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# BADLY APPROXIMABLE POINTS IN TWISTED DIOPHANTINE APPROXIMATION AND HAUSDORFF DIMENSION 

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#### Abstract

For any $j_{1}, \ldots, j_{n}>0$ with $\sum_{i=1}^{n} j_{i}=1$ and any $\theta \in$ $\mathbb{R}^{n}$, let $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$ denote the set of points $\eta \in \mathbb{R}^{n}$ for which $\max _{1 \leq i \leq n}\left(\left\|q \theta_{i}-\eta_{i}\right\|^{1 / j_{i}}\right)>c / q$ for some positive constant $c=c(\eta)$ and all $q \in \mathbb{N}$. These sets are the 'twisted' inhomogeneous analogue of $\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$ in the theory of simultaneous Diophantine approximation. It has been shown that they have full Hausdorff dimension in the non-weighted setting, i.e provided that $j_{i}=1 / n$, and in the weighted setting when $\theta$ is chosen from $\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$. We generalise these results proving the full Hausdorff dimension in the weighted setting without any condition on $\theta$. Moreover, we prove $\operatorname{dim}\left(\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right) \cap\right.$ $\operatorname{Bad}(1,0, \ldots, 0) \cap \ldots \cap \operatorname{Bad}(0, \ldots, 0,1))=n$.


## 1. Introduction

The classical result due to Dirichlet: for any real number $\theta$ there exist infinitely many natural numbers $q$ such that

$$
\begin{equation*}
\|q \theta\| \leq q^{-1} \tag{1}
\end{equation*}
$$

where $\|\cdot\|$ denotes the distance to the nearest integer, has higher dimension generalisations. Consider any $n$-tuple of real numbers $\left(j_{1}, \ldots, j_{n}\right)$ such that

$$
\begin{equation*}
j_{1}, \ldots, j_{n}>0 \quad \text { and } \quad \sum_{i=1}^{n} j_{i}=1 . \tag{2}
\end{equation*}
$$

Then, for any vector $\theta=\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}$, there exist infinitely many natural numbers $q$ such that

$$
\begin{equation*}
\max _{1 \leq i \leq n}\left(\left\|q \theta_{i}\right\|^{1 / j_{i}}\right) \leq q^{-1} . \tag{3}
\end{equation*}
$$

The two results above motivate the study of real numbers and real vectors $\theta \in \mathbb{R}^{n}$ for which the right hand side of (1) and (3) respectively cannot be improved by an arbitrary constant. They respectively constitute the sets Bad

[^0]of badly approximable numbers and $\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$ of $\left(j_{1}, \ldots, j_{n}\right)$-badly approximable numbers. Hence
$$
\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right):=\left\{\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}: \inf _{q \in \mathbb{N}} \max _{1 \leq i \leq n}\left(q^{j_{i}}\left\|q \theta_{i}\right\|\right)>0\right\}
$$

In the 1-dimensional case, it is well known that the set of badly approximable numbers has Lebesgue measure zero but maximal Hausdorff dimension. In the $n$-dimensional case, it is also a classical result that $\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$ has Lebesgue measure zero, and Schmidt proved in 1966 that the particular set $\operatorname{Bad}(1 / 2,1 / 2)$ has full Hausdorff dimension. But the result of maximal dimension in the weigthed setting hasn't been proved until almost 40 years later, by Pollington and Velani [21]. In the 2-dimensional case, An showed in [1] that $\operatorname{Bad}\left(j_{1}, j_{2}\right)$ is in fact winning for the now famous Schmidt games -see [22]. Thus he provided a direct proof of a conjecture of Schmidt stating that any countable intersection of sets $\operatorname{Bad}\left(j_{1}, j_{2}\right)$ is non empty -see also [2].

