left\| [\log(\mathbf{X}_{s,t}) - \mathbf{Y}_{s,t}]^i \right\|}{\omega(s,t)^{\theta}} < \infty. \tag{15}$$

Theorem 3.8 *Let* $p > 1$ *be a real number, let* $n := \lfloor p \rfloor$ *and suppose* $\mathbf{X}$ *is a* θ*-almost* p*-rough path that is* $\theta-$*approximately in* $G^{(n)}(E)$*. Then the unique* $p-$*rough path which is close to* $\mathbf{X}$ *is weakly geometric.*

Proof. Let $\mathbf{Y}$ be as in Definition 3.7 and $\hat{\mathbf{X}}_{s,t} := \exp(\mathbf{Y}_{s,t}) \in G^{(n)}$. The exponential function is locally Lipschitz with respect to the tensor norm on $T^{(n)}(E)$ and it therefore follows from (15) that

$$\max_{i=1,\ldots,n} \sup_{(s,t)\in\Delta} \frac{\left\| \mathbf{X}_{s,t}^i - \hat{\mathbf{X}}_{s,t}^i \right\|}{\omega(s,t)^{\theta}} < \infty \tag{16}$$

and up to a constant $K_\theta(\omega(0,T))$ that $\hat{\mathbf{X}}$ is controlled by ω. Also note that

$$\begin{aligned} \left(\mathbf{X}_{s,u}\hat{\mathbf{X}}_{u,t}\right)^i - \left(\hat{\mathbf{X}}_{s,u}\hat{\mathbf{X}}_{u,t}\right)^i &= \sum_{j=0}^{i} \mathbf{X}_{s,u}^j \hat{\mathbf{X}}_{u,t}^{i-j} - \sum_{j=0}^{i} \hat{\mathbf{X}}_{s,u}^j \hat{\mathbf{X}}_{u,t}^{i-j} \\ &= \sum_{j=0}^{i} \left(\mathbf{X}_{s,u}^j - \hat{\mathbf{X}}_{s,u}^j\right) \hat{\mathbf{X}}_{u,t}^{i-j} \end{aligned}$$

and therefore

$$\left\| \left(\mathbf{X}_{s,u}\hat{\mathbf{X}}_{u,t}\right)^i - \left(\hat{\mathbf{X}}_{s,u}\hat{\mathbf{X}}_{u,t}\right)^i \right\| \leq i K_\theta(\omega(0,T))\, \omega(s,u)^{\theta}\, \omega(0,T).$$

As a similar estimate also holds for the difference of $(\mathbf{X}_{s,u}\mathbf{X}_{u,t})^i$ and $\left(\mathbf{X}_{s,u}\hat{\mathbf{X}}_{u,t}\right)^i$ we may deduce using the triangle inequality that $\hat{\mathbf{X}}_{s,t}$ is also almost multiplicative and therefore is a θ -almost p -rough path. From Proposition 3.5 we see that the rough path $\tilde{\mathbf{X}}$ associated to $\hat{\mathbf{X}}$ is given by

$$\tilde{\mathbf{X}}_{s,t} = \lim_{|\Pi|\to 0} \prod_{\tau_j \in \Pi} \hat{\mathbf{X}}_{\tau_{j-1},\tau_j} \in G^{(n)}(E)$$

and the uniqueness part of the same theorem yields the claim. ■

Another fundamental result in rough path theory is *Lyons' extension theorem* (see *[13, Theorem 2.2.1]*). It tells us that any $p-$rough path $\mathbf{X} = 1 + \sum_{i=1}^{\lfloor p \rfloor} \mathbf{X}^i$ can, for any integer $m \geq p$, be uniquely (and, in fact, continuously) extended to a multiplicative functional $\mathbf{X}^{(m)} = 1 + \sum_{i=1}^{m} \mathbf{X}^i$ of finite p-variation. From Theorem 3.8 we now can deduce that if we apply this operation to a weakly geometric rough path then the extensions will be $G^{(m)}(E)$-valued.

