Article:

Katzman, M. orcid.org/0000-0001-7553-3520 (1999) Bipartite graphs whose edge algebras are complete intersections. Journal of Algebra, 220 (2). pp. 519-530. ISSN 0021-8693

BIPARTITE GRAPHS WHOSE EDGE ALGEBRAS ARE COMPLETE INTERSECTIONS

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ABSTRACT. Let R be monomial sub-algebra of $k[x_1, \ldots, x_N]$ generated by square free monomials of degree two. This paper addresses the following question: when is R a complete intersection?

For such a k-algebra we can associate a graph G whose vertices are $x_1, \ldots, x_N$ and whose edges are $\{(x_i, x_j) | x_i x_j \in R\}$. Conversely, for any graph G with vertices $\{x_1, \ldots, x_N\}$ we define the *edge algebra associated with* G as the sub-algebra of $k[x_1, \ldots, x_N]$ generated by the monomials $\{x_i x_j | (x_i, x_j)$ is an edge of $G\}$. We denote this monomial algebra by $k[G]$.

This paper describes all bipartite graphs whose edge algebras are complete intersections.

1. Introduction

For any graph G with vertices $\{x_1, \ldots, x_N\}$ we define the *edge algebra associated with* G as the sub-algebra of $k[x_1, \ldots, x_N]$ generated by the monomials

$$\{x_i x_j | (x_i, x_j) \text{ is an edge of } G\}.$$

We denote this monomial algebra by $k[G]$.

There has been a recent effort to relate the algebraic properties of $k[G]$ with the structure of G. For example, [7] and [4] give a criterion for the normality of $k[G]$ and the authors of the latter recently obtained a characterization of all bipartite graphs whose edge algebras are Gorenstein ([5].)

In this paper I follow this line of inquiry and I will present a characterization of all bipartite graphs whose edge algebras are complete intersections (theorem 3.5.)

1991 *Mathematics Subject Classification.* Primary 14M10 14M25 05E; Secondary 05C38 13P.

We shall denote by $\mathrm{E}(G)$ the set of edges of G and by $\mathrm{V}(G)$ its set of vertices. The cardinality of these sets will be denoted by $\mathrm{e}(G)$ and $\mathrm{v}(G)$ respectively.

We fix the following presentation of $k[G]$: define a map $\phi : k[\mathrm{E}(G)] \to k[G]$ by $\phi(x_r, x_s) = x_r x_s$ and let K_G be the kernel of ϕ. Note that K_G is a binomial prime ideal containing no monomials. We also recall that if G is connected then $\dim(k[G])$ equals $\mathrm{v}(G) - 1$ if G is bipartite, $\mathrm{v}(G)$ otherwise (corollary 7.3.1 in [8].)

The next section will describe a set of generators and a Gröbner basis for K_G. We will then obtain a number of straightforward corollaries linking the structure of G with that of $k[G]$. We will then restrict our attention to bipartite graphs, and we will characterize those graphs whose edge algebras are complete intersections.

2. A Gröbner basis for K_G

We first introduce some graph theoretical terminology:

Let G be a graph. A *walk of length l in G* is a sequence of edges

$$(v_1, v_2), (v_2, v_3), \dots, (v_{l-1}, v_l), (v_l, v_{l+1});$$

this walk is *closed* if $v_{l+1} = v_1$; if, in addition, $l \geq 3$ and every vertex in the walk occurs precisely twice this closed path is a *cycle of length l*.

A closed walk $e_1, e_2, \dots, e_l$ is *minimal* if no two consecutive (modulo l) edges are equal. A walk $e_1, e_2, \dots, e_{2l}$ is *trivial* if after a cyclic permutation of the edges we have $e_1 = e_2, e_3 = e_4, \dots e_{2l-1} = e_{2l}$.

A closed walk $e'_1, \dots, e'_r$ is *contained in* a closed walk $e_1, \dots, e_s$ if after a cyclic permutation of the edges of the walks we have $e_1 = e'_1, \dots, e_r = e'_r$. All other graph theoretical terminology in this paper conforms with [2].

