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Running Title: Nonlocal diffusion and nonlinear reaction

Identification of the population density of a species model with nonlocal diffusion and nonlinear reaction

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Abstract

The identification of the population density of a logistic equation backwards in time associated with nonlocal diffusion and nonlinear reaction, motivated by biology and ecology fields, is investigated. The diffusion depends on an integral average of the population density whilst the reaction term is a global or local Lipschitz function of the population density. After discussing the ill-posedness of the problem, we apply the quasi-reversibility method to construct stable approximation problems. It is shown that the regularized solutions stemming from such method not only depend continuously on the final data, but also strongly converge to the exact solution in L^2-norm. New error estimates together with stability results are obtained. Furthermore, numerical examples are provided to illustrate the theoretical results.

Keywords and phrases: Inverse problem; Nonlocal diffusion; Nonlinear reaction; Ill-posed problem; Population density; Quasi-reversibility method.
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1. Introduction

We consider the following nonlinear parabolic equation for the population density u:

$$u_t = \mathcal{D}(\ell_0(u)(t))\,\Delta u + \mathcal{R}(x,t,u), \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

where $T > 0$, Ω is an open, bounded and connected domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$, $\mathcal{R}$ is the reaction term and $\mathcal{D}$ is the diffusion which depends on a linear functional $\ell_0(u)$, as given in equation (2) below.

Many physical processes can be described by such a time-dependent parabolic equation with nonlocal diffusion and nonlinear reaction, see e.g. [10, 11, 37] and the references therein. These can be from the study of unsteady heat transfer phenomena in a solid, where we desire to know information of the thermal conductivity, or starting from the model of diffusion and reaction of active chemical species in predicting concrete corrosion. From biology and ecology perspectives that we consider in this paper, u represents the population density of an individual species at time t and the point x where the species stays. Also, studying a nonlocal term which is density dependent in diffusion $\mathcal{D}$, i.e.

$$\mathcal{D}(\ell_0(u)(t)) = \mathcal{D}\left(\int_\Omega f(x)\,u(x,t)\,dx\right), \quad \ell_0(u)(t) := \int_\Omega f(x)u(x,t)dx, \tag{2}$$

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where f is a given weight function, has a direct correlation to the development of population dynamics, and it is obviously meaningful if the density-related reaction $\mathcal{R}$ (often representing birth/death, immigration/emigration) is also present.

For simplicity, we consider homogeneous Neumann boundary conditions

$$\frac{\partial \phi}{\partial \nu}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

where ν is the outward unit normal to the boundary $\partial\Omega$, which physically mean that the boundary of the biological specimen is insulated. Homogeneous Dirichlet boundary conditions can also be considered instead of (3).

Inverse diffusion problems have been of continuous interest to researchers in a wide range of disciplines, and in the context of identifying parameters broadly popular, see e.g. [2, 13, 18] and references therein. However, our inverse problem considered here has a different formulation. In fact, we are interested in the question that if we do know the density of a certain biological species at a finite time $T > 0$, namely,

$$u(x,T) = g(x), \quad x \in \Omega, \tag{4}$$

then we have to determine the density at preceding times down to the initial time $t = 0$. Notice that the backward heat problem (1)-(4) is not well-posed in the sense that the solution does not depend continuously on data (4), i.e., from the small noise made in measurement data, the corresponding solution may generate itself large and undesired errors, and standard computational procedures are not stable.

If the diffusion coefficient is constant or time-dependent, [27], then equations (1)-(4) form the classical backward heat conduction problem (BHCP) which has been thoroughly investigated in many studies, see e.g. [1, 15, 16, 19, 20, 21, 31, 32, 36] to mention only a few. Further, even if the diffusion coefficient is density dependent, [2, 8, 13, 18], it is still not related to our study which considers the non-local diffusion expression (2).

Up-to-date, we believe we are the first to treat the general problem in this context: find a real unknown function $u(x,t)$ for $(x,t) \in \Omega \times (0,T)$ solution to the problem (P) given by equation (1) with the diffusion coefficient given by equation (2) and endowed with the conditions (3) and (4). Motivated by the aforementioned reasons, we shall employ a regularization method to find a stable approximate solution to this backwards in time determination of the population density. In particular, we are herein interested in the quasi-reversibility (QR) method.

Referring to the QR method, the work was commenced by Lattès and Lions [24] where this approach was first proposed to deal with the Cauchy problem for elliptic equations. The idea of the method is to construct a well-posed fourth-order problem that depends on a small regularization parameter, from the original ill-posed second-order problem. A dual-based QR method to solve the Cauchy problem in the presence of noisy data has been investigated in [6], whilst numerical finite element method has been implemented in [3, 9]. Various convergence rates for the QR-method have been established, e.g. Holder-type rate with the aid of Carleman estimates in [23], and of logarithmic-type in $C^{1,1}$ and Lipschitz domains in [4] and [5], respectively.

While many papers try to deal with ill-posed problems by performing regularization at the discretization level, in this paper we regularize the problem directly by the QR method. Furthermore, formal (computationally symbolic) iterative methods such as Picard's iteration, the decomposition method or the homotopy method, [12, 17, 25], which avoid discretization, are easy to use and in many cases enable to obtain an accurate solution within a few iterations. We will describe such an iterative procedure in Section 3.

The purpose of this paper is three-fold. First, we review in Section 2 basic facts about abstract settings working for our results, the forward problem, and understand the ill-posedness caused by the instability in the backward problem throughout a simple example. Second, the main results are achieved in Section 3 where we give the unique solvability, error and stability estimates. In particular, we investigate first the inverse problem without reaction, i.e. $\mathcal{R} = 0$, in Subsection 3.1, by the classical QR approach. Theorems 4 and 5 show the unique solvability of the backward heat problem (27) and its regularized counterpart (33), whilst Theorem 7 gives the error and stability estimates (44) and (47), respectively. Next we investigate the inverse problem with reaction in Subsection 3.3 by the modified QR approach. The regularized problem presented in (56) for the global Lipschitz reaction case, is shown to be uniquely solvable in Lemmas 11 and 12, whilst theorem 13, which contains the main results of the paper, gives the error and stability estimates. Furthermore, in the same spirit, Subsection 3.3.3 considers the case of local Lipschitz reaction, where the cut-off projection is applied and the obtained estimates are presented in Theorem 15. Third, we give in

Section 4 numerical examples to corroborate our theoretical results. Finally, Section 5 presents the conclusions of this research.

2. Preliminaries

2.1. Abstract settings

We begin this subsection by introducing some notations and assumptions that are needed for our analysis in the next sections. Let us first define

$$\mathbb{V} := \left\{ \phi \in H^1(\Omega) : \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

the closed subspace of $H^1(\Omega)$, and call $H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$. With a Banach space X we denote by $L^p(0,T;X)$, $C([0,T];X)$ and $C^1(0,T;X)$ the Banach spaces of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(\cdot,t)\|_X^p \, dt \right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess} \sup_{0<t<T} \|u(\cdot,t)\|_X < \infty, \quad p = \infty,$$

and

$$\|u\|_{C([0,T];X)} = \sup_{0\le t\le T} \|u(\cdot,t)\|_X < \infty, \quad \|u\|_{C^1(0,T;X)} = \|u\|_{C([0,T];X)} + \|u_t\|_{C([0,T];X)} < \infty.$$

Moreover, throughout this paper, we denote the L^2-norm by $\|\cdot\|$, and the inner product on $L^2(\Omega)$ by $\langle \cdot, \cdot \rangle$.

We now make the following assumptions:

(A_1) The measurable function $\mathcal{D} > 0$ is such that the mapping $\xi \mapsto \mathcal{D}(\xi)$ is continuous for $\xi \in \mathbb{R}$;

(A_2) There exist positive constants η_1 and η_2 such that

$$\eta_1 \le \mathcal{D}(\xi) \le \eta_2, \quad \forall\, \xi \in \mathbb{R};$$

(A_3) There exists a positive constant L such that

$$|\mathcal{D}(\xi_1) - \mathcal{D}(\xi_2)| \le \mathrm{L}\, |\xi_1 - \xi_2|, \quad \forall\, \xi_1, \xi_2 \in \mathbb{R};$$

(A_4) $f \in L^2(\Omega)$;

(A_5) $g \in L^2(\Omega)$ represents the exact data, whilst $g^\epsilon \in L^2(\Omega)$ represents the measured data with noise level $\epsilon > 0$ such that

$$\left\| g^\epsilon - g \right\| \le \epsilon.$$

Remark 1. The assumptions (A_3) and (A_4) imply that for any $u_1, u_2 \in L^\infty(0,T;L^2(\Omega))$ we have

$$|\mathcal{D}(\ell_0(u_1)(t)) - \mathcal{D}(\ell_0(u_2)(t))| \le \mathrm{L}\, \|f\|\, \|u_1(\cdot,t) - u_2(\cdot,t)\|, \quad t \in [0,T], \tag{5}$$

where we have used the Hölder inequality and the definition (2).

An important point to be made is that, see [14, Section 6.5], the eigenvalues of the operator $-\Delta$ on the open, bounded and connected domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary, subject to standard homogeneous boundary conditions (here we specify the zero Neumann conditions (3)) have the property that there exists an orthonormal basis of $L^2(\Omega)$, denoted by $\left\{\phi_p\right\}_{p\in\mathbb{N}}$, satisfying

$$\phi_p \in \mathbb{V} \cap C^\infty\left(\overline{\Omega}\right), \quad -\Delta\phi_p(x) = \lambda_p \phi_p(x),\ x \in \Omega, \quad 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ..., \text{and} \lim_{p\to\infty} \lambda_p = \infty, \tag{6}$$

where $\left\{\lambda_p\right\}_{p\in\mathbb{N}}$ is the discrete spectrum of the operator.

We also introduce the abstract Gevrey class of functions of order $\gamma > 0$ and index $\sigma > 0$, see *e.g.*, [7], defined by the spectrum of the Laplacian (a terminology previously used in the book [30]), as

$$\mathbb{G}_{\sigma,\gamma} = \left\{ v \in L^2(\Omega) : \sum_{p=0}^{\infty} \lambda_p^{\gamma} e^{2\sigma\lambda_p} \left| \left\langle v, \phi_p \right\rangle \right|^2 < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle v_1, v_2 \rangle_{\mathbb{G}_{\sigma,\gamma}} := \left\langle (-\Delta)^{\gamma/2} e^{\sigma\sqrt{-\Delta}} v_1, (-\Delta)^{\gamma/2} e^{\sigma\sqrt{-\Delta}} v_2 \right\rangle, \quad \forall v_1, v_2 \in \mathbb{G}_{\sigma,\gamma}$$

and the corresponding norm

$$\|v\|_{\mathbb{G}_{\sigma,\gamma}} = \sqrt{\sum_{p=0}^{\infty} \lambda_p^{\gamma} e^{2\sigma\lambda_p} \left| \left\langle v, \phi_p \right\rangle \right|^2} < \infty.$$

In what follows, we denote by (P) the main inverse problem given by equations (1)-(4). We mention that its full analysis will be given in Subsection 3.3 and the main results concerning error and stability estimates will be stated and proved in Theorems 13 and 15. But before that, in the next subsection the forward problem is discussed.

2.2. *The forward problem*

Let us take for the time being $\mathcal{R} = 0$ and consider the forward problem given by the homogeneous Neumann boundary condition (3), the partial differential equation

$$u_t = \mathcal{D}\left(\ell_0(u)(t)\right)\Delta u, \quad (x,t) \in \Omega \times (0,T), \tag{7}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{8}$$

Theorem 2. *Suppose that* (A_1)*-*(A_4) *hold. Then, for* $u_0 \in L^2(\Omega)$ *there exists a unique weak solution*

$$u \in L^2(0,T;\mathbb{V}) \cap C\left([0,T]; L^2(\Omega)\right), \tag{9}$$

satisfying (7) *in the weak sense, i.e.*

$$\frac{d}{dt}\langle u(\cdot,t), v\rangle = -\mathcal{D}\left(\ell_0(u)(t)\right)\langle \nabla u(\cdot,t), \nabla v\rangle, \quad \forall v \in \mathbb{V},\ t \in (0,T), \tag{10}$$

and (8).

Theorem 3. *Theorem 2 follows from the fact that the solution to the aforementioned forward problem can be represented as*

$$u(x,t) = \sum_{p=0}^{\infty} \exp\left(-\lambda_p \int_0^t \mathcal{D}\left(\ell_0(u)(s)\right) ds\right) u_{0p}\phi_p(x), \tag{11}$$

where $u_{0p} = \left\langle u_0, \phi_p \right\rangle$.

Proof. Letting $v = \phi_p$ in (10) in combination with plugging the solution u under the Fourier-mode

$$u(x,t) = \sum_{p=0}^{\infty} \left\langle u(x,t), \phi_p(x) \right\rangle \phi_p(x), \tag{12}$$

we then obtain

$$\begin{cases} \left\langle u_t(x,t), \phi_p(x) \right\rangle + \mathcal{D}(\ell_0(u)(t)) \lambda_p \left\langle u(x,t), \phi_p(x) \right\rangle = 0, \quad t \in (0,T), \\ \left\langle u(x,0), \phi_p(x) \right\rangle = u_{0p}. \end{cases} \tag{13}$$

Multiplying both sides of the first equation of (13) by the function $\exp\left(\lambda_p \int_0^t \mathcal{D}(\ell_0(u)(s))\, ds\right)$ and noticing that

$$\frac{d}{dt}\left(\int_0^t \mathcal{D}(\ell_0(u)(s))\, ds\right) = \mathcal{D}(\ell_0(u)(t)),$$

then integrating the obtained result with respect to t, and combining with the second equation in (13), it is straightforward to obtain that

$$\exp\left(\lambda_p \int_0^t \mathcal{D}(\ell_0(u)(s))\, ds\right) \left\langle u(x,t), \phi_p(x) \right\rangle = u_{0p},$$

which, via (12), leads to (11). It is also clear, by direct calculus, that u given by (11) satisfies (8) and (10).

Using the notation

$$E_{(a,b)}\left(\mathcal{D}(\ell_0(u)(s)), \lambda_p\right) := \exp\left(\lambda_p \int_a^b \mathcal{D}(\ell_0(u)(s))\, ds\right) \tag{14}$$

then, equation (11) can be written as

$$u(x,t) = \sum_{p=0}^{\infty} E_{(0,t)}\left(\mathcal{D}(\ell_0(u)(s)), -\lambda_p\right) u_{0p} \phi_p(x). \tag{15}$$

From this, the boundedness of $E_{(0,t)}\left(\mathcal{D}(\ell_0(u)(s)), -\lambda_p\right)$ (derived from (6) and (A_2)) and assumption (A_3) imply the well-posedness of (11) and hence of the forward problem, as follows.

Step 1. Define an operator $\mathcal{I}$ mapping from $C\left([0,T]; L^2(\Omega)\right)$ into itself by

$$\mathcal{I}(w)(x,t) := \sum_{p=0}^{\infty} E_{(0,t)}\left(\mathcal{D}(\ell_0(w)(s)), -\lambda_p\right) u_{0p} \phi_p(x).$$

We prove by induction that for $w_1, w_2 \in C\left([0,T]; L^2(\Omega)\right)$ and $n \in \mathbb{N}^*$

$$\left\| \mathcal{I}^n(w_1)(\cdot,t) - \mathcal{I}^n(w_2)(\cdot,t) \right\|^2 \le \left(T\mathrm{L}^2 \|f\|^2 \|u_0\|^2\right)^n \frac{t^n}{n!} \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}. \tag{16}$$

First, for $n = 1$ one easily has, by Parseval's relation, Holder's inequality and (5), that

$$\begin{aligned} \|\mathcal{I}(w_1)(\cdot,t) - \mathcal{I}(w_2)(\cdot,t)\|^2 &= \sum_{p=0}^{\infty} \left| E_{(0,t)}\left(\mathcal{D}(\ell_0(w_1)(s)), -\lambda_p\right) - E_{(0,t)}\left(\mathcal{D}(\ell_0(w_2)(s)), -\lambda_p\right) \right|^2 u_{0p}^2 \\ &\le T\mathrm{L}^2 \|f\|^2 \|u_0\|^2\, t\, \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}. \end{aligned}$$

Thus, (16) holds for $n = 1$. Now, supposing that this holds up to $n = k$, we then shall prove that it also holds for $n = k+1$. Indeed,

$$\begin{aligned} \left\| \mathcal{I}^{k+1}(w_1)(\cdot,t) - \mathcal{I}^{k+1}(w_2)(\cdot,t) \right\|^2 &\le T\mathrm{L}^2 \|f\|^2 \|u_0\|^2 \int_0^t \left\| \mathcal{I}^k(w_1)(\cdot,s) - \mathcal{I}^k(w_2)(\cdot,s) \right\|^2 ds \\ &\le T\mathrm{L}^2 \|f\|^2 \|u_0\|^2 \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))} \int_0^t \left(T\mathrm{L}^2 \|f\|^2 \|u_0\|^2\right)^k \frac{s^k}{k!} ds \\ &= \left(T\mathrm{L}^2 \|f\|^2 \|u_0\|^2\right)^{k+1} \frac{t^{k+1}}{(k+1)!} \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}. \end{aligned}$$

By the induction principle, we obtain (16). Since

$$\lim_{n\to\infty}\sqrt{\left(T\mathrm{L}^2\left\|f\right\|^2\left\|u_0\right\|^2\right)^n\frac{t^n}{n!}}=0,$$

there exists a number $n_0\in\mathbb{N}^*$ such that the expression under the limit is subunitary. This yields that $\mathcal{I}^{n_0}$ is a contraction mapping from $C\left([0,T];L^2(\Omega)\right)$ onto itself. Then, by the Banach fixed point theorem, there exists a unique solution in $C\left([0,T];L^2(\Omega)\right)$ to the equation $\mathcal{I}^{n_0}(w)=w$. In addition, one has $\mathcal{I}^{n_0}(\mathcal{I}(w))=\mathcal{I}(\mathcal{I}^{n_0})(w)=\mathcal{I}(w)$. Combining this with the uniqueness of the fixed point of $\mathcal{I}^{n_0}$, the equation $\mathcal{I}(w)=w$ admits a unique solution in $C\left([0,T];L^2(\Omega)\right)$.

Step 2. In this step we show that the solution u obtained at Step 1, also belongs to $L^2(0,T;\mathbb{V})$. First note that from (15),

$$\|u(\cdot,t)\|_{\mathbb{V}}^2=\sum_{p=0}^{\infty}\lambda_p\left\langle u(x,t),\phi_p\right\rangle^2=\sum_{p=0}^{\infty}\lambda_pE_{(0,t)}\left(\mathcal{D}\left(\ell_0(u)(s)\right),-2\lambda_p\right)u_{0p}^2,\quad \forall t\in[0,T].$$

Then, from this, (A_2) and (14) we obtain

$$\|u\|_{L^2(0,T;\mathbb{V})}^2=\int_0^T\|u(\cdot,t)\|_{\mathbb{V}}^2dt=\int_0^T\sum_{p=0}^{\infty}\lambda_pE_{(0,t)}\left(\mathcal{D}\left(\ell_0(u)(s)\right),-2\lambda_p\right)u_{0p}^2dt$$

$$\leq\int_0^T\sum_{p=0}^{\infty}\lambda_pu_{0p}^2\exp(-2\lambda_pt\eta_1)dt=\frac{1}{2\eta_1}\sum_{p=0}^{\infty}u_{0p}^2(1-\exp(-2\lambda_pT\eta_1))\leq\frac{1}{2\eta_1}\|u_0\|^2<\infty,$$

which implies that $u\in L^2(0,T;\mathbb{V})$ indeed. The proof is completed. □

2.3. Ill-posedness of the backward problem

Whilst in the previous subsection the forward problem was shown to be well-posed, in this subsection we investigate the ill-posedness of the backward problem. For time being, we postpone to the next section the proof the existence and uniqueness of solution in $C\left([0,T];L^2(\Omega)\right)\cap L^2(0,T;\mathbb{V})$, and investigate the solution's dependence on data. Similarly to the derivation of (15), one easily deduces the representation of solution to the backward problem (3), (4) and (7) as

$$u(x,t)=\sum_{p=0}^{\infty}E_{(t,T)}\left(\mathcal{D}\left(\ell_0(u)(s)\right),\lambda_p\right)g_p\phi_p(x),\tag{17}$$

where $g_p:=\left\langle g,\phi_p\right\rangle$.