Corollary 3.9 (Lyons' canonical extension theorem) *Let $p > 1$, $n := \lfloor p \rfloor$, and $m \in \mathbb{N}$ with $m > p$. If $\mathbf{X}$ is a weakly geometric $p-$rough path and $\mathbf{X}^{(m)}$ denotes the unique Lyons extension of $\mathbf{X}$ then*

$$\mathbf{X}_{s,t}^{(m)} \in G^{(m)}(E) \text{ for all } (s,t) \in \Delta.$$

Proof. Let $\theta := \frac{n+1}{p} > 1$. By assumption, $\mathbf{X}$ takes values in $G^{(n)}(E) \subset T^{(n)}(E) \subset T^{(m)}(E)$. Let us denote $\mathbf{X}$ by $\mathbf{Z}$ when we view $\mathbf{X}$ to be an element of $T^{(m)}(E)$, i.e.

$$\mathbf{Z}_{s,t}^{i} := \begin{cases} \mathbf{X}_{s,t}^{i} & \text{if } \quad 0 \leq i \leq n \\ 0 & \text{if } \quad n < i \leq m \end{cases}.$$

We do this because the multiplication in $T^{(n)}(E)$ is not consistent with the multiplication in $T^{(m)}(E)$. We can however easily verify, for $0 \leq s \leq u \leq t \leq T$, that

$$(\mathbf{Z}_{s,u}\mathbf{Z}_{u,t})^{i} = \begin{cases} \mathbf{X}_{st}^{i} = \mathbf{Z}_{s,t}^{i} & \text{if} \quad 0 \leq i \leq n \\ \sum_{r=i-n}^{n} \mathbf{X}_{s,u}^{r}\mathbf{X}_{u,t}^{i-r} & \text{if} \quad n < i \leq \min(2n,m) \\ 0 & \text{if} \quad \min(2n,m) < i \leq m \end{cases}.$$

Therefore, if ω is a control so that $\mathbf{X}$ is ω-controlled in the sense of (6), then for $n < i \leq m$,

$$\left\| [\mathbf{Z}_{s,u}\mathbf{Z}_{u,t} - \mathbf{Z}_{st}]^{i} \right\| \leq \sum_{r=i-n}^{n} \left\| \mathbf{X}_{s,u}^{r}\mathbf{X}_{u,t}^{i-r} \right\| \leq C\omega(s,t)^{i/p} \leq C'\omega(s,t)^{\theta}$$

while for all other i, $[\mathbf{Z}_{s,u}\mathbf{Z}_{u,t} - \mathbf{Z}_{st}]^{i} = 0$. This shows that i.e. $\mathbf{Z}$ is a $\theta-$ almost-multiplicative functional. With the aid of Eq. (17) below we may conclude $\mathbf{Z}$ is $\theta-$ approximately in $G^{(m)}(E)$. Using these facts, we may then apply Theorem 3.8 in order to show, for any q in $[m, m+1)$, that the unique $q-$rough path $\left(\mathbf{X}^{(m)}\right)$ associated to $\mathbf{Z}$ is in fact $G^{(m)}(E)$-valued.

Since $\mathbf{X}_{s,t} \in G^{(n)}(E)$, by Definition 2.9 there exists $\mathbf{Y}_{s,t} \in \overline{\mathrm{Lie}_{n}(E)}$ such that $\log^{(n)}(\mathbf{X}_{s,t}) = \mathbf{Y}_{s,t}$. Let $\tilde{\mathbf{Y}}_{s,t}$ denote $\mathbf{Y}_{st}$ thought of as an element of $T^{(m)}(E)$, i.e.

$$\tilde{\mathbf{Y}}_{s,t}^{i} = \begin{cases} \mathbf{Y}_{s,t}^{i} & \text{if } \quad 0 \leq i \leq n \\ 0 & \text{if } \quad n < i \leq m \end{cases}.$$

Using Remark 2.8 we have $\tilde{\mathbf{Y}}_{s,t} \in \overline{\mathrm{Lie}_{m}(E)}$ and so to show $\mathbf{Z}$ is $\theta-$ approximately in $G^{(m)}(E)$ it suffices to show

$$\left\| \log^{(m)}(\mathbf{Z}_{s,t}) - \tilde{\mathbf{Y}}_{s,t} \right\| \leq C\omega(s,t)^{\theta} \text{ for all } 0 \leq s \leq t \leq T. \tag{17}$$

However for $i \leq n$ we have,

$$\begin{aligned} \left[\log^{(m)}(\mathbf{Z}_{s,t})\right]^{(i)} &= \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} \left[(\mathbf{Z}_{s,t} - 1)^{k}\right]^{(i)} \\ &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \left[(\mathbf{X}_{s,t} - 1)^{k}\right]^{(i)} = \mathbf{Y}_{s,t}^{(i)} = \tilde{\mathbf{Y}}_{s,t}^{(i)}, \end{aligned}$$

wherein this equation, $[\dots]^{(i)}$ is temporarily being used to pick out the i^{th} – component of a tensor while $(\mathbf{Z}_{s,t}-1)^k$ indicates the k-fold product of $(\mathbf{Z}_{s,t}-1)$ with itself. For $i>n$, $\left[\log^{(m)}(\mathbf{Z}_{s,t})\right]^{(i)}$ is a finite linear combination of monomials of the form, $\mathbf{X}^{j_1}_{s,t}\dots\mathbf{X}^{j_k}_{s,t}$, with $j_1+\dots+j_k=i>n$ and each of these monomial satisfy the bound,

$$\left\|\mathbf{X}^{j_1}_{s,t}\dots\mathbf{X}^{j_k}_{s,t}\right\|\leq C'\omega(s,t)^{\frac{j_1+\dots+j_k}{p}}\leq C''\omega(s,t)^{\theta}.$$

