

This is a repository copy of *Nonlinear Stochastic Partial Differential Equations of hyperbolic type driven by Lévy-type noises*.

White Rose Research Online URL for this paper:
<https://eprints.whiterose.ac.uk/113577/>

Version: Accepted Version

Article:

Zhu, Jiahui and Brzezniak, Zdzislaw orcid.org/0000-0001-8731-6523 (2016) Nonlinear Stochastic Partial Differential Equations of hyperbolic type driven by Lévy-type noises. *Discrete & Continuous Dynamical Systems, Series B*. pp. 3269-3299. ISSN 1531-3492

<https://doi.org/10.3934/dcdsb.2016097>

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE DRIVEN BY LÉVY-TYPE NOISES

JIAHUI ZHU*

School of Science, Zhejiang University of Technology,
No. 288, Liuhe Road, Xihu District
Hangzhou, 310023, China

ZDZISŁAW BRZEŹNIAK

Department of Mathematics, University of York, Heslington
York, YO10 5DD, UK

ABSTRACT. We study a class of abstract nonlinear stochastic equations of hyperbolic type driven by jump noises, which covers both beam equations with nonlocal, nonlinear terms and nonlinear wave equations. We derive an Itô formula for the local mild solution which plays an important role in the proof of our main results. Under appropriate conditions, we prove the non-explosion and the asymptotic stability of the mild solution.

1. Introduction. In this paper we will study two nonlinear hyperbolic second order in time stochastic partial differential equations (SPDEs) driven by Lévy-type noises: the beam and the wave equations (with emphasis on the former). We propose a framework that covers both types of equations. Let us briefly describe the motivation for these equations.

Due to its widespread applications in structural and mechanical engineering, the subject of nonlinear vibrations of Euler-Bernoulli beam has been intensively studied by many authors (see [33] for the free oscillations of a damped beam with immovable ends under an axial force, [5] for the effect of a compressive axial load, [14] for the effect of an axial periodic load on the motion of a hinged beam and [23] for a model with a nonlinear friction force, to name just a few). A type of nonlinear stochastic beam equation describing nonlinear vibrations of an elastic panel subjected to random fluctuations (state dependent white noises) was investigated by Chow and Menaldi in [12]. By means of a stochastic energy equation, they proved the existence, uniqueness and regularity of solutions. Recently the second named author, Masłowski and Seidler in [8] studied a wide class of abstract stochastic nonlinear beam equations perturbed by a white noise in a Hilbert space which is applicable to the equation treated in [12]. With the help of Lyapunov functions, they proved non-explosion of the mild solution and established the asymptotic stability of the solution. Most of the previous studies on hyperbolic SPDEs considered Gaussianity perturbations. However a huge amount of experimental evidences have demonstrated that Lévy-type models possess properties which describe the physical, biological and financial phenomena more accurately than the pure diffusion-based models, see cf. [1, 4, 10, 13, 27]. For instance, the real asset prices move essentially by jumps and large, sudden movements may occur over the time scales. Also the growth and recruitment of planktonic fish larvae and the magnetization in ferromagnetic materials are modelled by means of jump processes. The problem of vibration of a road bridge under running vehicles is usually modeled as a simply supported beam structure subjected to moving loads. Because vehicle loads are

2010 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. stochastic nonlinear beam equation, Poisson random measure, local and global mild solution, Lyapunov function, Itô formula.

The authors are supported by NNSF of China 11501509. The first author is also supported by Zhejiang Provincial Natural Science Foundation of China LQ16A010008.

* Corresponding author: Jiahui Zhu.

random in nature and traffic flows may suddenly increase or decrease, this leads to discontinuous gaps between vehicle flows. Those features are unattainable by diffusion-based models but appear as prevailing in models with jumps. Lévy-type models are more sophisticated and not easily amenable to mathematical analysis. Many results achieved in diffusion models are thus deserve to be re-investigated when jumps are included and our paper could be seen as a modest contribution to theoretical underpinning of such applied research.

In our paper, we investigate the existence and uniqueness of global mild solutions to a stochastic model arising in the nonlinear theory of structural dynamics and aeroelasticity by focusing on the following abstract second order stochastic differential equation perturbed by jump noises

$$u_{tt} + A^2u + \nabla_u \Phi(u) + f(t, u, u_t) = \int_Z g(t, u, u_t, z) \tilde{N}(t, dz), \quad (1)$$

where u is an H -valued stochastic process with H being a Hilbert space, A is a self-adjoint operator on H such that $A \geq \mu I$ for some $\mu > 0$, $\Phi : D(A) \rightarrow [0, \infty)$ is an “energy” function (with the gradient $\nabla \Phi$ understood with respect to the Hilbert space structure on H) and \tilde{N} is a compensated Poisson random measure.

The two motivating examples of the function Φ are those corresponding to the beam and nonlinear wave equations. In the former case Φ is of the form

$$\Phi(u) = \frac{1}{2} m(|B^{\frac{1}{2}}u|^2), \quad u \in D(A), \quad (2)$$

where $m : [0, \infty) \rightarrow \infty$ is a C^1 class increasing function such that $m(0) = 0$ and its derivative is a locally Lipschitz continuous function, B is a self-adjoint operator such that $BA^{-1} \in \mathcal{L}(H)$. In the latter, more concrete, case, $H = L^2(D)$ for some domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, A^2 being the $-$ Laplacian with the Dirichlet boundary conditions, and, for suitable $p \geq 2$.

$$\Phi(u) = \frac{1}{p+1} \int_D |u(x)|^{p+1} dx, \quad u \in H^2(D) \cap H_0^1(D).$$

Note that in the former case

$$\nabla_u \Phi(u) = m'(|B^{\frac{1}{2}}u|^2)Bu, \quad u \in D(A)$$

while in the latter

$$\nabla_u \Phi(u) = |u|^{p-1}u, \quad u \in D(A).$$

We will first prove, see also [34] where the case of globally Lipschitz coefficients is carefully investigated, that under some natural local Lipschitz continuity conditions on the coefficients f and g , Equation 1 has a unique maximal local mild solution given by 11. More importantly, we will prove, see Theorem 2.6, that the maximal local solution is a global one and establish in Theorem 2.7 the ultimate boundedness and stability of such solutions. Let us stress that this is not a simple issue due to the presence of a nonlinear term involving the function m . The main ingredient of the proofs of these two theorems is a general version of the Itô Lemma for mild solutions. Note that the results about maximal local mild solution (Proposition 1) and the Itô Lemma (Lemma 5.2) are proved in a more general form for equation 10 than what is needed to establish the non-explosion of the mild solutions, i.e. for functions F and G not being of the special forms 7 or 9. On comparing with the method used in the case of stochastic beam equation driven by Wiener process, the factorization method used in showing the uniform L^p -convergence of the Yosida approximation for stochastic convolutions w.r.t. the Wiener noise, is not applicable in our case. Instead of considering the Yosida approximation A_n , we follow the approximation procedure introduced in [28] and [29]. We first apply the ordinary Itô formula to $D(\mathcal{A})$ -valued solution processes, then investigate its limit and obtain an Itô-type formula for the mild solution when the $D(\mathcal{A})$ -valued solution processes converges to the mild solution, here $D(\mathcal{A})$ is the domain of the generator \mathcal{A} . As a result, the Itô formula we established for mild solutions is sufficient to cover the two different cases: the existence of a global solution and its asymptotic boundedness. Compared with [8], where the two results were proved separately, we give a more efficient direct proof of the main results. As our major motivation, we also show that under some standard assumptions the results we have proved for problem 1 can be applied to a wider class of

models including stochastic nonlinear wave equations and stochastic nonlinear beam equations subject to either the periodic boundary condition or mixed hinged/clamped boundary conditions.

Stochastic PDE driven by discontinuous noise is a very new subject. So far mainly problems with Lipschitz coefficients have been investigated, see the recent monograph [24] by Peszat and Zabczyk and/or papers [18] by Hauseblas and [26] by Riedle. A type of stochastic PDEs with monotone and coercive coefficients, which is weaker than the usual Lipschitz and linear growth assumptions, driven by some discontinuous perturbations were studied by Gyöngy and Krylov in [16] for the finite-dimensional case and extended by Gyöngy to infinite-dimensional spaces in [17]. Recently, the authors and Liu in [3] established the existence and uniqueness of strong solutions for a large class of SPDEs with coercive and locally monotone coefficients driven by Lévy processes. Stochastic reaction diffusion equations driven by Lévy noises have been a subject of a recent paper [9], where also some comments on the existing literature can be found. The approach of the current paper is different as it does not use any compactness methods but instead follow a more natural route of constructing a maximal local solution and then proving that its lifespan is equal to infinity, see [8] and [6] for the gaussian noise case. To our best knowledge the present paper is the first one in which this approach is applied to SPDEs with non-Lipschitz coefficients and non-gaussian noise. Note however, that this method has been used and further developed in a joint paper by the current authors and Liu in [3]. It is a natural and interesting question to combine the results obtained in this paper together with those from [7] and prove the existence of an invariant measure for 11. However, contrary to the finite dimensional case, see e.g. [2], this is still an open problem and its resolution is postponed till the next publication.

The rest of the paper is organized as follows. Section 2 gives a detailed description of the problem, the main results and its hypotheses. Section 3 is devoted to proving a basic auxiliary result about stopped stochastic convolutions. Section 4 proves the existence and uniqueness of maximal local mild solution, while Section 5 proves the crucial Lemma, Itô Lemma. The proofs of the main theorems 2.6 and 2.7, are given in Section 6 and Section 7.

For the convenience of a reader let us describe the structure of the proof of Theorem 2.6 about the existence and uniqueness of a global solution to Problem 10. The existence and uniqueness of a local maximal solution is formulated in Proposition 1, which in turn is proved in section 4. In section 5 we formulate an infinite dimensional Itô Lemma for processes which are not semimartingales but are defined certain stochastic convolutions. This result is general enough to be applicable in the proof of both main theorems. In section 3 we formulate Lemma 3.1 about equality of two possible definitions of stopped stochastic convolution processes. To conclude the proof of Theorem 2.6 it is enough to show that the lifetime of the local maximal solution is equal to infinity. The proof of that fact is contained in section 2.6. For this we define auxiliary stopping times τ_n , by 51, and auxiliary processes $\tilde{f}_n, \tilde{g}_n, \tilde{F}_n$ and \tilde{G}_n defined by (52, 53, 54, 55) and consider processes \mathbf{v}_n which are global solutions of the auxiliary linear SPDE 56. We apply to \mathbf{v}_n the Itô formula from our fundamental section 5 and, since by Lemma 3.1, \mathbf{v}_n and u are equal up to τ_n we conclude the proof. The proof of the first part of Theorem 2.7 relies on the use of Itô formula from section 5 applied a modified energy function 58.

2. Framework and main results. Throughout the whole paper we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual hypotheses¹, and that (Z, \mathcal{Z}, ν) is a σ -finite measure space. We denote by \tilde{N} the compensated Poisson random measure on $[0, T] \times \Omega \times Z$ with the intensity measure ν , so that

$$\tilde{N}((0, t] \times B) = N((0, t] \times B) - t\nu(B), \text{ for } t \geq 0, B \in \mathcal{Z}.$$

We use \mathcal{P} to denote the predictable σ -field on $\mathbb{R}_+ \times \Omega$, that is the σ -field generated by all left continuous and \mathbb{F} -adapted real-valued processes. We write \mathcal{BF} for the σ -field of all \mathbb{F} -progressively measurable sets on $\mathbb{R}_+ \times \Omega$, i.e.

$$\mathcal{BF} = \{A \subset \mathbb{R}_+ \times \Omega : \forall t \in \mathbb{R}_+, A \cap ([0, t] \times \Omega) \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t\}.$$

¹i.e. \mathcal{F}_0 contains all sets of \mathbb{P} -measure zero and $\mathcal{F}_t = \mathcal{F}_{t+}$.

The two motivating stochastic evolution equations are the wave equation in \mathbb{R}^d ,

$$u_{tt} + (\gamma^2 - \Delta)u + |u|^{p-1}u = \beta u_t + \int_Z g(t, u(t-), u_t(t-), z) \tilde{N}(t, dz),$$

for $p > 1$ and the beam equation in $\mathcal{O} \subset \mathbb{R}^d$,

$$u_{tt} + \Delta^2 u - m'(|\nabla u|^2)\Delta u = \beta u_t + \int_Z g(t, u(t-), u_t(t-), z) \tilde{N}(t, dz),$$

where $m \in C^1(\mathbb{R}_+, \mathbb{R}_+)$.

Of course the above equations have to be supplemented by initial and boundary conditions. A common feature of these two problems is that they can be written as

$$u_{tt} + \nabla_u E(u) = \beta u_t + \int_Z g(t, u(t-), u_t(t-), z) \tilde{N}(t, dz), \quad (3)$$

where E is the energy defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u(x)|^2 + \gamma^2 |u(x)|^2] dx + \frac{1}{p+1} \int_{\mathbb{R}^3} |u(x)|^{p+1} dx \quad (4)$$

for the wave equation, and in the case of the beam equation,

$$E(u) = \frac{1}{2} \int_D [|\Delta u(x)|^2 + m(|\nabla u|^2)] dx, \quad (5)$$

and $\nabla_u E$ is the gradient of E with respect to the Hilbert space $L^2(\mathbb{R}^d)$ or $L^2(D)$. In fact, one consider the following generalised energy for the nonlinear beam equation

$$E(u) = \frac{1}{2} \int_{\mathcal{O}} [|\Delta u(x)|^2 + m(|\nabla u|^2)] dx + \frac{1}{p+1} \int_{\mathcal{O}} |u(x)|^{p+1} dx.$$

In such a case, the generalised energy for the nonlinear beam equation will take the following form

$$u_{tt} + \Delta^2 u - m'(|\nabla u|^2)\Delta u + |u|^{p-1}u = \beta u_t + \int_Z g(t, u(t-), u_t(t-), z) \tilde{N}(t, dz).$$

With an appropriate choice of the linear operators A and B , these problems can be written in a unified way

$$\begin{aligned} u_{tt} &= -A^2 u - f(t, u, u_t) - m'(|B^{\frac{1}{2}}u|^2)Bu + \int_Z g(t, u(t-), u_t(t-), z) \tilde{N}(t, dz), \\ u(0) &= u_0, \quad u_t(0) = u_1. \end{aligned} \quad (6)$$

As a byproduct of this approach we will see that Equation 3 with the energy function 5 has at least two non-equivalent formulations depending on the choice of the operator A (and hence the boundary conditions) while B is the Laplace operator with fixed boundary conditions.

Suppose that H is a real separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a corresponding norm $|\cdot|_H$. By $\mathcal{B}(H)$ we denote the Borel σ -field on H . Let A and B be self-adjoint operators in H . Suppose that $B \geq 0$ and $A \geq \mu I$ for some $\mu > 0$. We also assume that $BA^{-1} \in \mathcal{L}(H)$ and functions

$$f : \mathbb{R}_+ \times D(A) \times H \ni (t, \xi, \eta) \mapsto f(t, \xi, \eta) \in H$$

and

$$g : \mathbb{R}_+ \times D(A) \times H \times Z \ni (t, \xi, \eta) \mapsto g(t, \xi, \eta) \in L^2(Z, \nu; H)$$

are, respectively, $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(D(A)) \otimes \mathcal{B}(H)/\mathcal{B}(H)$ and $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(D(A)) \otimes \mathcal{B}(H) \otimes \mathcal{Z}/\mathcal{B}(H)$ measurable.