Recently, interest in the size of related sets, usually referred to as the 'twists' of the sets $\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$, has developed. The study of these new sets started in the 1-dimensional setting: we fix $\theta \in \mathbb{R}$ and consider the twist of Bad:

$$
\operatorname{Bad}_{\theta}:=\left\{\eta \in \mathbb{R}: \inf _{q \in \mathbb{N}} q\|q \theta-\eta\|>0\right\} .
$$

The set $\operatorname{Bad}_{\theta}$ has a palpable interpretation in terms of rotations of the unit circle. Identifying the circle with the unit interval $[0,1)$, the value $q \theta$ (modulo 1) may be thought of as the position of the origin after $q$ rotations by the angle $\theta$. If $\theta$ is rational, the rotation is periodic. If $\theta$ is irrational, a classical result of Weyl [25] implies that $q \theta$ (modulo 1 ) is equidistributed, so $q \theta$ visits any fixed subinterval of $[0,1)$ infinitely often. The natural question of what happens if the subinterval is allowed to shrink with time arises. Shrinking a subinterval corresponds to making its length decay according to some specified function. The set $\operatorname{Bad}_{\theta}$ corresponds to considering, for any $\epsilon>0$, the shrinking interval $(\eta-\epsilon / q, \eta+\epsilon / q)$ centred at the point $\eta$ and where the specified function is $\epsilon / q$. Khintchine showed in [14] that

$$
\begin{equation*}
\|q \theta-\eta\|<\frac{1+\delta}{\sqrt{5} q} \quad(\delta>0) \tag{4}
\end{equation*}
$$

is satisfied for infinitely many integers $q$, and Theorem III in Chapter III of Cassels' book [5] shows that the right hand side of (4) cannot be improved by an arbitrary constant for every irrational $\theta$ and every real $\eta$. This motivates the study of the set $\operatorname{Bad}_{\theta}$. Kim [16] proved in 2007 that it has Lebesgue measure zero, and later it was shown by Tseng [23] that it has full Hausdorff
dimension (actually Tseng proved that $\operatorname{Bad}_{\theta}$ has the stronger property of being winning for any $\theta \in \mathbb{R}$ ).

By generalising circle rotations to rotations on torus of higher dimensions, i.e. by considering the sequence $q \theta$ (modulo 1 ) in $[0,1)^{n}$ where $\theta=$ $\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}$, we obtain the 'twists' of the sets $\operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$ :

$$
\begin{equation*}
\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)=\left\{\left(\eta_{1}, \ldots, \eta_{n}\right) \in \mathbb{R}^{n}: \inf _{q \in \mathbb{N}} \max _{1 \leq i \leq n}\left(q^{j_{i}}\left\|q \theta_{i}-\eta_{i}\right\|\right)>0\right\} \tag{5}
\end{equation*}
$$

In [3] Bugeaud et al proved that the non-weighted set $\operatorname{Bad}_{\theta}(1 / n, \ldots$
$\ldots, 1 / n$ ) has full Hausdorff dimension. Recently, Einsiedler and Tseng [8] extended the results [3] and [23] by showing, among other results, that $\operatorname{Bad}_{\theta}(1 / n, \ldots, 1 / n)$ is also winning. It was shown in [18] that such results may be obtained by classical methods developed by Khintchine [15] and Jarník [12,13] and discussed in Chapter V of Cassels' book [5]. Unfortunately, these methods cannot be directly extended to the weighted setting. For the weighted setting, less has heretofore been known. Harrap did the first contribution [10] in the 2-dimensional case, by proving that $\operatorname{Bad}_{\theta}\left(j_{1}, j_{2}\right)$ has full Hausdorff dimension provided that the fixed point $\theta \in \mathbb{R}^{2}$ belongs to $\operatorname{Bad}\left(j_{1}, j_{2}\right)$, which is a significantly restrictive condition. Recently, under the hypothesis $\theta \in \operatorname{Bad}\left(j_{1}, \ldots, j_{n}\right)$, Harrap and Moshchevitin have extended to weighted linear forms in higher dimension and improved to winning the result in [10] (see [11]).

In this paper, we prove that the weighted set $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$ has full Hausdorff dimension for any $\theta \in \mathbb{R}^{n}$. Moreover, the following theorem holds.

Theorem 1.1. For any $\theta \in \mathbb{R}^{n}$ and all $j_{1}, \ldots, j_{n}>0$ with $\sum_{i=1}^{n} j_{i}=1$,

$$
\operatorname{dim}\left(\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right) \cap \operatorname{Bad}(1,0, \ldots, 0) \cap \ldots \cap \operatorname{Bad}(0, \ldots, 0,1)\right)=n
$$

The same type of theorem holds in the classical not twisted setting; it constitutes the work done in [21] (see Theorem 2).