These observations along with the triangle inequality,

$$\left\|\log^{(m)}(\mathbf{Z}_{s,t})-\tilde{\mathbf{Y}}_{s,t}\right\|\leq\sum_{n<i\leq m}\left\|\left[\log^{(m)}(\mathbf{Z}_{s,t})\right]^{i}\right\|,$$

verifies the truth of Eq. (17) for some $C<\infty$. ■

4 Rough integration

Let $\gamma>p>1$, $n=\lfloor p\rfloor$, W be another Banach space with norms on $\left\{W^{\otimes i}\right\}_{i=1}^{\infty}$ as describe for E, $\mathbf{X}\in W\Omega_p(E)$, and $\alpha:\{X_t:t\in[0,T]\}\to L(E,W)$ be a $Lip(\gamma-1)$ one-form on $\{X_t:t\in[0,T]\}\subset E$ with values in W. Recall from Definition 2.11 that α really comes with auxiliary components

$$\alpha^j:\{X_t:t\in[0,T]\}\to L\left(E^{\otimes j},L(E,W)\right)$$

with $j=1,\dots,\lfloor\gamma\rfloor-1$ which are required to model the structure of the j^{th} derivatives of a smooth function on E with values in W.

We will denote by X the E-valued path

$$X_t:=X_0+\mathbf{X}^1_{0,t}\text{ for }t\in[0,T].$$

The main goal of this section and of this paper is to construct the integral of α against $\mathbf{X}$ (denoted as $\int\alpha(d\mathbf{X})$) as an element in $W\Omega_p(W)$. Let us begin by recalling the discussion before Theorem 4.6 in [16] which requires the following notation.

Notation 4.1 (Ordered shuffles) *For $k_1,\dots,k_i\in\mathbb{N}$, let $K_0=0,K_l=k_1+\dots+k_l$ for $1\leq l\leq i$, and $OS(k_1,...,k_i)$ denote the set of ordered shuffles of $\{1,...,K_i\}$, i.e. those permutations of $\{1,...,K_i\}$ which preserve the ordering both within each of the "blocks," $\{K_{l-1}+1,\dots,K_l\}\subset\{1,...,K_i\}$ for $1\leq l\leq i$, and between the end-points of the different blocks. In other words, $\pi\in OS(k_1,...,k_i)$ iff π is a permutation of $\{1,...,K_i\}$ such that $\pi(j)<\pi(j+1)$ whenever $j,j+1\in\{K_{l-1}+1,\dots,K_l\}$ for some l, and such that $\pi(K_1)<\pi(K_2)<\dots<\pi(K_l)$. See [16, p.72] for more details.*

For the moment assume X_t is a smooth (or more generally finite variation path) in E, and that

$$\mathbf{X}_{st}^i = \int_{s<t_1<\cdots<t_i<t} dX_{t_1}\ldots dX_{t_i} \text{ for } 1 \le i \le n.$$

In order for the rough path theory to be consistent with smooth integration theory we should define (in this case)

$$\tilde{\mathbf{Y}}_{s,t}^1 := \int_s^t \alpha(X_\tau)\, dX_\tau,\ Z_t = \tilde{\mathbf{Y}}_{0,t}^1, \text{ and} \tag{18}$$

$$\tilde{\mathbf{Y}}_{s,t}^i = \int_{s<t_1<\cdots<t_i<t} dZ_{t_1}\ldots dZ_{t_i} \text{ for } 1 \le i \le n, \tag{19}$$

Using Taylor's approximation, it is argued in [16] that a good approximation to $\tilde{\mathbf{Y}}_{s,t}^1$ is given by

$$\tilde{\mathbf{Y}}_{s,t}^1 \cong \sum_{k=1}^n \alpha^{k-1}(X_s) \int_{s<t_1<\ldots<t_k<t} dX_{t_1}...dX_{t_k} = \sum_{k=1}^n \alpha^{k-1}(X_s)\, \mathbf{X}_{st}^k.$$