We follow a classical approach from the deterministic theory of second order (in time) equations based on introducing a new Hilbert space $\mathcal{H} := D(A) \times H$ with product norm $|(x, y)|_{\mathcal{H}}^2 := |Ax|_H^2 + |y|_H^2$, an operator \mathcal{A} defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A^2 & 0 \end{pmatrix}, \quad D(\mathcal{A}) = D(A^2) \times D(A),$$

which is a generator of a C_0 -unitary group (e^{tA}) , $t \in \mathbb{R}$, on \mathcal{H} , see [19], and functions

$$F : \mathbb{R}_+ \times D(A) \times H \ni (t, \xi, \eta) \mapsto -(0, f(t, \xi, \eta)) \in \mathcal{H}, \quad (7)$$

$$\mathbf{M} : D(A) \times H \ni (\xi, \eta) \mapsto -(0, m'(|B^{\frac{1}{2}}\xi|^2)B\xi) = -(0, \nabla_\xi \Phi(\xi)) \in \mathcal{H}, \quad (8)$$

$$G : \mathbb{R}_+ \times D(A) \times H \times Z \ni (t, \xi, \eta, z) \mapsto (0, g(t, \xi, \eta, z)) \in \mathcal{H}. \quad (9)$$

Then Equation 6 can be rewritten as a system of first order equations for the unknown function $\mathbf{u}(t) = (u(t), u_t(t))$ with respect to the time variable in the following form, with $\mathbf{u}_0 = (u_0, u_1) \in \mathcal{H}$,

$$\begin{aligned} du(t) &= \mathcal{A}u(t) dt + F(t, \mathbf{u}(t)) dt + \mathbf{M}(\mathbf{u}(t)) dt + \int_Z G(t, \mathbf{u}(t-), z) \tilde{N}(dt, dz), \quad t \geq 0, \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{aligned} \quad (10)$$

It is useful to notice that the above equation is more general than the problem 6. Indeed, the functions F and G do not need to be of the special form 7 and 9. We will formulate our local existence result, see Proposition 1, and the Itô Lemma, see Lemma 5.2, in this more general way, i.e. for general functions F and G . From that we will deduce a corresponding result for special functions F and G as in 7 and 9, however, with general functions f , m and g (of course satisfying some natural assumptions).

In the sequel, $\mathcal{M}_{loc}^2(\mathcal{BF} \otimes \mathcal{Z}, \mathcal{H})$ will denote the space of all $\mathcal{BF} \otimes \mathcal{Z}$ -progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{H}$ such that for all $T \geq 0$, $\mathbb{E} \int_0^T |\phi(t)|_{\mathcal{H}}^2 dt < \infty$, and $\mathcal{M}_{loc}^2(\mathcal{P} \otimes \mathcal{Z}, \mathcal{H})$ stands for the space of all $\mathcal{P} \otimes \mathcal{Z}$ -measurable functions $\varphi : \mathbb{R}_+ \times \Omega \times Z \rightarrow \mathcal{H}$ such that for all $T \geq 0$, $\mathbb{E} \int_0^T \int_Z |\varphi(t, z)|_{\mathcal{H}}^2 \nu(dz) dt < \infty$.

Definition 2.1. A strong solution to Equation 10 is a $D(A)$ -valued càdlàg \mathbb{F} -adapted stochastic process $\mathbf{u} = (\mathbf{u}(t))_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that

- (1) $\mathbf{u}(0) = \mathbf{u}_0$ a.s.;
- (2) the process ϕ defined by

$$\phi(t, \omega) = F(t, \mathbf{u}(t, \omega)) + \mathbf{M}(\mathbf{u}(t, \omega)) \in \mathbb{R}_+ \times \Omega$$

belongs to the space $\mathcal{M}_{loc}^2(\mathcal{BF} \otimes \mathcal{Z}, \mathcal{H})$ and the process φ defined by

$$\varphi(t, \omega, z) = G(t, \mathbf{u}(t-, \omega), z) \quad (t, \omega, z) \in \mathbb{R}_+ \times \Omega \times Z$$

belongs to $\mathcal{M}_{loc}^2(\mathcal{P} \otimes \mathcal{Z}, \mathcal{H})$;

- (3) for any $t \geq 0$, the following equality holds \mathbb{P} -a.s.

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathcal{A}\mathbf{u}(s) ds + \int_0^t [F(s, \mathbf{u}(s)) + \mathbf{M}(\mathbf{u}(s))] ds + \int_0^t \int_Z G(s, \mathbf{u}(s-), z) \tilde{N}(ds, dz).$$

Definition 2.2. A mild solution to Equation 10 is an \mathcal{H} -valued \mathbb{F} -adapted stochastic process $\mathbf{u} = (\mathbf{u}(t))_{t \geq 0}$ with² càdlàg paths defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that conditions (1) and (2) of Definition 2.1 are satisfied and

- (3') for any $t \geq 0$, the following equality holds \mathbb{P} -a.s.

$$\mathbf{u}(t) = e^{tA}\mathbf{u}_0 + \int_0^t e^{(t-s)A} [F(s, \mathbf{u}(s)) + \mathbf{M}(\mathbf{u}(s))] ds + \int_0^t \int_Z e^{(t-s)A} G(s, \mathbf{u}(s-), z) \tilde{N}(ds, dz).$$

Let us recall, see for instance [6], that a stopping time τ is called to be accessible if there exists an increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of stopping times such that $\tau_n < \tau$ and $\lim_{n \rightarrow \infty} \tau_n = \tau$ a.s. and this sequence $\{\tau_n\}_{n \in \mathbb{N}}$ will be called an approximating sequence for τ .

Using the above introduced notion we can state the main definition in this paper. One can fully understand it only after becoming familiar with the results presented in section 3.

²As before, one could add, to avoid a slight chance of ambiguity, “ \mathcal{H} -valued”-càdlàg.

Definition 2.3. A local mild solution to Equation 10 is an \mathcal{H} -valued, \mathbb{F} -adapted, càdlàg local process $\mathbf{u} = (\mathbf{u}(t))_{0 \leq t < \tau}$, where τ is an accessible stopping time with an approximating sequence $\{\tau_n\}_{n \in \mathbb{N}}$, such that for any $n \in \mathbb{N}$, the stopped process $\mathbf{u}_{\tau_n}(t) := \mathbf{u}(t \wedge \tau_n)$, $t \geq 0$, satisfies,

$$\begin{aligned} \mathbf{u}(t \wedge \tau_n) &= e^{(t \wedge \tau_n)\mathcal{A}}\mathbf{u}_0 + \int_0^{t \wedge \tau_n} e^{(t \wedge \tau_n - s)\mathcal{A}}[F(s, \mathbf{u}(s)) + \mathbf{M}(\mathbf{u}(s))] ds \\ &\quad + I_{\tau_n}(G(\mathbf{u}))(t \wedge \tau_n), \quad \mathbb{P}\text{-a.s. } t \geq 0, \end{aligned} \quad (11)$$

where $I_{\tau_n}(G(\mathbf{u}))$ is a process defined by

$$I_{\tau_n}(G(\mathbf{u}))(t) = \int_0^t \int_Z 1_{[0, \tau_n]}(s) e^{(t-s)\mathcal{A}} G(s, \mathbf{u}(s-), z) \tilde{N}(ds, dz), \quad t \geq 0. \quad (12)$$

For the future reference we specifically state the following important observation.

Remark 1. Suppose that X and Y are two càdlàg processes and τ is an \mathbb{F} -stopping time. If X and Y coincide on the open interval $[0, \tau)$, i.e. $X(s, \omega)1_{[0, \tau)}(s) = Y(s, \omega)1_{[0, \tau)}(s)$, for $(s, \omega) \in \mathbb{R}_+ \times \Omega$, we can infer

$$G(s, X(s-), z)1_{[0, \tau]}(s) = G(s, Y(s-), z)1_{[0, \tau]}(s), \quad \text{for } (s, \omega) \in \mathbb{R}_+ \times \Omega,$$

since the function $[0, \tau] \ni s \mapsto G(s, X(s-), z)$ depends only on the values of X on $[0, \tau)$.

Definition 2.4. A local mild solution $\mathbf{u} = (\mathbf{u}(t))_{0 \leq t < \tau}$ to Equation 10 is unique if for any other local mild solution $\tilde{\mathbf{u}} = \{\tilde{\mathbf{u}}_{0 \leq t < \tilde{\tau}}\}$ to Equation 10, the processes \mathbf{u} and $\tilde{\mathbf{u}}$ are equivalent on $[0, \tau \wedge \tilde{\tau})$.

A local mild solution $\mathbf{u} = (\mathbf{u}(t))_{0 \leq t < \tau}$ is called a maximal local mild solution if for any other local mild solution $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}(t))_{0 \leq t < \tilde{\tau}}$ satisfying $\tilde{\tau} \geq \tau$ a.s. and $\tilde{\mathbf{u}}|_{[0, \tau)}$ is equivalent to \mathbf{u} , one has $\tilde{\tau} = \tau$ a.s.. If $\mathbb{P}(\tau = +\infty) = 1$, the local mild solution \mathbf{u} is called a global mild solution to Equation 10.

Remark 2. If the local mild solution of Equation 10 is unique, then the uniqueness of the maximal local mild solution holds as well. In particular, a maximal local mild solution $\mathbf{u} = (\mathbf{u}(t))_{0 \leq t < \tau}$ is unique if and only if for any other local mild solution $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}(t))_{0 \leq t < \tilde{\tau}}$, we have $\tilde{\tau} \leq \tau$ a.s. and $\tilde{\mathbf{u}} = \mathbf{u}$ a.s. on $[0, \tau \wedge \tilde{\tau})$.

In order to show the existence and uniqueness of a global mild solution to Equation 10, it is sufficient to impose local Lipschitz continuity and some natural growth conditions on the functions F and G .

Condition (C.1).

The function $F : [0, \infty) \times \mathcal{H} \rightarrow \mathcal{H}$ is Lipschitz on balls in \mathcal{H} , locally uniformly w.r.t. t , i.e. for every $R > 0$ and $T > 0$, there exists a constant $L_{F, R, T} > 0$ such that for all $t \in [0, T]$ and $x, y \in B_{\mathcal{H}}(0, R)$,

$$|F(t, x) - F(t, y)|_{\mathcal{H}} \leq L_{F, R, T}|x - y|_{\mathcal{H}}.$$

There exists a constant $L_F > 0$ such that for all $t \in [0, T]$,

$$|F(t, 0)| \leq L_F.$$

Condition (C.2).

The function $G : [0, \infty) \times \mathcal{H} \rightarrow L^2(Z, \nu; \mathcal{H})$ is Lipschitz on balls in \mathcal{H} , locally uniformly w.r.t. t , i.e. for $T > 0$, there exists a constant $L_{G, R, T} > 0$ such that for all $t \in [0, T]$ and $x, y \in B_{\mathcal{H}}(0, R)$,

$$\int_Z |G(t, x, z) - G(t, y, z)|_{\mathcal{H}}^2 \nu(dz) \leq L_{G, R, T}|x - y|_{\mathcal{H}}^2.$$

There exists a constant $L_G > 0$ such that for all $t \in [0, T]$,

$$\int_Z |G(t, 0, z)|_{\mathcal{H}}^2 \nu(dz) \leq L_G.$$

Let us now formulate a basic result about the existence of a local maximal solution, see Definition 2.3.

Proposition 1. Suppose that conditions (C.1) and (C.2) are satisfied. Suppose also that the function $\mathbf{M} : \mathcal{H} \rightarrow \mathcal{H}$ is Lipschitz on balls. Then for every \mathcal{F}_0 -measurable initial data \mathbf{u}_0 , there exists a unique maximal local mild solution to Equation 10.

For the existence of global solutions we will additionally need the following classical assumptions on F and G .

Condition (C.3).

Function G is of linear growth on \mathcal{H} locally uniformly w.r.t. $t \in \mathbb{R}_+$, i.e. for every $T > 0$, there exists constants $K_{G,T}, R_{G,T} \geq 0$ such that for all $t \in [0, T]$,

$$\int_{\mathcal{Z}} |G(t, x, z)|_{\mathcal{H}}^2 \nu(dz) \leq K_{G,T} + R_{G,T} |x|_{\mathcal{H}}^2, \quad x \in \mathcal{H}. \quad (13)$$

Condition (C.4).

For every $T > 0$, there exists constants $K_{f,T}, R_{f,T} \geq 0$ such that for all $t \in [0, T]$,

$$-\langle x_2, f(t, x_1, x_2) \rangle_H \leq K_{f,T} + R_{f,T} |x|_{\mathcal{H}}^2, \quad x = (x_1, x_2) \in \mathcal{H} = D(A) \times H. \quad (14)$$

Condition (C.5).

The “energy” functional Φ is of the form 2, i.e.

$$\Phi(u) = \frac{1}{2} m(|B^{\frac{1}{2}} u|^2), \quad u \in D(A),$$

where, $m : [0, \infty) \rightarrow \infty$ is a C^1 class increasing function such that $m(0) = 0$ and the derivative m' is locally Lipschitz continuous.

Remark 3. Note that since m' is locally Lipschitz continuous and $B \in L(D(A), H)$, the function $D(A) \ni u \mapsto m'(|B^{\frac{1}{2}} u|^2) B u \in H$ is also locally Lipschitz continuous w.r.t. $u \in D(A)$. Hence we are confident that the next claim holds.

Lemma 2.5. Assume that $m : [0, \infty) \rightarrow \infty$ is a C^1 class increasing function such that $m(0) = 0$ and the derivative m' is locally Lipschitz continuous and that function \mathbf{M} is defined by formula 8, i.e.

$$\mathbf{M} : \mathcal{H} \ni (\xi, \eta) = -(0, \nabla_{\xi} \Phi(\xi)) = -(0, m'(|B^{\frac{1}{2}} \xi|^2) B \xi) \in \mathcal{H}.$$

Then the mapping M is Lipschitz on balls.

Let us now formulate two main results of our paper. The first one is about the existence of global solutions while the second is concerned with stability of solutions.

Theorem 2.6. Suppose that conditions (C.1)-(C.5) are satisfied. Then for every \mathcal{F}_0 -measurable \mathcal{H} -valued initial data u_0 , there exists a unique global mild solution to Equation 10.

Our last result is about asymptotic behaviour of solutions. This result is proved under more stringent conditions than the previous results. Let us formulate the relevant conditions.

Condition (C.6).

The function F is related to a function $f : \mathbb{R}_+ \times D(A) \times H \rightarrow H$ via formula 7 and the latter is of the following form. There exists $\delta > 0$ such that

$$f(t, x) = \delta x_2, \quad t \geq 0 \text{ and } x = (x_1, x_2) \in \mathcal{H} = D(A) \times H.$$

Condition (C.7).

There exists $\alpha > 0$ such that the function m' from (C.5) satisfies

$$m(z) \leq \frac{1}{\alpha} z m'(z), \quad z \geq 0. \quad (15)$$

Remark 4. An example of a function m' satisfying both conditions (C.5) and (C.7) is function $m(z) = z^{\alpha}$, $z \geq 0$ with $\alpha \geq 2$. Note that in this case, the function \mathbf{M} defined by 8 is Lipschitz on balls.

For the stability of the global solutions we will need the following stronger version of condition (C.3).

Condition (C.8).

Function G is of linear growth on \mathcal{H} uniformly w.r.t. $t \in \mathbb{R}_+$, i.e. there exists constants $K_G, R_G \geq 0$ such that for all $t \geq 0$,

$$\int_Z |G(t, x, z)|_{\mathcal{H}}^2 \nu(dz) \leq K_G + R_G |x|_{\mathcal{H}}^2, \quad x \in \mathcal{H}.$$

Theorem 2.7. Assume (C.2), (C.8) and (C.5)-(C.7). Assume furthermore that

$$R_G^2 - \beta < 0.$$

Then, the unique global mild solution $\mathbf{u} = (u, v)$ to Equation 10 with the initial data $\mathbf{u}_0 = (u_0, v_0)$ satisfying $\mathbb{E}[|u_0|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}u_0|_H^2)] < \infty$, satisfies the following estimate

$$\sup_{t \geq 0} \mathbb{E}[|\mathbf{u}(t)|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}\mathbf{u}(t)|_H^2)] < \infty.$$

Moreover, if G satisfies 13 with $K_{G,T} = 0$, i.e. there exists $R_{G,T} > 0$ such that for every $T > 0$,

$$\int_Z |g(t, x, z)|_H^2 \nu(dz) \leq R_{G,T} |x|_{\mathcal{H}}^2, \quad t \in [0, T], x \in \mathcal{H},$$

then there exist constants $C > 0$ and $\lambda > 0$ such that

$$\mathbb{E} |\mathbf{u}(t)|_{\mathcal{H}}^2 \leq \mathbb{E} \left[|\mathbf{u}(t)|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}\mathbf{u}(t)|_H^2) \right] \leq C e^{-\lambda t} \mathbb{E} \left[|u_0|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}u_0|_H^2) \right], \quad t \geq 0.$$

Note that Proposition 1 holds true for problem 10 in the general form, i.e. for the functions F and G satisfying conditions (C.1) and (C.2), and the \mathbf{M} satisfying the locally Lipschitz condition. Theorem 2.6 holds true for problem 10 in the special form since the functions \mathbf{M} , F and G are of special form 8, 7 and 9 respectively, and the corresponding functions m , f and g satisfy condition (C.5), (C.4) and (C.3). Finally, Theorem 2.7 holds for problem 10 in the above case when also functions \mathbf{M} , F and G are special form 8, 7 and 9 respectively, for functions m , f and g satisfying also Conditions (C.5), (C.7), (C.6) and (C.8).