Note that if $1, \theta_{1}, \ldots, \theta_{n}$ are linearly dependent over $\mathbb{Z}$, then Theorem 1.1 is obvious. Indeed, in this case $\{q \theta: q \in \mathbb{Z}\}$ is restricted to a hyperplane $H$ of $\mathbb{R}^{n}$, so $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right) \supset \mathbb{R}^{n} \backslash H$ is winning. Hence $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right) \cap$ $\operatorname{Bad}(1,0, \ldots, 0) \cap \ldots \cap \operatorname{Bad}(0, \ldots, 0,1)$ is winning and in particular has full dimension ${ }^{1}$. Therefore we suppose throughout the paper that $1, \theta_{1}, \ldots, \theta_{n}$ are linearly independent over $\mathbb{Z}$.

[^1]The strategy for the proof of Theorem 1.1 is as follows. We start by defining a set $\mathcal{V} \subset \operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$ related to the best approximations to the fixed point $\theta \in \mathbb{R}^{n}$. Then we construct a Cantor-type set $K(R)$ inside $\mathcal{V} \cap \operatorname{Bad}(1,0, \ldots, 0) \cap \ldots \cap \operatorname{Bad}(0, \ldots, 0,1)$. Finally we describe a probability measure supported on $K(R)$ to which we can apply the mass distribution principle and thus find a lower bound for the dimension of $K(R)$.

Best approximations are defined in Section 2. In Section 3 we define $\mathcal{V}$ and give the proof of the inclusion $\mathcal{V} \subset \operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$. We construct $K(R)$ in Section 4 and describe the probability measure in Section 5. Finally we compute the lower bound for the dimension of $K(R)$ in Section 6 .

In the following, we let $n \in \mathbb{N}$, fix an $n$-tuple $\left(j_{1}, \ldots, j_{n}\right) \in \mathbb{R}^{n}$ satisfying (2) and a vector $\theta=\left(\theta_{1}, \ldots, \theta_{n}\right) \in \mathbb{R}^{n}$ such that $1, \theta_{1}, \ldots, \theta_{n}$ are linearly independent over $\mathbb{Z}$. We denote by $x \cdot y$ the scalar product of two vectors $x$ and $y$ in $\mathbb{R}^{n}$, and by $\|\cdot\|$ the distance to the nearest integer.

## 2. Best approximations

Definition 2.1. An n-dimensional vector $m=\left(m_{1}, \ldots, m_{n}\right) \in \mathbb{Z}^{n} \backslash\{0\}$ is called a best approximation to $\theta$ if for all $v \in \mathbb{Z}^{n} \backslash\{0,-m, m\}$ the following implication holds:

$$
\max _{1 \leq i \leq n}\left(\left|v_{i}\right|^{1 / j_{i}}\right) \leq \max _{1 \leq i \leq n}\left(\left|m_{i}\right|^{1 / j_{i}}\right) \Longrightarrow\|v \cdot \theta\|>\|m \cdot \theta\|
$$

Note that the condition $1, \theta_{1}, \ldots, \theta_{n}$ are $\mathbb{Z}$-linearly independent allows us to demand a strict inequality in the right hand side of the implication above.

Note also that when $n=1$ the best approximations to a real number $x$ are, up to the sign, the denominators of the convergents to $x$.

Since $1, \theta_{1}, \ldots, \theta_{n}$ are $\mathbb{Z}$-linearly independent, we have an infinite number of best approximations to $\theta$. They can be arranged up to the sign -so that two vectors of opposite sign do not both appear- in an infinite sequence

$$
\begin{equation*}
m_{\nu}=\left(m_{\nu, 1}, \ldots, m_{\nu, n}\right) \quad \nu \geq 1 \tag{6}
\end{equation*}
$$

such that the values

$$
\begin{equation*}
M_{\nu}=\max _{1 \leq i \leq n}\left(\left|m_{\nu, i}\right|^{1 / j_{i}}\right) \tag{7}
\end{equation*}
$$

form a strictly increasing sequence, and the values

$$
\begin{equation*}
\zeta_{\nu}=\left\|m_{\nu} \cdot \theta\right\| \tag{8}
\end{equation*}
$$

form a strictly decreasing sequence. Hence each value $M_{\nu}$ corresponds to a single best approximation $m_{\nu}$. The quantity $M_{\nu}$ can be referred to as the 'height' of $m_{\nu}$.