If we fix any monomial order in $k[\mathrm{E}(G)]$ then given any closed walk of even length $w = e_1, \dots, e_{2l}$ we define

$$\psi(w) = \psi^+(w) - \psi^-(w) = \prod_{i=1}^{l} e_{2i-1} - \prod_{i=1}^{l} e_{2i} \in k[\mathrm{E}(G)]$$

where $\psi^+(w) \geq \psi^-(w)$. It is not hard to see that $\psi(w) \in K_G$ for all closed walks w of even length and it turns out that these generate K_G (lemma 1.1 in [3].)

Theorem 2.1. *Fix any lexicographic monomial order in* $k[\mathrm{E}(G)]$*. Let* $\mathcal{W}$ *be the set of minimal closed walks in* G *of even length and let* $\mathcal{G}_G = \{\psi(w) | w \in \mathcal{W}\}$*. Then there exists a subset of* $\mathcal{G}_G$ *which is a Gröbner basis for* K_G*.*

Proof. It is enough to show that any binomial in K_G reduces to zero with respect to $\mathcal{G}_G$. Pick a counterexample $A - B \in K_G$ with $A > B$ of minimal degree having disjoint support. Let e_1 be the largest variable occurring in A. If $e_1 = (v_1, v_2)$ then some variable $e_2 = (v_2, v_3)$ must occur in B. If $v_3 = v_1$ then $A - B = e_1(A/e_1 - B/e_1)$ where the second factor is a binomial in K_G of smaller degree and by the minimality of the degree of $A - B$ it reduces to zero, and we are done.

Assume now that $v_3 \neq v_1$. We can now pick a variable $e_3 = (v_3, v_4)$ occurring in A/e_1 and a variable $e_4 = (v_4, v_5)$ occurring in B/e_2. If $v_5 = v_1$ then $\psi^+(e_1, e_2, e_3, e_4)$ divides A and we are done. We may continue in this fashion until we produce a closed walk $w = e_1, e_2, \ldots, e_{2l}$ such that $\psi^+(w)$ divides A. □

Corollary 2.2. *If* G *has at most one odd cycle then*

$$\mathcal{G}_G = \{\psi(c) | c \textit{ is a even cycle in } G\}$$

is a Gröbner basis for K_G*.*

Proof. It is enough to show that if $w = (e_1, \ldots, e_{2l})$ is a minimal walk in G then $\psi^+(c)$ divides $\psi^+(w)$ for some even cycle c contained in w. Pick as a counterexample such a w with minimal length. Since w is minimal, there exists some cycle c' contained in w, say $c' = (e_1, e_2, \ldots, e_r)$. If r is odd then $(e_{r+1}, e_{r+2}, \ldots, e_{2l})$ is a closed walk of odd length, and, therefore, cannot be trivial and must contain an even cycle.

We have shown that w must contain an even cycle, say $c = (e_1, e_2, \ldots, e_{2s})$. If $s = l$ we are done, otherwise let w' be the even cycle $(e_{2s+1}, e_{2s+2}, \ldots, e_{2l})$. $\psi^+(w)$ must be divisible by $\psi^+(c)$ or by $\psi^+(w')$. If the former occurs we are done, if the latter occurs, the minimality of the length of w implies that there exists an even cycle c' in w' such that $\psi^+(c')$ divides $\psi^+(w')$. □

Corollary 2.3. *Let G have at most one odd cycle, and let $B_1, \dots, B_r$ be the blocks of G.*

(1) *$k[G]$ is a complete intersection if and only if $k[B_i]$ is a complete intersection for all $1 \leq i \leq r$.*
(2) *$k[G]$ is Gorenstein if and only if $k[B_i]$ is Gorenstein for all $1 \leq i \leq r$.*

Proof. Since K_G is generated by elements involving edges in one block we can write

$$k[G] \cong k[\mathrm{E}(G)]/K_G \cong k[\mathrm{E}(B_1)]/K_{B_1} \otimes_k \cdots \otimes_k k[\mathrm{E}(B_r)]/K_{B_r}$$

proving (1).