Since the stochastic nonlinear beam equation with either the hinged or the clamped boundary conditions can be treated in almost the same way as in [8], as only the the stochastic term will be different, we will discuss applications to the stochastic nonlinear beam equation with periodic boundary conditions. However, in order that a reader can easily spot the differences and similarities between our Example and section 4 from [8] we have decided to **keep** as much as possible the notation from that paper.

Definition 2.8. Suppose that Λ is a topological space and X_1, X_2 and Y are normed vector spaces. A Borel function

$$R : \Lambda \times X_1 \times X_2 \rightarrow Y$$

is called locally Lipschitz w.r.t. X_1 , globally Lipschitz w.r.t. X_2 , uniformly w.r.t. Λ iff for every $N \in \mathbb{N}$, there exist constants L_N and L such that for all $\lambda \in \Lambda$, $x'_1, x''_1 \in B_{X_1}(0, N)$, $x'_2, x''_2 \in X_2$, the following inequality holds

$$|R(\lambda, x''_1, x''_2) - R(\lambda, x'_1, x'_2)|_Y \leq L_N |x''_1 - x'_1|_{X_1} + L |x''_2 - x'_2|_{X_2}.$$

This definition can be given of various natural generalizations which we simply do not state.

Example 2.9 (Stochastic nonlinear beam equation with the periodic boundary condition). Assume that $L > 0$ and let \mathbb{T}^n be a n -dimensional torus of length L , i.e.

$$\mathbb{T}^n = \mathbb{R}^n / \sim_L,$$

where \sim_L an equivalence relation in \mathbb{R}^n defined by $x \sim_L y$ iff $\frac{y-x}{L} \in \mathbb{Z}^n$. It is well known that \mathbb{T}^n is a riemannian manifold (without boundary) and that functions defined on \mathbb{T}^n can be identified with functions defined on \mathbb{R}^n which are L -periodic in each coordinate (i.e. L^n -periodic). Partial differential equations on a torus are often used as the simplest model of PDES on manifolds where there is no need to introduce deep theory from Differential Geometry. A boundary or initial value problem with periodic boundary conditions is often identified as a corresponding problem on the torus. Somehow incorrectly, one can identify \mathbb{T}^n with the product on n intervals $[0, L)$ each equipped with a metric

$d(s, t) = \min\{|t - s|, |t - L - s|\}$. One can also identify (correctly) that \mathbb{T}^n with a cartesian product of copies of a circle of length L . In this way, one can see that \mathbb{T}^n is endowed with the Lebesgue measure m on the σ -fields of its Borel sets or of Lebesgue measurable sets. This gives rise to the family of Lebesgue spaces $L^p(\mathbb{T}^n)$ for each $p \in [1, \infty]$. With a bit of extra work one can define the scales of the Sobolev spaces $H^{k,p}(\mathbb{T}^n)$ for $k \in \mathbb{N}$ and $p \in [1, \infty]$. Every function $u \in H^{k,p}(\mathbb{T}^n)$ can be uniquely lifted to an L^n -periodic function $U \in H_{\text{loc}}^{k,p}(\mathbb{R}^n)$ so that u satisfies naturally periodicity conditions. These spaces satisfy the classical Gagliardo-Nirenberg inequalities, for instance if $k - \frac{n}{p} > 0$, then $u \in H^{k,p}(\mathbb{T}^n)$ has a unique representative \tilde{u} which belongs to $C(\mathbb{T}^n)$. Note also that the lift \tilde{U} of \tilde{u} belongs to $C(\mathbb{R}^n)$.

Define a linear operator B in $\mathbb{H} = L^2(\mathbb{T}^n)$ by

$$\begin{aligned} D(B) &= H^{2,2}(\mathbb{T}^n), \\ Bu &= -\Delta u, \quad u \in D(B). \end{aligned}$$

It is well known that B is a self-adjoint operator in \mathbb{H} .

Next let us now define an operator P by

$$\begin{aligned} D(P) &= H^{4,2}(\mathbb{T}^n), \\ Pu &= \Delta^2 u + u, \quad u \in D(P). \end{aligned}$$

It is well know that P is a strictly position self-adjoint operator on the Hilbert space $\mathbb{H} = L^2(\mathbb{T}^n)$ and

$$\langle Pu, u \rangle = \int_{\mathbb{T}^n} [|\Delta u|^2 + |u|^2] dm, \quad u \in D(P).$$

Let A_1 be the unique positive square root of P . It satisfies

$$D(A_1) = H^{2,2}(\mathbb{T}^n),$$

Obviously, since $D(P^{1/2})$ is equal to the complex interpolation space of order $\frac{1}{2}$ between $D(P)$ and $L^2(\mathbb{T}^n)$, $D(A_1) = H^{2,2}(\mathbb{T}^n) = D(B)$ and the A_1 -graph norm is equivalent to the $H^{2,2}$ -norm. In particular, $B \in \mathcal{L}(D(A_1), \mathbb{H})$.

Note that in view of the Gagliardo inequalities (or the Sobolev embeddings) if $u \in D(P)$ then $D^\alpha u \in L^2(\Gamma)$ for $\alpha \leq 3$, where $\Gamma = \partial([0, L]^n)$ is the boundary of the square of length L in \mathbb{R}^n .

Consider the following problem in $(0, L)^n$

$$\frac{\partial^2 u}{\partial t^2} - m \left(\int_{\mathcal{O}} |\nabla u|^2 dx \right) \Delta u + \gamma \Delta^2 u + \Upsilon \left(t, x, u, \nabla u, \frac{\partial u}{\partial t} \right) = \int_Z \Pi(t, x, u, \nabla u, \frac{\partial u}{\partial t}, z) \tilde{N}(t, dz) \quad (16)$$

with the periodic boundary conditions (written for simplicity in the case $n = 2$)

$$\begin{aligned} u(0, \cdot) &= u(L, \cdot), \quad u(\cdot, 0) = u(\cdot, L) \\ \frac{\partial u(0, \cdot)}{\partial x_1} &= \frac{\partial u(L, \cdot)}{\partial x_1}, \quad \frac{\partial u(\cdot, 0)}{\partial x_2} = \frac{\partial u(\cdot, L)}{\partial x_2} \end{aligned}$$

where $\gamma > 0$, $m \in \mathcal{C}^\alpha(\mathbb{R}_+, [0, \infty))$, a Borel function

$$\Upsilon : [0, T] \times D \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$$

is locally Lipschitz w.r.t. $\mathbb{R} \times \mathbb{R}^n$, globally Lipschitz w.r.t. \mathbb{R} , uniformly w.r.t. $[0, T] \times D$, a Borel function

$$\Pi : [0, T] \times D \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times Z \rightarrow \mathbb{R}$$

is such that the corresponding function

$$\tilde{\Pi} : [0, T] \times D \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow L^2(Z, \nu)$$

is locally Lipschitz w.r.t. $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ uniformly w.r.t. $[0, T] \times D$. Moreover we assume that the following growth conditions are satisfied.

1. There exists a constant K_Υ such that for all $t \in [0, T]$, $x \in D$, $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$c\Upsilon(t, x, a, b, c) \geq -K_\Upsilon(1 + |c|^2). \quad (17)$$

2. There exists a constant L_Π such that for all $t \in [0, T]$, $x \in D$, $a \in \mathbb{R}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$\int_Z |\Pi(t, x, a, b, c, z)|^2 \nu(dz) \leq L_\Pi(1 + |c|^2). \quad (18)$$

If $n = 1$ or $n \leq 3$ and the functions Υ and ψ depend only on the first and the second variables (i.e. on x and u), then there exists a unique maximal global mild solution to Equation 16.

Example 2.10 (Stochastic nonlinear beam equation with mixed hinged/clamped boundary conditions). Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain with a \mathcal{C}^∞ - boundary $\partial\mathcal{O}$ consisting of two parts (possibly disjoint) Γ_1 and Γ_2 . We assume that the common part $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, then it is a subset of a submanifold of $\partial(\mathcal{O})$ of dimension³ $\leq n - 2$. Let us denote by ν the unit exterior normal field to Γ_1 .

Let us introduce an operators B and P by

$$D(B) = W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O}), \quad B\psi = -\Delta\psi, \quad \psi \in D(B).$$

$$D(P) = \{\varphi \in H^{4,2}(\mathcal{O}) : \varphi = 0 \text{ on } \partial(\mathcal{O}), \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_1 \text{ and } \Delta\varphi = 0 \text{ on } \Gamma_2\},$$

$$P\varphi = \Delta^2\varphi, \text{ for } \varphi \in D(P).$$

It is well known that both B and P are self-adjoint, B is positive and P is non-negative. As in [8] we can check that it is also positive. Indeed, if $\varphi \in D(P)$ then by the Stokes Theorem and by [15, Lemma 9.17], since $D(P) \subset W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O})$,

$$\langle P\varphi, \varphi \rangle_H = \int_{\mathcal{O}} \Delta^2\varphi \cdot \varphi \, dx = \int_{\mathcal{O}} (\Delta\varphi, \Delta\varphi) dx = |\Delta\varphi|_H^2 \geq 0 \geq K|u|_H^2.$$

where the constant $K > 0$ is independent of φ . Set $A = P^{\frac{1}{2}}$. Then by [32], $D(A) = [H, D(P)]_{1/2} = \{\varphi \in W^{2,2}(\mathcal{O}) : \varphi = 0 \text{ on } \partial\mathcal{O} \text{ and } \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Gamma_1\}$. Thus we infer that $D(A) \subset D(B)$.

As in the previous Example 2.9, by adapting 17-18 on functions Υ and Π , we can verify that all the requirements on the functions f and g are fulfilled.

Therefore, Theorem 2.6 and 2.7 are applicable to Equation 16 with the mixed clamped/hinged boundary conditions

$$u = 0 \text{ on } \partial\mathcal{O}, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Gamma_1, \quad \Delta u = 0 \text{ on } \partial\Gamma_2. \quad (19)$$

Thus, as in the previous Example, we have the following Assertion: if $n = 1$ or $n \leq 3$ and the functions Υ and ψ depend only on the first and the second variables (i.e. on x and u), then there exists a unique maximal global mild solution to Equation 16 with the mixed clamped/hinged boundary conditions 19.

Example 2.11 (Nonlinear stochastic wave equations). We consider the following stochastic equation on \mathbb{R}^d

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \delta^2 \Delta u - \gamma^2 u + \Gamma(t, x, u, \nabla u, \frac{\partial u}{\partial t}) + \int_Z \Lambda(t, x, u, \nabla u, \frac{\partial u}{\partial t}, z) \tilde{N}(t, dz), \\ u(0) &= u_0, \quad u_t(0) = v_0 \end{aligned} \quad (20)$$

where $\gamma > 0$ and Γ and Λ are nonlinear terms. Let $H = L^2(\mathbb{R}^d)$. We consider two cases $d = 1$ and $d \geq 2$. We suppose that:

1. If $d = 1$, the function $\Gamma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz with respect to the 3rd variable, locally uniformly with respect to the 1st and uniformly with respect to the 2nd, 4th and 5th.
2. If $d \geq 2$, the function $\Gamma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz with respect to the 3rd, 4th and 5th variables, locally uniformly with respect to the 1st and uniformly with respect to the 2nd.
3. If $d = 1$, the function $\tilde{\Lambda} : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow L^2(Z, \nu)$ (associated with function $\Lambda(t, x, u, \nabla u, u_t)$) is locally Lipschitz is with respect to the 3rd variable and globally Lipschitz with respect to 4th and 5th, locally uniformly with respect to the 1st and uniformly with respect to the 2nd.

³For instance, if $n = 1$ and $\mathcal{O} = (a, b)$ then we can have $\Gamma_1 = \{a\}$ and $\Gamma_2 = \{b\}$.

4. If $d \geq 2$, the function $\tilde{\Lambda} : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow L^2(Z, \nu)$ (associated with function $\Lambda(t, x, u, \nabla u, u_t)$) is globally Lipschitz is with respect to the 3rd, 4th and 5th variables, locally uniformly with respect to the 1st and uniformly with respect to the 2nd.

Then Equation 20 has a unique local mild solution.

Moreover, if there exists $L > 0$ and L' and nonnegative functions $\kappa, \kappa' \in L^1(\mathbb{R}^d)$ such that for all $t \in [0, \infty)$, almost all $x \in \mathbb{R}^d$ and all $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^d$

$$\langle \Gamma(t, x, a, b, c), c \rangle \geq -L(\kappa(x) + |c|^2),$$

$$\int_Z |\Lambda(t, x, a, b, c, z)|^2 \nu(dz) \leq L'(\kappa'(x) + |a|^2 + |b|^2 + |c|^2).$$

then Equation 20 has a unique global mild solution.

Remark 5. As we will see below, the wave equation is a special case of the general problem with $m = 0$.

Proof of the Claim made in Example 2.11. Let A be positive square root of the strictly positive self-adjoint operator $-\delta^2 \Delta + \gamma^2 I$ on H . Then $A \geq \gamma I$, $D(A) = W^{1,2}(\mathbb{R}^d)$ and

$$|Au|_H^2 = \delta^2 |\nabla u|_{L^2}^2 + \gamma^2 |Au|_H^2.$$

Although, we could consider a more general case, we put $m = 0$.

Define f and g to be the Nemytski maps corresponding to functions Γ and Λ respectively, i.e.

$$f : [0, T] \times D(A) \times L^2(\mathbb{R}^d) \ni (t, \psi, \phi) \mapsto \Gamma(t, \cdot, \psi(\cdot), \nabla \psi(\cdot), \phi(\cdot)) \in L^2(\mathbb{R}^d),$$

$$g : [0, T] \times D(A) \times L^2(\mathbb{R}^d) \times Z \ni (t, \psi, \phi, z) \mapsto \Lambda(t, \cdot, \psi(\cdot), \psi(\cdot), \phi(\cdot), z) \in L^2(\mathbb{R}^d).$$

Put $C = \frac{1}{\min\{\delta^2, \gamma^2\}}$. Consider the case $d = 1$ as the other one is trivial. Let $\psi_1, \psi_2 \in D(A)$ with their norms being bounded by a fixed $R > 0$. Since by the Sobolev embedding Theorem, $W^{1,2}(\mathbb{R}) \subset C_b(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we can find $N > 0$ such that $|\psi_i|_{L^\infty} \leq N$. Denoting by L_N the Lipschitz constant of the function Γ on ball of radius N with respect to the 3rd variable and by L the Lipschitz constant of the function Γ with respect to the 4th and 5th variables we get, for all $t \in [0, \infty)$,

$$\begin{aligned} |f(t, \psi_1, \phi_1) - f(t, \psi_2, \phi_2)|_H^2 &= \int_{\mathbb{R}^d} |\Gamma(t, x, \psi_1(x), \nabla \psi_1(x), \phi_1(x)) - \Gamma(t, x, \psi_2(x), \nabla \psi_2(x), \phi_2(x))|^2 dx \\ &\leq L_N |\psi_1 - \psi_2|_H^2 + L \left(|\nabla \psi_1 - \nabla \psi_2|_H^2 + |\phi_1 - \phi_2|_H^2 \right) \\ &\leq C \left(L_N \gamma^2 |\psi_1 - \psi_2|_H^2 + L \delta^2 |\nabla \psi_1 - \nabla \psi_2|_H^2 \right) + L |\phi_1 - \phi_2|_H^2 \\ &= C \max\{L, L_N\} |A\psi_1 - A\psi_2|_H^2 + L |\phi_1 - \phi_2|_H^2, \end{aligned}$$

and analogously,

$$\int_Z |g(t, \psi_1, \phi_1) - g(t, \psi_2, \phi_2)|_H^2 \nu(dz) \leq C \max\{L, L_N\} |A\psi_1 - A\psi_2|_H^2 + L |\phi_1 - \phi_2|_H^2.$$

Similarly, in the general case, it's easy to verify that

$$\begin{aligned} \int_Z |g(t, \psi, \phi)|_H^2 \nu(dz) &= \int_Z \int_{\mathbb{R}^d} |\Lambda(t, \psi(x), \nabla \psi(x), \phi(x), z)| dx \nu(dz) \\ &= \int_{\mathbb{R}^d} \int_Z |\Lambda(t, \psi(x), \nabla \psi(x), \phi(x), z)| \nu(dz) dx \\ &\leq CL'(\gamma^2 |\psi|_H^2 + \delta^2 |\nabla \psi|_H^2) + L'(|\kappa'|_{L^1} + |\phi|_H^2) \\ &= CL'|A\psi|_H^2 + L'|\phi|_H^2 + L'|\kappa'|_{L^1}. \end{aligned}$$

$$\begin{aligned} \langle f(y, \psi, \phi), \phi \rangle &= \int_{\mathbb{R}^d} \langle \Gamma(t, x, \psi(x), \nabla \psi(x), \phi(x)), \phi(x) \rangle dx \\ &\geq -L \int_{\mathbb{R}^d} (\kappa(x) + |\phi(x)|^2) dx = -L|\kappa|_{L^1} - L|\phi|_H^2. \end{aligned}$$

□

Remark 6. As in [8] it is worth pointing out that our results are valid for bounded and unbounded domains.