Best approximations vectors have often been used in proofs, but not always explicitly. In particular, Voronoi [24] selected some points in a lattice that correspond exactly to the best approximation vectors (see also [7]). Similar constructions were introduced in [17] or Section 2 of [4]. Some important properties of the best approximation vectors are discussed in $[19,20]$ and a recent survey on the topic is due to Chevallier [6].

For each $\nu \geq 1$, it is easy to see that the region

$$
\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{R}^{n+1}: \max _{1 \leq i \leq n}\left(\left|x_{i}\right|^{1 / j_{i}}\right)<M_{\nu+1},\left|x_{0}+\sum_{i=1}^{n} x_{i} \theta_{i}\right|<\zeta_{\nu}\right\}
$$

does not contain any integer point different from 0 . Since this region has volume $2^{n+1} M_{\nu+1} \zeta_{\nu}$ (see Lemma 4 in Appendix B of [5]), it follows from Minkowski's convex body theorem that

$$
\begin{equation*}
\zeta_{\nu} M_{\nu+1} \leq 1 \tag{9}
\end{equation*}
$$

The inequality above will be used later as well as the following lemma, stating that the sequence of heights $M_{\nu}$ is lacunary.

Lemma 2.2. For every $\nu \geq 1$, we have

$$
M_{\nu+2 \cdot 3^{n}} \geq 2 M_{\nu}
$$

Proof. Given $\nu \geq 1$, we show that we have at most $2 \cdot 3^{n}$ vectors $m_{\nu+r}$ with $r \geq 0$ and $M_{\nu+r}<2 M_{\nu}$. The goal is to see that the 0 -symmetric region
(10) $\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{R}^{n+1}: \max _{1 \leq i \leq n}\left(\left|x_{i}\right|^{1 / j_{i}}\right)<2 M_{\nu},\left|x_{0}+\sum_{i=1}^{n} x_{i} \theta_{i}\right| \leq \zeta_{\nu}\right\}$
contains at most $4 \cdot 3^{n}$ integer points other than 0 . The region (10) is covered by sets of the form

$$
T(\xi)=\left\{\begin{aligned}
\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{R}^{n+1} & : \max _{1 \leq i \leq n}\left(\left|x_{i}-\xi_{i}\right|^{1 / j_{i}}\right) \leq M_{\nu} \\
& \text { and }\left|x_{0}-\xi_{0}+\sum_{i=1}^{n}\left(x_{i}-\xi_{i}\right) \theta_{i}\right| \leq \zeta_{\nu}
\end{aligned}\right\}
$$

with

$$
\begin{equation*}
\xi_{i} \in\left\{-2 M_{\nu}^{j_{i}}, 0,2 M_{\nu}^{j_{i}}\right\}, \quad \xi_{0}=-\sum_{i=1}^{n} \xi_{i} \theta_{i} \tag{11}
\end{equation*}
$$

Each region $T(\xi)$ is the translate by $\left(\xi_{0}, \ldots, \xi_{n}\right)$ of the set

$$
\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{R}^{n+1}: \max _{1 \leq i \leq n}\left(\left|x_{i}\right|^{1 / j_{i}}\right) \leq M_{\nu},\left|x_{0}+\sum_{i=1}^{n} x_{i} \theta_{i}\right| \leq \zeta_{\nu}\right\}
$$

which contains exactly three integer points: 0 and two best approximations with opposite sign. Hence each $T(\xi)$ contains at most four integer points. Since there are $3^{n}$ possible choices for $\left(\xi_{0}, \ldots, \xi_{n}\right)$ satisfying (11), the set (10) contains at most $4 \cdot 3^{n}$ integer points.