Approximating Z_u in Eq. (18) by $Y_u := \sum_{k=1}^n \alpha^{k-1}(X_s)\, \mathbf{X}_{su}^k$ leads, after some gymnastics with iterated integrals, to approximating $\tilde{\mathbf{Y}}_{s,t}^i$ by

$$\begin{aligned}
\mathbf{Y}_{s,t}^i &= \int_{s<t_1<\ldots<t_i<t} dY_{s,t_1}...dY_{s,t_i} \\
&= \sum_{k_1,..,k_i=1}^n \left(\bigotimes_{j=1}^i \alpha^{k_j-1}(X_s)\right)\left(\prod_{j=1}^i \int_{s<t_1^j<\ldots<t_{k_j}^j<t} d\mathbf{X}_{s,t_1^j}^{k_1}...d\mathbf{X}_{s,t_{k_j}^j}^{k_j}\right) \\
&= \sum_{k_1,..,k_i=1}^n \left(\bigotimes_{j=1}^i \beta^{k_j}\right) \sum_{\pi\in \mathrm{OS}(k_1,\ldots,k_i)} \pi^{-1} \int_{s<t_1<\ldots<t_{k_1+\ldots+k_i}.<t} dX_{t_1}...dX_{t_{k_1+\ldots+k_i}} \\
&= \sum_{k_1,..,k_i=1}^n \left(\bigotimes_{j=1}^i \beta^{k_j}\right) \sum_{\pi\in \mathrm{OS}(k_1,\ldots,k_i)} \pi^{-1}\mathbf{X}_{s,t}^{k_1+\ldots+k_i},
\end{aligned} \tag{20}$$

where $\beta^{k_j} := \alpha^{k_j-1}(X_s)$, the action of permutations on tensors is described in Eq. (1), and we use the ordering convention

$$\bigotimes_{j=1}^i \beta^{k_j} := \beta^{k_1} \otimes \beta^{k_2} \otimes \cdots \otimes \beta^{k_i}. \tag{21}$$

These computation motivate the following notation, again see [16, Theorem 4.6].

Notation 4.2 *Associated to a* $Lip(\gamma-1)$ *– function,* $\alpha : \{X_t : t\in[0,T]\} \to L(E,W)$ *and* $\mathbf{X} \in W\Omega_p(E)$, *let* $\mathbf{Y} = \mathbf{Y}(\alpha) := 1 + \sum_{i=1}^n \mathbf{Y}^i : \Delta \to T^{(n)}(W)$ *be defined by*

$$\mathbf{Y}_{s,t}^i = \sum_{k_1,\ldots,k_i=1}^n \bigotimes_{j=1}^i \alpha^{k_j-1}(X_s) \sum_{\pi\in OS(k_1,\ldots,k_i)} \pi^{-1}\mathbf{X}_{s,t}^{k_1+\ldots+k_i}. \tag{22}$$

[So in particular $\mathbf{Y}^1_{s,t} = \sum_{k=1}^{n} \alpha^{k-1}(X_s)\,\mathbf{X}^k_{s,t}$.*]*

Let us emphasize two points regarding $\mathbf{Y}$ defined above.

1. In Eq. (22) we use $\mathbf{X}^m$ for $m > n$ which is to be constructed using Lyons canonical extension of $\mathbf{X}$. Because of Corollary 3.9 we know that the extensions are still all weakly geometric.

2. The Lip-γ function $(\alpha^0, \ldots, \alpha^k)$ need only be defined on the trace of the rough path $\mathbf{X}$.

Theorem 4.3 ([16, Theorem 4.6]) *If* $\mathbf{X}$ *is assumed to be a geometric rough path (not just weakly geometric), then* $\mathbf{Y}$ *defined in Eq. (4.2) is a* (ω, p) *– almost rough path and therefore (by Theorem 3.3) there is a unique* (ω, p) *– rough path,* $\tilde{\mathbf{Y}}$*, close to* $\mathbf{Y}$.

Definition 4.4 *If* $\mathbf{X}$ *is a* ***geometric*** *rough path, the* (ω, p) *– rough path,* $\tilde{\mathbf{Y}}$*, associated to* $\mathbf{Y}$ *in Notation 4.2 is the integral of* α *against* $\mathbf{X}$ *and is denoted as* $\int \alpha(d\mathbf{X})$.

We may now precisely state the main result of this paper.