3. An auxiliary Lemma. Let $(e^{tA})_{t \in \mathbb{R}}$ be a C_0 -semigroup on a Hilbert space \mathcal{H} . Assume that τ is an accessible stopping time. Let $\varphi = (\varphi(t))$, $t \geq 0$ be an \mathcal{H} -valued process belonging to $\mathcal{M}_{loc}^2(\mathcal{P} \otimes \mathcal{Z}, \mathcal{H})$. Set

$$\begin{aligned} I(t) &= I(\varphi; t) = \int_0^t \int_Z e^{(t-s)A} \varphi(s, z) \tilde{N}(ds, dz), \quad t \geq 0, \\ I_\tau(t) &= I_\tau(\varphi; t) = \int_0^t \int_Z 1_{[0, \tau]}(s) e^{(t-s)A} \varphi(s \wedge \tau, z) \tilde{N}(ds, dz), \quad t \geq 0. \end{aligned}$$

Note that by the choice of process φ and the assumption about $(e^{tA})_{t \in \mathbb{R}}$, the stochastic convolution process $I(t)$, $t \geq 0$, is well defined. Also for any stopping time τ , the process $1_{[0, \tau]}(t, \omega)$ is predictable. In fact, the predictable σ -field is generated by the family of closed stochastic intervals $\{[0, T] : T \text{ is a stopping time}\}$, see [30]. This together with the predictability of φ implies that integrand of $I_\tau(t)$ is predictable. Thus the stochastic convolution $I_\tau(t)$ is well defined as well. Moreover, one can always assume that the stochastic convolution process $I(t)$ and $I_\tau(t)$, $t \geq 0$ are \mathcal{H} -valued càdlàg, see [35]. The following lemma verifies the definition 11 of a local mild solution. The proof below is mainly based on [8] (which in turn was provided by Martin Ondreját, see [22]). Let us point out a minor but important difference with [8]. We consider here closed random intervals $[0, \tau]$, while in the case of a Wiener process [8] we considered open random intervals $(0, \tau)$. Our formula 21 is also more general than formula [8, (A.4)] as we allow an additional time parameter r .

Lemma 3.1. Under the assumptions listed above, for any stopping time τ and for all $r \geq t \geq 0$,

$$e^{(r-t \wedge \tau)A} I(t \wedge \tau) = e^{(r-t)A} I_\tau(t), \quad \mathbb{P} - \text{a.s.} \quad (21)$$

In particular,

$$I(t \wedge \tau) = I_\tau(t \wedge \tau). \quad (22)$$

Remark 7. It is known, see [6] and [8] for the Wiener process case, that is if ξ is another process satisfying the same conditions as φ such that for some $\Omega_0 \in \mathcal{F}$,

$$\varphi = \xi \text{ Leb} \otimes \mathbb{P} \otimes \nu \text{ a.s. on } [0, \infty) \times \Omega_0 \times Z$$

then \mathbb{P} -a.s. on Ω_0 ,

$$\int_0^t \int_Z \varphi(s, z) \tilde{N}(ds, dz) = \int_0^t \int_Z \xi(s, z) \tilde{N}(ds, dz), \quad t \geq 0.$$

In particular, if $\Omega_0 \subset \{\omega \in \Omega : \xi(\cdot, \omega, \cdot) = \varphi(\cdot, \omega, \cdot) \text{ on } [0, \tau(\omega)]\}$, then the above equality implies that \mathbb{P} -a.s. on Ω_0

$$\int_0^{t \wedge \tau} \int_Z \varphi(s, z) \tilde{N}(ds, dz) = \int_0^{t \wedge \tau} \int_Z \xi(s, z) \tilde{N}(ds, dz), \quad t \geq 0.$$

One of the consequences of Lemma 3.1 is the following modification of the above assertions.

Corollary 1. Suppose that ξ is another process satisfying the same conditions as φ and

$$\varphi = \xi \text{ Leb} \otimes \mathbb{P} \otimes \nu \text{ a.s. on } [0, \tau] \times \Omega \times Z,$$

where $[0, \tau] \times \Omega \times Z := \{(t, \omega, z) \in [0, \infty) \times \Omega \times Z : 0 \leq t \leq \tau(\omega)\}$, then $t \geq 0$, \mathbb{P} -a.s.,

$$I(\varphi; t \wedge \tau) = I(\xi; t \wedge \tau).$$

Proof. Indeed, note that obviously, for $t \geq 0$, \mathbb{P} -a.s.,

$$I_\tau(\varphi; t) = I_\tau(\xi; t),$$

Thus, equality follows by applying 22. □

Remark 8. An alternative approach to Lemma 3.1 and Corollary 1 is via the method implemented by [7] and [11], i.e. maximal inequalities for stopped stochastic convolution processes. In our case, this would require to use Theorems 4.4, 4.5 and 5.1 from [35].

Proof of Lemma 3.1. It is enough to proof identity 21 as the equality 22 follows from the former by replacing the deterministic times $r = t$ by a stopping time $t \wedge \tau$. It is also enough to proof identity 21 when $r = t$.

We first verify it for deterministic time. Let $\tau = a$. If $t < a$, then

$$\begin{aligned} e^{(t-t\wedge a)\mathcal{A}}I(t\wedge a) &= I(t) = \int_0^T \int_Z 1_{[0,t]} e^{(t-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &= \int_0^T \int_Z 1_{[0,t]} 1_{[0,a]} e^{(t-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &= \int_0^t \int_Z 1_{[0,a]} e^{(t-s)\mathcal{A}} \varphi(s \wedge a, z) \tilde{N}(ds, dz) = I_a(t), \end{aligned}$$

where we used in the equality the fact that $1_{[0,a]}(s)\varphi(s, z) = 1_{[0,a]}(s)\varphi(s \wedge a, z)$.

If $t \geq a$, then

$$\begin{aligned} e^{(t-t\wedge a)\mathcal{A}}I(t\wedge a) &= e^{(t-a)\mathcal{A}}I(a) = e^{(t-a)\mathcal{A}} \int_0^a \int_Z e^{(a-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &= e^{(t-a)\mathcal{A}} \int_0^T \int_Z 1_{[0,a]}(s) e^{(a-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &\quad + e^{(t-a)\mathcal{A}} \int_0^T \int_Z 1_{(a,t]}(s) 1_{[0,a]}(s) e^{(a-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &= e^{(t-a)\mathcal{A}} \int_0^a \int_Z 1_{[0,a]}(s) e^{(a-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &\quad + e^{(t-a)\mathcal{A}} \int_a^t \int_Z 1_{[0,a]}(s) e^{(a-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \\ &= e^{(t-a)\mathcal{A}} \int_0^t \int_Z 1_{[0,a]}(s) e^{(a-s)\mathcal{A}} \varphi(s \wedge a, z) \tilde{N}(ds, dz) \\ &= \int_0^t \int_Z 1_{[0,a]}(s) e^{(t-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) = I_a(t). \end{aligned}$$

Thus equality 21 holds for any deterministic time.

Now let τ be an arbitrary stopping time. Define $\tau_n := 2^{-n}([2^n \tau] + 1)$, for each $n \in \mathbb{N}$. That is $\tau_n = \frac{k+1}{2^n}$ if $\frac{k}{2^n} \leq \tau < \frac{k+1}{2^n}$. Then $\tau_n \searrow \tau$ as $n \rightarrow \infty$ pointwise. Since equality 21 has been proved for each deterministic time $k2^{-n}$, in view of Remark 7 we infer that

$$\begin{aligned} e^{(t-t\wedge\tau_n)\mathcal{A}}I(t\wedge\tau_n) &= \sum_{k=0}^{\infty} 1_{\{k2^{-n} \leq \tau < (k+1)2^{-n}\}} e^{(t-t\wedge(k+1)2^{-n})\mathcal{A}} I(t \wedge (k+1)2^{-n}) \\ &= \sum_{k=0}^{\infty} 1_{\{k2^{-n} \leq \tau < (k+1)2^{-n}\}} I_{(k+1)2^{-n}}(t) = I_{\tau_n}(t). \end{aligned} \tag{23}$$

Since $\tau_n \searrow \tau$, by the \mathbb{P} -a.s. right-continuity of the process I , see [35], the random variable $I(t \wedge \tau_n)$ converges pointwise on Ω to $I(t \wedge \tau)$ as $n \rightarrow \infty$ for every $t \geq 0$ \mathbb{P} -a.s. Also, observe that

$$\begin{aligned} &\left| e^{(t-t\wedge\tau_n)\mathcal{A}}I(t\wedge\tau_n) - e^{(t-t\wedge\tau)\mathcal{A}}I(t\wedge\tau) \right| \\ &\leq \left| e^{(t-t\wedge\tau_n)\mathcal{A}}(I(t\wedge\tau_n) - I(t\wedge\tau)) \right| + \left| \left(e^{(t-t\wedge\tau_n)\mathcal{A}} - e^{(t-t\wedge\tau)\mathcal{A}} \right) I(t\wedge\tau) \right| \\ &\leq |I(t\wedge\tau_n) - I(t\wedge\tau)| + \left| \left(e^{(t-t\wedge\tau_n)\mathcal{A}} - e^{(t-t\wedge\tau)\mathcal{A}} \right) I(t\wedge\tau) \right|. \end{aligned}$$

converges to 0 as $n \rightarrow \infty$. Thus we conclude that $e^{(t-t\wedge\tau_n)\mathcal{A}}I(t \wedge \tau_n)$ converges to $e^{(t-t\wedge\tau)\mathcal{A}}I(t \wedge \tau)$, for each $t \geq 0$, \mathbb{P} -a.s. For the term $I_{\tau_n}(t)$, by the Itô isometry we find out that

$$\begin{aligned} \mathbb{E}|I_{\tau_n}(t) - I_{\tau}(t)|^2 &= \mathbb{E} \left| \int_0^t \int_Z (1_{[0, \tau_n]}(s) - 1_{[0, \tau]}(s)) e^{(t-s)\mathcal{A}} \varphi(s, z) \tilde{N}(ds, dz) \right|^2 \\ &= \mathbb{E} \int_0^t \int_Z \left| (1_{[0, \tau_n]}(s) - 1_{[0, \tau]}(s)) e^{(t-s)\mathcal{A}} \varphi(s, z) \right|^2 \nu(dz) ds. \end{aligned}$$

Recall that that $\tau_n \downarrow \tau$ as $n \rightarrow \infty$. So $1_{[0, \tau_n]}$ converges to $1_{[0, \tau]}$ as $n \rightarrow \infty$. Obviously, the integrand is bounded by $|\varphi(s, z)|^2$ for all n . Thus by the Lebesgue dominated convergence theorem it follows that

$$\lim_{n \rightarrow \infty} \mathbb{E}|I_{\tau_n}(t) - I_{\tau}(t)|^2 \rightarrow 0.$$

Hence we can always find a subsequence which is convergent a.s. Finally, Letting $n \rightarrow \infty$ in both sides of 23 yields 21. This completes the proof. \square

4. Proofs of Proposition 1.

Proof of Proposition 1. Set $\tilde{F}(t, x) = F(t, x) + \mathbf{M}(x)$, for $t \geq 0$, $x \in \mathcal{H}$. Since functions F , \mathbf{M} and G are Lipschitz on closed balls in \mathcal{H} , for every $n \in \mathbb{N}$ we may find globally Lipschitz functions $\tilde{F}_n : \mathcal{H} \rightarrow \mathcal{H}$ and $G_n : \mathcal{H} \rightarrow \mathcal{H}$ such that $\tilde{F}_n = \tilde{F}$ and $G_n = G$ on $\bar{B}_{\mathcal{H}}(0, n)$, the closed ball in \mathcal{H} of radius n and centered at the origin. By using a classical argument we infer that there exists a unique mild solution $(\mathbf{u}_n(t))_{t \geq 0}$ to problem 10 with \tilde{F} replaced by \tilde{F}_n and G replaced by G_n , see e.g. Theorem 4.1.10 in [34]. By the càdlàg property of the process \mathbf{u}_n , a random variable τ_n defined by

$$\tau_n := \inf\{t \geq 0 : |\mathbf{u}_n(t)|_{\mathcal{H}} \geq n\}$$

is a stopping time. So,

$$\tilde{F}_n(s, \mathbf{u}_n(s)) = \tilde{F}(s, \mathbf{u}_n(s)) \text{ and } G_n(s, \mathbf{u}_n(s), z) = G(s, \mathbf{u}(s), z) \text{ on } [0, \tau_n).$$

It follows that on $[0, \tau_n)$ we have

$$\mathbf{u}_n(t) = e^{t\mathcal{A}}\mathbf{u}_0 + \int_0^t e^{(t-s)\mathcal{A}} \tilde{F}(s, \mathbf{u}_n(s)) ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}} G(s, \mathbf{u}_n(s-), z) \tilde{N}(ds, dz). \quad (24)$$

Let $\Phi(\mathbf{u}_n)$ denote the right hand side of 24. Note that the value of $\Phi(\mathbf{u}_n)$ at τ_n depends only on the values of \mathbf{u}_n on $[0, \tau_n)$, i.e.

$$\Delta\Phi(\mathbf{u}_n)(\tau_n) := \Phi(\mathbf{u}_n)(\tau_n) - \Phi(\mathbf{u}_n)(\tau_n-) = \int_Z G(\tau_n, \mathbf{u}_n(\tau_n-), z) \tilde{N}(\{\tau_n\}, dz).$$

Hence we may extend the process \mathbf{u}_n from the interval $[0, \tau_n)$ to the closed interval $[0, \tau_n]$ by setting

$$\mathbf{u}_n(\tau_n) = \Phi(\mathbf{u}_n)(\tau_n) = e^{\tau_n\mathcal{A}}\mathbf{u}_0 + \int_0^{\tau_n} e^{(\tau_n-s)\mathcal{A}} \tilde{F}(s, \mathbf{u}_n(s)) ds + I_{\tau_n}(G(\mathbf{u}_n))(\tau_n), \quad (25)$$

where $I_{\tau_n}(G(\mathbf{u}_n))$ is a process defined by 12 with \mathbf{u} replaced by \mathbf{u}_n , i.e.

$$I_{\tau_n}(G(\mathbf{u}_n))(t) = \int_0^t \int_Z 1_{[0, \tau_n]} e^{(t-s)\mathcal{A}} G(s, \mathbf{u}_n(s-), z) \tilde{N}(ds, dz), \quad t \geq 0.$$