## 3. The set $\mathcal{V}$ included in $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$

The following proposition allows us to work with a set defined by the best approximations to $\theta$ instead of working directly with $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$.

Proposition 3.1. If $\eta \in \mathbb{R}^{n}$ satisfies

$$
\begin{equation*}
\inf _{\nu}\left\|m_{\nu} \cdot \eta\right\|>0 \tag{12}
\end{equation*}
$$

then $\eta \in \operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right)$.
Proof. Let $\eta=\left(\eta_{1}, \ldots, \eta_{n}\right) \in \mathbb{R}^{n}$ satisfy

$$
\left\|m_{\nu} \cdot \eta\right\|>\gamma \quad \forall \nu \geq 1
$$

for some $\gamma>0$. For all $q \in \mathbb{N}$ and $\nu \geq 1$, we have the identity

$$
m_{\nu} \cdot \eta=m_{\nu} \cdot(\eta-q \theta)+q m_{\nu} \cdot \theta
$$

from which we obtain the inequalities

$$
\begin{equation*}
\gamma<\left\|m_{\nu} \cdot \eta\right\| \leq n \max _{1 \leq i \leq n}\left(\left|m_{\nu, i}\right| \cdot\left\|\eta_{i}-q \theta_{i}\right\|\right)+q \zeta_{\nu} \tag{13}
\end{equation*}
$$

Since $\zeta_{\nu}$ is strictly decreasing and $\zeta_{\nu} \rightarrow 0$ as $\nu \rightarrow \infty$, there exists $\nu \geq 1$ such that

$$
\begin{equation*}
\frac{\gamma}{2 \zeta_{\nu}} \leq q \leq \frac{\gamma}{2 \zeta_{\nu+1}} \tag{14}
\end{equation*}
$$

On the one hand, from the inequalities (13) and the upper bound in (14), we deduce that

$$
\begin{equation*}
\max _{1 \leq i \leq n}\left(\left\|\eta_{i}-q \theta_{i}\right\| \cdot\left|m_{\nu+1, i}\right|\right)>\frac{\gamma}{2 n} \tag{15}
\end{equation*}
$$

On the other hand, from the lower bound in (14) and the inequality (9), it follows that

$$
q \geq \frac{\gamma}{2} M_{\nu+1} .
$$

We deduce that

$$
\begin{equation*}
q^{j_{i}} \geq c\left|m_{\nu+1, i}\right| \quad \forall i=1, \ldots, n, \tag{16}
\end{equation*}
$$

where

$$
c=\min _{1 \leq i \leq n}\left(\left(\frac{\gamma}{2}\right)^{j_{i}}\right)
$$

Finally, by combining (15) and (16), we have that

$$
\max _{1 \leq i \leq n}\left(\left\|\eta_{i}-q \theta_{i}\right\| q^{j_{i}}\right)>\frac{\gamma c}{2 n} .
$$

This concludes the proof of the proposition.
We define the set

$$
\mathcal{V}:=\left\{\eta \in \mathbb{R}^{n}: \inf _{\nu \geq 1}\left\|m_{\nu} \cdot \eta\right\|>0\right\}
$$

Clearly

$$
\begin{equation*}
\mathcal{V} \subset \operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right) \tag{17}
\end{equation*}
$$

## 4. The Cantor-type set $K(R)$

In this section we construct the Cantor-type set $K(R)$ inside $\operatorname{Bad}_{\theta}\left(j_{1}, \ldots, j_{n}\right) \cap \operatorname{Bad}(1,0, \ldots, 0) \cap \ldots \cap \operatorname{Bad}(0, \ldots, 0,1)$. In order to lighten the notation, throughout this section we denote by $\mathcal{M}$ the set of best approximations in the sequence (6), and for each $m \in \mathcal{M}$, by $M_{m}$ the quantity defined by (7), i.e.

$$
M_{m}=\max _{1 \leq i \leq n}\left(\left|m_{i}\right|^{1 / j_{i}}\right)
$$

Hence

$$
\mathcal{V}=\left\{\eta \in \mathbb{R}^{n}: \inf _{m \in \mathcal{M}}\|m \cdot \eta\|>0\right\}
$$