Theorem 4.5 *Let* $\gamma > p > 1$, E *and* W *be Banach spaces. Suppose* $\mathbf{X} \in W\Omega_p(E)$ *and* $\alpha : \{X_t : t \in [0,T]\} \to L(E, W)$ *is a Lip* $(\gamma - 1)$ *one-form, then* $\mathbf{Y} = \mathbf{Y}(\alpha)$ *described in Notation 4.2 is almost multiplicative and therefore (by Theorem 3.3) there is a unique* (ω, p) *– rough path* $\tilde{\mathbf{Y}}$ *close to* $\mathbf{Y}$*. Moreover,* $\tilde{\mathbf{Y}} \in W\Omega_p(W)$*, i.e.* $\int \alpha(d\mathbf{X})$ *exists and is again a weakly geometric rough path.*

Proof. The key points of the proof are to; (i) show $\mathbf{Y}$ in Eq. (22) takes values in $G^{(n)}(W)$ (see Proposition 4.8), and (ii) show $\mathbf{Y}$ is almost multiplicative. (Assertion (ii) is a consequence of Lemma 4.11 below and the comments preceding the lemma.) Assuming these results, Corollary 3.6 implies there is a unique element $\int \alpha(d\mathbf{X}) := \tilde{\mathbf{Y}} \in W\Omega_p(W)$ which is close to $\mathbf{Y}$, where $\tilde{\mathbf{Y}}$ is as described in Proposition 3.5. ■

Remark 4.6 *In other words, Theorem 4.5 states the property of being weakly geometric is stable under rough integration. Strictly speaking one could even argue that the integral of a weakly geometric path had not yet even been constructed in the general Banach space setting as the geometric assumption was previously used to show that the approximations to* $\int \alpha(\mathbf{dX})$ *are almost multiplicative.*

Recall that the smooth approximation argument stating that weakly geometric and geometric rough paths are essentially equivalent are unproven in infinite dimensions. Thus we must prove the two assertions used in the proof of Theorem 4.5 **without** using such smooth approximation arguments. The next lemma is elementary but nevertheless holds the key to sidestepping the use of smooth approximation arguments in infinite dimensions.

Lemma 4.7 *If $g \in G^{(N)}_{alg}(E)$, then there exists a finite dimensional subspace $E_0 \subset E$ such that $g \in G^{(N)}(E_0)$. [Notice that $G^{(N)}(E_0) = G^{(N)}_{alg}(E_0)$ is now a finite dimensional nilpotent Lie group.]*

Proof. From the definition of the algebraic group, $G^{(N)}_{\mathrm{alg}}(E)$, for $g \in G^{(N)}_{\mathrm{alg}}(E)$ there exists $y \in \mathrm{Lie}_N(E)$ such that $g = \exp^{(N)}(y)$. Since $y \in \mathrm{Lie}_N(E) \subset \oplus_{k=1}^{N} E^{\otimes_{\mathrm{alg}} k}$, y may be written as a finite linear combination of expressions of the form $v_1 \otimes \cdots \otimes v_l$ where $v_i \in E$. We then may take E_0 to be the linear span of all vectors $\{v_i\} \subset E$ which appear in the indecomposable tensors, $v_1 \otimes \cdots \otimes v_l$, used to describe y. ■

Proposition 4.8 *Let $p > 1$ and $n = \lfloor p \rfloor$. Assume that E and W are Banach spaces and $\beta^k \in L\left(E^{\otimes k}, W\right)$ for $k = 1, ..., n$ are continuous linear maps. Suppose $\mathbf{X}$ is a weakly geometric $p-$rough path. For fixed $0 \le s < t \le T$ define $\mathbf{Y}_{s,t} := 1 + \sum_{i=1}^{n} \mathbf{Y}^i_{s,t} \in T^{(n)}(W)$ by*

$$\mathbf{Y}^i_{s,t} = \sum_{k_1,\ldots,k_i=1}^{n} \bigotimes_{j=1}^{i} \beta^{k_j} \sum_{\pi \in OS(k_1,\ldots,k_i)} \pi^{-1} \mathbf{X}^{k_1+\ldots+k_i}_{s,t}, \tag{23}$$

wherein $\mathbf{X}^m$, for $m > n$, is obtained from the canonical Lyons extension of $\mathbf{X}$ (cf. Corollary 3.9). Then $\mathbf{Y}_{s,t}$ takes values in $G^{(n)}(W)$.