Therefore, combining equalities 24 and 25, we deduce that the stopped process $\mathbf{u}_n(\cdot \wedge \tau_n)$ satisfies

$$\mathbf{u}_n(t \wedge \tau_n) = e^{(t \wedge \tau_n)\mathcal{A}}\mathbf{u}_0 + \int_0^{t \wedge \tau_n} e^{(t \wedge \tau_n - s)\mathcal{A}} \tilde{F}(s, \mathbf{u}_n(s)) ds + I_{\tau_n}(G(\mathbf{u}_n))(t \wedge \tau_n), \quad t \geq 0.$$

For $n < m$, set $\tau_{n,m} = \tau_n \wedge \tau_m$. It follows that $|\mathbf{u}_n(t)| \leq n < m$ and $|\mathbf{u}_m(t)| \leq m$ for $t \in [0, \tau_{n,m})$. So $\tilde{F}_n(s, \mathbf{u}_n(s)) = \tilde{F}(s, \mathbf{u}_n(s))$ and $\tilde{F}_m(s, \mathbf{u}_m(s)) = \tilde{F}(s, \mathbf{u}_m(s))$ for $s \in [0, \tau_{n,m})$. Also $G_n(s, \mathbf{u}_n(s)) = G(s, \mathbf{u}_n(s))$ and $G_m(s, \mathbf{u}_m(s)) = G(s, \mathbf{u}_m(s))$ for $s \in [0, \tau_{n,m})$. Therefore, both \mathbf{u}_n and \mathbf{u}_m , solve on $[0, \tau_{n,m})$ the same equation 10. Hence by the uniqueness of mild solutions, see [34, Theorem 4.1.10], we

have $\mathbf{u}_n(t) = \mathbf{u}_m(t)$, for $t \in [0, \tau_{n,m})$, a.s. Since $\Delta \mathbf{u}_i(\tau_{n,m}) = \int_Z G(\tau_{n,m}, \mathbf{u}_i(\tau_{n,m}-), z) \tilde{N}(\{\tau_{n,m}\}, dz)$, for $i = n, m$ and, by Remark 1, $G(s, \mathbf{u}_n(s-), z)$ and $G(s, \mathbf{u}_m(s-), z)$ coincide on $[0, \tau_{n,m}]$, we infer that

$$\mathbf{u}_n = \mathbf{u}_m \text{ on } [0, \tau_{n,m}]. \quad (26)$$

Hence, arguing by contradiction, we can show that a.s.

$$\tau_n \leq \tau_m \text{ if } n < m.$$

So the limit $\lim_{n \rightarrow \infty} \tau_n =: \tau_\infty$ exists a.s. Let us denote $\Omega_0 = \{\omega : \lim_{n \rightarrow \infty} \tau_n = \tau_\infty\}$ and note that $\mathbb{P}(\Omega_0) = 1$.

Now define a local process $(\mathbf{u}_t)_{0 \leq t < \tau_\infty}$ as follows. If $\omega \notin \Omega_0$, set $\mathbf{u}(t, \omega) = 0$, for $0 \leq t < \tau_\infty$. If $\omega \in \Omega_0$, then for every $t < \tau_\infty(\omega)$, there exists a number $n \in \mathbb{N}$ such that $t \leq \tau_n(\omega)$ and we set $\mathbf{u}(t, \omega) = \mathbf{u}_n(t, \omega)$. In view of 26 this process is well defined and it satisfies

$$\mathbf{u}(t \wedge \tau_n) = e^{(t \wedge \tau_n) \mathcal{A}} \mathbf{u}_0 + \int_0^{t \wedge \tau_n} e^{(t \wedge \tau_n - s) \mathcal{A}} \tilde{F}(s, \mathbf{u}(s)) ds + I_{\tau_n}(G(\mathbf{u}))(t \wedge \tau_n), \quad t \geq 0,$$

where we used the fact because that of 26, for all $t \geq 0$,

$$I_{\tau_n}(G(\mathbf{u}_n))(t) = I_{\tau_n}(G(\mathbf{u}))(t).$$

Furthermore, by the definition of the sequence $\{\tau_n\}_{n=1}^\infty$ we infer that a.s. on the set $\{\tau_\infty < \infty\}$,

$$\lim_{t \nearrow \tau_\infty(\omega)} |\mathbf{u}(t, \omega)|_{\mathcal{H}} = \lim_{n \nearrow \infty} |\mathbf{u}(\tau_n(\omega), \omega)|_{\mathcal{H}} \geq \lim_n n = \infty. \quad (27)$$

Next we will show that the process $\mathbf{u}(t)$, $0 \leq t < \tau_\infty$ is a maximal local mild solution to Problem 10. Let us suppose that $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}(t))_{0 \leq t < \tilde{\tau}}$ is another local mild solution to Problem 10 such that $\tilde{\tau} \geq \tau_\infty$ a.s. and $\tilde{\mathbf{u}}|_{[0, \tau_\infty) \times \Omega}$ is equivalent to \mathbf{u} . It follows from 27 and the \mathbb{P} -equivalence of \mathbf{u} and $\tilde{\mathbf{u}}$ on $[0, \tau_\infty)$ that

$$\lim_{t \nearrow \tau_\infty(\omega)} |\tilde{\mathbf{u}}(t, \omega)|_{\mathcal{H}} = \lim_{t \nearrow \tau_\infty(\omega)} |\mathbf{u}(t, \omega)|_{\mathcal{H}} = \infty. \quad (28)$$

It remains to show that $\mathbb{P}(\tilde{\tau} > \tau_\infty) = 0$. To prove this, assume the contrary, namely $\mathbb{P}(\tilde{\tau} > \tau_\infty) > 0$. Since $\tilde{\mathbf{u}}$ is a local mild solution, there exists a sequence $\{\tilde{\tau}_n\}$ of increasing stopping times such that $\tilde{\mathbf{u}}$ is a mild solution on the interval $[0, \tilde{\tau}_n]$, i.e. the equation 11 is satisfied. Define a new family of stopping times by

$$\sigma_{n,k} := \tilde{\tau}_n \wedge \inf\{t : |\tilde{\mathbf{u}}(t)| > k\}; \quad \sigma_k := \sup \sigma_{n,k}.$$

Since $\sigma_{n,k} \leq \tilde{\tau}_n \leq \tilde{\tau}$, $\sigma_k \leq \tilde{\tau}$. Also, observe that $\lim_k \sigma_k = \tilde{\tau}$. Since $\sigma_k \nearrow \tilde{\tau}$ and $\mathbb{P}(\tilde{\tau} > \tau_\infty) > 0$, there exists a number k such that $\mathbb{P}(\sigma_k > \tau_\infty) > 0$. Hence, we have $|\tilde{\mathbf{u}}(t, \omega)|_{\mathcal{H}} \leq k$ for $t \in [\tau_\infty(\omega), \sigma_k(\omega))$ contradicting the earlier observation 28. Moreover the uniqueness of the solution follows immediately from the above construction of the solution \mathbf{u} . \square

The proof that the maximal local solution is a global one is based on the use of the Khashminski test, see [20] and [8]. The essence of this method is to prove first the existence and uniqueness of a local maximal solution, then to find an appropriate Lyapunov function, with the help of which prove the life span of that local maximal solution is equal to ∞ . This method had been earlier used in the parabolic case in [11].

In order to prove the main result we need the following auxiliary standard result whose proof can be found in [8]. The function V satisfying conditions below is called a Lyapunov function for Equation 10.

Lemma 4.1. (Khashminski's test for non-explosion) Let V be a continuous function for which

$$\mathbb{E}V(\mathbf{u}_0) < \infty \text{ and } V(x) \rightarrow +\infty \text{ as } |x|_{\mathcal{H}} \rightarrow \infty. \quad (29)$$

Let $\mathbf{u}(t)$, $0 \leq t < \tau_\infty$ be a maximal local mild solution to Equation 10 with an approximating sequence $\{\tau_n\}_{n \in \mathbb{N}}$. Let $T > 0$. Suppose that there exists constants $C_i > 0$, $i = 1, 2$, such that for every $t \in [0, T]$, and $n \in \mathbb{N}$,

$$\mathbb{E}V(\mathbf{u}(t \wedge \tau_n)) \leq \mathbb{E}V(\mathbf{u}_0) + \int_0^t (C_1 + C_2 \mathbb{E}(V(\mathbf{u}(s \wedge \tau_n)))) ds. \quad (30)$$

Then $\tau_\infty \wedge T = T$, \mathbb{P} -a.s..

5. **An Itô Lemma.** In this section we will formulate a general form of an Itô Lemma which in the next two sections will be used to prove the existence of a global solution and the asymptotic boundedness of the solutions. The novelty of our approach is that we prove one result general enough to cover the two different cases. In the case of stochastic beam equation driven by a Wiener process, see [8], the authors formulated and proved two separate results. Our proof would yield, had we considered that case here, those results as corollaries.

We begin with formulation of the general assumptions and the Itô Lemma. This will be proceeded by two examples when these general assumptions are satisfied. The section will be finished with the proof of the Itô Lemma.

Assumption 5.1. Assume that Q is linear self-adjoint, strictly positive and bounded operator on \mathcal{H} such that the quadratic form

$$D(\mathcal{A}) \ni x \mapsto \langle Ax, Qx \rangle_{\mathcal{H}} \in \mathbb{R}$$

has a unique extension from $D(\mathcal{A})$ to a bounded and symmetric quadratic form

$$\Gamma : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$$

on the whole \mathcal{H} .

Put, for $x = (x_1, x_2) \in \mathcal{H}$,

$$V_0(x) = \frac{1}{2} \langle x, Qx \rangle_{\mathcal{H}}, \quad V_1(x) = m(|B^{\frac{1}{2}}x_1|_{\mathcal{H}}^2),$$

and define a function $V : \mathcal{H} \rightarrow \mathbb{R}_+$ by

$$V(x) = V_0(x) + V_1(x) = \frac{1}{2} \langle x, Qx \rangle_{\mathcal{H}} + m(|B^{\frac{1}{2}}x_1|_{\mathcal{H}}^2), \quad x \in \mathcal{H}. \quad (31)$$

An obvious consequence of the above definition is that $V(x) = V_0(x)$ whenever $\pi_1 x = 0$ i.e.

$$V(x) = \frac{1}{2} \langle x, Qx \rangle_{\mathcal{H}}, \quad \text{if } x = (0, x_2) \in \mathcal{H}.$$

However, we also have two important equalities, for $x \in \mathcal{H}$ and $y = (0, y_2) \in \mathcal{H}$:

$$V(x + y) - V(x) = \frac{1}{2} \langle y, Qy \rangle_{\mathcal{H}} + \langle Qx, y \rangle_{\mathcal{H}} \quad (32)$$

$$V'(x)(y) = \langle Qx, y \rangle_{\mathcal{H}} \quad (33)$$

Note that since we assume that m is a C^1 -class function with locally Lipschitz derivative, the function V is also of C^1 -class on \mathcal{H} , with the Fréchet derivative of V being Lipschitz on balls of \mathcal{H} . See also Lemma 5.4 for further important consequences of this regularity assumption. But it seems that in order to be able to apply our Itô Lemma, it would be sufficient to assume that the derivative functions m' is only locally Hölder continuous.

Let us also note the following inequalities comparing the behaviour of $V(x)$ and $|x|_{\mathcal{H}}$.

$$\mu_0 |x|_{\mathcal{H}}^2 \leq V(x) \leq \mu_1 |x|_{\mathcal{H}}^2 + m(C|x|_{\mathcal{H}}^2) \quad (34)$$

for some $C > 0$, where $\mu_0 > 0$ and $\mu_1 > 0$ are such that

$$\mu_0 |x|_{\mathcal{H}}^2 \leq \langle Qx, x \rangle_{\mathcal{H}} \leq \mu_1 |x|_{\mathcal{H}}^2, \quad x \in \mathcal{H}. \quad (35)$$

In order to prove 34 let us note that in view of 35 it is sufficient to prove the second inequality in 34 for V_1 . Since by assumptions: the function m is increasing and, for some $C > 0$, $|By| \leq C^{\frac{1}{2}}|y|$, for $y \in D(A)$ (what makes sense since we also assume that $D(A) \subset D(B)$), we have for $x = (x_1, x_2) \in \mathcal{H} = D(A) \times H$,

$$\begin{aligned} V_1(x) &= m(|B^{\frac{1}{2}}x_1|_{\mathcal{H}}^2) = m(\langle Bx_1, x_1 \rangle) \leq m\left(\frac{1}{2}[|Bx_1|^2 + |x_1|^2]\right) \\ &\leq m\left(\frac{\max\{C, 1\}}{2}[|Ax_1|^2 + |x_1|^2]\right) = m\left(\frac{\max\{C, 1\}}{2}|x|_{\mathcal{H}}^2\right) \end{aligned}$$

Let us note that it follows from 34 that V is a bounded function on bounded subsets of \mathcal{H} and that it satisfies condition 29 of the Khasminski Lemma 4.1.

Now we are ready to formulate the announced Itô Lemma, the main result in this section.

Lemma 5.2. Assume that the operator Q satisfies Assumption 5.1 and the function m satisfies Assumption (C.5). Let the function V be defined by formula 31. Assume that $\mathbf{u} = (u, v)$ be a global mild solution of the problem 10 and let σ be a bounded stopping time such that the processes $\mathbf{u}(r)$, $F(r, \mathbf{u}(r))$ and $G(r, \mathbf{u}(r))$ are uniformly bounded on $[0, \sigma] \times \Omega$. Then we have \mathbb{P} -a.s.,

$$\begin{aligned} V(\mathbf{u}(\sigma))e^{\lambda\sigma} - V(\mathbf{u}(s))e^{\lambda s} &= \int_s^\sigma e^{\lambda r} \left[\lambda V(\mathbf{u}(r)) + \Gamma(\mathbf{u}(r), \mathbf{u}(r)) + 2m'(|B^{\frac{1}{2}}u(r)|^2) \langle Bu(r), v(r) \rangle_H \right. \\ &\quad \left. + V'(\mathbf{u}(r))(F(r, \mathbf{u}(r)) + \mathbf{M}(\mathbf{u}(r))) \right] dr \\ &+ \int_s^\sigma \int_Z e^{\lambda r} \left[V(\mathbf{u}(r) + G(r, \mathbf{u}(r), z)) - V(\mathbf{u}(r)) - V'(\mathbf{u}(r))(G(r, \mathbf{u}(r), z)) \right] \nu(dz) dr \\ &\quad + \int_s^\sigma \int_Z e^{\lambda r} \left[V(\mathbf{u}(r-) + G(r, \mathbf{u}(r), z)) - V(\mathbf{u}(r-)) \right] \tilde{N}(dr, dz) \end{aligned} \quad (36)$$

The main difficulty in proving the above version of the Itô Lemma stems from facts: (i) the operator \mathcal{A} unbounded, and (ii) the C_0 -semigroup generated by the operator \mathcal{A} is not analytic. Indeed, all three nonlinear maps F , \mathbf{M} and G are smooth maps defined on the state space \mathcal{H} .

Example 5.3. Let $\beta \geq 0$. Define a linear self-adjoint bounded operator $Q = Q_\beta$ on \mathcal{H} by

$$Q_\beta := \begin{pmatrix} \beta^2 A^{-2} + 2I & \beta A^{-2} \\ \beta I & 2I \end{pmatrix}. \quad (37)$$

Note that $Q_0 = 2I$.

The following properties of Q_β have been established in the last cited paper.