We can find a system of parameters for $k[\mathrm{E}(G)]/K_G$ where each parameter is in some $k[B_i]$. Killing these parameters gives us a zero-dimensional k-algebra whose socle is the tensor product of r non-zero vector spaces. Thus the type of $k[G]$ is one if and only if all these vector spaces are one dimensional. □

Corollary 2.4. *Let G be a connected graph and let $d = \dim k[G]$. Let $(1, h_1, h_2, \dots)$ be the h-vector of $k[G]$. If $2L$ is the length of the smallest minimal even closed walk in G then $h_i = \binom{\mathrm{e}(G) - d + i - 1}{\mathrm{e}(G) - d - 1}$ for all $0 \leq i < L$ while $\binom{\mathrm{e}(G) - d + L - 1}{\mathrm{e}(G) - d - 1} - h_L$ is the number of (minimal) closed walks of length $2L$ in G.*

Proof. Let $\mathrm{H}(i)$ be the Hilbert function of $k[\mathrm{E}(G)]/K_G$ (where the degrees of the variables are one,) and consider the short exact sequence

$$0 \to K_G \to K[\mathrm{E}(G)] \to k[\mathrm{E}(G)]/K_G \to 0.$$

Since the minimal degree of a generator of K_G is L we have $\mathrm{H}(i) = \binom{\mathrm{e}(G)+i-1}{\mathrm{e}(G)-1}$ for all $i < L$ while $\mathrm{H}(L) = \binom{\mathrm{e}(G)+L-1}{\mathrm{e}(G)-1} - \gamma$ where γ is the number of closed walks of length γ in G.

Now h_i is the coefficient of t^i in $(1-t)^d \sum_{j=0}^{i} \mathrm{H}(j)t^j$, and for $i < L$ this is the coefficient of t^i in $\dfrac{(1-t)^d}{(1-t)^{\mathrm{e}(G)}}$, i.e., $h_i = \dbinom{\mathrm{e}(G)-d+i-1}{\mathrm{e}(G)-d-1}$. On the other hand, h_L is the coefficient of t^L in

$$(1-t)^d \left(\sum_{j=0}^{L} \binom{\mathrm{e}(G)+j-1}{\mathrm{e}(G)-1} t^j - \gamma t^L \right) = \frac{(1-t)^d}{(1-t)^{\mathrm{e}(G)}} - (1-t)^d \gamma t^L$$

and, therefore, $h_L = \dbinom{\mathrm{e}(G)+L-1}{\mathrm{e}(G)-1} - \gamma$. □

3. Bipartite graphs whose edge algebra is a complete intersection

We begin this section by producing a minimal set of generators for $k[G]$ where G is bipartite (i.e., all cycles in G are even.) We shall assume that we fixed some unspecified monomial order in $k[\mathrm{E}(G)]$ so that for any closed walk w of even length in G, $\psi(w)$ is well defined.

Definition 3.1. A bipartite graph G is a *CI graph* if any two cycles with no chords have at most one edge in common.

For any graph G we will denote the set of cycles in G with no chords by $\mathcal{C}(G)$.

The following observation, also proved in [6], provides a link between the structures of G and $k[G]$.

Theorem 3.2. *If G is a bipartite graph then*

$$\mathcal{S} = \{\psi(c) | c \in \mathcal{C}(G)\}$$

is a minimal set of generators of K_G.

Proof. We first show that $\mathcal{S}$ generates K_G. Pick as a counterexample a cycle $c = e_1, \ldots, e_{2l}$ of minimal length such that $\psi(c)$ is not contained in $< \mathcal{S} >$. Then c must have a chord e and we obtain after a cyclic permutation of the edges of c two cycles