While the term $E_{(0,t)}\left(\mathcal{D}\left(\ell_0(u)(s)\right),-\lambda_p\right)$ in the forward problem solution (15) can be definitely bounded by a constant, for the term $E_{(t,T)}\left(\mathcal{D}\left(\ell_0(u)(s)\right),\lambda_p\right)$ in our backward problem solution (17), we have the following inequality:

$$\left|E_{(t,T)}\left(\mathcal{D}(\xi_1),\lambda_p\right)-E_{(t,T)}\left(\mathcal{D}(\xi_2),\lambda_p\right)\right|^2\leq\max\left\{\frac{\lambda_p^2\left|\int_t^T\left[\mathcal{D}(\xi_1)-\mathcal{D}(\xi_2)\right]ds\right|^2}{E_{(t,T)}\left(\mathcal{D}(\xi_1),-2\lambda_p\right)},\frac{\lambda_p^2\left|\int_t^T\left[\mathcal{D}(\xi_1)-\mathcal{D}(\xi_2)\right]ds\right|^2}{E_{(t,T)}\left(\mathcal{D}(\xi_2),-2\lambda_p\right)}\right\},\tag{18}$$

for any $u_1,u_2\in C\left([0,T];L^2(\Omega)\right)\cap L^2(0,T;\mathbb{V})$, where we have denoted

$$\xi_i:=l_0(u_i)(s)=\int_\Omega f(x)u_i(x,s)dx,\quad E_{(t,T)}\left(\mathcal{D}(\xi_i),\lambda_p\right)=\exp\left(\lambda_p\int_t^T\mathcal{D}(\xi_i)ds\right),\quad i=1,2,$$

and we have used the elementary inequality:

$$\left|e^a-e^b\right|\leq\max\left\{|a-b|\,e^a,|a-b|\,e^b\right\},\quad\forall a,b\geq 0.\tag{19}$$

Combining this with (A_2) and using Hölder's inequality, yield

$$\begin{aligned}\left|E_{(t,T)}\left(\mathcal{D}(\xi_1),\lambda_p\right)-E_{(t,T)}\left(\mathcal{D}(\xi_2),\lambda_p\right)\right|^2 &\le \exp\left(2\eta_2\lambda_p(T-t)\right)\lambda_p^2\left|\int_t^T[\mathcal{D}(\xi_1)-\mathcal{D}(\xi_2)]\,ds\right|^2\\ &\le \exp\left(2\eta_2\lambda_p(T-t)\right)\lambda_p^2(T-t)\int_t^T|\mathcal{D}(\xi_1)-\mathcal{D}(\xi_2)|^2\,ds. \qquad (20)\end{aligned}$$

In order to illustrate the ill-posedness of the backward problem through an example, let $g\in\mathbb{G}_{\sigma,\gamma}$ for $\sigma\ge\eta_2 T,\gamma=2$ (this condition is essentially needed to prove the existence and uniqueness of solution) and an explicitly defined function $g^\epsilon\in L^2(\Omega)$ given by

$$g^\epsilon(x)=g(x)+\frac{1}{\lambda_n}\phi_n(x),$$

where $\epsilon:=\epsilon(n)=\frac{1}{\lambda_n}$ (for some positive integer n) represents the possible measurement noise which obviously satisfies (A_5). Then, the corresponding solution to the backward problem (3), (4) and (7) with such a noisy final data can be represented as

$$u^\epsilon(x,t)=\frac{E_{(t,T)}(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_n)}{\lambda_n}\phi_n(x)+\sum_{p=0}^{\infty}E_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p\right)g_p\phi_p(x), \qquad (21)$$

where we have used the orthonormal property of the eigenfunctions $\{\phi_p\}_{p\in\mathbb{N}}$. It yields from (17) and (21), by the triangle inequality, that

$$\begin{aligned}\left\|u^\epsilon(\cdot,t)-u(\cdot,t)\right\| &\ge \left\|\frac{E_{(t,T)}(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_n)}{\lambda_n}\phi_n(x)\right\|\\ &-\left\|\sum_{p=0}^{\infty}\left[E_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p\right)-E_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)),\lambda_p\right)\right]g_p\phi_p(x)\right\|. \qquad (22)\end{aligned}$$

The next step is to estimate the second norm on the right-hand side of (22) (here it is denoted by $\mathcal{J}_1$). Using (5), (20) and Parseval's relation, we have

$$\begin{aligned}\mathcal{J}_1^2 &\le \mathrm{L}^2\|f\|^2 T\left(\sum_{p=0}^{\infty}\lambda_p^2\exp\left(2\eta_2\lambda_p T\right)g_p^2\right)\int_t^T\left\|u^\epsilon(\cdot,s)-u(\cdot,s)\right\|^2 ds\\ &\le T\mathrm{L}^2\|f\|^2\|g\|^2_{\mathbb{G}_{\sigma,\gamma}}(T-t)\left\|u^\epsilon-u\right\|^2_{C([0,T];L^2(\Omega))}, \qquad (23)\end{aligned}$$

or,

$$\mathcal{J}_1\le T\mathrm{L}\|f\|\|g\|_{\mathbb{G}_{\sigma,\gamma}}\left\|u^\epsilon-u\right\|_{C([0,T];L^2(\Omega))}. \qquad (24)$$

Combining (22), (24) and (A_2), we obtain

$$\left\|u^\epsilon(\cdot,t)-u(\cdot,t)\right\|+T\mathrm{L}\|f\|\|g\|_{\mathbb{G}_{\sigma,\gamma}}\left\|u^\epsilon-u\right\|_{C([0,T];L^2(\Omega))}\ge\frac{e^{\eta_1(T-t)\lambda_n}}{\lambda_n}. \qquad (25)$$

Now, since we assume that the functions u^ϵ and u belong to the space $C\left([0,T];L^2(\Omega)\right)$, from (25) we obtain that

$$\left\|u^\epsilon-u\right\|_{C([0,T];L^2(\Omega))}\ge\frac{e^{\eta_1(T-t)\lambda_n}}{\lambda_n\left(1+T\mathrm{L}\|f\|\|g\|_{\mathbb{G}_{\sigma,\gamma}}\right)},\quad t\in[0,T].$$

Therefore, we conclude that for any $t\in[0,T)$

$$\lim_{n\to\infty}\left\|u^\epsilon-u\right\|\ge\lim_{n\to\infty}\frac{e^{\eta_1(T-t)\lambda_n}}{\lambda_n\left(1+T\mathrm{L}\|f\|\|g\|_{\mathbb{G}_{\sigma,\gamma}}\right)}=\infty. \qquad (26)$$

So, (26) shows that that even if the noise level $\epsilon=1/\lambda_n$ goes to zero, as $n\to\infty$, the instability always happens backwards in time. Hence, the need for a regularization method has been ascertained.

3. Quasi-reversibility (QR) based iterative approximation

In recent years, considerable attention has been given to accommodate the QR method, originally introduced in the book of Lattès and Lions [24], as a regularization approach for obtaining stable numerical solutions of ill-posed boundary value problems for partial differential equations. The main idea of such method is to basically add a perturbed term into the main equation or in the input conditions such that its new solution is stable and convergent to the exact solution, as the perturbation becomes decreasingly small, [3, 23]. In the present paper, we apply this general idea of the QR method for solving problem (P) given by equations (1)-(4).

Included below, we first try to apply the QR method to the problem without reaction term, i.e. $\mathcal{R} = 0$ in (1). Then, global and local Lipschitz nonlinearity reactions will be studied using a modified QR approach.

3.1. The inverse problem without reaction

In this subsection, we take $\mathcal{R} = 0$ in (1) and consider the backward heat problem given by

$$\begin{cases} u_t = \mathcal{D}(\ell_0(u)(t))\Delta u, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,T) = g(x), & x \in \Omega. \end{cases} \tag{27}$$

As mentioned before in subsection 2.3, it is now worth proving the existence and uniqueness of solution to the problem (27), as given by the following theorem.

Theorem 4. *Assume* (A_1)-(A_5) *hold and let* $g \in \mathbb{G}_{\sigma,\gamma}$ *for* $\sigma \geq \eta_2 T, \gamma = 2$. *Then the integral equation* (17) *has a unique solution in* $C\left([0,T]; L^2(\Omega)\right) \cap L^2(0,T;\mathbb{V})$*, which is also the unique solution of the problem (27).*

Proof. The proof is divided into three steps. First, we prove, by the contraction principle, that equation (17) has a unique solution in $C\left([0,T]; L^2(\Omega)\right)$. Second, we prove that this solution also belongs to $L^2(0,T;\mathbb{V})$. Thirdly, we prove that the problem (27) is equivalent to the integral equation (17).

Step 1. For $w \in C\left([0,T]; L^2(\Omega)\right)$, we define an operator $\mathcal{G}$ mapping from $C\left([0,T]; L^2(\Omega)\right)$ into itself by

$$\mathcal{G}(w)(x,t) := \sum_{p=0}^{\infty} E_{(t,T)}\left(\mathcal{D}(\ell_0(w)(s)), \lambda_p\right) g_p \phi_p(x). \tag{28}$$

We prove by induction that for $w_1, w_2 \in C\left([0,T]; L^2(\Omega)\right)$ and $n \in \mathbb{N}^*$

$$\left\|\mathcal{G}^n(w_1)(\cdot,t) - \mathcal{G}^n(w_2)(\cdot,t)\right\|^2 \leq \left(T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right)^n \frac{(T-t)^n}{n!} \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}. \tag{29}$$

First, for $n = 1$ one easily has, by Parseval's relation, that

$$\|\mathcal{G}(w_1)(\cdot,t) - \mathcal{G}(w_2)(\cdot,t)\|^2 = \sum_{p=0}^{\infty} \left|E_{(t,T)}\left(\mathcal{D}(\ell_0(w_1)(s)), \lambda_p\right) - E_{(t,T)}\left(\mathcal{D}(\ell_0(w_2)(s)), \lambda_p\right)\right|^2 g_p^2.$$

In a similar manner to that used to get (23), we obtain

$$\|\mathcal{G}(w_1)(\cdot,t) - \mathcal{G}(w_2)(\cdot,t)\|^2 \leq T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}} (T-t) \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}.$$

Thus, (29) holds for $n = 1$. Now, supposing that (29) holds up to $n = k$, we then shall prove that it also holds for $n = k+1$. Indeed, it is straightforward to estimate (29) at that level, by following the derivation of (23) using (19), to obtain

$$\begin{aligned} \left\|\mathcal{G}^{k+1}(w_1)(\cdot,t) - \mathcal{G}^{k+1}(w_2)(\cdot,t)\right\|^2 &\leq T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}} \int_t^T \left\|\mathcal{G}^k(w_1)(\cdot,s) - \mathcal{G}^k(w_2)(\cdot,s)\right\|^2 ds \\ &\leq T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}} \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))} \int_t^T \left(T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right)^k \frac{(T-s)^k}{k!} ds \\ &= \left(T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right)^{k+1} \frac{(T-t)^{k+1}}{(k+1)!} \|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}. \end{aligned}$$

By the induction principle, we obtain (29) or, in another form by taking the square root

$$\left\|\mathcal{G}^n (w_1)(\cdot,t) - \mathcal{G}^n (w_2)(\cdot,t)\right\| \le \sqrt{\left(T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right)^n \frac{(T-t)^n}{n!}} \|w_1 - w_2\|_{C([0,T];L^2(\Omega))}. \tag{30}$$

Since

$$\lim_{n\to\infty} \sqrt{\left(T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right)^n \frac{(T-t)^n}{n!}} = 0,$$

there exists a number $n_0 \in \mathbb{N}^*$ such that

$$\sqrt{\left(T\mathrm{L}^2 \|f\|^2 \|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right)^{n_0} \frac{(T-t)^{n_0}}{n_0!}} < 1,$$

which from (30) it yields that $\mathcal{G}^{n_0}$ is a contraction mapping from $C\left([0,T];L^2(\Omega)\right)$ onto itself. Then, by the Banach fixed point theorem, there exists a unique solution in $C\left([0,T];L^2(\Omega)\right)$ to the equation $\mathcal{G}^{n_0}(w) = w$. In addition, one has $\mathcal{G}^{n_0}(\mathcal{G}(w)) = \mathcal{G}(\mathcal{G}^{n_0})(w) = \mathcal{G}(w)$. Combining this with the uniqueness of the fixed point of $\mathcal{G}^{n_0}$, the equation $\mathcal{G}(w) = w$ admits a unique solution in $C\left([0,T];L^2(\Omega)\right)$.

Step 2. In this step we show that the solution u obtained at Step 1, also belongs to $L^2(0,T;\mathbb{V})$. First note that from (17),

$$\|u(\cdot,t)\|^2_{\mathbb{V}} = \sum_{p=0}^{\infty} \lambda_p \left\langle u(x,t),\phi_p\right\rangle^2 = \sum_{p=0}^{\infty} \lambda_p E_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)), 2\lambda_p\right) g_p^2, \quad \forall t\in[0,T].$$

From the assumption on g, we have that $\sum_{p=0}^{\infty} \lambda_p^2 e^{2\sigma\lambda_p} g_p^2 < \infty$. Then using the above, (A_2) and $\sigma \ge \eta_2 T$, we obtain

$$\|u\|^2_{L^2(0,T;\mathbb{V})} = \int_0^T \|u(\cdot,t)\|^2_{\mathbb{V}} dt = \int_0^T \sum_{p=0}^{\infty} \lambda_p E_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)), 2\lambda_p\right) g_p^2 dt$$

$$\le \int_0^T \sum_{p=0}^{\infty} \lambda_p \exp(2\eta_2\lambda_p(T-t)) g_p^2 dt \le T\sum_{p=0}^{\infty} \lambda_p e^{2\sigma\lambda_p} g_p^2 \le \frac{T}{\lambda_1}\sum_{p=0}^{\infty} \lambda_p^2 e^{2\sigma\lambda_p} g_p^2 < \infty,$$

which implies that $u \in L^2(0,T;\mathbb{V})$ indeed.

Step 3. If $u \in C\left([0,T];L^2(\Omega)\right) \cap L^2(0,T;\mathbb{V})$ satisfies (17), then by direct calculus we obtain that u is a solution of (27). Thus, it remains to prove that if u satisfies (27), it must be a solution of (17). Indeed, the proof can be established by using the representation of solution to the forward problem. Observe that from (11), one can find a relationship between u_{0p} and g_p. Letting $t = T$ in (11) and taking the action with ϕ_p we obtain

$$g_p = \left\langle u(x,T), \phi_p(x)\right\rangle = E_{(0,T)}\left(\mathcal{D}(\ell_0(u)(s)), -\lambda_p\right) u_{0p}$$

or,

$$u_{0p} = E_{(0,T)}\left(\mathcal{D}(\ell_0(u)(s)), \lambda_p\right) g_p. \tag{31}$$

Combining (11) and (31), we conclude that u is a solution of (17). The proof of the theorem is completed. □

So far, from Theorem 4 we have proved the existence and uniqueness of solution to the problem (27) and, from subsection 2.3, the discontinuous dependence of such a solution on the data. It is now essential to develop a regularization approach in order to obtain a stable solution. In fact, we will construct an approximation problem to (27), denoted by (P_1), which will be used to establish our regularized solution (denoted by u^ϵ), whose existence and uniqueness carry out as in Theorem 4. Obviously, here we take the noisy measured final data g^ϵ into account.

By an analogue of (14), let us denote, for $\alpha > 0$,

$$E^{\alpha}_{(a,b)}\left(\mathcal{D}(\ell_0(u)(s)), \lambda_p\right) := \exp\left(\int_a^b \frac{\mathcal{D}(\ell_0(u)(s))\lambda_p}{1+\alpha\mathcal{D}(\ell_0(u)(s))\lambda_p} ds\right), \tag{32}$$

and let us consider the regularized problem (P_1) given by

$$\begin{cases} u_t^\epsilon = \mathcal{D}\left(\ell_0\left(u^\epsilon\right)(t)\right)\Delta u^\epsilon + \alpha\mathcal{D}\left(\ell_0\left(u^\epsilon\right)(t)\right)\Delta u_t^\epsilon, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u^\epsilon}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u^\epsilon(x,T) = g^\epsilon(x), & x \in \Omega, \end{cases} \tag{33}$$

where $\alpha = \alpha(\epsilon)$ plays the role of a regularization parameter. Next we prove that this problem has a unique solution, as follows.

Theorem 5. *Assume* (A_1)-(A_5) *hold. Then the integral equation*

$$u^\epsilon(x,t) = \sum_{p=0}^{\infty} E_{(t,T)}^{\alpha}\left(\mathcal{D}\left(\ell_0\left(u^\epsilon\right)(s)\right), \lambda_p\right) g_p^\epsilon \phi_p(x), \tag{34}$$

where $g_p^\epsilon := \left\langle g^\epsilon, \phi_p \right\rangle$, *has a unique solution in* $C\left([0,T];L^2(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$, *which is also the unique solution of the problem* (33).

Proof. As in the proof of Theorem 4 we have three steps.