We define the following partition of $\mathcal{M}$ :

$$
\begin{equation*}
\mathcal{M}_{k}:=\left\{m \in \mathcal{M}: R^{k-1} \leq M_{m}<R^{k}\right\} \quad(k \geq 0) \tag{18}
\end{equation*}
$$

Note that $\mathcal{M}_{0}=\emptyset$. We have that $\mathcal{M}=\bigcup_{k=0}^{\infty} \mathcal{M}_{k}$.
We also need, for each $1 \leq i \leq n$, the following partitions of $\mathbb{N}$ :

$$
\begin{equation*}
\mathcal{Q}_{k}^{(i)}:=\left\{q \in \mathbb{N}: R^{(k-1) j_{i} / 2} \leq q<R^{k j_{i} / 2}\right\} \quad(k \geq 0) \tag{19}
\end{equation*}
$$

Note that $\mathcal{Q}_{0}^{(i)}=\emptyset$ and for each $1 \leq i \leq n$, we have that $\mathbb{N}=\bigcup_{k=0}^{\infty} \mathcal{Q}_{k}^{(i)}$.

At the heart of the construction of $K(R)$ is constructing a collection $\mathcal{F}_{k}$ of hyperrectangles $H_{k}$ inside the hypercube $[0,1]^{n}$ that satisfy the following $n$ conditions:
(0) $|m \cdot \eta+p| \geq \epsilon \quad \forall \eta \in H_{k}, \forall m \in \mathcal{M}_{k-1}, \forall p \in \mathbb{Z}$;
(1) $q\left|q \eta_{1}-p\right| \geq \epsilon \quad \forall \eta \in H_{k}, \forall q \in \mathcal{Q}_{k-1}^{(1)}, \forall p \in \mathbb{Z}$;
(n) $q\left|q \eta_{n}-p\right| \geq \epsilon \quad \forall \eta \in H_{k}, \forall q \in \mathcal{Q}_{k-1}^{(n)}, \forall p \in \mathbb{Z}$
for some $\epsilon>0$.
We start by constructing a collection $\left(\mathcal{G}_{k}^{(0)}\right)_{k \geq 0}$ of hyperrectangles satisfying condition (0). This construction is done by induction. Then we define a subcollection $\mathcal{G}_{k}^{(1)} \subset \mathcal{G}_{k}^{(0)}$ of hyperrectangles that also satisfy condition (1), a subcollection $\mathcal{G}_{k}^{(2)} \subset \mathcal{G}_{k}^{(1)}$ that also satisfies condition (2), etc. This
process ends with a subcollection $\mathcal{G}_{k}^{(n)}$ that satisfies the $n$ conditions above. We would like to quantify $\# \mathcal{G}_{k}^{(n)}$. We can give a lower bound, but we cannot quantify the exact cardinal. So we refine the collection $\mathcal{G}_{k}^{(n)}$ by choosing a right and final subcollection $\mathcal{F}_{k}$ that we can quantify.

Let

$$
j_{\min }=\min _{1 \leq i \leq n}\left(j_{i}\right), \quad j_{\max }=\max _{1 \leq i \leq n}\left(j_{i}\right) .
$$

Let $R>4^{1 / j_{\text {min }}}$ and $\epsilon>0$ be such that

$$
\begin{equation*}
\epsilon<\frac{1}{2 R^{2 j_{\max }}} . \tag{20}
\end{equation*}
$$

The parameter $R$ will be chosen later to be sufficiently large in order to satisfy various conditions.
4.1. The collection $\mathcal{G}_{k}^{(0)}$. For each $m \in \mathcal{M}$ and $p \in \mathbb{Z}$, let

$$
\Delta(m, p):=\left\{x \in \mathbb{R}^{n}:|m \cdot x+p|<\epsilon\right\} .
$$

Geometrically, $\Delta(m, p)$ is the thickening of a hyperplane of the form

$$
\begin{equation*}
\mathcal{L}(m, p):=\left\{x \in \mathbb{R}^{n}: m \cdot x+p=0\right\} \tag{21}
\end{equation*}
$$

with width $2 \epsilon / m_{i}$ in all the $x_{i}$-coordinate directions.