$c_1 = e, e_1, \dots, e_{2r-1}$ and $c_2 = e, e_{2r}, \dots, e_{2l}$ in G whose length is smaller than the length of c. By the minimality of c we have $\psi(c_1), \psi(c_2) \in < \mathcal{S} >$ but since

$$e_2 e_4 \dots e_{2r-2}(e e_{2r+1} e_{2r+3} \dots e_{2l-1} - e_{2r} e_{2r+2} \dots e_{2l}) -$$

$$e_{2r+1} e_{2r+3} \dots e_{2l-1}(e e_2 e_4 \dots e_{2r-2} - e_1 e_3 \dots e_{2r-1}) =$$

$$e_1 e_3 \dots e_{2r-1} e_{2r+1} \dots e_{2l-1} - e_2 e_4 \dots e_{2r-2} e_{2r} \dots e_{2l}$$

$\psi(c)$ is in the ideal generated by $\psi(c_1)$ and $\psi(c_2)$, a contradiction.

Assume now that for some $c = e_1 \dots e_{2l} \in \mathcal{C}$ we have $\psi(c) \in < \mathcal{S} - \{\psi(c)\} >$. In this case there is a monomial in one of the generators of $< \mathcal{S} - \{\psi(c)\} >$ which divides a monomial in $\psi(c)$, i.e., there exists $d = f_1 \dots f_{2r} \in \mathcal{C}$ such that after a cyclic permutation of the edges of d we have $f_1 = e_{2i_1 - 1}, f_3 = e_{2i_2 - 1}, \dots, f_{2r-1} = e_{2i_r - 1}$. But then if any of $f_2, f_4, \dots f_{2r}$ is not an edge in c then it must be a chord of c and, therefore, all the edges of d are in c, implying that $c = d$, a contradiction. □

Lemma 3.3. *Let G be a graph with no triangles with the property that any two cycles with no chords in G share at most one edge.*

(1) *If two cycles with no chords have a common edge then there is no edge connecting them other than the common edge.*
(2) *There exists an edge in at most one cycle with no chords.*
(3) *If e is an edge on precisely one cycle with no chords then e is not a chord of any cycle.*
(4) *If G is connected then there are precisely $\mathrm{e}(G) - \mathrm{v}(G) + 1$ cycles with no chords in G.*

Proof. It is enough to prove the lemma for all the connected components of G, so we will assume henceforth that G is connected.

(1) Let $c_1, c_2 \in \mathcal{C}(G)$ have the edge (v_1, v_2) as a common edge. Write

$$c_1 = (v_1, v_2)(v_2, u_3) \dots (u_r, v_1) \qquad (r > 3)$$

and

$$c_2 = (v_1, v_2)(v_2, w_3) \dots (w_s, v_1) \qquad (s > 3).$$

If there is an edge other than (v_1, v_2) connecting c_1 and c_2, since c_1 and c_2 have no chords we can pick $3 \le i \le r$ minimal such that there exists an edge in G connecting u_i with a vertex of c_2, and we can pick $3 \le j \le s$ minimal such that (u_i, w_j) is an edge.

We cannot have $i = j = 3$ otherwise we would have a triangle in G, and we may assume that $i > 3$. The cycle

$$c_3 = (v_2, u_3)(u_3, u_4) \dots (u_{i-1}, u_i)(u_i, w_j)(w_j, w_{j-1}) \dots (w_3, v_2)$$

has no chords and $\#(c_3 \cap c_1) > 1$, a contradiction.

(2) Let $\mathfrak{G}$ be the bipartite graph whose vertices are

$$\mathcal{C}(G) \cup \{e \in \mathrm{E}(G) | e \text{ is in some } c \in \mathcal{C}(G)\}$$

and whose edges are

$$\{(e, c) | e \in \mathrm{E}(G), c \in \mathcal{C}(G) \text{ and } e \text{ is an edge of } c\}.$$

If any edge is in at least two cycles with no chords then the degree of the vertices of $\mathfrak{G}$ is at least two and we can pick a minimal cycle $c_1, e_1, c_2, e_2, \dots, c_r, e_r$ in $\mathfrak{G}$, i.e., we produce a sequence $(c_1, \dots, c_r) \subset \mathcal{C}(G)$ together with a sequence of edges $e_1, \dots, e_r$ such that for all $1 \le i < r$ we have $e_i \in c_i \cap c_{i+1}$ and $e_r \in c_r \cap c_1$ and in addition only consecutive (modulo r) cycles in this sequence have a common edge.