Step 1. Let us define the mapping $\mathcal{G}_\alpha : C\left([0,T];L^2(\Omega)\right) \to C\left([0,T];L^2(\Omega)\right)$ by

$$\mathcal{G}_\alpha(w)(x,t) = \sum_{p=0}^{\infty} E_{(t,T)}^{\alpha}\left(\mathcal{D}\left(\ell_0(w)(s)\right), \lambda_p\right) g_p^\epsilon \phi_p(x), \quad \forall w \in C\left([0,T];L^2(\Omega)\right). \tag{35}$$

For $w_1, w_2 \in C\left([0,T];L^2(\Omega)\right)$ and $n \in \mathbb{N}^*$, we proceed using induction to prove that

$$\left\|\mathcal{G}_\alpha^n(w_1)(\cdot,t) - \mathcal{G}_\alpha^n(w_2)(\cdot,t)\right\|^2 \le \left(\frac{T\mathrm{L}^2\|f\|^2\|g^\epsilon\|^2 e^{2T/\alpha}}{\eta_1^2\alpha^2}\right)^n \frac{(T-t)^n}{n!}\|w_1 - w_2\|_{C([0,T];L^2(\Omega))}^2. \tag{36}$$

As in the proof of Theorem 4, though the technicalities are slightly different, we first estimate the difference

$$\mathcal{J}_2 := \left|E_{(t,T)}^{\alpha}\left(\mathcal{D}\left(\ell_0(w_1)(s)\right), \lambda_p\right) - E_{(t,T)}^{\alpha}\left(\mathcal{D}\left(\ell_0(w_2)(s)\right), \lambda_p\right)\right|^2, \tag{37}$$

by the following:

$$\begin{aligned} \mathcal{J}_2 &\le \left|\int_t^T \left(\frac{\mathcal{D}\left(\ell_0(w_1)(s)\right)\lambda_p}{1 + \alpha\mathcal{D}\left(\ell_0(w_1)(s)\right)\lambda_p} - \frac{\mathcal{D}\left(\ell_0(w_2)(s)\right)\lambda_p}{1 + \alpha\mathcal{D}\left(\ell_0(w_2)(s)\right)\lambda_p}\right) ds\right|^2 \\ &\quad \times \max\left\{\left(E_{(t,T)}^{\alpha}\right)^2\left(\mathcal{D}\left(\ell_0(w_1)(s)\right), \lambda_p\right), \left(E_{(t,T)}^{\alpha}\right)^2\left(\mathcal{D}\left(\ell_0(w_2)(s)\right), \lambda_p\right)\right\} \\ &\le e^{2(T-t)/\alpha}\left|\int_t^T \left(\frac{\mathcal{D}\left(\ell_0(w_1)(s)\right)\lambda_p}{1 + \alpha\mathcal{D}\left(\ell_0(w_1)(s)\right)\lambda_p} - \frac{\mathcal{D}\left(\ell_0(w_2)(s)\right)\lambda_p}{1 + \alpha\mathcal{D}\left(\ell_0(w_2)(s)\right)\lambda_p}\right) ds\right|^2, \end{aligned} \tag{38}$$

where we have used the elementary inequality (19) and that

$$E_{(t,T)}^{\alpha}\left(\mathcal{D}\left(\ell_0(w)(s)\right), \lambda_p\right) \le e^{(T-t)/\alpha}, \quad \forall w \in C\left([0,T];L^2(\Omega)\right), \tag{39}$$

noticing that $\frac{\mathcal{D}(\ell_0(w_i)(s))\lambda_p}{1+\alpha\mathcal{D}(\ell_0(w_i)(s))\lambda_p} \le \frac{1}{\alpha}$ for $i = 1,2$. Moreover, if $\mathcal{J}_3$ denotes the integral on the last right-hand side of (38), it is due to (A_2) that

$$\begin{aligned} \mathcal{J}_3 &\le (T-t)\int_t^T \left|\frac{\left[\mathcal{D}\left(\ell_0(w_1)(s)\right) - \mathcal{D}\left(\ell_0(w_2)(s)\right)\right]\lambda_p}{\left(1 + \alpha\mathcal{D}\left(\ell_0(w_1)(s)\right)\lambda_p\right)\left(1 + \alpha\mathcal{D}\left(\ell_0(w_2)(s)\right)\lambda_p\right)}\right|^2 ds \\ &\le \frac{T-t}{\eta_1^2\alpha^2}\int_t^T \left|\mathcal{D}\left(\ell_0(w_1)(s)\right) - \mathcal{D}\left(\ell_0(w_2)(s)\right)\right|^2 ds, \end{aligned} \tag{40}$$

where we have applied the Hölder inequality and that $1 + \alpha\mathcal{D}(\ell_0(w)(s))\lambda_p \geq 1$.

Combining (35), (37), (38), (40) coupled with (5), and thanks to Parseval's relation, we obtain

$$\begin{aligned} \|\mathcal{G}_\alpha(w_1)(\cdot,t) - \mathcal{G}_\alpha(w_2)(\cdot,t)\|^2 &\leq \frac{T\mathrm{L}^2\|f\|^2 e^{2(T-t)/\alpha}}{\eta_1^2\alpha^2}\left(\sum_{p=0}^{\infty}\left|g_p^\epsilon\right|^2\right)\int_t^T \|w_1(\cdot,s) - w_2(\cdot,s)\|^2 ds \\ &\leq \frac{T\mathrm{L}^2\|f\|^2\|g^\epsilon\|^2 e^{2(T-t)/\alpha}}{\eta_1^2\alpha^2}(T-t)\|w_1 - w_2\|^2_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{41}$$

This shows that (36) holds for $n = 1$.

As in the proof of Theorem 4, one easily passes through the inductive step to obtain (36) by induction principle. Following from such an argument in combination with the Banach fixed-point theorem, we conclude the existence and uniqueness of solution $u^\epsilon \in C\left([0,T]; L^2(\Omega)\right)$ to the equation $\mathcal{G}_\alpha(w) = w$.

Step 2. In this step, we prove that the solution obtained in Step 1 is also in $L^2\left(0,T;L^2(\Omega)\right)$. This follows, using (39), from

$$\|u\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \sum_{p=0}^{\infty} E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)), 2\lambda_p\right)(g_p^\epsilon)^2 dt \leq Te^{2T/\alpha}\sum_{p=0}^{\infty}(g_p^\epsilon)^2 = Te^{2T/\alpha}\|g^\epsilon\|^2 < \infty.$$

Step 3. It suffices to prove that if $u^\epsilon \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$ satisfies (33), then it is a solution of (34). In fact, by taking the action of the first equation of (33) with ϕ_p and using (6), we obtain

$$\left\langle u_t^\epsilon(x,t), \phi_p(x)\right\rangle + \frac{\mathcal{D}(\ell_0(u^\epsilon)(t))\lambda_p}{1 + \alpha\mathcal{D}(\ell_0(u^\epsilon)(t))\lambda_p}\left\langle u^\epsilon(x,t), \phi_p(x)\right\rangle = 0. \tag{42}$$

Multiplying both sides of (42) by the function $E^\alpha_{(0,t)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)), \lambda_p\right)$, integrating the obtained result with respect to s from t to T, and using that u^ϵ can be represented by the Fourier series as

$$u^\epsilon(x,t) = \sum_{p=0}^{\infty}\left\langle u^\epsilon(x,t), \phi_p(x)\right\rangle\phi_p(x),$$

we obtain the representation (34). The proof of the theorem is completed. □

Remark 6. It is clear from the boundedness of $E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(w)(s)), \lambda_p\right)$ by (39) that for each regularization parameter α, dependent on the noise level ϵ, the solution $u^\epsilon \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$ of (34) depends continuously on the measured final data $g^\epsilon \in L^2(\Omega)$. This, in combination with the existence and uniqueness results in Theorem 5, implies that the problem (P_1) given by (33) is well-posed. It is also remarkable that even though we have been proving Theorem 5 in a way that is similar to Theorem 4, the emphasis is essentially put on the regularity of the final data. In fact, g^ϵ here belongs to $L^2(\Omega)$ only, whereas g needs to be in a class of Gevrey spaces, which is a closed subset of $L^2(\Omega)$. In addition, if g^ϵ is also in $\mathbb{G}_{\sigma,\gamma}$ for $\sigma \geq \eta_2 T, \gamma = 2$, then $u^\epsilon \in C\left([0,T]; L^2(\Omega)\right) \cap L^2(0,T;\mathbb{V})$ and also, there is another way to estimate (38), (40) and (41), namely,

$$\mathcal{J}_2 \leq e^{2\eta_2(T-t)\lambda_p}\left|\int_t^T\left(\frac{\mathcal{D}(\ell_0(w_1)(s))\lambda_p}{1 + \alpha\mathcal{D}(\ell_0(w_1)(s))\lambda_p} - \frac{\mathcal{D}(\ell_0(w_2)(s))\lambda_p}{1 + \alpha\mathcal{D}(\ell_0(w_2)(s))\lambda_p}\right)ds\right|^2,$$

$$\mathcal{J}_3 \leq (T-t)\lambda_p^2\int_t^T|\mathcal{D}(\ell_0(w_1)(s)) - \mathcal{D}(\ell_0(w_2)(s))|^2 ds,$$

which imply

$$\|\mathcal{G}_\alpha(w_1)(\cdot,t) - \mathcal{G}_\alpha(w_2)(\cdot,t)\|^2 \leq T\mathrm{L}^2\|f\|^2\left\|g^\epsilon\right\|^2_{\mathbb{G}_{\sigma,\gamma}}\int_t^T\|w_1(\cdot,s) - w_2(\cdot,s)\|^2 ds. \tag{43}$$

The advantage of (43) will be seen later on.

At this stage, we consider the convergence analysis. To ensure this, let us note that we must also impose the regularity of u in the space $L^\infty\left(0,T;H^4(\Omega)\right)$. We then have the convergence result given by the following theorem.

Theorem 7. *Suppose that* (A_1)*-*(A_5) *hold and let* u^ϵ *be the solution of* (P_1) *given by (33). Assume that* $g \in \mathbb{G}_{\sigma,\gamma}$ *for* $\sigma \geq \eta_2 T, \gamma = 2$ *holds. If the solution* u *of the problem* (27) *belongs to* $C^1\left(0,T;L^2(\Omega)\right) \cap L^\infty\left(0,T;H^4(\Omega)\right)$*, then we have the following error estimate:*

$$\left\|u^\epsilon(\cdot,t) - u(\cdot,t)\right\| \leq \sqrt{3\left[e^{2(T-t)/\alpha}\epsilon^2 + \eta_2^4(T-t)^2\alpha^2\|u\|^2_{L^\infty(0,T;H^4(\Omega))}\right]}\,\exp\left(\frac{3}{2}T(T-t)\mathrm{L}^2\|f\|^2\|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right), \quad t \in [0,T]. \tag{44}$$

By choosing $\alpha := \alpha(\epsilon) > 0$ *such that*

$$\lim_{\epsilon\to0} e^{T/\alpha}\epsilon = \lim_{\epsilon\to0}\alpha = 0, \tag{45}$$

this implies that $\|u^\epsilon(\cdot,t) - u(\cdot,t)\| \to 0$ *as* $\epsilon \to 0$*, for every* $t \in [0,T]$*. Moreover, if the regularization parameter* α *is explicitly given by*

$$\alpha = \frac{T}{\ln\left(\frac{1}{\epsilon^{1-\omega}}\right)}, \quad \textit{for some } \omega \in (0,1), \tag{46}$$

which satisfies (45)*, then from* (44) *we obtain the stability estimate*

$$\left\|u^\epsilon(\cdot,t) - u(\cdot,t)\right\| \leq \sqrt{3\left[\epsilon^{2\omega} + \frac{\eta_2^4T^2(T-t)^2}{\ln^2\left(\frac{1}{\epsilon^{1-\omega}}\right)}\|u\|^2_{L^\infty(0,T;H^4(\Omega))}\right]}\,\exp\left(\frac{3}{2}T(T-t)\mathrm{L}^2\|f\|^2\|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right), \quad t \in [0,T]. \tag{47}$$

Proof. Let us first define $w^\epsilon(x,t) = u^\epsilon(x,t) - u(x,t)$ which, based on (17) and (34), it can be expressed as

$$\begin{aligned}
w^\epsilon(x,t) &= \sum_{p=0}^{\infty} E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p\right)\left(g^\epsilon_p - g_p\right)\phi_p(x) \\
&\quad + \sum_{p=0}^{\infty}\left[E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p\right) - E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)),\lambda_p\right)\right]g_p\phi_p(x) \\
&\quad + \sum_{p=0}^{\infty}\left[E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)),\lambda_p\right) - E_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)),\lambda_p\right)\right]g_p\phi_p(x) \\
&= \mathcal{J}_4 + \mathcal{J}_5 + \mathcal{J}_6.
\end{aligned} \tag{48}$$

It is now immediate to observe that using the inequality (39) together with Parseval's relation and (A_5), give us the following inequality:

$$\|\mathcal{J}_4\|^2 = \sum_{p=0}^{\infty}\left|E^\alpha_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p\right)\right|^2\left|g^\epsilon_p - g_p\right|^2 \leq e^{2(T-t)/\alpha}\left\|g^\epsilon - g\right\|^2 \leq e^{2(T-t)/\alpha}\epsilon^2. \tag{49}$$

Estimating $\mathcal{J}_5$ is totally similar to (43) with the aid of Parseval's relation, and we thus obtain

$$\|\mathcal{J}_5\|^2 \leq T\mathrm{L}^2\|f\|^2\|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\int_t^T\left\|u^\epsilon(\cdot,s) - u(\cdot,s)\right\|^2 ds. \tag{50}$$

From (17) we obtain that

$$\sum_{p=0}^{\infty}\lambda_p^4u_p^2(t) \leq \|u\|^2_{L^\infty(0,T;H^4(\Omega))}, \quad t \in [0,T], \tag{51}$$

where $u_p^2(t) := E_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)), 2\lambda_p\right) g_p^2$. Then using (A$_2$) and (51), as in (18), the term $\mathcal{J}_6$ can be estimated as follows:

$$\begin{aligned}\|\mathcal{J}_6\|^2 &\le \sum_{p=0}^{\infty}\left|\int_t^T \left(\frac{\mathcal{D}(\ell_0(u)(s))\lambda_p}{1+\alpha\mathcal{D}(\ell_0(u)(s))\lambda_p} - \mathcal{D}(\ell_0(u)(s))\lambda_p\right) ds\right|^2 E_{(t,T)}\left(\mathcal{D}(\ell_0(u)(s)), 2\lambda_p\right) g_p^2 \\ &= \sum_{p=0}^{\infty}\alpha^2\lambda_p^4\left(\int_t^T \frac{\mathcal{D}^2(\ell_0(u)(s))}{1+\alpha\mathcal{D}(\ell_0(u)(s))\lambda_p} ds\right)^2 u_p^2(t) \le \eta_2^4(T-t)^2\alpha^2\|u\|^2_{L^\infty(0,T;H^4(\Omega))}. \end{aligned} \tag{52}$$

Combining (48)-(50) and (52), we obtain (using that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$),

$$\left\|w^\epsilon(\cdot,t)\right\|^2 \le 3e^{2(T-t)/\alpha}\epsilon^2 + 3\eta_2^4(T-t)^2\alpha^2\|u\|^2_{L^\infty(0,T;H^4(\Omega))} + 3T\mathrm{L}^2\|f\|^2\|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\int_t^T\left\|w^\epsilon(\cdot,s)\right\|^2 ds.$$

Finally, using Gronwall's inequality, we conclude that

$$\left\|w^\epsilon(\cdot,t)\right\|^2 \le 3\left[e^{2(T-t)/\alpha}\epsilon^2 + \eta_2^4(T-t)^2\alpha^2\|u\|^2_{L^\infty(0,T;H^4(\Omega))}\right]\exp\left(3T(T-t)\mathrm{L}^2\|f\|^2\|g\|^2_{\mathbb{G}_{\sigma,\gamma}}\right),$$

and taking the square root we obtain (44). This completes the proof of the theorem. □

Remark 8. Theorem 7 points out that our unique and stable solution of the problem (P_1) (given by equation (33)) is strongly convergent to the exact solution of the problem (27) in the L^2-norm. Furthermore, the error estimate (47) is a combination of the orders of ϵ^ω and $-\frac{1}{\ln(\epsilon^{1-\omega})}$ for some $\omega \in (0,1)$, or we can say it is $O(-1/\ln(\epsilon))$.

3.2. A computational tool

Our current objective is to solve the fixed point equation

$$u^\epsilon(x,t) = \mathcal{G}_\alpha(u^\epsilon)(x,t), \tag{53}$$

where, for each $\alpha > 0$, the operator $\mathcal{G}_\alpha$ defined by (35) has a unique fix point in $C\left([0,T];L^2(\Omega)\right)$, according to the proof of Theorem 5 (Step 1). We then seek the solution of (53) in the formal series form

$$u^\epsilon(x,t) = \sum_{m=0}^{\infty} u^\epsilon_{[m]}(x,t), \tag{54}$$

where the components $u^\epsilon_{[m]}$ are obtained from the following recurrence scheme, [12],

$$\begin{cases} u^\epsilon_{[m+1]}(x,t) = \mathcal{G}_\alpha\left(\sum_{j=0}^m u^\epsilon_{[j]}\right)(x,t) - \mathcal{G}_\alpha\left(\sum_{j=0}^{m-1} u^\epsilon_{[j]}\right)(x,t), \quad m = 1,2,... \\ u^\epsilon_{[1]}(x,t) = \mathcal{G}_\alpha\left(u^\epsilon_{[0]}\right)(x,t), \quad u^\epsilon_{[0]}(x,t) = 0. \end{cases} \tag{55}$$

3.3. The inverse problem with reaction

In this subsection, we consider the fully nonlinear case for the problem (P) (given by equations (1)-(4)) with the density-dependent reaction term $\mathcal{R}(x,t,u)$ included. This is more challenging since the regularization methods via the nonlinear spectral theory based on Fourier series are somewhat difficult to compute in this case. The impediment lies not only in the fact that we have to define an additional functional space reasonably covering our analysis, but also other technicalities, where our existence and uniqueness results can be derived by fixed-point arguments, are required.

For $\beta := \beta(\epsilon) > 0$ by the modified QR method, let us consider the problem (P_2) given by

$$\begin{cases} u^\epsilon_t + \mathcal{D}(\ell_0(u^\epsilon)(t))\Delta^\epsilon u^\epsilon = \mathcal{R}(x,t,u^\epsilon), & (x,t) \in \Omega\times(0,T), \\ \frac{\partial u^\epsilon}{\partial\nu}(x,t) = 0, & (x,t) \in \partial\Omega\times(0,T), \\ u^\epsilon(x,T) = g^\epsilon(x), & x \in \Omega, \end{cases} \tag{56}$$

where

$$u_p(t) := \left\langle u(\cdot,t), \phi_p \right\rangle, \quad \Delta^\epsilon u := \sum_{p=0}^{\infty} \lambda_p^{(\epsilon)} u_p(t)\,\phi_p(x), \quad u_p^\epsilon(t) := \left\langle u^\epsilon(\cdot,t), \phi_p \right\rangle, \quad \Delta^\epsilon u^\epsilon := \sum_{p=0}^{\infty} \lambda_p^{(\epsilon)} u_p^\epsilon(t)\,\phi_p(x), \tag{57}$$

and

$$\lambda_p^{(\epsilon)} = -\frac{1}{\eta_2 T} \ln\left(\beta + e^{-\eta_2 T \lambda_p}\right). \tag{58}$$

Note that from (57) and (58), we have

$$\left\|\Delta^\epsilon u^\epsilon(\cdot,t)\right\| \le \frac{\ln\left(\frac{1}{\beta}\right)}{\eta_2 T} \left\|u^\epsilon(\cdot,t)\right\|. \tag{59}$$

At this stage, a brief comment between the problems (P_1) and (P_2) is appropriate. Of course, the main idea of the QR method is to approximate an unbounded operator, say A, by a bounded (regularized) one, A^ϵ. Note that the problem (P_2) with $\mathcal{R} = 0$ does not reduce to problem (P_1) for the following reasons. In the previous problem (P_1) without reaction, we have applied the classical QR method, as in (33), to obtain the convergence rates given in Theorem 7 under the higher smoothness of the exact solution in $L^\infty\left(0,T;H^4(\Omega)\right)$. However, if we apply this classical QR method to the problem (1)-(4) with reaction, we were not able to estimate the error between the exact and the regularized solutions. Therefore, we employ a modified QR-type approach, [33, 34], as in (56), for which we are able to prove the error estimates given in Theorems 13 and 15 below.

3.3.1. Global Lipschitz reaction

In combination with (A_1)-(A_5), we make an additional Lipschitz assumption on the reaction term as follows: (A_6) $\mathcal{R} \in L^\infty(\Omega\times(0,T)\times\mathbb{R})$ and there exists a positive constant K such that

$$|\mathcal{R}(x,t,\xi_1) - \mathcal{R}(x,t,\xi_2)| \le K|\xi_1 - \xi_2|, \quad \forall (x,t) \in \Omega\times(0,T),\ \forall \xi_1,\xi_2 \in \mathbb{R}.$$

Notice that from (A_6), we can deduce that

$$\|\mathcal{R}(\cdot,t,w_1) - \mathcal{R}(\cdot,t,w_2)\| \le K\|w_1(\cdot,t) - w_2(\cdot,t)\|, \quad \forall w_1,w_2 \in C\left([0,T];L^2(\Omega)\right), \tag{60}$$

which leads to

$$\|\mathcal{R}(\cdot,t,w) - \mathcal{R}(\cdot,t,0)\| \le K\|w(\cdot,t)\|, \quad \forall w \in C\left([0,T];L^2(\Omega)\right).$$

Thus, we have

$$\|\mathcal{R}(\cdot,t,w)\|^2 \le (K\|w(\cdot,t)\| + \|\mathcal{R}(\cdot,t,0)\|)^2 \le \left(K^2+1\right)\left(\|w(\cdot,t)\|^2 + \|\mathcal{R}(\cdot,t,0)\|^2\right). \tag{61}$$

Given a constant $\beta \in (0,1)$ (which will be assumed from now on) and a function $w \in C\left([0,T];L^2(\Omega)\right)$, we denote the scaling with β as follows:

$$\|w\|_{\beta,0}(t) := \beta^{-\frac{t}{T}}\|w\|_{C([0,T];L^2(\Omega))}, \quad \|w\|_{\beta,\infty} := \sup_{0\le t\le T}\|w\|_{\beta,0}(t). \tag{62}$$

It is obvious to see that $\|w(\cdot,t)\| \le \|w\|_{\beta,0}(t) \le \|w\|_{\beta,\infty}$ for $t \in [0,T]$.