Proof. Fix $(s,t) \in \Delta$. We first note that (23) when $i = n$ involves terms $\mathbf{X}^m$ up to the maximum degree $m = n^2 =: N$. It follows from Corollary 3.9 that the Lyons extension $\mathbf{X}^{(N)}$ of the weakly geometric rough path $\mathbf{X}$ takes values in $G^{(N)}(E)$. Henceforth will denote this extension $\mathbf{X}^{(N)}$ simply by $\mathbf{X}$. By Definition 2.9 and Corollary 2.6 we have $G^{(N)}(E) = \overline{G^{(N)}_{\mathrm{alg}}(E)}$, hence there exists a sequence $(\mathbf{X}_{s,t}(r))_{r=1}^{\infty} \subset G^{(N)}_{\mathrm{alg}}(E)$ which converges to $\mathbf{X}_{s,t}$ as $r \to \infty$. From Lemma 4.7, there exists finite dimensional subspaces $\{E_r\}_{r=1}^{\infty}$ of E such that $\mathbf{X}_{s,t}(r) \in G^{(N)}(E_r)$ for each r. Thus by the Chow connectivity theorem, see e.g. [6, Theorem 0.4] there exist absolutely continuous paths $x(r) : [s,t] \to E_r$ such that

$$\mathbf{X}_{s,t}(r) = \sum_{k=0}^{N} \int_{s<t_1<\ldots<t_k<t} dx_{t_1}(r) \ldots dx_{t_k}(r) =: \sum_{k=0}^{N} \mathbf{X}_{s,t}(r)^k. \tag{24}$$

We now define $\mathbf{Y}_{s,t}(r) := 1 + \sum_{i=1}^{n} \mathbf{Y}_{s,t}(r)^i : \Delta \to T^{(n)}(W)$ by replacing $\mathbf{X}_{s,t}$ by $\mathbf{X}_{s,t}(r)$ in Eq. (23), i.e.

$$[\mathbf{Y}_{s,t}(r)]^i := \sum_{k_1,\ldots,k_i=1}^{n} \bigotimes_{j=1}^{i} \beta^{k_j} \sum_{\pi \in \mathrm{OS}(k_1,\ldots,k_i)} \pi^{-1} \mathbf{X}_{s,t}(r)^{k_1+\ldots+k_i}.$$

The continuity of $\beta^1, ..., \beta^n$ and the fact that $\mathbf{X}_{s,t}(r) \to \mathbf{X}_{s,t}$ in $T^{(N)}(E)$ as $r \to \infty$ gives immediately that $\mathbf{Y}_{s,t}(r) \to \mathbf{Y}_{s,t}$ in $T^{(n)}(W)$ as $r \to \infty$. We will prove that

$$\mathbf{Y}_{s,t}(r) \in G^{(n)}(W) \text{ for every } r, \tag{25}$$

which, since $G^{(N)}(W)$ is a closed group, will show that the limit $\mathbf{Y}_{s,t}$ must also belong to it.

To verify the assertion (25) we introduce, for each r, a finite dimensional subspace W_r^β of W given by the span of

$$\left(\bigotimes_{j=1}^{i} \beta^{k_j}\right) \pi^{-1} v$$

over all $i \in \{1, ..., n\}$, $k_1, ..., k_i \in \{1, ..., n\}$, $\pi \in$OS$(k_1, ..., k_i)$ and $v \in T_1^{(k_1+...+k_i)} E_r$. We then define the sequence of smooth paths $y_{s,\cdot}(r) : [s,t] \to W_r$ defined by

$$y_u(r) = \sum_{k=1}^{n} \beta^k \int_{s<t_1<...<t_k<u} dx_{t_1}(r) ... dx_{t_k}(r), \text{ for } r \in \mathbb{N}.$$

Evaluating at $u = t$ we observe that $y_{s,t}(r) = \mathbf{Y}_{s,t}(r)^1$. The same calculation as in Eq. (20) together with (24) shows that for all $i = 2, ...n$

$$[\mathbf{Y}_{s,t}(r)]^i = \int_{s<t_1<...<t_i<t} dy_{t_1}(r) ... dy_{t_i}(r), \tag{26}$$

which in turn implies

$$\mathbf{Y}_{s,t}(r) = \sum_{k=0}^{n} \int_{s<t_1<...<t_k<t} dy_{t_1}(r) ... dy_{t_k}(r) \in G^{(n)}(W_r) \subset G^{(n)}(W) \tag{27}$$

proving (25). ■

Corollary 4.9 *Let $\gamma > p$, $\alpha : E \to L(E, W)$ a Lip$(\gamma - 1)$ one-form and $\mathbf{Y} : \Delta \to T^{(n)}(W)$ be defined by (22) then $\mathbf{Y}_{s,t} \in G^{(n)}(W)$ for every $(s,t) \in \Delta$.*

Proof. Fix $(s,t) \in \Delta$ and apply the previous proposition with

$$\beta^1 := \alpha(X_s), \text{ and } \beta^k := \alpha^{k-1}(X_s) \text{ for } k = 1, ..., n.$$

■

Remark 4.10 *Note that the linear maps β^k in Proposition 4.8 do not need to satisfy the symmetry requirements imposed on the α^{k-1} which are modeled along the derivatives of a one form.*