We first note that there is no edge connecting two vertices in different c_i, c_j other than one of $e_1, \dots, e_r$; if there were such an edge e then by part (1) of the lemma i and j are not consecutive (modulo r.) After a cyclic permutation of the cycles we may assume that $1 \le i < j < r$ and write $e = (v_1, v_2)$ with $v_1 \in c_i$ and $v_2 \in c_j$. We can find a path p between v_1 and v_2 lying in $(c_i \cup \dots \cup c_j) - \{e_i, e_{i+1}, \dots, e_{j-1}\}$; add to this path the edge e to obtain a cycle c. If c has chords replace it with another cycle with no chords containing a sub-path of p and an edge e' connecting two vertices in $c_{i'}$ and $c_{j'}$ with $i \leq i' < j' \leq j$. Thus we may assume that c has no chords and we may replace $c_1, \dots, c_r$ with a possibly shorter sequence c_i, c_{i+1}, c implying that $r = 3$. But when $r = 3$ any two cycles are consecutive and we are done by the first part of this lemma.

Consider the graph $H = (c_1 \cup \cdots \cup c_r) - \{e_1, \dots, e_r\}$; H has at most two connected components, one of which must be a cycle c (one of these components may be a single vertex, but not both.)

Assume first that $H = c$. For any e_i there is a path p in H connecting the endpoints of e_i, and if we pick this path to have minimal length, the cycle c' obtained by concatenating p and e_i has no chords. But p must have an edge in common with either c_{i-1} or with c_i, and, therefore, c' must share at least two edges with c_{i-1} or with c_i.

Consider now the case where H has two connected components, one of which is the cycle c. We have shown that this cycle cannot have a chord, i.e., $c \in \mathcal{C}(G)$, and, therefore, $\#(c \cap c_i) \leq 1$ for all $1 \leq i \leq r$. But since every edge in every c_i except two are in H we must have $\#(c \cap c_i) = 1$ for all $1 \leq i \leq r$. This immediately shows that both connected components of H are cycles and also that $r > 3$ because G has no triangles.

Let the two connected components of H be $f_1, f_2, \dots, f_r$ and $g_1, g_2, \dots, g_r$ where $f_i, g_i \in c_i$ for all $1 \leq i \leq r$ and consider the cycles

$$c' = e_r, g_1, g_2, e_2, f_3, f_4, \dots, f_r$$

and

$$c'' = g_1, g_2, \dots, g_r.$$

These cycles have no chords and their intersection is $\{g_1, g_2\}$, a contradiction.

(3) Any chord is an edge of at least two cycles with no chords.

(4) We proceed by induction on $\mathrm{e}(G) - \mathrm{v}(G)$. If $\mathrm{e}(G) - \mathrm{v}(G) = -1$ then G is a tree and the claim is trivial. Assume that $\mathrm{e}(G) - \mathrm{v}(G) \geq 0$ and pick an edge e precisely in one $c \in \mathcal{C}(G)$. Consider the graph $H = G - \{e\}$; by removing the edge e we removed from G one cycle with no chords, and since e is not a chord of any cycle in G, removing e does not add any new cycles with no chords. Thus H has one less cycle with no chords than G and by the induction hypothesis H has $\mathrm{e}(G) - \mathrm{v}(G)$ such cycles.

□

Lemma 3.4. *Let G be any graph.*

(1) $\#\mathcal{C}(G) \geq \mathrm{e}(G) - \mathrm{v}(G) + 1$.

(2) *If G has two cycles with no chords with more than one common edge then* $\#\mathcal{C}(G) > \mathrm{e}(G) - \mathrm{v}(G) + 1$.

Proof. (1) If e is any edge in G we denote by $G_{(e)}$ the graph obtained from G by "shrinking" e, i.e., by removing the edge e and identifying its endpoints. We also denote by Δ_e the number of triangles in G of which e is an edge.