Before tackling the main results, we give the following lemmas.

Lemma 9. *Suppose that (A_1)-(A_5) hold. Then, for $0 \le t \le T_* \le T$, the self-mapping $\mathcal{H}$ on $C\left([0,T];L^2(\Omega)\right)$ defined by*

$$\mathcal{H}(w)(x,t) := \sum_{p=0}^{\infty} E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w)(s)), \lambda_p^{(\epsilon)}\right) g_p^\epsilon \phi_p(x), \quad \forall w \in C\left([0,T];L^2(\Omega)\right), \tag{63}$$

can be estimated by

$$\|\mathcal{H}(w)\|_{\beta,0}(t) \le \beta^{-\frac{T_*}{T}}\left\|g^\epsilon\right\|, \tag{64}$$

and for all $w_1, w_2 \in C\left([0,T];L^2(\Omega)\right)$, we have

$$\|\mathcal{H}(w_1)-\mathcal{H}(w_2)\|_{\beta,0}(t) \le \frac{\mathrm{L}(T_*-t)\ln\left(\frac{1}{\beta}\right)\|f\|\,\|g^\epsilon\|}{\eta_2 T}\|w_1-w_2\|_{\beta,\infty}. \tag{65}$$

Proof. From (14), (58) and Parseval's relation one observes that

$$\begin{aligned}\|\mathcal{H}(w)(\cdot,t)\|^2 &= \sum_{p=0}^{\infty} E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w)(s)),2\lambda_p^{(\epsilon)}\right)\left|g_p^\epsilon\right|^2\\ &\le \exp\left(\frac{2}{\eta_2 T}\ln\left(\frac{1}{\beta}\right)\eta_2(T_*-t)\right)\sum_{p=0}^{\infty}\left|g_p^\epsilon\right|^2 = \beta^{\frac{2(t-T_*)}{T}}\left\|g^\epsilon\right\|^2.\end{aligned}$$

It follows, from the definition of the norm $\|\cdot\|_{\beta,0}$, that (64) holds.

To prove (65), let us first estimate the difference

$$\mathcal{J}_7 := \left|E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w_1)(s)),\lambda_p^{(\epsilon)}\right)-E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w_2)(s)),\lambda_p^{(\epsilon)}\right)\right|^2. \tag{66}$$

As in (18), we have

$$\begin{aligned}\mathcal{J}_7 &\le \left|\lambda_p^{(\epsilon)}\right|^2(T_*-t)\int_t^{T_*}|\mathcal{D}(\ell_0(w_1)(s))-\mathcal{D}(\ell_0(w_2)(s))|^2\,ds\\ &\quad\times\max\left\{E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w_1)(s)),2\lambda_p^{(\epsilon)}\right),E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w_2)(s)),2\lambda_p^{(\epsilon)}\right)\right\}.\end{aligned}$$

Using ($\mathrm{A}_\mathbf{2}$), (5) and (58) we obtain

$$\begin{aligned}\mathcal{J}_7 &\le \left|\lambda_p^{(\epsilon)}\right|^2(T_*-t)\exp\left(2\lambda_p^{(\epsilon)}\eta_2(T_*-t)\right)\mathrm{L}^2\|f\|^2\int_t^{T_*}\|w_1(\cdot,s)-w_2(\cdot,s)\|^2\,ds\\ &\le \frac{\mathrm{L}^2(T_*-t)^2\ln^2\left(\frac{1}{\beta}\right)\|f\|^2}{\eta_2^2T^2}\beta^{\frac{2(t-T_*)}{T}}\|w_1-w_2\|^2_{C([0,T];L^2(\Omega))},\end{aligned} \tag{67}$$

where in the last inequality we have used that the function $\ln^2(\beta)\beta^{2(t-T_*)/T}$ is decreasing, as a function of $\beta\in(0,1)$. It implies from (66), (67) together with Parseval's relation that

$$\|\mathcal{H}(w_1)(\cdot,t)-\mathcal{H}(w_2)(\cdot,t)\|^2 = \sum_{p=0}^{\infty}\mathcal{J}_7\left|g_p^\epsilon\right|^2 \le \frac{\mathrm{L}^2(T_*-t)^2\ln^2\left(\frac{1}{\beta}\right)\|f\|^2\|g^\epsilon\|^2}{\eta_2^2T^2}\beta^{\frac{2(t-T_*)}{T}}\|w_1-w_2\|^2_{C([0,T];L^2(\Omega))}. \tag{68}$$

Taking (68) over the scaling $\|\cdot\|_{\beta,0}$ and using that $\|w\|_{\beta,0}(T_*)\le\|w\|_{\beta,\infty}$, one deduces (65). This completes the proof. □

Lemma 10. *Suppose that* (A_1)*-*(A_6) *hold. Then, for* $0\le t\le T_*\le T$*, the self-mapping* $\bar{\mathcal{H}}$ *on* $C([0,T];L^2(\Omega))$ *defined by*

$$\bar{\mathcal{H}}(w)(x,t)=\sum_{p=0}^{\infty}\left[\int_t^{T_*}E_{(t,s)}\left(\mathcal{D}(\ell_0(w)(\tau)),\lambda_p^{(\epsilon)}\right)\mathcal{R}_p(w)(s)\,ds\right]\phi_p(x),\quad \forall w\in C([0,T];L^2(\Omega)) \tag{69}$$

is bounded by

$$\left\|\bar{\mathcal{H}}(w)\right\|_{\beta,0}(t)\le(T_*-t)(K+1)\left(\|w\|_{\beta,\infty}+\|\mathcal{R}(0)\|_{\beta,\infty}\right), \tag{70}$$

and for all $w_1,w_2\in C\left([0,T];L^2(\Omega)\right)$*, we have*

$$\left\|\bar{\mathcal{H}}(w_1)-\bar{\mathcal{H}}(w_2)\right\|_{\beta,0}(t)\le\ \sqrt{2}\,(T_*-t)\,(K+1)\,\|w_1-w_2\|_{\beta,\infty}\left[\frac{\mathrm{L}\ln\left(\frac{1}{\beta}\right)\beta^{\frac{t}{T}}}{\eta_2}\,\|f\|\left(\|w_1\|_{\beta,\infty}+\|\mathcal{R}(0)\|_{\beta,\infty}\right)+1\right]. \tag{71}$$

In (69), we have used the notation $\mathcal{R}_p(w)(s):=\left\langle\mathcal{R}(\cdot,s,w),\phi_p\right\rangle$, whilst in (70), the notation $\mathcal{R}(0):=\mathcal{R}(\cdot,\cdot,0)$.

Proof. In the same way as in the proof of Lemma 9, by Parseval's relation and Hölder's inequality in combination with (A_2) and (58), one has

$$\left\|\bar{\mathcal{H}}(w)(\cdot,t)\right\|^2=\sum_{p=0}^{\infty}\left|\int_t^{T_*}E_{(t,s)}\left(\mathcal{D}(\ell_0(w)(\tau)),\lambda_p^{(\epsilon)}\right)\mathcal{R}_p(w)(s)\,ds\right|^2$$

$$\le(T_*-t)\int_t^{T_*}\exp\left(\frac{2}{\eta_2 T}\ln\left(\frac{1}{\beta}\right)\eta_2(s-t)\right)\sum_{p=0}^{\infty}\left|\mathcal{R}_p(w)(s)\right|^2ds\le(T_*-t)\int_t^{T_*}\beta^{\frac{2(t-s)}{T}}\|\mathcal{R}(\cdot,s,w)\|^2\,ds. \tag{72}$$

Combining this with (61) we obtain (70).

Let us now show (71). Consider the difference $D(x,t):=\bar{\mathcal{H}}(w_1)(x,t)-\bar{\mathcal{H}}(w_2)(x,t)$, as follows:

$$D(x,t)=\sum_{p=0}^{\infty}\left[\int_t^{T_*}\left[E_{(t,s)}\left(\mathcal{D}(\ell_0(w_1)(\tau)),\lambda_p^{(\epsilon)}\right)-E_{(t,s)}\left(\mathcal{D}(\ell_0(w_2)(\tau)),\lambda_p^{(\epsilon)}\right)\right]\mathcal{R}_p(w_1)(s)\,ds\right]\phi_p(x)$$
$$+\sum_{p=0}^{\infty}\left[\int_t^{T_*}E_{(t,s)}\left(\mathcal{D}(\ell_0(w_2)(\tau)),\lambda_p^{(\epsilon)}\right)\left[\mathcal{R}_p(w_1)(s)-\mathcal{R}_p(w_2)(s)\right]ds\right]\phi_p(x).$$

Then, we get

$$\begin{aligned}\|D(\cdot,t)\|^2&\le2\sum_{p=0}^{\infty}\left|\int_t^{T_*}\left[E_{(t,s)}\left(\mathcal{D}(\ell_0(w_1)(\tau)),\lambda_p^{(\epsilon)}\right)-E_{(t,s)}\left(\mathcal{D}(\ell_0(w_2)(\tau)),\lambda_p^{(\epsilon)}\right)\right]\mathcal{R}_p(w_1)(s)\,ds\right|^2\\&\quad+2\sum_{p=0}^{\infty}\left|\int_t^{T_*}E_{(t,s)}\left(\mathcal{D}(\ell_0(w_2)(\tau)),\lambda_p^{(\epsilon)}\right)\left[\mathcal{R}_p(w_1)(s)-\mathcal{R}_p(w_2)(s)\right]ds\right|^2\\&=\|\mathcal{J}_8\|^2+\|\mathcal{J}_9\|^2.\end{aligned} \tag{73}$$

To estimate $\mathcal{J}_8$, we use (5), (19) and (61) together with (A_2), similarly as (67) was derived:

$$\|\mathcal{J}_8\|^2\le2(T_*-t)\sum_{p=0}^{\infty}\int_t^{T_*}\left|E_{(t,s)}\left(\mathcal{D}(\ell_0(w_1)(\tau)),\lambda_p^{(\epsilon)}\right)-E_{(t,s)}\left(\mathcal{D}(\ell_0(w_2)(\tau)),\lambda_p^{(\epsilon)}\right)\right|^2|\mathcal{R}_p(w_1)(s)|^2ds$$
$$\le\frac{2(T_*-t)\mathrm{L}^2}{\eta_2^2T^2}\ln^2\left(\frac{1}{\beta}\right)\|f\|^2\|w_1-w_2\|^2_{C([0,T];L^2(\Omega))}\int_t^{T_*}(s-t)^2\beta^{\frac{2(t-s)}{T}}\|\mathcal{R}(\cdot,s,w_1)\|^2ds$$
$$\le\frac{2(T_*-t)\mathrm{L}^2}{\eta_2^2T^2}\ln^2\left(\frac{1}{\beta}\right)\|f\|^2(K^2+1)\beta^{\frac{2t}{T}}\|w_1-w_2\|^2_{C([0,T];L^2(\Omega))}\int_t^{T_*}(s-t)^2\left(\|w_1\|^2_{\beta,0}(s)+\|\mathcal{R}(0)\|^2_{\beta,0}(s)\right)ds.$$

Taking the scaling (62) we obtain

$$\|\mathcal{J}_8\|^2_{\beta,0}(t)\le\frac{2(T_*-t)^4\mathrm{L}^2}{3\eta_2^2T^2}\ln^2\left(\frac{1}{\beta}\right)\|f\|^2(K^2+1)\beta^{\frac{2t}{T}}\|w_1-w_2\|^2_{\beta,\infty}\left(\|w_1\|^2_{\beta,\infty}+\|\mathcal{R}(0)\|^2_{\beta,\infty}\right). \tag{74}$$

Next, the term $\mathcal{J}_9$ can be estimated in the same way as (72) was derived. Thereby, we have

$$\|\mathcal{J}_9\|^2\le2(T_*-t)K^2\int_t^{T_*}\beta^{\frac{2(t-s)}{T}}\|w_1(\cdot,s)-w_2(\cdot,s)\|^2ds,$$

where we have also used (60). It thus follows that

$$\|\mathcal{J}_9\|_{\beta,0}^2(t) \le 2(T_*-t)^2 K^2 \|w_1-w_2\|_{\beta,\infty}^2. \tag{75}$$

Hence, combining (73)-(75) we obtain

$$\|D\|_{\beta,0}^2(t) \le \frac{2(K^2+1)\mathrm{L}^2(T_*-t)^4\ln^2\left(\frac{1}{\beta}\right)}{3\eta_2^2T^2}\beta^{\frac{2t}{T}}\|f\|^2\left(\|w_1\|_{\beta,\infty}^2+\|\mathcal{R}(0)\|_{\beta,\infty}^2\right)\|w_1-w_2\|_{\beta,\infty}^2+2(T_*-t)^2K^2\|w_1-w_2\|_{\beta,\infty}^2,$$

which leads to

$$\begin{aligned}
&\left\|\bar{\mathcal{H}}(w_1)-\bar{\mathcal{H}}(w_2)\right\|_{\beta,0}(t)\\
&\le \sqrt{2}(T_*-t)(K+1)\|w_1-w_2\|_{\beta,\infty}\left[\frac{\mathrm{L}\ln\left(\frac{1}{\beta}\right)\beta^{\frac{t}{T}}(T_*-t)}{\sqrt{3}\eta_2T}\|f\|\left(\|w_1\|_{\beta,\infty}+\|\mathcal{R}(0)\|_{\beta,\infty}\right)+1\right]\\
&\le \sqrt{2}(T_*-t)(K+1)\|w_1-w_2\|_{\beta,\infty}\left[\frac{\mathrm{L}\ln\left(\frac{1}{\beta}\right)\beta^{\frac{t}{T}}}{\eta_2}\|f\|\left(\|w_1\|_{\beta,\infty}+\|\mathcal{R}(0)\|_{\beta,\infty}\right)+1\right],
\end{aligned}$$

by the elementary inequality

$$\sqrt{a^2(c^2+d^2)+b^2} \le a\sqrt{c^2+d^2}+b \le a(c+d)+b, \quad \forall a,b,c,d \ge 0.$$

This shows that (71) holds, and the proof of Lemma 10 is completed. □

In Lemma 10, notice that (69) can be rewritten as

$$\bar{\mathcal{H}}(w)(x,t) = \sum_{p=0}^{\infty}\left[E_{(t,T_*)}\left(\mathcal{D}(\ell_0(w)(s)),\lambda_p^{(\epsilon)}\right)\int_t^{T_*}E_{(s,T_*)}\left(\mathcal{D}(\ell_0(w)(\tau)),-\lambda_p^{(\epsilon)}\right)\mathcal{R}_p(w)(s)\,ds\right]\phi_p(x). \tag{76}$$

Then, combining the operators $\mathcal{H}$ given by (63) (with measured final data g^ϵ and regularization element $\lambda_p^{(\epsilon)}$ taken into account) and $\bar{\mathcal{H}}$ given by (76) leads us to the following lemma.

Lemma 11. *Suppose that* (A_1)*-*(A_6) *hold. Then the problem* (P_2) *given by (56) has a solution* $u^\epsilon \in C^1\left(0,T;L^2(\Omega)\right)$.

Proof. Our proof is based on the ideas given in [34]. We divide the proof into three parts.

Part (a). *The nonlinear integral equation*

$$\begin{aligned}
&u^\epsilon(x,t) = \mathcal{H}(w)(x,t)-\bar{\mathcal{H}}(w)(x,t)\\
&= \sum_{p=0}^{\infty}E_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p^{(\epsilon)}\right)\left[g_p^\epsilon-\int_t^T E_{(s,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(\tau)),-\lambda_p^{(\epsilon)}\right)\mathcal{R}_p(u^\epsilon)(s)\,ds\right]\phi_p(x),
\end{aligned} \tag{77}$$

admits a unique solution in $C\left([0,T];L^2(\Omega)\right)$*, which is also a solution of the problem* (P_2) *given by (56).*

Proof of part (a). We prove first the second statement of part (a), namely, if u^ϵ satisfies (77), then it is also a solution to the problem (56). Indeed, if we differentiate (77) with respect to t, and use (57), and (77) again, it is straightforward to see that

$$\begin{aligned}
u_t^\epsilon(x,t) &= -\sum_{p=0}^{\infty}\lambda_p^{(\epsilon)}\mathcal{D}(\ell_0(u^\epsilon)(t))E_{(t,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(s)),\lambda_p^{(\epsilon)}\right)\\
&\quad\times\left[g_p^\epsilon-\int_t^T E_{(s,T)}\left(\mathcal{D}(\ell_0(u^\epsilon)(\tau)),-\lambda_p^{(\epsilon)}\right)\mathcal{R}_p(u^\epsilon)(s)\,ds\right]\phi_p(x)+\sum_{p=0}^{\infty}\mathcal{R}_p(u^\epsilon)(t)\phi_p(x)\\
&= -\mathcal{D}(\ell_0(u^\epsilon)(t))\sum_{p=0}^{\infty}\lambda_p^{(\epsilon)}u_p^\epsilon(t)\phi_p(x)+\mathcal{R}(x,t,u^\epsilon) = -\mathcal{D}(\ell_0(u^\epsilon)(t))\Delta^\epsilon u^\epsilon+\mathcal{R}(x,t,u^\epsilon).
\end{aligned}$$

Furthermore, $u^{\epsilon}(x,T) = g^{\epsilon}(x)$ is also valid and $\frac{\partial u}{\partial \nu}(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,T)$ since $\phi_p \in \mathbb{V}$. This completes the proof of the second statement of parta (a).

For $g^{\epsilon} \in L^2(\Omega)$, set

$$\tilde{M} := \sqrt{(\|g^{\epsilon}\|^2 + 2TQ^2)\exp\left(2\ln\left(\frac{1}{\beta}\right) + (2K^2+1)T\right)} \geq 0, \quad \text{with } Q := \sup_{t\in[0,T]} \|\mathcal{R}(\cdot,t,0)\|, \tag{78}$$

and letting $r > \beta^{-1}\tilde{M}$, we define

$$N := \left[\frac{T(K+1)\left(r + \|\mathcal{R}(0)\|_{\beta,\infty}\right)}{r - \beta^{-1}\tilde{M}}\right] + 1 + \left[\frac{2\sqrt{2}\max\{K,1\}}{\eta_2}\left(\mathrm{L}\ln\left(\frac{1}{\beta}\right)\|f\|\left((1+T)r + T\|\mathcal{R}(0)\|_{\beta,\infty}\right) + \eta_2 T\right)\right],$$
$$h = \frac{T}{N}, \tag{79}$$

where $[x]$ denotes the integer part of the real number x.