To finish the proof of Theorem 4.5 it remains to verify that $\mathbf{Y}$ almost multiplicative. This is proved in [16] again under the hypothesis that $\mathbf{X}$ is *geometric* by showing the following identity

$$\mathbf{Y}_{s,u}\mathbf{Y}_{u,t} - \mathbf{Y}_{s,t} = \mathbf{Y}_{s,u}\mathbf{N}_{s,u,t}. \tag{28}$$

Here $\mathbf{N}_{s,u,t}=\sum_{i=1}^{n}\mathbf{N}_{s,u,t}^{i}$ is given for all $0\leq s\leq u\leq t\leq T$ by

$$\mathbf{N}_{s,u,t}^{i}=\sum_{\substack{k_1,...,k_i\in\{1,...,n\}\\ \epsilon_1,...,\epsilon_i\in\{0,1\},\\ \epsilon_l=0\text{ for some }l\in\{1,...,i\}}}\otimes_{j=1}^{i}F_{k_j-1}^{\epsilon_j}\left(X_s,X_u\right)\sum_{\pi\in\mathrm{OS}(k_1,...,k_i)}\pi^{-1}\mathbf{X}_{u,t}^{k_1+...+k_i}, \tag{29}$$

$F_l^0,F_l^1:E\times E\rightarrow L\left(E^{\otimes l},W\right)$ denote the functions

$$F_l^0\left(v_1,v_2\right)=R_l\left(v_1,v_2\right)\text{ and }F_l^1\left(v_1,v_2\right)=-\alpha^l\left(v_1\right)\ ,$$

and $R_0,R_1,...,R_{n-1}$ are the remainder functions introduced in Definition 2.11 in the Taylor-like expansions satisfied by $\alpha,\alpha^1,...,\alpha^{n-1}$. Note that together with the observation that $\left\|\mathbf{N}_{s,u,t}^{i}\right\|\leq C\omega\left(s,t\right)^{\frac{\gamma}{p}}$ (compare [16, p.77]) identity (28) immediately implies that $\mathbf{Y}$ is almost multiplicative.

Although the proof of this identity in [16, Lemma 4.8] makes use of smooth approximations, we will show that this is not necessary and the following lemma asserts the identity when $\mathbf{X}$ is only weakly geometric. The spirit is the same as the previous proof: take a sequence of algebraic approximations, this time for two fixed consecutive time intervals $[s,u]$ and $[u,t]$, and use Chow's theorem together with classical calculus for the iterated integrals of smooth paths.

Lemma 4.11 *Let $\gamma>p>1$. Suppose $\mathbf{X}$ is in $W\Omega_p\left(E\right)$ and $\alpha:E\rightarrow L\left(E,W\right)$ is a a Lip$\left(\gamma-1\right)$ one-form. Let $\mathbf{Y}:=1+\sum_{i=1}^{n}\mathbf{Y}^i:\Delta\rightarrow T^{(n)}\left(W\right)$ denote the functional in (22), then $\mathbf{Y}$ satisfies the identity*

$$\mathbf{Y}_{s,u}\mathbf{Y}_{u,t}-\mathbf{Y}_{s,t}=\mathbf{Y}_{s,u}\mathbf{N}_{s,u,t}\quad\text{for all}\quad 0\leq s\leq u\leq t\leq T,$$

where $\mathbf{N}$ is given by (29).

Proof. We begin by fixing s,u and t such that $0\leq s\leq u\leq t\leq T$. Then, as in the proof of Proposition 4.8, we $n=\lfloor p\rfloor$, $N=n^2$ and relabel $\mathbf{X}$ to denote the unique Lyons extension of the given rough path, which will then take values in $G^{(N)}\left(E\right)$. We let $\left(\mathbf{X}_{s,u}\left(r\right)\right)_{r=1}^{\infty}$ and $\left(\mathbf{X}_{u,t}\left(r\right)\right)_{r=1}^{\infty}$ be two sequences in $G_{\text{alg}}^{(N)}\left(E\right)$ which converge respectively to $\mathbf{X}_{s,u}$ and $\mathbf{X}_{u,t}$ as $r\rightarrow\infty$. Again, by the definition of the algebraic group and Chow's connectivity theorem (cf. Remark 2.10), we can find for each r a finite dimensional subspace, $E_r\subset E$, and two continuous paths of finite variation $x^{s,u}\left(r\right):[s,u]\rightarrow E_r$ and $x^{u,t}\left(r\right):[u,t]\rightarrow E_r$ whose level-N truncated signatures are respectively equal to the group elements $\mathbf{X}_{s,u}\left(r\right)$ and $\mathbf{X}_{u,t}\left(r\right)$; cf. formula (24). If we let $x^{t,u}\left(r\right)=x^{s,u}\left(r\right)*x^{u,t}\left(r\right):[s,t]\rightarrow E_r$ denote the concatenation of $x^{s,u}$ and $x^{u,t}$, then an easy calculation shows that