We have $\mathrm{e}(G_{(e)}) = \mathrm{e}(G) - \Delta_e - 1$ and $\mathrm{v}(G_{(e)}) = \mathrm{v}(G) - 1$. We also have $\#\mathcal{C}(G_{(e)}) = \#\mathcal{C}(G) - \Delta_e - \epsilon_e$ where $\epsilon_e \geq 0$ is the number of cycles with no chords in G which acquire a chord after shrinking e.

We can now use induction on $\mathrm{e}(G)$:

$$\#\mathcal{C}(G) = \#\mathcal{C}(G_{(e)}) + \Delta_e + \epsilon_e \geq \mathrm{e}(G_{(e)}) - \mathrm{v}(G_{(e)}) + 1 + \Delta_e =$$
$$\mathrm{e}(G) - \Delta_e - 1 - \mathrm{v}(G) + 1 + 1 + \Delta_e = \mathrm{e}(G) - \mathrm{v}(G) + 1.$$

(2) Pick $c_1, c_2 \in \mathcal{C}(G)$ such that $\#(c_1 \cap c_2) > 1$. We can find a path

$$p = (u, w_1), (w_1, w_2), \ldots, (w_l, v)$$

where u, v are vertices in c_1 and $w_1, \ldots, w_l$ are vertices in $c_2 - c_1$. Note that (u, v) cannot be an edge in G, otherwise, since c_2 has no chords, c_2 would be the concatenation of p and (u, v) and would have only one edge in common with c_1.

We can now shrink G successively at all edges of p but one. After this shrinking c_1 will acquire a chord, thus at least one of the ϵ_e's obtained in this process will be positive, and the inequality follows.

□

Theorem 3.5. *Let G be a bipartite graph. $k[G]$ is a complete intersection if and only if G is a CI graph.*

Proof. If G_1 and G_2 are two disjoint graphs then $k[G_1 \cup G_2] = k[G_1] \otimes_k k[G_2]$, thus we may assume that G is connected.

$k[G]$ is a complete intersection if and only if K_G is generated by $\mathrm{e}(G) - \dim(k[G]) = \mathrm{e}(G) - \mathrm{v}(G) + 1$ elements (cf. corollary 7.3.1 in [8]) and theorem 3.2 implies that $k[G]$ is a complete intersection if and only if $\#\mathcal{C}(G) = \mathrm{e}(G) - \mathrm{v}(G) + 1$; the result now follows from lemmas 3.3(4) and 3.4.

□

Example 3.6. Consider the graph G_n with vertices $\{x, y, u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ and edges

$$\{(x, y)\} \cup \{(x, u_1), \ldots, (x, u_n)\} \cup \{(y, v_1), \ldots, (y, v_n)\} \cup \{(u_1, v_1), \ldots, (u_n, v_n)\}.$$

G_n is bipartite with $\#\mathcal{C}(G_n) = n$ and since $\mathrm{e}(G_n) - \mathrm{v}(G_n) + 1 = 3n + 1 - (2n + 2) + 1 = n$ we conclude that $k[G_n]$ is a complete intersection. Notice, however, that if H_n is the graph obtained from G_n by removing the edge (x, y) we have $\#\mathcal{C}(H_n) = \binom{n}{2}$ cycles with no chords, and, therefore, K_{H_n} is a prime ideal of height n which is $\binom{n}{2}$–generated.

Theorem 3.7. *Let G be a graph as in lemma 3.3. Then G is planar.*

Proof. The following proof is based on the proof of lemma 11.13(a) in [2].

Pick a counterexample G with minimal $\mathrm{e}(G)$; G will necessarily be a block and we may pick an edge $e = (u_1, u_4) \in \mathrm{E}(G)$ lying in a unique $c \in \mathcal{C}(G)$. We may shrink the edge e in G without affecting the hypothesis of the theorem unless c is a cycle of length four; we shall assume henceforth that $c = (u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_1)$.

Let $H = G - \{e\}$; note that H satisfies the hypothesis of the theorem and that u_1 and u_4 must lie in different blocks B_1 and B_2 of H thus we may pick a cutpoint in all paths in H from u_1 to u_4 and with no loss of generality we may take this cutpoint to be u_2.