Let $\tilde{g}^{\epsilon}_j \in L^2(\Omega)$ for $j = \overline{0,(N-1)}$, be such that $\|\tilde{g}^{\epsilon}_j\| \leq \tilde{M}$, and define

$$t_j = T - jh, \quad j = \overline{0,N},$$
$$\mathbb{W}_j := \left\{ v \in C\left([t_{j+1},t_j];L^2(\Omega)\right) : v(x,t_j) = \tilde{g}^{\epsilon}_j(x), \forall x \in \Omega, \sup_{t_{j+1}\leq t\leq t_j} \|v\|_{\beta,0}(t) \leq r \right\}, \quad j = \overline{0,(N-1)},$$
$$\mathcal{Z}_j(w)(x,t) := \mathcal{H}_j(w)(x,t) - \bar{\mathcal{H}}_j(w)(x,t), \quad t \in [t_{j+1},t_j], \quad j = \overline{0,(N-1)},$$

where we have used the notation $\mathcal{Z}_j(w)(x,t)$ for $\mathcal{Z}(w)(x,t)$ restricted to the time subinterval $[t_{j+1},t_j]$ (and similarly $\mathcal{H}_j(w)(x,t)$ and $\bar{\mathcal{H}}_j(w)(x,t)$).

We investigate a local self-mapping defined on a family of closed subsets in $C\left([0,T];L^2(\Omega)\right)$ and proceed piecewisely backwards in time from the final layer $[T-h = t_1, t_0 = T]$ with $\tilde{g}^{\epsilon}_0 = g^{\epsilon}$ backwards. Thus, let us fix a generic $j = \overline{0,(N-1)}$ and show that the operator $\mathcal{Z}_j$ is a contraction on the space $\mathbb{W}_j \subset C\left([t_{j+1},t_j];L^2(\Omega)\right)$.

Combining (64) and (70) (for $T_* = t_j$), for $w \in \mathbb{W}_j$ we have

$$\begin{aligned}\left\|\mathcal{Z}_j(w)\right\|_{\beta,0}(t) &\leq \beta^{-\frac{t_j}{T}}\left\|\tilde{g}^{\epsilon}_j\right\| + \left(t_j - t\right)(K+1)\left(\|w\|_{\beta,\infty} + \|\mathcal{R}(0)\|_{\beta,\infty}\right)\\ &\leq \beta^{-1}\tilde{M} + h(K+1)(r + \|\mathcal{R}(0)\|_{\beta,\infty}), \quad t \in [t_{j+1},t_j].\end{aligned}$$

From (79), one deduces that

$$h \leq \frac{r - \beta^{-1}\tilde{M}}{(K+1)\left(r + \|\mathcal{R}(0)\|_{\beta,\infty}\right)},$$

which implies that $\left\|\mathcal{Z}_j(w)\right\|_{\beta,0}(t) \leq r$. And it is obvious from (63) and (69) that $\mathcal{Z}(w)(x,T) = \mathcal{H}_j(w)(x,T) - \bar{\mathcal{H}}_j(w)(x,T) = \sum_{p=0}^{\infty} \tilde{g}^{\epsilon}_{jp}\phi_p(x) - 0 = \tilde{g}^{\epsilon}_j(x)$. As a consequence, we have $\mathcal{Z}_j\left(\mathbb{W}_j\right) \subset \mathbb{W}_j$.

By triangle's inequality coupled with (65) and (71), after some rearrangements one has that for any $w_1, w_2 \in \mathbb{W}_j$,

$$\begin{aligned}\left\|\mathcal{Z}_j(w_1) - \mathcal{Z}_j(w_2)\right\|_{\beta,0}(t) &\leq \left\|\mathcal{H}_j(w_1) - \mathcal{H}_j(w_2)\right\|_{\beta,0}(t) + \left\|\bar{\mathcal{H}}_j(w_1) - \bar{\mathcal{H}}_j(w_2)\right\|_{\beta,0}(t)\\ &\leq \frac{\mathrm{L}(T-t)\ln\left(\frac{1}{\beta}\right)\|f\|\left\|\tilde{g}^{\epsilon}_j\right\|}{\eta_2 T}\|w_1 - w_2\|_{\beta,\infty}\\ &\quad + \sqrt{2}(K+1)(T-t)\left[\frac{\mathrm{L}\ln\left(\frac{1}{\beta}\right)\beta^{\frac{t}{T}}}{\eta_2}\|f\|\left(\|w_1\|_{\beta,\infty} + \|\mathcal{R}(0)\|_{\beta,\infty}\right) + 1\right]\|w_1 - w_2\|_{\beta,\infty}\\ &\leq \frac{h}{\eta_2 T}\left[\mathrm{L}\ln\left(\frac{1}{\beta}\right)\|f\|\left(r + \sqrt{2}(K+1)T\left(r + \|\mathcal{R}(0)\|_{\beta,\infty}\right)\right) + \sqrt{2}(K+1)\eta_2 T\right]\|w_1 - w_2\|_{\beta,\infty},\end{aligned}$$

where we have used that, since $\|\mathcal{Z}_j(w)\|_{\beta,0}(t) \le r$ and $\mathcal{Z}_j(x,t_j) = \tilde{g}_j^\epsilon(x)$, we have $\beta^{-\frac{t_j}{T}}\|\tilde{g}_j^\epsilon\| \le r$. Afterwards, we have also used that $\beta^{\frac{t}{T}} \le 1$, as $\beta \in (0,1)$.

Observe (79) once again to see that

$$\begin{aligned}\frac{1}{h} &\geq \frac{2\sqrt{2}\max\{K,1\}}{\eta_2 T}\left[\mathrm{L}\ln\left(\frac{1}{\beta}\right)\|f\|\left((1+T)r + T\|\mathcal{R}(0)\|_{\beta,\infty}\right) + \eta_2 T\right] \\ &> \frac{\mathrm{L}\ln\left(\frac{1}{\beta}\right)\|f\|\left(r + \sqrt{2}(K+1)T\left(r + \|\mathcal{R}(0)\|_{\beta,\infty}\right)\right)}{\eta_2 T} + \sqrt{2}(K+1),\end{aligned}$$

where we have used the simple inequality $2\max\{K,1\} \geq K+1$.

So, from the above it is evident to claim that there exists $c \in [0,1)$ such that $\left\|\mathcal{Z}_j(w_1) - \mathcal{Z}_j(w_2)\right\|_{\beta,0}(t) \le c\|w_1 - w_2\|_{\beta,\infty}$ which implies that $\mathcal{Z}_j$ is a contraction mapping on $\mathbb{W}_j \subset C\left([t_{j+1},t_j];L^2(\Omega)\right)$. Thus, the existence and uniqueness arguments are obtained by the Banach fixed-point theorem. It can be observed that if the regularization parameter β goes to zero, then h becomes smaller and smaller, whereas the space $\mathbb{W}_j$ expands increasingly in size. Therefore, our construction is reasonable.

Once the solution has been obtained uniquely over the layer $[t_{j+1},t_j]$, provided that is possible to obtain an upper bound for the norm of u^ϵ, we can proceed the same way to the next layer $[t_{j+2},t_{j+1}]$ and so on, backwards in time up to the initial time $t=0$. In the next part, we establish such an upper bound.

Part (b). *Upper bound of the norm of solution u^ϵ of problem (56).*

Proof of part (b). Let $0 \le \tau \le T$ and assume that $u^\epsilon \in C^1\left(\tau,T;L^2(\Omega)\right)$ satisfies

$$\begin{cases} u_t^\epsilon + \mathcal{D}(\ell_0(u^\epsilon)(t))\Delta^\epsilon u^\epsilon = \mathcal{R}(x,t,u^\epsilon), & (x,t)\in\Omega\times(\tau,T), \\ \frac{\partial u^\epsilon}{\partial \nu}(x,t) = 0, & (x,t)\in\partial\Omega\times(\tau,T), \\ u^\epsilon(x,T) = g^\epsilon(x), & x\in\Omega, \end{cases} \tag{80}$$

One has

$$\langle u_t^\epsilon, u^\epsilon\rangle + \langle \mathcal{D}(\ell_0(u^\epsilon)(t))\Delta^\epsilon u^\epsilon, u^\epsilon\rangle = \langle \mathcal{R}(x,t,u^\epsilon), u^\epsilon\rangle. \tag{81}$$

Using the Lipschitz property (A_6), we have

$$\begin{aligned}|\langle \mathcal{R}(x,t,u^\epsilon), u^\epsilon\rangle| &\le \|\mathcal{R}(\cdot,t,u^\epsilon)\|\,\|u^\epsilon(\cdot,t)\| \le \frac{1}{2}\left[\|\mathcal{R}(\cdot,t,u^\epsilon) - \mathcal{R}(\cdot,t,0)\| + \|\mathcal{R}(\cdot,t,0)\|\right]^2 + \frac{1}{2}\|u^\epsilon(\cdot,t)\|^2 \\ &\le \frac{1}{2}\left[K\|u^\epsilon(\cdot,t)\| + \|\mathcal{R}(\cdot,t,0)\|\right]^2 + \frac{1}{2}\|u^\epsilon(\cdot,t)\|^2 \le \left(K^2 + \frac{1}{2}\right)\|u^\epsilon(\cdot,t)\|^2 + \|\mathcal{R}(\cdot,t,0)\|^2.\end{aligned} \tag{82}$$

Using (A_2) and (80)-(82) we obtain

$$-\left[\left(K^2+\frac{1}{2}\right)\|u^\epsilon(\cdot,t)\|^2 + \|\mathcal{R}(\cdot,t,0)\|^2\right] \le \frac{1}{2}\frac{d}{dt}\|u^\epsilon(\cdot,t)\|^2 + \eta_2|\langle\Delta^\epsilon u^\epsilon, u^\epsilon\rangle|.$$

Integrating this inequality from t to T we obtain

$$-\int_t^T\left(K^2+\frac{1}{2}\right)\|u^\epsilon(\cdot,s)\|^2 ds - \int_t^T \|\mathcal{R}(\cdot,s,0)\|^2 ds \le \frac{1}{2}\|u^\epsilon(\cdot,T)\|^2 - \frac{1}{2}\|u^\epsilon(\cdot,t)\|^2 + \eta_2\int_t^T |\langle\Delta^\epsilon u^\epsilon, u^\epsilon\rangle| ds.$$

So, we obtain

$$\begin{aligned}\|u^\epsilon(\cdot,t)\|^2 &\le \|g^\epsilon\|^2 + 2\int_t^T \|\mathcal{R}(\cdot,s,0)\|^2 ds + 2\int_t^T\left[\eta_2|\langle\Delta^\epsilon u^\epsilon,u^\epsilon\rangle| + \left(K^2+\frac{1}{2}\right)\|u^\epsilon(\cdot,s)\|^2\right]ds \\ &\le \|g^\epsilon\|^2 + 2TQ^2 + 2\left[\frac{\ln\left(\frac{1}{\beta}\right)}{T} + \left(K^2+\frac{1}{2}\right)\right]\int_t^T \|u^\epsilon(\cdot,s)\|^2 ds,\end{aligned}$$

where, from (57) and (58), we have used that

$$|\langle \Delta^\epsilon u^\epsilon, u^\epsilon \rangle| \le \frac{\ln\left(\frac{1}{\beta}\right)}{\eta_2 T} \|u^\epsilon(\cdot,t)\|^2. \tag{83}$$

Finally, Gronwall's inequality and (78) give

$$\|u^\epsilon(\cdot,t)\|^2 \le \left[\|g^\epsilon\|^2 + 2TQ^2\right] \exp\left(2\ln\left(\frac{1}{\beta}\right) + (2K^2+1)T\right) = \tilde{M}^2,$$

or,

$$\|u^\epsilon(\cdot,t)\| \le \tilde{M}. \tag{84}$$

Part (c). *The existence of a solution* $u^\epsilon \in C^1\left(0,T;L^2(\Omega)\right)$ *of problem (56).*

Proof of part (c). This part connects parts (a) and (b). We shall prove by induction that problem (56) has a solution on $[t_j, T]$ for $j = 0,1,...,N$. In fact, for $j = 0$, we set $g_0^\epsilon = g^\epsilon$. From part (a), we know that there is a unique $u_0^\epsilon \in \mathbb{W}_0 \subset C\left([t_1,T];L^2(\Omega)\right)$ such that $\mathcal{Z}_0(u_0^\epsilon) = u_0^\epsilon$ and we can verify that u_0^ϵ satisfies the problem (56) on $[t_1, t_0 = T]$. From the Lipschitz property (A_1) of $\mathcal{R}$, triangle inequality and (83), we obtain

$$\left\|\frac{\partial u_0^\epsilon}{\partial t}\right\| \le \|\mathcal{D}\left(\ell_0\left(u_0^\epsilon\right)(t)\right)\Delta^\epsilon u_0^\epsilon\| + \|\mathcal{R}(\cdot,t,u_0^\epsilon)\| \le \eta_2 \frac{\ln\left(\frac{1}{\beta}\right)}{\eta_2 T}\|u_0^\epsilon\| + K\|u_0^\epsilon\| + \|\mathcal{R}(\cdot,t,0)\| \le \left(\frac{1}{T}\ln\left(\frac{1}{\beta}\right) + K\right) r + Q.$$

This implies that $u_0^\epsilon \in C^1\left(t_1,T;L^2(\Omega)\right)$. Now, we assume that the problem (56) has a solution $u^\epsilon \in C^1\left([t_k,T);L^2(\Omega)\right)$, for $0 \le k \le N-1$ with $u^\epsilon(\cdot,T) = g^\epsilon(\cdot)$. We shall prove that we can extend this solution to the interval $[t_{k+1},T]$. In fact, from part (b), we have that (84) holds for $t_k \le t \le T$. Set $g_k^\epsilon = u^\epsilon(x,t_k)$. From part (a), there is a unique $u_k^\epsilon \in \mathbb{W}_k \subset C\left([t_{k+1},t_k];L^2(\Omega)\right)$ such that $\mathcal{Z}_k(u_k^\epsilon) = u_k^\epsilon$ and we can verify that u_k^ϵ satisfies the problem (56) on $[t_{k+1},t_k]$ with $u_k^\epsilon(\cdot,t_k) = u^\epsilon(\cdot,t_k)$. And we also obtain that $u_k^\epsilon \in C^1\left(t_{k+1},t_k;L^2(\Omega)\right)$. So, we can extend u^ϵ to $[t_{k+1},T]$ by putting $u^\epsilon(\cdot,t) = u_k^\epsilon(\cdot,t)$ for $t \in [t_{k+1},t_k]$. By induction, we complete the proof of part (c).

Finally, parts (a)-(c) when grouped together conclude the proof of Lemma 11. □

Lemma 12. *Suppose that* (A_1)*-*(A_6) *hold. Then, the solution of the problem* (P_2) *given by (56) is unique in* $C^1\left(0,T;L^2(\Omega)\right)$.

Proof. Lemma 11 has proved the existence of a solution $u^\epsilon \in C^1\left(0,T;L^2(\Omega)\right)$ of problem (P_2). To prove uniqueness, let u^ϵ and v^ϵ be two solutions to the problem (P_2) in $C^1\left(0,T;L^2(\Omega)\right)$, where u^ϵ satisfies (77). Then, if we define $d^\epsilon(x,t) = e^{q(t-T)}\left(u^\epsilon(x,t) - v^\epsilon(x,t)\right)$ for some $q > 0$, from (56) we obtain that

$$\begin{aligned} d_t^\epsilon &= qe^{q(t-T)}\left(u^\epsilon - v^\epsilon\right) + e^{q(t-T)}\left(u_t^\epsilon - v_t^\epsilon\right) \\ &= qd^\epsilon + e^{q(t-T)}\left[\mathcal{R}(x,t,u^\epsilon) - \mathcal{R}(x,t,v^\epsilon) - \mathcal{D}(\ell_0(u^\epsilon)(t))\Delta^\epsilon u^\epsilon + \mathcal{D}(\ell_0(v^\epsilon)(t))\Delta^\epsilon v^\epsilon\right]. \end{aligned}$$

Thus, one has

$$d_t^\epsilon + \mathcal{D}(\ell_0(v^\epsilon)(t))\Delta^\epsilon d^\epsilon - qd^\epsilon = e^{q(t-T)}\left(\mathcal{R}(x,t,u^\epsilon) - \mathcal{R}(x,t,v^\epsilon)\right) - e^{q(t-T)}\left[\mathcal{D}(\ell_0(u^\epsilon)(t)) - \mathcal{D}(\ell_0(v^\epsilon)(t))\right]\Delta^\epsilon u^\epsilon. \tag{85}$$

Taking the action of (85) with d^ϵ gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|d^\epsilon(\cdot,t)\right\|^2 - q\left\|d^\epsilon(\cdot,t)\right\|^2 &= e^{q(t-T)}\langle \mathcal{R}(\cdot,t,u^\epsilon) - \mathcal{R}(\cdot,t,v^\epsilon), d^\epsilon(\cdot,t)\rangle \\ -e^{q(t-T)}\langle (\mathcal{D}(\ell_0(u^\epsilon)(t)) - \mathcal{D}(\ell_0(v^\epsilon)(t)))\Delta^\epsilon u^\epsilon(\cdot,t), d^\epsilon(\cdot,t)\rangle &- \mathcal{D}(\ell_0(v^\epsilon)(t))\langle \Delta^\epsilon d^\epsilon(\cdot,t), d^\epsilon(\cdot,t)\rangle. \end{aligned}$$

Integrating in time yields

$$\left\|d^{\epsilon}(\cdot,T)\right\|^{2}-\left\|d^{\epsilon}(\cdot,t)\right\|^{2}+2\int_{t}^{T}\mathcal{D}\left(\ell_{0}\left(v^{\epsilon}\right)(s)\right)\left\langle\Delta^{\epsilon}d^{\epsilon}(\cdot,s),d^{\epsilon}(\cdot,s)\right\rangle ds=2q\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}ds+\mathcal{J}_{10}+\mathcal{J}_{11},\quad(86)$$

where $\mathcal{J}_{10}$ and $\mathcal{J}_{11}$ are defined by

$$\begin{aligned}\mathcal{J}_{10}&=2\int_{t}^{T}e^{q(s-T)}\left\langle\mathcal{R}\left(\cdot,s,u^{\epsilon}\right)-\mathcal{R}\left(\cdot,s,v^{\epsilon}\right),d^{\epsilon}(\cdot,s)\right\rangle ds,\\ \mathcal{J}_{11}&=-2\int_{t}^{T}e^{q(s-T)}\left\langle\left(\mathcal{D}\left(\ell_{0}\left(u^{\epsilon}\right)(s)\right)-\mathcal{D}\left(\ell_{0}\left(v^{\epsilon}\right)(s)\right)\right)\Delta^{\epsilon}u^{\epsilon}(\cdot,s),d^{\epsilon}(\cdot,s)\right\rangle ds.\end{aligned}$$

First, let us estimate $\mathcal{J}_{10}$, as follows:

$$\left|\mathcal{J}_{10}\right|\le2\int_{t}^{T}e^{q(s-T)}\left\|\mathcal{R}\left(\cdot,s,u^{\epsilon}\right)-\mathcal{R}\left(\cdot,s,v^{\epsilon}\right)\right\|\left\|d^{\epsilon}(\cdot,s)\right\|ds\le2K\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}ds,\quad(87)$$

where Hölder's inequality, the Lipschitz property (60) of reaction term and the definition of d^{ϵ} have been applied.