$$\begin{aligned}\mathbf{X}_{s,t}\left(r\right)&:=\sum_{k=0}^{n}\int_{s<t_1<...<t_k<t}dx_{t_1}^{t,u}\left(r\right)...dx_{t_1}^{t,u}\left(r\right)\\&=\mathbf{X}_{s,u}\left(r\right)\mathbf{X}_{u,t}\left(r\right)\rightarrow\mathbf{X}_{s,u}\mathbf{X}_{u,t}=\mathbf{X}_{s,t}\text{ as }r\rightarrow\infty.\end{aligned}$$

For each subinterval $[a,b] \in \{[s,u],[u,t],[s,t]\}$ and every $r \in \mathbb{N}$, we then define the path $y^{a,b}(r) : [a,b] \to E_r$ by setting

$$y_v^{a,b}(r) = \sum_{k=1}^{n} \alpha^{k-1}(X_a) \int_{a<t_1<...<t_k<v} dx_{t_1}^{a,b}(r) ... dx_{t_k}^{a,b}(r) \text{ for } v \in [a,b].$$

We have again shown in the proof of Proposition 4.8 that the level$-n$ truncated signature of $y^{a,b}(r)$ equals

$$\mathbf{Y}_{a,b}(r) := 1 + \sum_{i=1}^{n} \sum_{k_1,...,k_i=1}^{n} \otimes_{j=1}^{i} \alpha^{k_j}(X_a) \sum_{\pi \in \mathrm{OS}(k_1,...,k_i)} \pi^{-1} \mathbf{X}_{a,b}^{k_1+...+k_i}(r).$$

On the other hand, Lemma 4.7 of [16] gives that

$$y_v^{u,t}(r) = y_v^{s,t}(r) + \sum_{l=1}^{n} R_{l-1}(X_s, X_u) \int_{u<t_1<...<t_l<v} dx_{t_1}^{u,t}(r) \otimes ... \otimes dx_{t_1}^{u,t}(r)$$

for all $v \in [u,t]$, where $R_0, R_1, ..., R_{n-1}$ are the remainder functions in the Taylor-like expansions satisfied by $\alpha, \alpha^1, ..., \alpha^{n-1}$. From this expression a straight forward calculation yields

$$\begin{aligned}\mathbf{Y}_{u,t}^m(r) &= \int_{u<t_1<...<t_m<t} dy_{t_1}^{u,t}(r) ... dy_{t_m}^{u,t}(r) \\ &= \int_{u<t_1<...<t_m<t} dy_{t_1}^{s,t}(r) ... dy_{t_m}^{s,t}(r) + \mathbf{N}_{s,u,t}^m(r),\end{aligned}$$

where $\mathbf{N}_{s,u,t}^m(r)$ is defined as in (29) with $\mathbf{X}_{u,t}$ replaced with $\mathbf{X}_{u,t}(r)$ on the right hand side. With this expression in hand we deduce for every $r \in \mathbb{N}$ and every $i = 1, ..., n$ that

$$\begin{aligned}&\sum_{m=1}^{i} \mathbf{Y}_{s,u}^m(r) \mathbf{Y}_{u,t}^{i-m}(r) \\ &= \sum_{m=1}^{i} \int_{s<t_1<...<t_m<u} dy_{t_1}^{s,u}(r) ... dy_{t_m}^{s,u}(r) \int_{u<t_{m+1}<...<t_i<t} dy_{t_{m+1}}^{s,t}(r) ... dy_{t_i}^{s,t}(r) \\ &+ \sum_{m=1}^{i} \mathbf{Y}_{s,u}^m(r) \mathbf{N}_{s,u,t}^{i-m}(r) \\ &= \int_{s<t_1<...<t_i<t} dy_{t_1}^{s,t}(r) ... dy_{t_i}^{s,t}(r) + \sum_{m=1}^{i} \mathbf{Y}_{s,u}^m(r) \mathbf{N}_{s,u,t}^{i-m}(r) \\ &= \mathbf{Y}_{s,t}^i(r) + \sum_{m=1}^{i} \mathbf{Y}_{s,u}^m(r) \mathbf{N}_{s,u,t}^{i-m}(r),\end{aligned}$$

where the second line follows from the fact that $y^{s,u}(r) = y^{s,t}(r)$ on $[s,u]$. Letting $r \to \infty$ then gives the stated result. ■

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