Let $B_2' = B_2 \cup \{(u_2, u_4), (u_2, u_3)\}$ and let $B_2'' = B_2 \cup \{(u_1, u_4), (u_1, u_2), (u_2, u_3)\}$ (note that the edge (u_2, u_3) may have already been present in B_2.) Clearly, B_2'' contains no triangles and since the only cycle of B_2'' not in B_2 is c, we see that B_2'' satisfies the hypothesis of the theorem.

If $B_2'' \neq G$ we may deduce that it is planar, and B_2', being homeomorphic to it, must also be planar. We may then embed $H \cup \{(u_2, u_4)\}$ in the plane in such a way that (u_1, u_2) and (u_2, u_4) are exterior edges; adding now the edge (u_1, u_4) will not affect the planarity of the graph, and we conclude that G is planar.

Assume now that $B_2'' = G$. If u_2 and u_4 belong to different blocks of $F = B_2 \cup \{(u_2, u_3)\}$ then so do the edges (u_2, u_3) and (u_3, u_4) and we can embed F in the plane so that these edges bound the exterior face. We can then add the edges (u_1, u_2) and (u_1, u_4) without affecting the planarity. If u_2 and u_4 lie in the same block of F we can find minimal path p in $F - \{u_3\}$ connecting u_2 with u_4. The cycle obtained by concatenating p with (u_1, u_2) and (u_1, u_4) has no chords and is different from c, contradicting the fact that (u_1, u_4) lies in a unique cycle with no chords. □

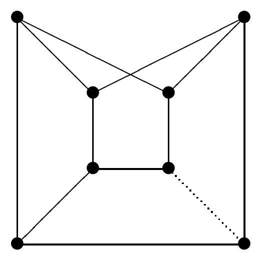
Corollary 3.8. *Let G be a connected CI graph. Then either G is a single edge or* $\mathrm{e}(G) \leq 2(\mathrm{v}(G) - 2)$.

Proof. Since G must be planar and with no triangles, the result follows easily from Euler's formula for planar graphs (see also corollary 11.17(b) in [2].) □

Remark 3.9.

(1) It is not hard to see that a bipartite outerplanar graph is a CI graph but the reverse inclusion does not hold, e.g. the graph G_n in example 3.6 is not outerplanar for $n \geq 3$ since it contains a subgraph homeomorphic to $K_{2,3}$. Therefore the family of CI-graphs is strictly between the families of bipartite outerplanar graphs and bipartite planar graphs.

(2) When G is not bipartite, $k[G]$ may be a complete intersection without G being planar. For example let G be the following graph:



A computation with Macaulay2 ([1]) shows that $k[G]$ is a complete intersection; the solid lines show a subgraph of G homeomorphic to $K_{3,3}$.

4. Algorithmic applications and some examples

In this section we will generalize theorem 3.2 which will result in an algorithm for computing $\mathcal{C}(G)$. Throughout this section we shall assume that $k[\mathrm{E}(G)]$ is equipped with a monomial order so that for any closed walk w of even length $\psi(w)$ is well defined.

Theorem 4.1. *The elements of* $\{\psi(c) | c \in \mathcal{C}(G)$ *is an even cycle*$\}$ *form part of a minimal set of generators for* K_G.

Proof. Let W be a set of closed walks of even length such that $\{\psi(w) | w \in W\}$ is a minimal set of generators for K_G and let $c \in \mathcal{C}(G)$. We will show that $c \in W$.

Since $\psi(c) \in K_G$ there exists a $w \in W$ and a monomial in $\psi(w)$ which divides $\psi^+(c)$.

If w contains no odd cycles then the proof of theorem 3.2 shows that $w \in \mathcal{C}(G)$ and that $w = c$.