Second, using (5) and (59), $\mathcal{J}_{11}$ can be estimated by

$$\begin{aligned}\left|\mathcal{J}_{11}\right|&\le2\int_{t}^{T}e^{q(s-T)}\left|\mathcal{D}\left(\ell_{0}\left(u^{\epsilon}\right)(s)\right)-\mathcal{D}\left(\ell_{0}\left(v^{\epsilon}\right)(s)\right)\right|\left\|\Delta^{\epsilon}u^{\epsilon}(\cdot,s)\right\|\left\|d^{\epsilon}(\cdot,s)\right\|ds\\ &\le2\mathrm{L}\left\|f\right\|\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}\left\|\Delta^{\epsilon}u^{\epsilon}(\cdot,s)\right\|ds\le\frac{2\mathrm{L}\ln\left(\frac{1}{\beta}\right)\left\|f\right\|}{\eta_{2}T}\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}\left\|u^{\epsilon}(\cdot,s)\right\|ds.\end{aligned}\quad(88)$$

Now let us estimate $\|u^{\epsilon}(\cdot,s)\|$. It follows from (61), (77), Parseval's relation, and technicalities such as (72) and $(a\pm b)^{2}\le2(a^{2}+b^{2})$, that

$$\begin{aligned}\left\|u^{\epsilon}(\cdot,t)\right\|^{2}&\le2\sum_{p=0}^{\infty}\exp\left(\frac{2(T-t)}{T}\ln\left(\frac{1}{\beta}\right)\right)\left|g_{p}^{\epsilon}\right|^{2}+2(T-t)\int_{t}^{T}\exp\left(\frac{2(s-t)}{T}\ln\left(\frac{1}{\beta}\right)\right)\sum_{p=0}^{\infty}\left|\mathcal{R}_{p}\left(u^{\epsilon}\right)(s)\right|^{2}ds\\ &\le2\beta^{\frac{2(t-T)}{T}}\left\|g^{\epsilon}\right\|^{2}+2T\left(K^{2}+1\right)\int_{t}^{T}\beta^{\frac{2(t-s)}{T}}\left(\left\|u^{\epsilon}(\cdot,s)\right\|^{2}+\|\mathcal{R}(\cdot,s,0)\|^{2}\right)ds\\ &\le2\beta^{\frac{2(t-T)}{T}}\left(\left\|g^{\epsilon}\right\|^{2}+T^{2}\left(K^{2}+1\right)\|\mathcal{R}(0)\|_{C([0,T];L^{2}(\Omega))}^{2}\right)+2T\left(K^{2}+1\right)\int_{t}^{T}\beta^{\frac{2(t-s)}{T}}\left\|u^{\epsilon}(\cdot,s)\right\|^{2}ds,\end{aligned}$$

which implies that

$$\beta^{-\frac{2t}{T}}\left\|u^{\epsilon}(\cdot,t)\right\|^{2}\le2\beta^{-2}\left(\left\|g^{\epsilon}\right\|^{2}+T^{2}\left(K^{2}+1\right)\|\mathcal{R}(0)\|_{C([0,T];L^{2}(\Omega))}^{2}\right)+2T\left(K^{2}+1\right)\int_{t}^{T}\beta^{-\frac{2s}{T}}\left\|u^{\epsilon}(\cdot,s)\right\|^{2}ds.$$

Applying Gronwall's inequality to this yields

$$\beta^{-\frac{2t}{T}}\left\|u^{\epsilon}(\cdot,t)\right\|^{2}\le2\beta^{-2}\left(\left\|g^{\epsilon}\right\|^{2}+T^{2}\left(K^{2}+1\right)\|\mathcal{R}(0)\|_{C([0,T];L^{2}(\Omega))}^{2}\right)\exp\left(2T(T-t)\left(K^{2}+1\right)\right).$$

Therefore, for all $t\in[0,T]$ we have

$$\left\|u^{\epsilon}(\cdot,t)\right\|\le\sqrt{2}\beta^{-1}\left(\left\|g^{\epsilon}\right\|+T(K+1)\|\mathcal{R}(0)\|_{C([0,T];L^{2}(\Omega))}\right)\exp\left(T^{2}\left(K^{2}+1\right)\right).\quad(89)$$

Combining (88) and (89), we obtain

$$\left|\mathcal{J}_{11}\right|\le\frac{2\sqrt{2}\mathrm{L}\ln\left(\frac{1}{\beta}\right)\|f\|}{\eta_{2}T}\beta^{-1}\left(\left\|g^{\epsilon}\right\|+T(K+1)\|\mathcal{R}(0)\|_{C([0,T];L^{2}(\Omega))}\right)\exp\left(T^{2}\left(K^{2}+1\right)\right)\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}ds.\quad(90)$$

From (86), (87) and (90), we can now obtain that

$$\left\|d^{\epsilon}(\cdot,T)\right\|^{2}-\left\|d^{\epsilon}(\cdot,t)\right\|^{2}+2\int_{t}^{T}\mathcal{D}\left(\ell_{0}\left(v^{\epsilon}\right)(s)\right)\left\langle\Delta^{\epsilon}d^{\epsilon}(\cdot,s),d^{\epsilon}(\cdot,s)\right\rangle ds\geq 2\left(q-\bar{q}\right)\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}ds, \tag{91}$$

where $\bar{q} := K + \frac{2\sqrt{2}\mathrm{L}\ln\left(\frac{1}{\beta}\right)\|f\|}{\eta_2 T}\beta^{-1}\left(\|g^{\epsilon}\| + T(K+1)\|\mathcal{R}(0)\|_{C([0,T];L^2(\Omega))}\right)\exp\left(T^2\left(K^2+1\right)\right)$ is a positive constant. We observe that

$$\left\langle\Delta^{\epsilon}d^{\epsilon}(\cdot,t),d^{\epsilon}(\cdot,t)\right\rangle=\left\langle\sum_{p=0}^{\infty}\lambda_{p}^{(\epsilon)}d_{p}^{\epsilon}(t)\phi_{p},\sum_{p=0}^{\infty}d_{p}^{\epsilon}(t)\phi_{p}\right\rangle=\sum_{p=0}^{\infty}\lambda_{p}^{(\epsilon)}\left|d_{p}^{\epsilon}(t)\right|^{2}\leq\frac{1}{\eta_{2}T}\ln\left(\frac{1}{\beta}\right)\left\|d^{\epsilon}(\cdot,t)\right\|^{2}, \tag{92}$$

where we have used $0 \leq \lambda_p^{(\epsilon)} \leq \frac{1}{\eta_2 T}\ln\left(\frac{1}{\beta}\right)$ for $\beta \in \left(0, 1-e^{-\eta_2 T\lambda_1}\right)$. Combining (92) with ($\mathrm{A}_2$) into (91) we obtain

$$\left\|d^{\epsilon}(\cdot,T)\right\|^{2}-\left\|d^{\epsilon}(\cdot,t)\right\|^{2}\geq 2\left(q-\bar{q}-\frac{1}{T}\ln\left(\frac{1}{\beta}\right)\right)\int_{t}^{T}\left\|d^{\epsilon}(\cdot,s)\right\|^{2}ds.$$

So our objective is well-accomplished. Indeed, by choosing $q \geq \bar{q} + \frac{1}{T}\ln\left(\frac{1}{\beta}\right)$, it becomes clear by the fact that for all $t \in [0,T]$ we have $\|d^{\epsilon}(\cdot,t)\| \leq \|d^{\epsilon}(\cdot,T)\| \equiv 0$ which leads to the uniqueness of the solution in $C^1\left(0,T;L^2(\Omega)\right)$. This completes the proof of the lemma. □

Let us summarise what we have obtained in Lemmas 9-12 prior to investigating the convergence rate. Lemmas 11 and 12 showed that the approximate problem (P_2) given by (56) admits a unique solution $u^{\epsilon} \in C^1\left(0,T;L^2(\Omega)\right)$. Like the results obtained in the case without reaction in subsection 3.1, the regularized solution, based on the nonlinear spectral theory, which can be represented by the integral equation (77), is unique as well. Therefore, we can set up a similar computational procedure as that of subsection 3.2 once again. Another important point which should be mentioned here is that from Lemmas 9 and 10 one easily observes the stability of the regularized solution, and hence the well-posedness of the regularized problem (P_2) given by (56).

Theorem 13. *Assume* (A_1)-(A_6) *hold, and suppose that the solution* u *of the problem* (P) *given by equations* (1)-(4) *belongs to* $C\left([0,T];\mathbb{G}_{\sigma,\gamma}\right)$ *for* $\sigma \geq \eta_2 T, \gamma = 2$, *and* $u_t \in C\left([0,T];L^2(\Omega)\right)$. *Then the* L^2*-error estimate between* u^{ϵ} *in (77), solution to problem* (P_2) *given by* (56), *and the exact solution* u *is given by*

$$\left\|u^{\epsilon}(\cdot,t)-u(\cdot,t)\right\|\leq \mathrm{P}\,\beta^{\frac{t}{T}}\exp\left(\mu(T-t)\right),\quad t\in[0,T], \tag{93}$$

where

$$\mathrm{P} := \sqrt{\beta^{-2}\epsilon^{2}+\frac{(1+\lambda_{1}^{2})\|u\|_{C([0,T];\mathbb{G}_{\sigma,\gamma})}^{2}}{\lambda_{1}^{2}T^{2}}}\quad\text{and}\quad \mu := K+\frac{1}{2}+\frac{\mathrm{L}}{\lambda_{1}\eta_{2}T}\|f\|\|u\|_{C([0,T];\mathbb{G}_{\sigma,\gamma})}>0.$$

By choosing $\beta := \beta(\epsilon) \in (0,1)$ *such that*

$$\begin{cases}\lim_{\epsilon\to0^{+}}\beta(\epsilon)=0,\\ \lim_{\epsilon\to0^{+}}\frac{\epsilon}{\beta(\epsilon)}=\textit{finite},\end{cases} \tag{94}$$

this implies that $\|u^{\epsilon}(\cdot,t)-u(\cdot,t)\| \to 0$, *as* $\epsilon \to 0$, *for every* $t \in (0,T]$. *Moreover, for* $\epsilon > 0$ *small enough there exists* $t^{\epsilon} \in (0,T)$ *such that* $\lim_{\epsilon\to0^{+}} t^{\epsilon} = 0$ *and*

$$\|u^{\epsilon}(\cdot,t^{\epsilon})-u(\cdot,0)\|\leq\left(\mathrm{P}\exp(\mu T)+B\right)\sqrt{\frac{T}{\ln\left(\frac{1}{\beta}\right)}}, \tag{95}$$

where $B = \sup_{t\in[0,T]}\|u_t(\cdot,t)\|$.

Proof. Let us define $\bar{d}^{\epsilon}(x,t) := e^{q(t-T)}(u^{\epsilon}(x,t) - u(x,t))$ for some $q > 0$. Observing the problems (P) and (P_2) given by equations (1)-(4) and (56), respectively, as in the proof of Lemma 12, one deduces that

$$\begin{aligned}\bar{d}_t^{\epsilon} &= q e^{q(t-T)}(u^{\epsilon} - u) + e^{q(t-T)}(u_t^{\epsilon} - u_t) \\ &= q\bar{d}^{\epsilon} + e^{q(t-T)}\left[\mathcal{R}(x,t,u^{\epsilon}) - \mathcal{R}(x,t,u) - \mathcal{D}(\ell_0(u^{\epsilon})(t))\Delta^{\epsilon}u^{\epsilon} + \mathcal{D}(\ell_0(u)(t))\Delta u\right],\end{aligned}$$

or,

$$\begin{aligned}\bar{d}_t^{\epsilon} + \mathcal{D}(\ell_0(u^{\epsilon})(t))\Delta^{\epsilon}\bar{d}^{\epsilon} - q\bar{d}^{\epsilon} = e^{q(t-T)}\left[\mathcal{R}(x,t,u^{\epsilon}) - \mathcal{R}(x,t,u) - (\mathcal{D}(\ell_0(u^{\epsilon})(t)) - \mathcal{D}(\ell_0(u)(t)))\Delta u\right. \\ \left. - \mathcal{D}(\ell_0(u^{\epsilon})(t))(\Delta^{\epsilon}u - \Delta u)\right]. \qquad (96)\end{aligned}$$

Taking the action of (96) with $\bar{d}^{\epsilon}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\bar{d}^{\epsilon}(\cdot,t)\right\|^2 - q\left\|\bar{d}^{\epsilon}(\cdot,t)\right\|^2 = e^{q(t-T)}\left\langle \mathcal{R}(\cdot,t,u^{\epsilon}) - \mathcal{R}(\cdot,t,u), \bar{d}^{\epsilon}(\cdot,t)\right\rangle \\ - e^{q(t-T)}(\mathcal{D}(\ell_0(u^{\epsilon})(t)) - \mathcal{D}(\ell_0(u)(t)))\left\langle \Delta u(\cdot,t), \bar{d}^{\epsilon}(\cdot,t)\right\rangle - e^{q(t-T)}\mathcal{D}(\ell_0(u^{\epsilon})(t))\left\langle \Delta^{\epsilon}u(\cdot,t) - \Delta u(\cdot,t), \bar{d}^{\epsilon}(\cdot,t)\right\rangle \\ - \mathcal{D}(\ell_0(u^{\epsilon})(t))\left\langle \Delta^{\epsilon}\bar{d}^{\epsilon}(\cdot,t), \bar{d}^{\epsilon}(\cdot,t)\right\rangle.\end{aligned}$$

Integrating in time yields

$$\left\|\bar{d}^{\epsilon}(\cdot,T)\right\|^2 - \left\|\bar{d}^{\epsilon}(\cdot,t)\right\|^2 + 2\int_t^T \mathcal{D}(\ell_0(u^{\epsilon})(s))\left\langle \Delta^{\epsilon}\bar{d}^{\epsilon}(\cdot,s), \bar{d}^{\epsilon}(\cdot,s)\right\rangle ds = 2q\int_t^T \left\|\bar{d}^{\epsilon}(\cdot,s)\right\|^2 ds + \mathcal{J}_{12} + \mathcal{J}_{13} + \mathcal{J}_{14}, \qquad (97)$$

where

$$\begin{aligned}\mathcal{J}_{12} &:= 2\int_t^T e^{q(s-T)}\left\langle \mathcal{R}(\cdot,s,u^{\epsilon}) - \mathcal{R}(\cdot,s,u), \bar{d}^{\epsilon}(\cdot,s)\right\rangle ds, \\ \mathcal{J}_{13} &:= -2\int_t^T e^{q(s-T)}(\mathcal{D}(\ell_0(u^{\epsilon})(s)) - \mathcal{D}(\ell_0(u)(s)))\left\langle \Delta u(\cdot,s), \bar{d}^{\epsilon}(\cdot,s)\right\rangle ds, \\ \mathcal{J}_{14} &:= -2\int_t^T e^{q(s-T)}\mathcal{D}(\ell_0(u^{\epsilon})(s))\left\langle \Delta^{\epsilon}u(\cdot,s) - \Delta u(\cdot,s), \bar{d}^{\epsilon}(\cdot,s)\right\rangle ds.\end{aligned}$$

Next, we derive upper bounds for the absolute values of $\mathcal{J}_i$, $i = 12, 13, 14$. First, notice that we obtain here the same result as in (87) for $\mathcal{J}_{12}$, namely,

$$|\mathcal{J}_{12}| \le 2K\int_t^T \|\bar{d}^{\epsilon}(\cdot,s)\|^2 ds. \qquad (98)$$

Also, as in (88), we obtain

$$|\mathcal{J}_{13}| \le 2\mathrm{L}\,\|f\|\int_t^T \left\|\bar{d}^{\epsilon}(\cdot,s)\right\|^2 \|\Delta u(\cdot,s)\|\,ds \le \frac{2\mathrm{L}}{\lambda_1\eta_2 T}\|f\|\,\|u\|_{C([0,T];\mathbb{G}_{\sigma,\gamma})}\int_t^T \left\|\bar{d}^{\epsilon}(\cdot,s)\right\|^2 ds, \qquad (99)$$

where we have recalled the spectral representation (6) and the elementary inequality $a < e^a, \forall a > 0$, (in the form $e^{\eta_2 T\lambda_p} > \eta_2 T\lambda_p \ge \eta_2 T\lambda_1$ for $p \ge 1$) to yield

$$\Delta u(x,t) = -\sum_{p=0}^{\infty}\lambda_p u_p(t)\phi_p(x), \quad \|\Delta u(\cdot,t)\| \le \sqrt{\sum_{p=0}^{\infty}\frac{\lambda_p^2 e^{2\eta_2 T\lambda_p}\left|u_p(t)\right|^2}{\lambda_1^2\eta_2^2 T^2}} \le \frac{1}{\lambda_1\eta_2 T}\|u(\cdot,t)\|_{\mathbb{G}_{\sigma,\gamma}}.$$

For $\mathcal{J}_{14}$, applying Hölder's inequality and using (A_2) coupled with the basic inequality $\ln(1+a) \le a, \forall a > 0$ and the argument which reads

$$(\Delta^{\epsilon} - \Delta)u(x,t) = \sum_{p=0}^{\infty}\left(\lambda_p^{(\epsilon)} + \lambda_p\right)u_p(t)\phi_p(x) = \frac{1}{\eta_2 T}\sum_{p=0}^{\infty}\ln\left(\frac{e^{\eta_2 T\lambda_p}}{e^{-\eta_2 T\lambda_p} + \beta}\right)u_p(t)\phi_p(x),$$

we obtain by Parseval's relation that

$$
\begin{aligned}
|\mathcal{J}_{14}| &\le \eta_2^2 \int_t^T \left\|(\Delta^\epsilon - \Delta)\, u(\cdot,s)\right\|^2 ds + \int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2 ds \\
&\le \eta_2^2 \int_t^T \left[\frac{1}{\eta_2^2 T^2}\sum_{p=0}^{\infty} \ln^2\left(\frac{e^{\eta_2 T\lambda_p}}{e^{-\eta_2 T\lambda_p}+\beta}\right)\left|u_p(s)\right|^2\right] ds + \int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2 ds \\
&\int_t^T \left[\frac{\beta^2}{T^2}|u_0(s)|^2 + \frac{1}{\lambda_1^2 T^2}\sum_{p=1}^{\infty} \lambda_p^2 e^{2\eta_2 T\lambda_p}\beta^2 \left|u_p(s)\right|^2\right] ds + \int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2 ds \\
&\le \frac{(1+\lambda_1^2)\beta^2}{\lambda_1^2 T^2}(T-t)\,\|u\|^2_{C([0,T];\mathbb{G}_{\sigma,\gamma})} + \int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2 ds,
\end{aligned} \tag{100}
$$

where we have also used that

$$
\ln^2\left(\frac{e^{\eta_2 T\lambda_p}}{e^{-\eta_2 T\lambda_p}+\beta}\right) = \ln^2\left(\frac{\beta e^{\eta_2 T\lambda_p}+1}{e^{2\eta_2 T\lambda_p}}\right) \le \ln^2\left(\beta e^{\eta_2 T\lambda_p}+1\right) \le \beta^2 e^{2\eta_2 T\lambda_p}
$$

and note that $|u_0(s)|^2 \le \|u\|^2_{C([0,T];\mathbb{G}_{\sigma,\gamma})}$ for $s \in [0,T]$. Combining (97)-(100) gives

$$
\begin{aligned}
&\left\|\bar{d}^\epsilon(\cdot,T)\right\|^2 - \left\|\bar{d}^\epsilon(\cdot,t)\right\|^2 + 2\int_t^T \mathcal{D}\left(\ell_0\left(u^\epsilon\right)(s)\right)\left\langle \Delta^\epsilon \bar{d}^\epsilon(\cdot,s), \bar{d}^\epsilon(\cdot,s)\right\rangle ds \\
&\ge \left[2q - 2K - \frac{2L}{\lambda_1\eta_2 T}\|f\|\,\|u\|_{C([0,T];\mathbb{G}_{\sigma,\gamma})} - 1\right]\int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2 ds - \frac{(1+\lambda_1^2)\beta^2}{\lambda_1^2 T^2}(T-t)\,\|u\|^2_{C([0,T];\mathbb{G}_{\sigma,\gamma})}.
\end{aligned} \tag{101}
$$

By using (A_2) and (92) we also have

$$
\int_t^T \mathcal{D}\left(\ell_0\left(u^\epsilon\right)(s)\right)\left\langle \Delta^\epsilon \bar{d}^\epsilon(\cdot,s), \bar{d}^\epsilon(\cdot,s)\right\rangle ds \le \frac{1}{T}\ln\left(\frac{1}{\beta}\right)\int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2. \tag{102}
$$

By choosing $q = \frac{1}{T}\ln\left(\frac{1}{\beta}\right) > 0$ and noticing that $\left\|\bar{d}^\epsilon(\cdot,T)\right\| = \|g^\epsilon - g\| \le \epsilon$ by (A_5), we introduce (102) into (101) to get

$$
\left\|\bar{d}^\epsilon(\cdot,t)\right\|^2 \le \epsilon^2 + \frac{(1+\lambda_1^2)\beta^2}{\lambda_1^2 T^2}(T-t)\,\|u\|^2_{C([0,T];\mathbb{G}_{\sigma,\gamma})} + \left(2K + \frac{2\mathrm{L}}{\lambda_1\eta_2 T}\|f\|\,\|u\|_{C([0,T];\mathbb{G}_{\sigma,\gamma})} + 1\right)\int_t^T \left\|\bar{d}^\epsilon(\cdot,s)\right\|^2 ds.
$$

Using Gronwall's inequality we obtain

$$
\left\|\bar{d}^\epsilon(\cdot,t)\right\| \le \sqrt{\epsilon^2 + \frac{(1+\lambda_1^2)\beta^2}{\lambda_1^2 T^2}\|u\|^2_{C([0,T];\mathbb{G}_{\sigma,\gamma})}}\exp\left(\mu(T-t)\right),
$$

which can be rewritten, by the expression of q, as (93).