The operator Q_β is an isomorphism of \mathcal{H} and

$$|Q_\beta|_{\mathcal{L}(\mathcal{H})}^{-1} \langle Q_\beta \mathbf{u}, \mathbf{u} \rangle_{\mathcal{H}} \leq |\mathbf{u}|_{\mathcal{H}}^2 \leq \langle Q_\beta \mathbf{u}, \mathbf{u} \rangle_{\mathcal{H}}, \quad \mathbf{u} \in \mathcal{H}; \quad (38)$$

$$\langle (0, -\delta u_2), Q_\beta \mathbf{u} \rangle_{\mathcal{H}} = -\beta \delta \langle u_1, u_2 \rangle - 2\delta |u_2|^2 \quad \mathbf{u} = (u_1, u_2) \in \mathcal{H}, \delta \in \mathbb{R};$$

$$\langle \mathcal{A} \mathbf{u}, Q_\beta \mathbf{u} \rangle_{\mathcal{H}} = -\beta |Au_1|_H^2 + \beta^2 \langle u_1, u_2 \rangle + \beta |u_2|^2, \quad \mathbf{u} = (u_1, u_2) \in D(\mathcal{A}). \quad (39)$$

and hence the quadratic form

$$\langle \mathcal{A} \mathbf{u}, Q_\beta \mathbf{u} \rangle_{\mathcal{H}}, \quad \mathbf{u} \in D(\mathcal{A})$$

has a unique extension from $D(\mathcal{A})$ to a bounded quadratic form on the whole \mathcal{H} and thus the quadratic form Γ_β satisfies

$$\Gamma_\beta(\mathbf{u}, \mathbf{u}) = \beta \left[\beta \langle u_1, u_2 \rangle + |u_2|^2 - |Au_1|_H^2 \right], \quad \mathbf{u} = (u_1, u_2) \in \mathcal{H}. \quad (40)$$

In particular, Q_β satisfies Assumption 5.1. In sections 6 and resp. 7 we will use Lemma 5.2 with $Q = Q_\beta$ for $\beta = 0$, resp. $\beta > 0$.

To our best of our knowledge, the operator Q appeared for the first time in the paper [25] in connection with a stability analysis of linear deterministic hyperbolic equations and was applied to stochastic hyperbolic problems in [21]. The above result was also used in [8] for similar purposes as in the current paper.

Remark 9. As noted in Example 5.3 above, if $\beta = 0$ then $Q = 2I$, i.e. $Q_0 = 2I$. Then, by 40, $\Gamma_0 = 0$ and $\langle Q_0 x, y \rangle_{\mathcal{H}} = 2 \langle x_2, y_2 \rangle$ for all $x = (x_1, x_2) \in \mathcal{H}$ and $y = (0, y_2) \in \mathcal{H}$. Thus, invoking the definitions

7, 8 and 9 of maps F , \mathbf{M} and G in terms of maps f , m and g respectively, we infer that the following is thus a special case of our equality 36, where we also put $\lambda = 0$,

$$\begin{aligned} V(\mathbf{u}(\sigma)) - V(\mathbf{u}_0) &= \int_0^\sigma \left[-2\langle \pi_2 \mathbf{u}(r), f(r, z) \rangle_H + \int_Z |g(r, z)|_H^2 \nu(dz) \right] dr \\ &\quad + \int_s^\sigma \int_Z \left[2\langle \pi_2 \mathbf{u}(r-), g(r, z) \rangle_H + |g(r, z)|_H^2 \right] \tilde{N}(ds, dz) \end{aligned} \quad (41)$$

Indeed, when $Q = Q_\beta$, by 32 and 33, we have for $x = (x_1, x_2) \in \mathcal{H}$ and $y = (0, y_2) \in \mathcal{H}$

$$V(x + y) - V(x) = |y_2|_H^2 + \langle \beta x_1 + 2x_2, y_2 \rangle_H \quad (42)$$

$$V'(x)(y) = \langle \beta x_1 + 2x_2, y_2 \rangle_H. \quad (43)$$

Note however, that in our proof, contrary to the corresponding proof in [8], we will not prove the above special formula 41 but the general one 36

Note that every Hilbert space is 2-smooth. The following result is an immediate consequence of [31, Lemma 2.1 and Lemma 2.2].

Lemma 5.4. For every $r > 0$, there exists $C = C(r) > 0$ such that

$$|V(y) - V(x) - V'(x)(y - x)|_{\mathcal{H}} \leq C|y - x|_{\mathcal{H}}^2, \text{ for all } x, y \in B_r(0, \mathcal{H}),$$

Indeed, function V is of C^2 class and it's first Frechét derivative is Lipschitz on balls.

Proof of Lemma 5.2. We start the proof with constructing a sequence of global strong solutions which converges to the global mild solution uniformly. To do this, let us set, see [29], for $l \in \mathbb{N}$, $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $z \in Z$,

$$\begin{aligned} \mathbf{u}_l(0) &= l(lI - \mathcal{A})^{-1} \mathbf{u}(0), \\ F_l(t, \omega) &= l(lI - \mathcal{A})^{-1} \left[F(t, \mathbf{u}(t, \omega)) + \mathbf{M}(\mathbf{u}(t, \omega)) \right] \text{ and} \\ G_l(t, \omega, z) &= l(lI - \mathcal{A})^{-1} G(t, \mathbf{u}(t, \omega), z). \end{aligned}$$

We will apply the standard version of Itô formula for each fixed l and then a limit when $l \rightarrow \infty$ will be taken.

Let us introduce the following two canonical linear projections:

$$\pi_1 : \mathcal{H} \ni (x, y) \mapsto x \in D(\mathcal{A}) \text{ and } \pi_2 : \mathcal{H} \ni (x, y) \mapsto y \in H.$$

Let us also observe that

$$\pi_1 \mathcal{A} = \pi_2 \text{ on } D(\mathcal{A}) \text{ and } \pi_1 \mathbf{M} = 0 \text{ on } \mathbb{R}_+ \times \Omega. \quad (44)$$

Our approach here differs from the one used in [8], where instead the Yosida approximation of the operator A was used.

The following result can be applied to the above approximations with Y equal to \mathcal{H} and S equal to either $(0, T) \times \Omega$ or $(0, T) \times \Omega \times Z$.

Lemma 5.5. Suppose that Y is a separable Banach space, (S, \mathcal{S}, σ) a measure space, $p \in [1, \infty)$ and $\xi : S \rightarrow Y$ a Borel function such that

$$\int_S |\xi(s)|_Y^p d\sigma(s) < \infty.$$

Let \mathcal{A} be the infinitesimal generator of a contraction C_0 -semigroup on Y . Then

$$\lim_{l \rightarrow \infty} \int_S |\xi(s) - (lI - \mathcal{A})^{-1} \xi(s)|_Y^p d\sigma(s) = 0.$$

Proof. The proof is straightforward, since by the Hille-Yosida Theorem, we know $|(lI - \mathcal{A})^{-1}| \leq \frac{1}{l}$ and for every $s \in S$, $(lI - \mathcal{A})^{-1} \xi(s) \rightarrow \xi(s)$ pointwisely. \square

Since by definition, the processes F_l and G_l take values in $D(\mathcal{A})$, we infer $F_l \in \mathcal{M}_{loc}^2(\mathcal{BF} \otimes \mathcal{Z}; D(\mathcal{A}))$ and $G_l \in \mathcal{M}_{loc}^2(\mathcal{P} \otimes \mathcal{Z}; D(\mathcal{A}))$, for $l \in \mathbb{N}$. Therefore, the equation

$$\begin{aligned} du_l(t) &= \mathcal{A}u_l(t)dt + F_l(t)dt + \int_Z G_l(t, z)\tilde{N}(dt, dz), \quad t \geq 0 \\ u_l(0) &= u_l(0). \end{aligned}$$

has a unique global strong solution u_l given by

$$u_l(t) = u_l(0) + \int_0^t [\mathcal{A}u_l(r) + F_l(r, u(r))]dr + \int_0^t \int_Z G_l(r, u(r-), z)\tilde{N}(dr, dz), \quad t \geq 0.$$

Now we can apply Itô formula to the process u_l and the function $e^{\lambda t}V(x)$ to get

$$\begin{aligned} V(u_l(\sigma))e^{\lambda\sigma} - V(u_l(s))e^{\lambda s} &= \int_s^\sigma e^{\lambda r} [\lambda V(u_l(r)) + V'(u_l(r))(\mathcal{A}u_l(r) + F_l(r, u(r)))]dr \\ &+ \int_s^\sigma \int_Z e^{\lambda r} [V(u_l(r) + G_l(r, z)) - V(u_l(r)) - V'(u_l(r))(G_l(r, z))] \nu(dz) dr \\ &+ \int_s^\sigma \int_Z e^{\lambda r} [V(u_l(r-) + G_l(r, z)) - V(u_l(r-))] \tilde{N}(dr, dz). \end{aligned} \quad (45)$$

We next prove the following auxiliary result.

Lemma 5.6. For every $T > 0$ and every $n \in \mathbb{N}$,

$$\lim_{l \rightarrow \infty} \mathbb{E} \sup_{t \in [0, T]} |u_l(t) - u(t)|_{\mathcal{H}}^2 = 0.$$

Proof. Let us fix $T > 0$. Then we have

$$\begin{aligned} u_l(t) - u(t) &= \int_0^t e^{(t-s)\mathcal{A}} ([F(s, u(s)) + \mathbf{M}(u(t, \omega))] - F_l(s)) ds \\ &+ \int_0^t \int_Z e^{(t-s)\mathcal{A}} (G(s, u(s), z) - G_l(s, z)) \tilde{N}(ds, dz), \quad t \in [0, T]. \end{aligned}$$

The result follows by applying the Cauchy-Schwarz inequality, Lemma 5.5 to processes F and G and the Davis inequality for stochastic convolution processes, see [35]. \square

Therefore, by taking a subsequence we can deduce that \mathbb{P} -a.s., for every $T > 0$,

$$\lim_{l \rightarrow \infty} \sup_{t \in [0, T]} |u_l(t) - u(t)|_{\mathcal{H}}^2 = 0. \quad (46)$$

Since \mathcal{H} is a Hilbert space, the Frechét derivative V' of function V can be identified with the gradient DV of V which satisfy (we continue to use notation $x = (x_1, x_2)$ and $y = (y_1, y_2)$)

$$\begin{aligned} V'(x)(y) &= \langle Qx, y \rangle_{\mathcal{H}} + 2m'(|B^{\frac{1}{2}}x_1|_H^2) \langle Bx_1, y_1 \rangle_H, \quad x, y \in \mathcal{H}, \\ V'(x)(\mathcal{A}x) &= \langle Qx, \mathcal{A}x \rangle_{\mathcal{H}} + 2m'(|B^{\frac{1}{2}}x_1|_H^2) \langle Bx_1, x_2 \rangle_H, \quad x \in D(\mathcal{A}), \end{aligned} \quad (47)$$

see [8]. By taking into account 44 and Assumption 5.1 we infer that

$$\begin{aligned} V'(u_l(r))(\mathcal{A}u_l(r) + F_l(r)) &= \Gamma(u_l(r), u_l(r)) + 2m'(|B^{\frac{1}{2}}\pi_1 u_l(r)|_H^2) \langle B\pi_1 u_l(r), \pi_2 u_l(r) \rangle_H \\ &+ \langle Qu_l(r), F_l(r) \rangle_{\mathcal{H}} + 2m'(|B^{\frac{1}{2}}\pi_1 u_l(r)|_H^2) \langle B\pi_1 u_l(r), \pi_1 F_l(r) \rangle_H. \end{aligned} \quad (48)$$

Since $m \in C^1$ and $B \in \mathcal{L}(D(\mathcal{A}), H)$, we infer that \mathbb{P} -a.s. uniformly in $[0, \sigma(\omega)]$, $m'(|B^{\frac{1}{2}}\pi_1 u_l(s)|_H^2) \rightarrow m'(|B^{\frac{1}{2}}\pi_1 u(s)|_H^2)$, $\pi_2 u_l(s) \rightarrow \pi_2 u(s)$ and $B\pi_1 u_l(s) \rightarrow B\pi_1 u(s)$, as $l \rightarrow \infty$. Moreover, as the maps $\Gamma; \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $Q : \mathcal{H} \rightarrow \mathcal{H}$ are continuous, we infer that $\Gamma(u_l(r), u_l(r)) \rightarrow \Gamma(u(r), u(r))$ and $\langle Qu_l(r), F_l(r) \rangle_{\mathcal{H}} \rightarrow \langle Qu(r), F(r, u(r)) + \mathbf{M}(u(r)) \rangle_{\mathcal{H}}$. Since the process u_l is càdlàg, we infer that for every $l \in \mathbb{N}$, the set $\{u_l(t, \omega) : t \in [0, T]\}$ is relatively compact subset of \mathbb{R} for almost all ω and the sequence

$\{\mathbf{u}_l\}_{l \in \mathbb{N}}$ converges uniformly to v , \mathbb{P} -a.s., we infer that the set $\{\mathbf{u}_l(s), s \in [0, T], l \in \mathbb{N}\}$ is bounded in \mathcal{H} , \mathbb{P} -a.s. Therefore, \mathbb{P} -a.s.,

$$\langle Q\mathbf{u}_l(r), F_l(r) \rangle_{\mathcal{H}} \leq C|F_l(r)|_{\mathcal{H}} \sup_{0 \leq r \leq T} |\mathbf{u}_l(r)|_{\mathcal{H}} \leq C|F(r, \mathbf{u}(r)) + \mathbf{M}(\mathbf{u}(r))|_{\mathcal{H}}, \quad s \in [0, T].$$

Hence, by the Lebesgue DCT, we conclude that

$$\int_s^\sigma \langle Q\mathbf{u}_l(r), F_l(r) \rangle_{\mathcal{H}} dr \rightarrow \int_s^\sigma \langle Q\mathbf{u}(r), F(r, \mathbf{u}(r)) + \mathbf{M}(\mathbf{u}(r)) \rangle_{\mathcal{H}} dr \quad \mathbb{P}\text{-a.s.} \quad (49)$$

Analogously, because of 46 and 44 we can argue as in [8] and deduce that \mathbb{P} -a.s.

$$\begin{aligned} \int_s^\sigma 2m'(|B^{\frac{1}{2}}\pi_1\mathbf{u}_l(r)|_H^2) \langle B\pi_1\mathbf{u}_l(r), \pi_2\mathbf{u}_l(r) \rangle_H dr &\rightarrow \int_s^\sigma 2m'(|B^{\frac{1}{2}}\pi_1\mathbf{u}(r)|_H^2) \langle B\pi_1\mathbf{u}(r), \pi_2\mathbf{u}(r) \rangle_H dr \\ \int_s^\sigma 2m'(|B^{\frac{1}{2}}\pi_1\mathbf{u}_l(r)|_H^2) \langle B\pi_1\mathbf{u}_l(r), \pi_1 F_l(r) \rangle_H dr &\rightarrow \int_s^\sigma 2m'(|B^{\frac{1}{2}}\pi_1\mathbf{u}(r)|_H^2) \langle B\pi_1\mathbf{u}(r), \pi_1(F(r, \mathbf{u}(r)) \\ &\quad + \mathbf{M}(\mathbf{u}(r))) \rangle_H dr. \end{aligned} \quad (50)$$

Taking 48, 49, 50 and also 47 into account, the first term on the right side of inequality 45 converges \mathbb{P} -a.s. as $l \rightarrow \infty$ to

$$\begin{aligned} \int_s^\sigma e^{\lambda r} \left[\lambda V(\mathbf{u}(r)) + \Gamma(\mathbf{u}(r), \mathbf{u}(r)) + 2m'(|B^{\frac{1}{2}}\pi_1\mathbf{u}(r)|_H^2) \langle B\pi_1\mathbf{u}(r), \pi_2\mathbf{u}(r) \rangle_H \right. \\ \left. + V'(\mathbf{u}(r))(F(r, \mathbf{u}(r)) + \mathbf{M}(\mathbf{u}(r))) \right] dr. \end{aligned}$$

Now Set $X(\omega) = \{\mathbf{u}_l(s, \omega) : s \in [0, T], l \in \mathbb{N}\}$, for $\omega \in \Omega$. As we have noticed before, $X(\omega)$ is a bounded subset of \mathcal{H} for almost all $\omega \in \Omega$. Since V is of C^2 class on \mathcal{H} , by Lemma 5.4, we have

$$|V(\mathbf{u}_l(r) + G_l(r, z)) - V(\mathbf{u}_l(r)) - V'(\mathbf{u}_l(r))(G_l(r, z))| \leq C|G_l(r, z)|^2$$

By using above results, 46, along with the Lebesgue DCT, we infer that \mathbb{P} -a.s.

$$\begin{aligned} \lim_{l \rightarrow \infty} \int_s^\sigma \int_Z e^{\lambda r} \left[V(\mathbf{u}_l(r) + G_l(r, z)) - V(\mathbf{u}_l(r)) - V'(\mathbf{u}_l(r))(G_l(r, z)) \right] \nu(dz) dr \\ = \int_s^\sigma \int_Z e^{\lambda r} \left[V(\mathbf{u}(r) + G(r, \mathbf{u}(r), z)) - V(\mathbf{u}(r)) - V'(\mathbf{u}(r))(G(r, \mathbf{u}(r), z)) \right] \nu(dz) dr \end{aligned}$$

For the last term on the right side of inequality 45, we observe that

$$\begin{aligned} \left| V(\mathbf{u}_l(s) + G_l(r, z)) - V(\mathbf{u}_l(r)) - \left(V(\mathbf{u}(s) + G(r, \mathbf{u}(r), z)) - V(\mathbf{u}(r)) \right) \right|^2 \\ \leq 2 \sup_{x \in X(\omega)} |V'(x)|^2 |G(\mathbf{u}(r), z)|_{\mathcal{H}}^2. \end{aligned}$$

By applying the Itô isometry property of the stochastic integral, see [34] and then the the Lebesgue DCT we get

$$\begin{aligned} \lim_{l \rightarrow \infty} \mathbb{E} \left| \int_s^\sigma \int_Z e^{\lambda r} \left[V(\mathbf{u}_l(r-) + G_l(r, z)) - V(\mathbf{u}_l(r-)) \right] \tilde{N}(dr, dz) \right. \\ \left. - \int_s^\sigma \int_Z e^{\lambda r} \left[V(\mathbf{u}(r-) + G(r, \mathbf{u}(r), z)) - V(\mathbf{u}(r-)) \right] \tilde{N}(dr, dz) \right|_{\mathcal{H}}^2 = 0. \end{aligned}$$

Therefore, by passing a subsequence in 45, it is evident to see that the Itô formula 36 holds. \square

6. Proof of Theorem 2.6. We will now prove Theorem 2.6 with the help of the two lemmas from the preceding sections. In particular, as announced earlier, we will use Lemma 5.2 with the operator $Q = Q_0 = 2I$ and the parameter $\beta = 0$, see Example 5.3 and Remark 9.