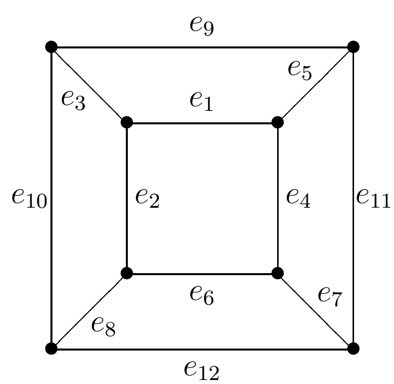
If $w = (e_1, e_2, \ldots, e_{2l})$ contains an odd cycle, say $(e_1, e_2, \ldots, e_{2r+1})$ then each of $\psi^+(w)$ and $\psi^-(w)$ is divisible by one of $e_1 e_{2r+1}$ or $e_1 e_{2r+2}$. But this is impossible since $\psi^+(c)$ is not divisible by any two edges sharing a common vertex. □

As a corollary we obtain an algorithm for producing $\mathcal{C}(G)$ as follows: given a graph G construct the ideal I_G generated by

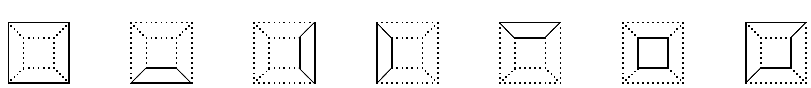
$$\{e - uv | u, v \in V(G), e = (u, v) \in \mathrm{E}(G)\} \subset R = k[\mathrm{V}(G), \mathrm{E}(G)].$$

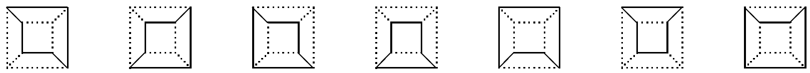
Using a lexicographic order in R with $v > e$ for any $v \in \mathrm{V}(G)$ and $e \in \mathrm{E}(G)$ compute a Gröbner basis for I_G and eliminate the variables corresponding to vertices of G. The resulting set will contain a minimal subset of generators for K_G; we can now pick those corresponding to $\mathcal{C}(G)$.

Example 4.2. Let G be the following graph:



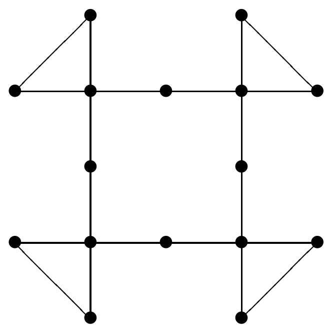
Applying the algorithm above using a lexicographical order in which $e_1 > e_2 > \cdots > e_{12}$ we obtain a Gröbner basis for K_G corresponding to the cycles:



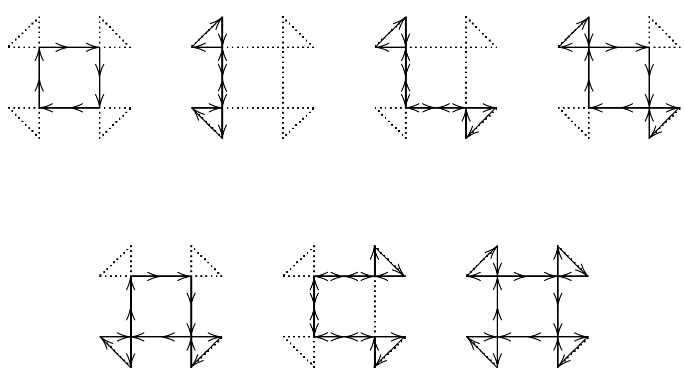


The first ten elements give us $\mathcal{C}(G)$.

Example 4.3. The minimal generators of K_G when G is not bipartite can correspond to quite complicated paths. Let G be the following graph:



There are twenty minimal generators of K_G corresponding to the following walks (up to symmetry):



Notice that the last generator corresponds to the Euler path in G. It is possible to generalize this example to obtain Eulerian graphs in which the Euler paths correspond to minimal generators of K_G and where there are minimal generators of K_G corresponding to closed walks containing an arbitrarily large number of odd cycles.

Acknowledgment

I would like to express my gratitude to Victor Neumann-Lara for our pleasant discussions on Graph Theory and his many useful suggestions.

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