In order to show (95), remark that due to the continuity of u_t, for $\epsilon > 0$ small enough we have

$$
\|u^\epsilon(\cdot,t) - u(\cdot,0)\| \le \|u^\epsilon(\cdot,t) - u(\cdot,t)\| + \|u(\cdot,t) - u(\cdot,0)\| \le \mathrm{P}\beta^{\frac{t}{T}}\exp(\mu T) + t\|u_t\|_{C([0,T];L^2(\Omega))}, \quad t \in [0,T], \tag{103}
$$

where use has been made of (93).

Now, for every $\epsilon > 0$ small, let us take t^ϵ be the unique solution in $(0,T)$ of the equation $t = \beta^{\frac{t}{T}}$, where obviously $\beta = \beta(\epsilon)$. This implies that $\frac{\ln(t^\epsilon)}{t^\epsilon} = \frac{\ln(\beta)}{T}$ and using the inequality $\ln(t) > -\frac{1}{t}$ for all $t > 0$, we obtain that

$$
\beta^{\frac{t^\epsilon}{T}} = t^\epsilon < \sqrt{\frac{T}{\ln\left(\frac{1}{\beta}\right)}}.
$$

Clearly, since $\lim_{\epsilon\to 0^+}\beta(\epsilon) = 0$ this implies that $\lim_{\epsilon\to 0^+} t^\epsilon = 0$, and taking $t = t^\epsilon$ in (103) we obtain (95). Finally, remark that the estimate (95) also gives the stability at $t = 0$, as from it we have that $\lim_{\epsilon\to 0^+}\|u(\cdot,t^\epsilon) - u(\cdot,0)\| = 0$. The proof of Theorem 13 is completed. □

3.3.2. A computational tool

Let us remind that in subsection 3.1.1, a computational tool was developed to approximate the solution of (53), of the inverse problem without reaction. As an extension, we here enter the integral equation (53) in such a manner that it stems from (77). To do so, we consider $u^{\epsilon}(x,t) = \bar{u}(x,t) + \tilde{u}(x,t)$, where $\bar{u}(x,t)$ and $\tilde{u}(x,t)$ are, respectively, the solutions to the following integral equations:

$$\bar{u}(x,t) = \mathcal{G}_1(\bar{u})(x,t) := \sum_{p=0}^{\infty} E_{(t,T)}\left(\mathcal{D}(\ell_0(\bar{u})(s)), \lambda_p^{(\epsilon)}\right) g_p^{\epsilon} \phi_p(x),$$

$$\tilde{u}(x,t) = \mathcal{G}_2(\tilde{u})(x,t) := -\sum_{p=0}^{\infty} \left(\int_t^T E_{(t,s)}\left(\mathcal{D}(\ell_0(\tilde{u})(\tau)), \lambda_p^{(\epsilon)}\right) \mathcal{R}_p(\tilde{u})(s)\, ds \right) \phi_p(x). \tag{104}$$

Consequently, two corresponding iterative schemes can be unified by the following system:

$$\begin{cases} \bar{u}_{[m+1]}(x,t) = \mathcal{G}_1\left(\sum_{j=0}^{m} \bar{u}_{[j]}\right)(x,t) - \mathcal{G}_1\left(\sum_{j=0}^{m-1} \bar{u}_{[j]}\right)(x,t), \\ \tilde{u}_{[m+1]}(x,t) = \mathcal{G}_2\left(\sum_{j=0}^{m} \tilde{u}_{[j]}\right)(x,t) - \mathcal{G}_2\left(\sum_{j=0}^{m-1} \tilde{u}_{[j]}\right)(x,t), \quad m = 1,2,... \\ \bar{u}_{[1]}(x,t) = \mathcal{G}_1(\bar{u}_{[0]})(x,t), \quad \tilde{u}_{[1]}(x,t) = \mathcal{G}_2(\tilde{u}_{[0]})(x,t), \\ \bar{u}_{[0]}(x,t) = \tilde{u}_{[0]}(x,t) = 0. \end{cases} \tag{105}$$

Then, the series $\sum_{i=0}^{\infty} (\bar{u}_{[i]} + \tilde{u}_{[i]})(x,t)$ yields the unique solution u^{ϵ} of (77).

3.3.3. Local Lipschitz reaction

Much of the analysis of subsection 3.2.1 can be extended to the case when the reaction term $\mathcal{R}(\cdot,\cdot,u)$ is only a local Lipschitz function instead of a global one, using the techniques recently developed by the authors in [35].

Keeping (A_1)-(A_5), we replace the global Lipschitz assumption (A_6) by the local Lipschitz one:
(A_7) $\mathcal{R} \in L^{\infty}(\Omega \times (0,T) \times \mathbb{R})$ and for each $M > 0$, there exists $K(M) \in (0,\infty)$ such that

$$|\mathcal{R}(x,t,\xi_1) - \mathcal{R}(x,t,\xi_2)| \le K(M)|\xi_1 - \xi_2|, \quad \forall (x,t) \in \Omega \times (0,T), \ \forall \xi_1, \xi_2 \in [-M, M].$$

In general, we can take

$$K(M) := \sup\left\{ \left| \frac{\mathcal{R}(x,t,\xi_1) - \mathcal{R}(x,t,\xi_2)}{\xi_1 - \xi_2} \right| : (x,t) \in \Omega \times [0,T], \xi_1, \xi_2 \in [-M,M], \xi_1 \neq \xi_2 \right\} < +\infty,$$

be continuous increasing function with $\lim_{M \to 0^+} K(M) = 0$ and $\lim_{M \to \infty} K(M) = \infty$.

Now, we outline our ideas of constructing a regularization method for the problem (P) given by equations (1)-(4). For any $M > 0$, we approximate $\mathcal{R}$ by $\mathcal{R}_M$ defined by

$$\mathcal{R}_M(x,t,u) := \begin{cases} \mathcal{R}(x,t,M), & u > M, \\ \mathcal{R}(x,t,u), & -M \le u \le M, \\ \mathcal{R}(x,t,-M), & u < -M. \end{cases} \tag{106}$$

From (A_7) and (106) we immediately obtain the following lemma.

Lemma 14. *For $\mathcal{R} \in L^{\infty}(\Omega \times (0,T) \times \mathbb{R})$, we have*

$$|\mathcal{R}_M(x,t,\xi_1) - \mathcal{R}_M(x,t,\xi_2)| \le K(M)|\xi_1 - \xi_2|, \quad \forall (x,t) \in \Omega \times (0,T), \ \forall \xi_1, \xi_2 \in \mathbb{R}.$$

For each $\epsilon > 0$, we consider a sequence $M^{\epsilon} \to +\infty$ as $\epsilon \to 0^+$ and let u^{ϵ} be the solution of the following problem (P_3):

$$\begin{cases} u_t^{\epsilon} + \mathcal{D}(\ell_0(u^{\epsilon})(t))\, \Delta^{\epsilon} u^{\epsilon} = \mathcal{R}_{M^{\epsilon}}(x,t,u^{\epsilon}), & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u^{\epsilon}}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u^{\epsilon}(x,T) = g^{\epsilon}(x), & x \in \Omega. \end{cases} \tag{107}$$

Theorem 15. *Assume (A_1)-(A_5) and (A_7) hold, and suppose that the solution u of the problem (P) given by equations* (1)-(4) *belongs to $C\left([0,T]\,;\mathbb{G}_{\sigma,\gamma}\right)\cap L^{\infty}\left([0,T];L^2(\Omega)\right)$ for $\sigma\geq\eta_2 T$, $\gamma=2$, and $u_t\in C\left([0,T];L^2(\Omega)\right)$. Then, for $\beta=\beta(\epsilon)\in(0,1)$ such that $\lim_{\epsilon\to 0^+}\beta(\epsilon)=0$, assuming that we can choose a sequence $M^\epsilon>0$ such that $\lim_{\epsilon\to 0^+}M^\epsilon=\infty$ and satisfying*

$$K(M^\epsilon)\leq\frac{1}{T}\ln\left(\ln^m\left(\frac{1}{\beta}\right)\right),\quad \textit{for some } m>0, \tag{108}$$

the L^2-norm of the error between the solution u^ϵ of the problem (P_3) given by (107), and the exact solution u is estimated by

$$\left\|u^\epsilon(\cdot,t)-u(\cdot,t)\right\|\leq \mathrm{P}\exp(\tilde{\mu}(T-t))\beta^{\frac{t}{T}}\ln^m\left(\frac{1}{\beta}\right),\quad t\in[0,T], \tag{109}$$

where $\tilde{\mu}:=\frac{1}{2}+\frac{\mathrm{L}}{\lambda_1\eta_2 T}\|f\|\,\|u\|_{C([0,T];\mathbb{G}_{\sigma,\gamma})}$.

If $\beta:=\beta(\epsilon)\in(0,1)$ satisfies (94) *then* (109) *implies that $\|u^\epsilon(\cdot,t)-u(\cdot,t)\|\to 0$, as $\epsilon\to 0^+$, for all $t\in(0,T]$. Moreover, for $\epsilon>0$ small enough there exists $t^\epsilon\in(0,T)$ such that $\lim_{\epsilon\to 0^+}t^\epsilon=0$ and*

$$\|u^\epsilon(\cdot,t^\epsilon)-u(\cdot,0)\|\leq\left(\mathrm{P}\exp(\tilde{\mu}T)\ln^m\left(\frac{1}{\beta}\right)+B\right)\sqrt{\frac{T}{\ln\left(\frac{1}{\beta}\right)}}. \tag{110}$$

Remark that if $m\in(0,1/2)$ the right-hand side of (110) *tends to zero, as $\epsilon\to 0^+$.*

Proof. As in the proof of Theorem 13, using (108) we obtain that

$$\begin{aligned}\left\|u^\epsilon(\cdot,t)-u(\cdot,t)\right\|&\leq \mathrm{P}\ \exp(\tilde{\mu}(T-t))\beta^{\frac{t}{T}}\exp\left(K(M^\epsilon)(T-t)\right)\\ \leq \mathrm{P}\ \exp(\tilde{\mu}(T-t))\beta^{\frac{t}{T}}\exp\left(K(M^\epsilon)T\right)&\leq \mathrm{P}\ \exp(\tilde{\mu}(T-t))\beta^{\frac{t}{T}}\ln^m\left(\frac{1}{\beta}\right),\quad t\in[0,T].\end{aligned} \tag{111}$$

This proves the estimate (109). In deriving (111), all the arguments made in the proof of Theorem 13 remain valid for $\mathcal{R}$ replaced by $\mathcal{R}_{M^\epsilon}$. Since $\lim_{\epsilon\to 0^+}M^\epsilon=+\infty$, for a sufficiently small $\epsilon>0$, there is an $M^\epsilon>0$ such that $M^\epsilon\geq\|u\|_{L^\infty([0,T];L^2(\Omega))}$. Then, for this M^ϵ we have $\mathcal{R}_{M^\epsilon}(x,t,u)=\mathcal{R}(x,t,u)$ and we can use the Lipschitz property of $\mathcal{R}_M$ globally, as given by Lemma 14.

The estimate (110) is established the same way as (95) and using (108). Finally, remark that this also gives the stability at $t=0$, as from (110) for $m\in(0,1/2)$ we have that $\lim_{\epsilon\to 0^+}\|u(\cdot,t^\epsilon)-u(\cdot,0)\|=0$. The proof of theorem is completed. □

4. Numerical results and discussion

In this section, we give applications of the proposed QR iterative scheme in computing solutions of model problems. Our two examples are over the region $(x,t)\in\Omega\times(0,T)=(0,\pi)\times(0,1)$, where the exact solutions $u_{ex}(x,t)$ are explicitly available and are compared with the regularized solutions in order to assess their accuracy and stability.

An orthonormal eigenbasis in $L^2(0,\pi)$ satisfying (6) is $\phi_p(x)=\sqrt{\frac{2}{\pi}}\cos(px)$ and $\lambda_p=p^2$, $p\in\mathbb{N}$, is the corresponding eigenvalue.

The final exact observation (4) is measured by a noisy function $g^\epsilon(x)$ as

$$g^\epsilon(x)=A_\epsilon g(x),\quad x\in\Omega=(0,\pi),\quad A_\epsilon:=1+\frac{\epsilon\ \mathrm{rand}(\epsilon)}{\|g\|} \tag{112}$$

with maximum error ϵ and $\mathrm{rand}(\epsilon)$ is as a random number between $[-1,1]$.

At the discretization level, a uniform grid of mesh-points (x_k,t_n) is used, where $x_k=k\Delta x$ and $t_n=n\Delta t$ for $k=\overline{0,\mathrm{K}}$, $n=\overline{0,N}$ and $\Delta x=\frac{|\Omega|}{\mathrm{K}}=\frac{\pi}{\mathrm{K}}$; $\Delta t=\frac{T}{N}=\frac{1}{N}$. We shall seek the unknowns $u_k^n:=u(x_k,t_n)$ at which the regularized solution is computed.

To numerically illustrate our theoretical results in L^2-norm, we use the following discrete norm of the ℓ^2-error, approximated using the trapezoidal rule as,

$$\mathcal{E}(t_n) := \left\|u_h^n - u_{ex}(\cdot,t_n)\right\|_{\ell^2(0,\pi)} = \sqrt{\Delta x\left(\sum_{k=1}^{K-1}\left|u_k^n - u_{ex}(x_k,t_n)\right|^2 + \frac{1}{2}\left[\left|u_0^n - u_{ex}(0,t_n)\right|^2 + \left|u_K^n - u_{ex}(\pi,t_n)\right|^2\right]\right)},$$
$$n = \overline{0,N}. \quad (113)$$

We also define the rate of convergence between two amounts of noise $\epsilon_1 > \epsilon_2$ as

$$\mathrm{Rate}(\epsilon_1 > \epsilon_2; t_n) := \ln\left(\frac{\mathcal{E}(t_n)|_{\epsilon_1}}{\mathcal{E}(t_n)|_{\epsilon_2}}\right)\Big/\ln\left(\frac{\epsilon_1}{\epsilon_2}\right), \quad n = \overline{0,N}. \quad (114)$$

Example 1. (A generalized KiSS model).
Our first example is a generalized critical patch model for plankton concentration where the population is assumed to inhabit a finite region with a lethal exterior, named KiSS which is an acronym for Kierstead and Slobodkin [22] and Skellam [28] for their independent pioneering studies. The model predicts the size of nutrient patches needed to sustain phytoplankton blooms for which an exponentially growing population disperses intrinsically within and out of a patch into lethal habitat. Here its generalization takes the form

$$u_t = \mathcal{D}\left(\ell_0\left(u\right)(t)\right)u_{xx} + cu + \mathcal{F}\left(x,t\right), \quad (x,t) \in \Omega \times (0,T), \quad (115)$$

where $\mathcal{F}$ is an external source, the intrinsic growth rate of the population is c and the diffusion coefficient $\mathcal{D}$ is given by

$$\mathcal{D}\left(\ell_0\left(u\right)(t)\right) = \omega + \int_\Omega u\left(x,t\right)dx, \quad f \equiv 1, \quad \mathcal{D}(\xi) = \omega + \xi, \quad (116)$$

where ω is a given positive constant. Comparing equations (115) and (1) one can identify the reaction term as $\mathcal{R}(x,t,u) = cu + \mathcal{F}(x,t)$ which is a global Lipschitz function in u.

What we shall compute is u^ϵ, approximation of the population density u from data (112) that has been corrupted by noise. For implementation, we take $\omega = 3$, $c = 1$, $g(x) = e^{-T}\cos x$ (here $T = 1$ and $\Omega = (0,\pi)$) and external source $\mathcal{F}\left(x,t\right) = e^{-t}\cos x$. The exact solution is then given by $u_{ex}\left(x,t\right) = e^{-t}\cos x$.

Observe that from (105), with $p = 1$ in (104), the approximate solution $u_{[m]}^\epsilon\left(x,t\right)$ can be step by step found, as follows:

$$u_{[1]}^\epsilon(x,t) = \left[A_\epsilon\exp\left(\lambda_1^{(\epsilon)}\omega\left(T-t\right)-T\right) - \frac{1}{\lambda_1^{(\epsilon)}\omega - 1}\left(\exp\left(\lambda_1^{(\epsilon)}\omega\left(T-t\right)-T\right) - \exp\left(-t\right)\right)\right]\cos x$$
$$= U_1\left(t\right)\cos x, \quad (117)$$

$$u_{[2]}^\epsilon(x,t) = -c\left[\left(A_\epsilon - \frac{1}{\lambda_1^{(\epsilon)}\omega - 1}\right)\left(T-t\right)\exp\left(\lambda_1^{(\epsilon)}\omega\left(T-t\right)-T\right)\right.$$
$$\left. + \frac{1}{\left(\lambda_1^{(\epsilon)}\omega - 1\right)^2}\left(\exp\left(\lambda_1^{\epsilon}\omega\left(T-t\right)-T\right) - \exp\left(-t\right)\right)\right]\cos x = -U_2\left(t\right)\cos x, \quad (118)$$

$$u_{[3]}^\epsilon(x,t) = c^2\left[\left(A_\epsilon - \frac{1}{\lambda_1^{(\epsilon)}\omega - 1}\right)\frac{(T-t)^2}{2}\exp\left(\lambda_1^{(\epsilon)}\omega\left(T-t\right)-T\right)\right.$$
$$\left. + \frac{T-t}{\left(\lambda_1^{(\epsilon)}\omega - 1\right)^2}\exp\left(\lambda_1^{(\epsilon)}\omega\left(T-t\right)-T\right) - \frac{1}{\left(\lambda_1^{(\epsilon)}\omega - 1\right)^3}\left(\exp\left(\lambda_1^{(\epsilon)}\omega\left(T-t\right)-T\right) - \exp\left(-t\right)\right)\right]\cos x$$
$$= U_3\left(t\right)\cos x. \quad (119)$$

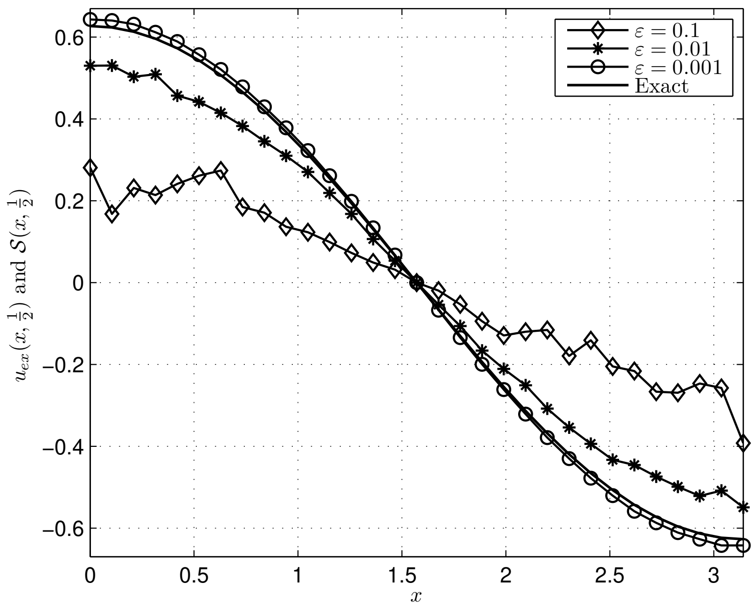

Figure 1: The exact solution $u_{ex}(x,t) = e^{-t}\cos(x)$ and the regularized solution $\mathcal{S}(x,t)$ at $t = 1/2$, for various amounts of noise $\epsilon \in \{0.001, 0.01, 0.1\}$, for Example 1.