Proof of Theorem 2.6. Let $\mathbf{u}(t)$, $0 \leq t < \tau_\infty$, be a maximal local mild solution to problem 10. Our aim is to prove that $\tau_\infty = \infty$. Before we continue with the proof let us observe that it is sufficient to prove that for any $T > 0$, $\tau_\infty \geq T$. This is of particular importance because our time dependent coefficients satisfy Conditions (C.3) and (C.4). For this purpose, we fix a positive time $T > 0$ and we replace the stopping time τ_∞ by $\tau_\infty \wedge T$. We will prove that $\tau_\infty \wedge T \geq T$. Whenever we will speak in this section about the infimum of an empty set we will define it to be equal to T .

Define a sequence of stopping times by

$$\tau_n = \inf \{t \in [0, T] : |\mathbf{u}(t)|_{\mathcal{H}} \geq n\}, \quad n \in \mathbb{N}. \quad (51)$$

As in the proof of Proposition 1, we can show that $\{\tau_n\}_{n \in \mathbb{N}}$ is an approximating sequence of the accessible stopping time $\tau_\infty \wedge T$.

Let us set $\beta = 0$. We have $Q = 2I$. Then $V(x) = |x|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}x_1|_H^2)$. It is clear that for every $x \in \mathcal{H}$, $V(x) \geq 0$. Let us define $q_R = \inf\{V(x) : |x|_{\mathcal{H}} \geq R\}$. Since $2q_R \geq \inf\{|x|_{\mathcal{H}}^2 : |x|_{\mathcal{H}} \geq R\} = R^2$ we infer that that $q_R \rightarrow +\infty$. Moreover, we have

$$\mathbb{E}(V(\mathbf{u}_0)) = \mathbb{E}|\mathbf{u}_0|_{\mathcal{H}}^2 + \mathbb{E}m(|B^{\frac{1}{2}}\mathbf{u}_0|_H) < \infty.$$

Now it remains to verify condition 30 from Lemma 4.1. Notice that the solution \mathbf{u} to Equation 10 is a process with possibly finite lifespan. We therefore introduce a sequence of globally defined processes \tilde{F}_n and \tilde{G}_n , $n \in \mathbb{N}$, such that, roughly speaking, up to the stopping time τ_n , the solution \mathbf{u} agrees with a solution \mathbf{v}_n of a certain linear stochastic evolution equation. The Itô formula is applied to the process \mathbf{v}_n and then a limit when $n \rightarrow \infty$ is taken.

Let us now show details of this program. We begin with fixing $n \in \mathbb{N}$. Then we introduce the following processes, for $t \in [0, T]$,

$$\tilde{f}_n(t) = 1_{[0, \tau_n)}(t)f(t, \mathbf{u}(t \wedge \tau_n)), \quad (52)$$

$$\tilde{g}_n(t, z) = 1_{[0, \tau_n)}(t)g(t, \mathbf{u}(t \wedge \tau_n -), z), \quad \text{for } z \in Z, \quad (53)$$

$$\tilde{F}_n(t) = \left(0, -\tilde{f}_n(t) - m'(|B^{\frac{1}{2}}\mathbf{u}(t \wedge \tau_n)|_H^2)B\mathbf{u}(t \wedge \tau_n)1_{[0, \tau_n)}(t)\right), \quad (54)$$

$$\tilde{G}_n(t, z) = (0, \tilde{g}_n(t, z)), \quad \text{for } z \in Z. \quad (55)$$

One can see that both processes \tilde{F}_n and \tilde{G}_n are bounded. Consider the following linear equation

$$d\mathbf{v}_n(t) = \mathcal{A}\mathbf{v}_n(t)dt + \tilde{F}_n(t)dt + \int_Z \tilde{G}_n(t, z)\tilde{N}(dt, dz), \quad t \in [0, T], \quad \mathbf{v}_n(0) = \mathbf{u}(0). \quad (56)$$

There exists a unique global mild solution of this equation, which is given by

$$\mathbf{v}_n(t) = e^{t\mathcal{A}}\mathbf{u}(0) + \int_0^t e^{(t-s)\mathcal{A}}\tilde{F}_n(s)ds + \int_0^t \int_Z e^{(t-s)\mathcal{A}}\tilde{G}_n(s, z)\tilde{N}(ds, dz), \quad t \in [0, T].$$

Furthermore, the stopped process $\mathbf{v}_n(\cdot \wedge \tau_n)$ satisfies

$$\mathbf{v}_n(t \wedge \tau_n) = e^{(t \wedge \tau_n)\mathcal{A}}\mathbf{u}(0) + \int_0^{t \wedge \tau_n} e^{(t \wedge \tau_n - s)\mathcal{A}}\tilde{F}_n(s)ds + I_{\tau_n}(\tilde{G}_n)(t \wedge \tau_n), \quad t \in [0, T],$$

where as before $I_{\tau_n}(\tilde{G}_n)(t) = \int_0^t \int_Z 1_{[0, \tau_n]}(s) e^{(t-s)\mathcal{A}} \tilde{G}_n(s, z) \tilde{N}(ds, dz)$, $t \in [0, T]$. By (54-55), we have

$$\begin{aligned} I_{\tau_n}(\tilde{G}_n)(t) &= \int_0^t \int_Z 1_{[0, \tau_n]}(s) e^{(t-s)\mathcal{A}} \tilde{G}_n(s, z) \tilde{N}(ds, dz) \\ &= \int_0^t \int_Z 1_{[0, \tau_n]}(s) e^{(t-s)\mathcal{A}} G(s, \mathbf{u}(s \wedge \tau_n -), z) \tilde{N}(ds, dz) \\ &= \int_0^t \int_Z 1_{[0, \tau_n]}(s) e^{(t-s)\mathcal{A}} G(s, \mathbf{u}(s-), z) \tilde{N}(ds, dz) = I_{\tau_n}(G(\mathbf{u}))(t), \quad t \in [0, T]. \end{aligned}$$

On the basis of Lemma 21, we see that for each $n \in \mathbb{N}$ and every $t \in [0, T]$

$$\begin{aligned} \mathbf{v}_n(t \wedge \tau_n) &= e^{(t \wedge \tau_n)\mathcal{A}} \mathbf{u}(0) + \int_0^{t \wedge \tau_n} e^{(t \wedge \tau_n - s)\mathcal{A}} \tilde{F}_n(s) ds + I_{\tau_n}(\tilde{G}_n)(t \wedge \tau_n) \\ &= e^{(t \wedge \tau_n)\mathcal{A}} \mathbf{u}(0) + \int_0^{t \wedge \tau_n} e^{(t \wedge \tau_n - s)\mathcal{A}} 1_{[0, \tau_n]}(s) F(s, \mathbf{u}(s \wedge \tau_n)) ds + I_{\tau_n}(G(\mathbf{u}))(t \wedge \tau_n) \\ &= \mathbf{u}(t \wedge \tau_n) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Applying Itô's Lemma 5.2 to \mathbf{v}_n and Remark 9, it follows that

$$\begin{aligned} V(\mathbf{v}_n(t \wedge \tau_n)) - V(\mathbf{u}(0)) &= \int_0^{t \wedge \tau_n} -2\langle \pi_2 \mathbf{v}(r), f(r, z) \rangle_H dr + \int_0^{t \wedge \tau_n} \int_Z |g(r, z)|_H^2 \nu(dz) dr \\ &\quad + \int_0^{t \wedge \tau_n} \int_Z \left[2\langle \pi_2 \mathbf{v}(r-), g(r, z) \rangle_H + |g(r, z)|_H^2 \right] \tilde{N}(ds, dz) \end{aligned}$$

Finally, according to the fact that \mathbf{u} coincides with \mathbf{v} \mathbb{P} -a.s. up to time $t \wedge \tau_n$, by applying conditions 13 and 14, we get, for $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}V(\mathbf{u}(t \wedge \tau_n)) &= \mathbb{E}V(\mathbf{u}_0) - 2\mathbb{E} \int_0^t \langle u_t(s), f(\mathbf{u}(s)) \rangle_{\mathcal{H}} 1_{[0, \tau_n]} ds + \mathbb{E} \int_0^t \int_Z |g(s, \mathbf{u}(s), z)|_H^2 1_{[0, \tau_n]} \nu(dz) ds \\ &\leq \mathbb{E}V(\mathbf{u}_0) + 2\mathbb{E} \int_0^t [K_f + R_f |\mathbf{u}(s)|_{\mathcal{H}}^2] 1_{[0, \tau_n]} ds + \mathbb{E} \int_0^t \int_Z [K_g + R_g |\mathbf{u}(s)|_{\mathcal{H}}^2] 1_{[0, \tau_n]} \nu(dz) ds \\ &\leq \mathbb{E}V(\mathbf{u}_0) + \int_0^t \left((2K_{f,T} + K_{G,T}) + (2R_{f,T} + R_{G,T}) \mathbb{E}V(\mathbf{u}(s \wedge \tau_n)) \right) ds. \end{aligned}$$

This implies inequality 30 with $C_1 = 2K_{f,T} + K_{G,T}$ and $C_2 = 2R_{f,T} + R_{G,T}$.

In conclusion, we proved that V is a Lyapunov function and hence we can apply Lemma 4.1 to deduce that $\tau_\infty \wedge T = T$. \square

7. Proof of Theorem 2.7. Let us fix $\beta > 0$ and $Q = Q_\beta$ defined in 37. Let us remind that in this section we consider functions F , \mathbf{M} and G defined recollectively by equalities 7, 8 and 9. The following proof follows the lines of [8]. We will try to keep it self consistent but also to pay attention to new elements (due to the different type of noise). In the Itô formula we encounter expressions of the form $\langle Q_\beta \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}}$, where $v_1 \pi_1 \mathbf{v} = 0$. These will play an exceptionally important rôle below and hence let us write down explicitly that for $u = (u_1, u_2)$, $\mathbf{v} = (0, v_2)$, $\mathfrak{z} = (0, z_2) \in \mathcal{H}$

$$\begin{aligned} \langle Q_\beta \mathbf{u}, \mathbf{v} \rangle_{\mathcal{H}} &= \langle \beta u_1 + 2u_2, v_2 \rangle, \\ \langle Q_\beta \mathfrak{z}, \mathbf{v} \rangle_{\mathcal{H}} &= 2\langle z_2, v_2 \rangle \end{aligned}$$

In particular, in view of 7, 8 and 9 we get, for $\mathbf{u} = (u_1, u_2)$,

$$\begin{aligned} \langle Q_\beta \mathbf{u}, F(t, \mathbf{u}) + \mathbf{M}(\mathbf{u}) \rangle_{\mathcal{H}} &= -\langle \beta u_1 + 2u_2, f(t, u_1, u_2) + m'(|B^{\frac{1}{2}} u_1|^2) B u_1 \rangle, \\ \langle Q_\beta \mathbf{u}, G(t, \mathbf{u}, \cdot) \rangle_{\mathcal{H}} &= \langle \beta u_1 + 2u_2, g(u_1, u_2, \cdot) \rangle, \\ \langle Q_\beta G(t, \mathbf{u}, \cdot), G(t, \mathbf{u}, \cdot) \rangle_{\mathcal{H}} &= 2|g(t, u_1, u_2, \cdot)|^2. \end{aligned}$$

Thus, from 40 and 42 we infer that for $\mathbf{u} = (u_1, u_2) \in \mathcal{H}$,

$$\begin{aligned}
\Gamma_\beta(\mathbf{u}, \mathbf{u}) &+ 2m'(|B^{\frac{1}{2}}u_1|^2)\langle Bu_1, u_2 \rangle + V'(\mathbf{u})(F(t, \mathbf{u}) + \mathbf{M}(\mathbf{u})) \\
&= \beta^2\langle u_1, u_2 \rangle + \beta|u_2|^2 - \beta|Au_1|^2 + 2m'(|B^{\frac{1}{2}}u_1|^2)\langle Bu_1, u_2 \rangle \\
&- \beta\langle u_1, f(t, u_1, u_2) \rangle - \beta m'(|B^{\frac{1}{2}}u_1|^2)\langle u_1, Bu_1 \rangle \\
&- 2\langle u_2, f(t, u_1, u_2) \rangle - 2m'(|B^{\frac{1}{2}}u_1|^2)\langle u_2, Bu_1 \rangle \\
&= \beta^2\langle u_1, u_2 \rangle + \beta|u_2|^2 - \beta|Au_1|^2 - \beta\langle u_1, f(t, u_1, u_2) \rangle \\
&- \beta m'(|B^{\frac{1}{2}}u_1|^2)\langle u_1, Bu_1 \rangle - 2\langle u_2, f(t, u_1, u_2) \rangle.
\end{aligned}$$

Now, if we recall that by Assumption (C.6), for some $\delta > 0$,

$$f(t, \mathbf{u}) = \delta u_2, \text{ for } t \geq 0, \mathbf{u} = (u_1, u_2) \in \mathcal{H},$$

then we infer

$$\begin{aligned}
\Gamma_\beta(\mathbf{u}, \mathbf{u}) &+ 2m'(|B^{\frac{1}{2}}u_1|^2)\langle Bu_1, u_2 \rangle + V'(\mathbf{u})(F(t, \mathbf{u}) + \mathbf{M}(\mathbf{u})) \\
&= \beta^2\langle u_1, u_2 \rangle + \beta|u_2|^2 - \beta|Au_1|^2 - \beta\delta\langle u_1, u_2 \rangle \\
&- \beta m'(|B^{\frac{1}{2}}u_1|^2)\langle u_1, Bu_1 \rangle - 2\delta\langle u_2, u_2 \rangle
\end{aligned}$$

We see that if we put $\beta = \delta$, then $\beta^2\langle u_1, u_2 \rangle - \beta\delta\langle u_1, u_2 \rangle = 0$ we get some cancelation in the equality above, i.e.

$$\begin{aligned}
\Gamma_\delta(\mathbf{u}, \mathbf{u}) &+ 2m'(|B^{\frac{1}{2}}u_1|^2)\langle Bu_1, u_2 \rangle + V'(\mathbf{u})(F(t, \mathbf{u}) + \mathbf{M}(\mathbf{u})) \\
&= \delta|u_2|^2 - \delta|Au_1|^2 - \delta m'(|B^{\frac{1}{2}}u_1|^2)\langle u_1, Bu_1 \rangle - 2\delta\langle u_2, u_2 \rangle \\
&= -\delta|u_2|^2 - \delta|Au_1|^2 - \delta m'(|B^{\frac{1}{2}}u_1|^2)|B^{\frac{1}{2}}u_1|^2 \\
&= -\delta \left[|u|_{\mathcal{H}}^2 + m'(|B^{\frac{1}{2}}u_1|^2)|B^{\frac{1}{2}}u_1|^2 \right]
\end{aligned} \tag{57}$$