Now we simply stop at $m = 3$ and consider numerical results when ϵ goes smaller and smaller. The series obtained by (117)-(119) is

$$\mathcal{S}(x,t) = (U_1(t) - U_2(t) + U_3(t))\cos x. \tag{120}$$

We take the regularization parameter $\beta(\epsilon) = \epsilon$ in (93), for which we expect the order of error estimate to be $O\left(\epsilon^{t/T}\right)$. From (58), we obtain $\lambda_1^{(\epsilon)} = -\frac{1}{\eta_2 T}\ln\left(\beta + e^{-\eta_2 T\lambda_1}\right) = -\frac{1}{3}\ln\left(\epsilon + e^{-3}\right)$ for $T = 1$, $\beta(\epsilon) = \epsilon$, $\eta_2 = \omega = 3$ and $\lambda_1 = 1$. We also take K $= N = 30$ in (113).

The numerical results for the solution (120) for various amounts of noise $\epsilon \in \{0.001, 0.01, 0.1\}$ are compared with the exact solution $u_{ex}(x,t) = e^{-t}\cos(x)$ at a selected time $t = T/2 = 1/2$ in Figure 1. From this figure it can be seen that the numerical results are accurate and stable for $\epsilon \in \{0.001, 0.01\}$, but for the larger amount of noise $\epsilon = 0.1$ they seem to deteriorate and start deviating significantly from the exact solution $u_{ex}(x, 1/2) = e^{-1/2}\cos(x)$.

Table 1 shows the ℓ^2-error (113) at various times $t \in \{1/3, 1/2, 2/3\}$, for various amounts of noise $\epsilon \in \{0.001, 0.01, 0.1\}$. The numerical rates of convergence (defined in (114)) for two consecutive amounts of noise $0.01 > 0.001$ and $0.1 > 0.01$ are also included. From Table 1 the following conclusions can be drawn:
(i) $\mathcal{E}(t_n)$ decreases as n increases, i.e. the error increases as t decreases from $t = T$ to $t = 0$. Also, as expected, $\mathcal{E}(t)$ decreases as the amount of noise ϵ decreases;
(ii) the rate/speed of convergence for $u(x, 2/3)$ is very good for the relatively low amount of noise $\epsilon = 0.001$ and it does decrease as t decreases from $t = 2/3$ to $t = 1/2$ and then to $t = 1/3$, as predicted by the theoretical results. However, for the medium amount of noise $\epsilon = 0.01$, the rate changes its monotonic behaviour with respect to t, by increasing as t decreases.

Example 2. (Fisher's model).
The second example is the classical reaction-diffusion model of ecological import which is modelled by the logistic population growth plus Brownian random dispersal, [28],

$$u_t = \mathcal{D}u_{xx} + r_1 u\left(1 - \frac{u}{C}\right) + \mathcal{F}(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{121}$$

where $r_1 \geq 0$ is the population intrinsic rate of growth and $C > 0$ is the carrying capacity. Fisher's model has been used to make predictions of range expansion using microscale data on individual movement for a variety of animals,

Table 1: The error (113) and the rate (114) at various times $t \in \{1/3, 1/2, 2/3\}$, for various amounts of noise $\epsilon \in \{0.001, 0.01, 0.1\}$, for Example 1.

ϵ	$\mathcal{E}(1/3)$	Rate	$\mathcal{E}(1/2)$	Rate	$\mathcal{E}(2/3)$	Rate
0.001	0.1132	0.11	0.0208	0.79	0.0027	1.46
0.01	0.1487	0.66	0.1314	0.54	0.0799	0.51
0.1	0.6856	—	0.4662	—	0.2617	—

working well with cabbage butterflies, muskrats, grey squirrels and neolithic farmers.

Here we implement the model with constant diffusion $\mathcal{D} = \mathcal{D}_0 = 1/8$, the growth rate $r_1 = 1/4$ and the capacity $C = 2$. We take $g(x) = T\cos(4x)$ (here $\Omega = (0, \pi)$) and the external source $\mathcal{F}(x,t) = \left(\frac{t^2}{8}\cos(4x) + \frac{7t}{4} + 1\right)\cos(4x)$. Then the exact solution is given by $u_{ex}(x,t) = t\cos(4x)$.

In (121), the nonlinearity $\mathcal{R}(x,t,u) = r_1 u\left(1 - \frac{u}{C}\right) + \mathcal{F}(x,t)$ represents a locally Lipschitz reaction with the Lipschitz constant $K(M) = r_1\left(1 + \frac{2M}{C}\right)$. Then (108) implies (taking $\beta(\epsilon) = \epsilon$ and $m = 0.49 \in (0, 1/2)$) that we can choose

$$M^\epsilon = \frac{C}{2}\left(\frac{1}{r_1 T}\ln\left(\ln^{0.49}\left(\frac{1}{\epsilon}\right)\right) - 1\right), \tag{122}$$

as the truncation level in (106).

We compute the regularized solution up to $p = 10$ and only retain $u^\epsilon_{[1]}(x,t)$. As in the previous example, from (105) and (112), the form of the obtained series can be presented as

$$\mathcal{S}(x,t) = (U_1(t) + U_2(t))\cos(4x) + (U_3(t) - U_4(t))\cos(8x), \tag{123}$$

where

$$\begin{aligned}
U_1(t) &= A_\epsilon T e^{\mathcal{D}_0(T-t)\lambda_4^{(\epsilon)}},\\
U_2(t) &= -e^{\mathcal{D}_0(T-t)\lambda_4^{(\epsilon)}}\left(r_1 A_\epsilon T(T-t) + \frac{1}{(\mathcal{D}_0\lambda_4^{(\epsilon)})^2}\left(\frac{7}{4}(\mathcal{D}_0\lambda_4^{(\epsilon)}T - 1) + \mathcal{D}_0\lambda_4^{(\epsilon)}\right)\right)\\
&\quad + \frac{1}{(\mathcal{D}_0\lambda_4^{(\epsilon)})^2}\left(\frac{7}{4}(\mathcal{D}_0\lambda_4^{(\epsilon)}t - 1) + \mathcal{D}_0\lambda_4^{(\epsilon)}\right),\\
U_3(t) &= \frac{r_1 A_\epsilon^2 T^2}{2C\mathcal{D}_0(\lambda_8^{(\epsilon)} - 2\lambda_4^{(\epsilon)})}\left(e^{\mathcal{D}_0(T-t)\lambda_8^{(\epsilon)}} - e^{2\mathcal{D}_0(T-t)\lambda_4^{(\epsilon)}}\right),\\
U_4(t) &= \frac{1}{16(\mathcal{D}_0\lambda_8^{(\epsilon)})^3}\left[e^{\mathcal{D}_0(T-t)\lambda_8^{(\epsilon)}}\left((\mathcal{D}_0\lambda_8^{(\epsilon)}T - 1)^2 + 1\right) - (\mathcal{D}_0\lambda_8^{(\epsilon)}t - 1)^2 - 1\right].
\end{aligned}$$

From (58), we obtain $\lambda_4^{(\epsilon)} = -\frac{1}{\eta_2 T}\ln\left(\beta + e^{-\eta_2 T\lambda_4}\right) = -2\ln\left(\epsilon + e^{-1/2}\right)$ and $\lambda_8^{(\epsilon)} = -\frac{1}{\eta_2 T}\ln\left(\beta + e^{-\eta_2 T\lambda_8}\right) = -2\ln\left(\epsilon + e^{-2}\right)$ for $T = 1/4$, $\beta(\epsilon) = \epsilon$, $\eta_2 = \mathcal{D}_0 = 1/8$, $\lambda_4 = 16$ and $\lambda_8 = 64$. We also take K = 60 and $N = 30$ in (113).

The numerical solution (123) for various amounts of noise $\epsilon \in \{0.001, 0.01, 0.1\}$ are compared in Figure 2 with the exact solution $u_{ex} = t\cos(4x)$ at a selected time $t = 3T/4 = 3/16$. From this figure it can be seen that the numerical approximations become more accurate as ϵ decreases. Graphically, there is excellent agreement between the numerical and exact solutions for $\epsilon \in \{0.001, 0.01\}$ and moreover, we report that the errors (113) are decreasing from 0.0535 for $\epsilon = 0.1$ to 0.0052 for $\epsilon = 0.01$ and then to 0.0005 for $\epsilon = 0.001$. From these errors one can also easily calculate the rates (114) to be Rate$(0.01 > 0.001; 3/16) = 1.02$ and Rate$(0.1 > 0.01; 3/16) = 1.00$. We also report that for the small amount of noise $\epsilon = 0.001$ we obtain that the errors (113) are $0.0005 = \mathcal{E}(3/16) < \mathcal{E}(1/8) = 0.0037 < \mathcal{E}(1/16) = 0.0101$, showing that, as expected, the errors increase with decreasing the time t from the final time $t = T = 1/4$, i.e. the problem looses its stability as we proceed backwards in time.

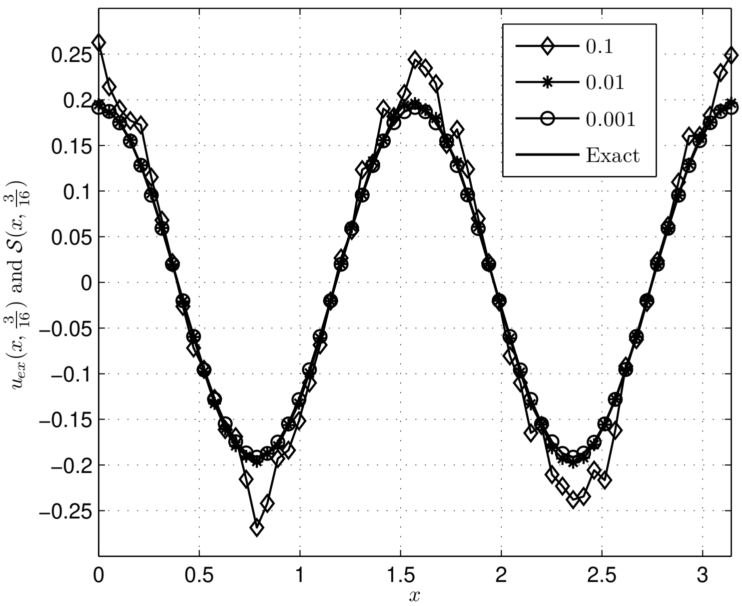

Figure 2: Exact solution $u_{ex}(x,t) = t\cos(4x)$ and the regularized solutions $\mathcal{S}(x,t)$ at $t = 3/16$, for various amounts of noise $\epsilon \in \{0.001, 0.01, 0.1\}$, for Example 2.

Other significant models which are not illustrated in this section but which can be investigated using the analysis of this paper include catalist models describing chemical reactions in cells, [29], with

$$\mathcal{D}(\ell_0(u)(t)) = \omega + \int_\Omega \exp\left(-\frac{x}{|\Omega|}\right) u(x,t)\,dx, \quad \mathcal{R}(x,t,u) = \mathcal{F}(x,t) - u(u-0.5)(u-1),$$

and many other models of molecural interactions with power-law reaction rates, [26].

5. Conclusions

The ill-posed backward continuation in time for the nonlinear parabolic equation (1) with nonlocal diffusion (2) giving the population density of a species has been investigated. The governing partial differential equation (1) also incorporates a source term $\mathcal{R}(x,t,u)$ which models a global or local Lipschitz reaction.

The quasi-reversibility regularized solutions that has been proposed has been shown to depend continuously on the measured final data (4) and to strongly converge in the L^2-norm to the exact solution, if it exists. Throughout the paper, novel and new error estimates together with stability results have been obtained.

Furthermore, a computational tool for symbolic based solution has been developed, followed by the Picard-type iteration. It is worth noting that this algorithm is led by the fixed-point argument which applies for proving the existence and uniqueness of the approximate solutions. In cases where the complexity of the truncated series such as (120) or (123) increases beyong purpose, standard numerical discretisation methods would be preferable. Numerical results presented and discussed for a couple of physical models illustrate the convergence and stability of the regularized solution for the backward parabolic problem with nonlocal diffusion and nonlinear reaction term.

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References

[1] K.A. Ames and J.F. Epperson. A kernel-based method for the approximate solution of backward parabolic problems. *SIAM J. Numer. Anal.*, 34(4):127–145, 1997.

[2] H.T. Banks and K.A. Murphy. Estimation of nonlinearities in parabolic models for growth, predation, and dispersal of populations. *J. Math. Anal. Appl.*, 141:580–602, 1989.

[3] L. Bourgeois. A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation. *Inverse Problems*, 21(3):1087–1104, 2005.

[4] L. Bourgeois. About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$ domains. *ESAIM: Math. Model. Numer. Anal.*, 44:715-735, 2010.

[5] L. Bourgeois and J. Darde. About stability and regularization of ill-posed elliptic Cauchy problems: the case of Lipschitz domains. *Appl. Anal.*, 89:1745-1768, 2009.

[6] L. Bourgeois and J. Darde. A duality-based method of quasi-reversibility to solve the Cauchy problem in the presence of noisy data. *Inverse Problems*, 26:095016 (21 pp), 2010.

[7] C. Cao, M.A. Rammaha, and E.S. Titi. The Navier-Stokes equations on the rotating 2-D sphere: Gevrey regularity and asymptotic degrees of freedom. *Z. Angew. Math. Phys.*, 50:341–360, 1999.

[8] H. Cao and S.V. Pereverzev. Natural linearization for the identification of a diffusion coefficient in a quasi-linear parabolic system from short-time observations. *Inverse Problems*, 22:2311–2330, 2006.

[9] H. Cao, M.V. Klibanov and S.V. Pereverzev. A Carleman estimate and the balancing principle in the quasi-reversibility method for solving the Cauchy problem for the Laplace equation. *Inverse Problems*, 25:035005 (21pp), 2009.

[10] T. Caraballo, M. Herrera-Cobos and P. Marin-Rubio. Long-time behavior of a non-autonomous parabolic equation with nonlocal diffusion and sublinear terms. *Nonlinear Anal.*, 121:3–18, 2015.

[11] N.-H. Chang and M. Chipot. On some mixed boundary value problems with nonlocal diffusion. *Adv. Math. Sci. Appl.*, 14:1–24, 2004.

[12] V. Daftardar-Gejji and H. Jafari. An iterative method for solving nonlinear functional equations. *J. Math. Anal. Appl.*, 316:753–763, 2006.

[13] P. Duchateau, R. Thelwell, and G. Butters. Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient. *Inverse Problems*, 20:601–625, 2004.

[14] L. C. Evans. *Partial Differential Equations*, Vol. 19. American Mathematical Society, Providence, Rhode Island, 1998.

[15] R.E. Ewing. The approximation of certain parabolic equations backward in time by Sobolev equation. *SIAM J. Math. Anal.*, 6(2):283–294, 1975.

[16] C.L. Fu, X.X. Tuan, and Z. Qian. Fourier regularization for a backward heat equation. *J. Math. Anal. Appl.*, 331(1):472–480, 2007.

[17] M.A. Golberg. A note on the decomposition method for operator equations. *Appl. Math. Comput.*, 106:215–220, 1999.

[18] M. Hanke and O. Scherzer. Error analysis of an equation error method for the identification of the diffusion coefficient in a quasi-linear parabolic differential equation. *SIAM J. Appl. Math.*, 59(3):1012–1027, 1999.

[19] D.N. Hao, N. Van Duc and D. Lesnic. Regularization of parabolic equations backward in time by a non-local boundary value problem. *IMA J. Appl. Math.*, 75:291–315, 2010.

[20] D.N. Hao and N. Van Duc. Stability results for a backward parabolic equation with time dependent coefficient. *Inverse Problems*, 27(2), 025003, 2011.

[21] A. Hasanov and J.L. Mueller A numerical method for backward parabolic problems with non-selfadjoint elliptic operators. *Appl. Numer. Math.*, 37:55–78, 2001.

[22] H. Kierstead and L. Slobodkin. The size of water masses containing plankton bloom. *J. Marine Res.*, 12:141–147, 1953.

[23] M.V. Klibanov and F. Santosa. A computational quasi-reversibility method for Cauchy problems for Laplace's equation. *SIAM J. Appl. Math.*, 51:1653–1675, 1991.

[24] R. Lattès and J.L. Lions. *Methode de Quasi-Reversibility et Application*. Dunod: Gauthier–Villars, Paris, 1967.

[25] D. Lesnic and L. Elliott. The decomposition approach to inverse heat conduction. *J. Math. Anal. Appl.*, 232:82–98, 1999.

[26] C.V. Pao. *Nonlinear Parabolic and Elliptic Equations*. Plenum Press, New York, 1992.

[27] P.H. Quan, D.D. Trong, L.M. Triet and N.H. Tuan. A modified quasi-boundary value method for regularizing a backward problem with time-dependent coefficient. *Inverse Problems Sci. Eng.*, 19:409-423, 2011.

[28] J. G. Skellam. Random dispersal in theoretical populations. *Biometrika*, 38:196–218, 1951.

[29] F. Schlogl, C. Escher and R.S. Berry. Fluctuations in the interface between two phases. *Physical Review A*, 27:2698-2704, 1983.

[30] H. Triebel. *Fourier Analysis and Function Spaces*. Teubner: Leipzig, 1977.

[31] D.D. Trong and N.H. Tuan. Regularization and error estimates for nonhomogeneous backward heat problem. *Electron. J. Diff. Eqns.*, 4:1–10, 2006.

[32] N.H. Tuan and D.D. Trong. A nonlinear parabolic equation backward in time: regularization with new error estimates. *Nonlinear Anal.*, 73(6):1842–1852, 2010.

[33] N.H. Tuan. Stability estimates for a class of semi-linear ill-posed problems. *Nonlinear Anal.: Real World Applications*, 14:1203–1215, 2013.

[34] N.H. Tuan and D.D. Trong. On a backward parabolic problem with local Lipschitz source. *J. Math. Anal. Appl.*, 414:678–692, 2014.

[35] N.H. Tuan, L.D. Thang, and D. Lesnic. A new general filter regularization method for Cauchy problems for elliptic equations with a locally Lipschitz nonlinear source. *J. Math. Anal. Appl.*, 434:1376–1393, 2016.

[36] B. Yildiz, H. Yetis, and A. Sever. A stability estimate on the regularized solution of the backward heat problem. *Appl. Math. Comput.*, 135:561–567, 2003.

[37] S. Zheng and M. Chipot. Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms. *Asymptotic Anal.*, 45:301–312, 2005.