Proof of Theorem 2.7. Let us fix $s \geq 0$ and let \mathbf{u} be the solution to Equation 10 with the approximating sequence $\{\tau_n\}_{n \in \mathbb{N}}$. Fix $t \geq 0$ and $n \in \mathbb{N}$. By applying Lemma 5.2 with function V defined by 31

$$V(\mathbf{u}) = \frac{1}{2}\langle Q_\delta \mathbf{u}, \mathbf{u} \rangle_{\mathcal{H}} + m(|B^{\frac{1}{2}}u_1|^2), \mathbf{u} = (u_1, u_2) \in \mathcal{H}, \tag{58}$$

and then using 57, 42 and 43, we infer that \mathbb{P} -a.s.

$$\begin{aligned}
V(\mathbf{u}(t \wedge \tau_n))e^{\lambda(t \wedge \tau_n)} &= V(\mathbf{u}(s))e^{\lambda s} + \int_s^{t \wedge \tau_n} e^{\lambda r} \left[\lambda V(\mathbf{u}(r)) - \delta|u(r)|_{\mathcal{H}}^2 - \delta m'(|B^{\frac{1}{2}}u(r)|^2)|B^{\frac{1}{2}}u(r)|^2 \right. \\
&\quad \left. + \int_Z |g(r, \mathbf{u}(r), z)|^2 \nu(dz) \right] dr \\
&+ \int_s^{t \wedge \tau_n} \int_Z e^{\lambda r} \left[\langle \delta u(r-) + 2v(r-), g(r, \mathbf{u}(r), z) \rangle + |g(r, \mathbf{u}(r), z)|^2 \right] \tilde{N}(dr, dz).
\end{aligned}$$

Now applying (C.8) yields

$$\begin{aligned}
&V(\mathbf{u}(t \wedge \tau_n))e^{\lambda(t \wedge \tau_n)} \\
&\leq V(\mathbf{u}(s))e^{\lambda s} + \int_s^{t \wedge \tau_n} e^{\lambda r} \left[\frac{\lambda}{2}\langle Q_\delta \mathbf{u}(r), \mathbf{u}(r) \rangle_{\mathcal{H}} + \lambda m(|B^{\frac{1}{2}}u(r)|^2) + (R_G - \delta)|u(r)|_{\mathcal{H}}^2 \right. \\
&\quad \left. - \delta m'(|B^{\frac{1}{2}}u(r)|^2)|B^{\frac{1}{2}}u(r)|^2 + K_G \right] dr \\
&+ \int_s^{t \wedge \tau_n} \int_Z e^{\lambda r} \left[\langle \delta u(r-) + 2v(r-), g(r, \mathbf{u}(r), z) \rangle_{\mathcal{H}} + |g(r, \mathbf{u}(r), z)|^2 \right] \tilde{N}(dr, dz).
\end{aligned}$$

We continue by applying inequalities 38 and 15, the latter from (C.7), to infer that there exists $\alpha > 0$ such that

$$\begin{aligned} & V(\mathbf{u}(t \wedge \tau_n))e^{\lambda(t \wedge \tau_n)} \\ & \leq V(\mathbf{u}(s))e^{\lambda s} + \int_s^{t \wedge \tau_n} e^{\lambda r} \left[\left(\frac{\lambda}{2} \|Q_\delta\|_{\mathcal{L}(H)} + R_G - \delta \right) |\mathbf{u}(r)|_{\mathcal{H}}^2 \right. \\ & \quad \left. + \left(\frac{\lambda}{\alpha} - \delta \right) m'(|B^{\frac{1}{2}}u(r)|^2) |B^{\frac{1}{2}}u(r)|^2 + K_G \right] dr \\ & \quad + \int_s^{t \wedge \tau_n} \int_Z e^{\lambda r} \left[\langle \delta u(r-) + 2v(r-), g(r, \mathbf{u}(r), z) \rangle + |g(r, \mathbf{u}(r), z)|^2 \right] \tilde{N}(dr, dz). \end{aligned}$$

Now let us take the limits when $n \rightarrow \infty$. Since by Theorem 2.6, $\tau_\infty = \infty$, we infer

$$\begin{aligned} V(\mathbf{u}(t))e^{\lambda t} & \leq V(\mathbf{u}(s))e^{\lambda s} + \int_s^t e^{\lambda r} \left[\left(\frac{\lambda}{2} \|Q_\delta\|_{\mathcal{L}(H)} + R_G - \delta \right) |\mathbf{u}(r)|_{\mathcal{H}}^2 \right. \\ & \quad \left. + \left(\frac{\lambda}{\alpha} - \delta \right) m'(|B^{\frac{1}{2}}u(r)|_H^2) |B^{\frac{1}{2}}u(r)|_H^2 + K_G \right] dr \\ & \quad + \int_s^t \int_Z e^{\lambda r} \left[\langle \delta u(r-) + 2u_t(r-), g(r, \mathbf{u}(r), z) \rangle + |g(r, \mathbf{u}(r), z)|_H^2 \right] \tilde{N}(dr, dz), \quad 0 \leq s \leq t < \infty. \end{aligned}$$

Because of our assumptions it is possible to find $\lambda > 0$ such that

$$\frac{\lambda}{\alpha} - \delta < 0 \quad \text{and} \quad \frac{\lambda}{2} \|Q_\delta\|_{\mathcal{L}(H)} + R_G - \delta < 0.$$

Therefore,

$$\begin{aligned} V(\mathbf{u}(t))e^{\lambda t} & \leq \int_s^t \int_Z e^{\lambda r} \left[\langle \delta u(r-) + 2u_t(r-), g(r, \mathbf{u}(r), z) \rangle_{\mathcal{H}} + |g(r, \mathbf{u}(r), z)|_H^2 \right] \tilde{N}(dr, dz) \\ & \quad + V(\mathbf{u}(s))e^{\lambda s} + \int_s^t e^{\lambda r} K_G dr, \quad 0 \leq s \leq t < \infty. \end{aligned} \tag{59}$$

Next, we first consider the case when $K_G = 0$. Then by taking the conditional expectation with respect to \mathcal{F}_s to both sides of 59 we get for $0 \leq s \leq t < \infty$,

$$\begin{aligned} \mathbb{E}(V(\mathbf{u}(t))e^{\lambda t} | \mathcal{F}_s) & \leq \mathbb{E}(V(\mathbf{u}(s))e^{\lambda s} | \mathcal{F}_s) \\ & \quad + \mathbb{E} \left(\int_s^t \int_Z e^{\lambda r} \left[\langle \beta u(r-) + 2v(r-), g(r, \mathbf{u}(r), z) \rangle_{\mathcal{H}} + |g(r, \mathbf{u}(r), z)|_H^2 \right] \tilde{N}(dr, dz) \middle| \mathcal{F}_s \right) \\ & = V(\mathbf{u}(s))e^{\lambda s}. \end{aligned}$$

This proves that the process $\Phi(\mathbf{u}(t))e^{\lambda t}$, $t \geq 0$, is a supermartingale. Therefore,

$$\mathbb{E}|\mathbf{u}(t)|_{\mathcal{H}}^2 \leq \mathbb{E}\langle Q_\delta \mathbf{u}(t), \mathbf{u}(t) \rangle_{\mathcal{H}} \leq 2\mathbb{E}V(\mathbf{u}(t)) \leq 2e^{-\lambda t} \mathbb{E}V(\mathbf{u}(0)),$$

where the first inequality follows from 38, the last inequality follows from the supermartingale property of $\Phi(\mathbf{u}(t))e^{\lambda t}$, $t \geq 0$. Also, note that

$$\mathbb{E}V(\mathbf{u}(0)) = \mathbb{E} \left[\frac{1}{2} \langle Q_\delta \mathbf{u}(0), \mathbf{u}(0) \rangle_{\mathcal{H}} + m(|B^{\frac{1}{2}}u(0)|) \right] \leq \left(\frac{1}{2} \|Q_\delta\|_{\mathcal{L}(\mathcal{H})} + 1 \right) \mathbb{E} \left[|\mathbf{u}(0)|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}u(0)|) \right]. \tag{60}$$

We conclude that with $C = \|Q\|_{\mathcal{L}(\mathcal{H})} + 2$,

$$\mathbb{E}|\mathbf{u}(t)|_H^2 \leq C e^{-\lambda t} \mathbb{E} \left[|\mathbf{u}(0)|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}u(0)|) \right], \quad t \geq 0,$$

which shows the exponentially mean-square stability of the mild solution.

In the case $K_G \neq 0$, taking first the expectation to both sides of 59 and then setting $s = 0$ gives

$$\mathbb{E}V(\mathbf{u}(t)) \leq e^{-\lambda t} \mathbb{E}V(\mathbf{u}(0)) + \frac{K_G}{\lambda} (1 - e^{-\lambda t}), \quad t \geq 0.$$

Thus, applying the inequality $|x|_H^2 \leq \langle x, Qx \rangle_H$ deduces

$$\mathbb{E}|u(t)|_H^2 \leq \mathbb{E}\langle u(t), Q_\delta u(t) \rangle_{\mathcal{H}} \leq \mathbb{E}V(u(t)) \leq 2e^{-\lambda t} \mathbb{E}V(u(0)) + \frac{2K_G}{\lambda}, \quad t \geq 0.$$

Therefore, combining with 60 we obtain

$$\sup_{t \geq 0} \mathbb{E}|u(t)|_H^2 \leq (\|Q\|_{\mathcal{L}(\mathcal{H})} + 2) \mathbb{E}\left[|u(0)|_{\mathcal{H}}^2 + m(|B^{\frac{1}{2}}u(0)|)\right] + \frac{2K_G}{\lambda} < \infty,$$

which completes the proof of Theorem 2.7. \square

Acknowledgments. The authors would like to thank our friends, Professors Pao-Liu Chow, Jan Seidler and Jerzy Zabczyk for reading the earlier draft of the paper and for providing valuable comments that greatly improved our presentation. We would also like to thank anonymous referees for careful reading of the manuscript.

REFERENCES

- [1] Aït-Sahalia, Telling from discrete data whether the underlying continuous-time model is a diffusion, *Journal of Finance*, **57** (2002), 2075–2112.
- [2] S. Albeverio, Z. Brzeźniak and J.-L. Wu, Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients, *Journal of Mathematical Analysis and Applications*, **371.1** (2010), 309–322.
- [3] Z. Brzeźniak, W. Liu and J. Zhu, Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise, *Nonlinear Analysis: Real World Applications*, **17** (2014), 283–310.
- [4] W.E. Baylis and J. Huschilt, Energy balance with the Landau-Lifshitz equation, *Phys. Lett. A*, **301.1** (2002), 7–12.
- [5] D. Burgreen, Free Vibrations of a Pin-Ended Column with Constant Distance Between Pin Ends, *No. PIBAL-166. POLYTECHNIC INST OF BROOKLYN NY*, 1950.
- [6] Z. Brzeźniak and K.D. Elworthy, Stochastic differential equations on Banach manifolds, *Methods Funct. Anal. Topology* **6.1** (2000), 43–84.
- [7] Z. Brzeźniak and D. Gałtarek, Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces, *Stochastic processes and their applications*, **84.2** (1999), 187–225.
- [8] Z. Brzeźniak, B. Masłowski, J. Seidler, Stochastic nonlinear beam equations, *Probability Theory and Related Fields* **132.1** (2005), 119–149.
- [9] Z. Brzeźniak, E. Hausenblas and P. Razafimandimby, Martingale solutions for stochastic equation of reaction diffusion type driven by Lévy noise or Poisson random measure, preprint, [arXiv:1010.5933](https://arxiv.org/abs/1010.5933).
- [10] J.F Burrow, P.D. Baxter, J.W Pitchford, Lévy processes, saltatory foraging, and superdiffusion, *Mathematical Modelling of Natural Phenomena* **3.3** (2008), 115–130.
- [11] A. Carroll, *The stochastic nonlinear heat equation*, Ph. D. Thesis, The University of Hull, 1999.
- [12] P. L. Chow, J. L. Menaldi, Stochastic PDE for nonlinear vibration of elastic panels, *Differential Integral Equations* **12.3** (1999), 419–434.
- [13] S.R. Das, The surprise element: jumps in interest rates, *Journal of Econometrics*, **106.1** (2002), 27–65.
- [14] J. G. Easley, Nonlinear vibration of beams and rectangular plates, *Zeitschrift fr angewandte Mathematik und Physik ZAMP*, **15** (1964), 167–175.
- [15] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [16] I. Gyöngy and N. V. Krylov, On stochastic equations with respect to semimartingales. I. *Stochastics: An International Journal of Probability and Stochastic Processes*, **4.1** (1980), 1–21.
- [17] I. Gyöngy, On stochastic equations with respect to semimartingale III, *Stochastics: An International Journal of Probability and Stochastic Processes*, **7.4** (1982), 231–254.
- [18] E. Hausenblas, Existence, uniqueness and regularity of parabolic SPDEs driven by Poisson random measure, *Electron. J. Probab*, **10** (2005), 1496–1546.
- [19] P. D. Lax and R. S. Phillips, *Scattering theory*, Pure and Applied Mathematics, Vol. 26 Academic Press, New York-London, 1967.
- [20] R. Z. Khas'minskii, Stability of systems of differential equations under random perturbations of their parameters, *M: Nauka*, 1969.
- [21] B. Masłowski, J. Seidler and I. Vrkoč, Integral continuity and stability for stochastic hyperbolic equations, *Differential Integral Equations*, **6** (1993), 355–382.
- [22] M. Ondreját, a private communication to [8]
- [23] S. K. Patcheu, On a global solution and asymptotic behaviour for the generalized damped extensible beam equation, *Journal of Differential Equations*, **135.2** (1997), 299–314.

- [24] S. Peszat and J. Zabczyk, *Stochastic partial differential equations with Lévy noise: An evolution equation approach*, Encyclopedia of Mathematics and its Applications, 113. Cambridge University Press, Cambridge, 2007.
- [25] A. J. Pritchard and J. Zabczyk, Stability and stabilizability of infinite-dimensional systems, *Siam Review*, **23.1** (1981), 25–52.
- [26] M. Riedle, Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes, *Potential Analysis*, **42.4** (2015), 809–838.
- [27] T. Russo, P. Baldi, A. Parisi, G. Magnifico, S. Mariani and S. Cataudella, Lévy processes and stochastic von Bertalanffy models of growth, with application to fish population analysis, *Journal of theoretical biology*, **258.4** (2009), 521–529.
- [28] L. Tubaro, On abstract stochastic differential equation in Hilbert spaces with dissipative drift, *Stochastic Analysis and Applications*, **1.2** (1983), 205–214.
- [29] L. Tubaro, An estimate of Burkholder type for stochastic processes defined by the stochastic integral, *Stochastic Analysis and Applications*, **2.2** (1984), 187–192.
- [30] M. Métivier, *Semimartingales, A course on stochastic processes*, de Gruyter Studies in Mathematics, 2. Walter de Gruyter & Co., Berlin-New York, 1982.
- [31] J. Van Neerven, J. Zhu A maximal inequality for stochastic convolutions in 2-smooth Banach spaces, *Electron. Commun. Probab*, **16** (2011), 689–705.
- [32] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York, 1978.
- [33] S. Woinowsky-Krieger, The effect of an axial force on the vibration of hinged bars, *J. Appl. Mech*, **17** (1950), 35–36.
- [34] J. Zhu, *A Study of SPDEs w.r.t. Compensated Poisson Random Measures and Related Topics*, Ph. D. Thesis, University of York, 2010.
- [35] J. Zhu, Z. Brzeźniak and E. Hausenblas, Maximal inequality of stochastic convolution driven by compensated Poisson random measures in Banach spaces, preprint, [arXiv:1005.1600](https://arxiv.org/abs/1005.1600).

E-mail address: jiahuizhu@zjut.edu.cn

E-mail address: zdzislaw.brzezniak@york.ac.uk