Next we describe the induction procedure in order to define the collection $\left(\mathcal{G}_{k}^{(0)}\right)_{k \geq 0}$. We work within the closed hypercube $H_{0}=[0,1]^{n}$ and set $\mathcal{G}_{0}^{(0)}=$ $\left\{H_{0}\right\}$. For $k \geq 0$, we divide each $H_{k} \in \mathcal{G}_{k}^{(0)}$ into new hyperrectangles $H_{k+1}$ of size

$$
R^{-(k+1) j_{1}} \times \ldots \times R^{-(k+1) j_{n}}
$$

Note that if $R^{j_{i}} \notin \mathbb{Z}$ for some $1 \leq i \leq n$, the division will not be exact, in the sense that the new hyperrectangles will not cover $H_{k}$. This division gives at least $\prod_{i=1}^{n}\left[R^{j_{i}}\right]>R-\sum_{i=1}^{n} R^{j_{i}}$ new hyperrectangles. Among these new hyperrectangles, we denote by $\mathcal{G}^{(0)}\left(H_{k}\right)$ the collection of hyperrectangles $H_{k+1} \subset H_{k}$ satisfying

$$
H_{k+1} \cap \Delta(m, p)=\emptyset \quad \forall m \in \mathcal{M}_{k}, \forall p \in \mathbb{Z}
$$

We define

$$
\mathcal{G}_{k+1}^{(0)}:=\bigcup_{H_{k} \in \mathcal{G}_{k}^{(0)}} \mathcal{G}^{(0)}\left(H_{k}\right)
$$

Hence $\mathcal{G}_{k+1}^{(0)}$ is nested in $\mathcal{G}_{k}^{(0)}$ and it is a collection of 'good' hyperrectangles with respect to all the best approximations $m$ satisfying $M_{m}<R^{k}$ and all the integers $p$. The collection $\mathcal{G}^{(0)}\left(H_{k}\right)$ is the collection of 'good' hyperrectangles that we obtain from the division of $H_{k}$.

Next we give a lower bound for $\# \mathcal{G}_{k}^{(0)}$. Actually, for a fixed hyperrectangle $H_{k} \in \mathcal{G}_{k}^{(0)}$, we give a lower bound for the number of hyperrectangles $H_{k+1} \in \mathcal{G}^{(0)}\left(H_{k}\right)$. Alternatively, we give an upper bound for the number of 'bad' hyperrectangles in $H_{k}$; these are the hyperrectangles $H_{k+1} \subset H_{k}$ that intersect the thickening $\Delta(m, p)$ of some hyperplane $\mathcal{L}(m, p)$ with $m \in \mathcal{M}_{k}$. Fact 1 and Fact 2 bound the number of thickenings $\Delta(m, p)$ with $m \in \mathcal{M}_{k}$ and $p \in \mathbb{Z}$ that intersect $H_{k}$. Fact 3 bounds the number of hyperrectangles $H_{k+1} \subset H_{k}$ that are intersected by a thickening $\Delta(m, p)$ with $m \in \mathcal{M}_{k}$ and $p \in \mathbb{Z}$.

Fact 1. We show that for each $k \geq 1$, the set $\mathcal{M}_{k}$ contains at most $2 \cdot 3^{n}\left(1+\log _{2}(R)\right)$ best approximations. Indeed, lemma 2.2 implies that

$$
\begin{aligned}
M_{\nu+2 \cdot 3^{n}\left(1+\log _{2}(R)\right)} & \geq 2^{1+\log _{2}(R)} M_{\nu} \\
& \stackrel{(18)}{\geq} 2^{1+\log _{2}(R)} R^{k-1} \\
& >R^{k} .
\end{aligned}
$$

Therefore, there are at most $2 \cdot 3^{n}\left(1+\log _{2}(R)\right)$ best approximations in $\mathcal{M}_{k}$.

Fact 2. Fix $m \in \mathcal{M}_{k}$. We show that there are at most $2^{n} n$ thickenings $\Delta(m, p)$ that intersect $H_{k}$. Indeed, suppose that two different thickenings $\Delta(m, p)$ and $\Delta\left(m, p^{\prime}\right)$ intersect the same edge of $H_{k}$. This edge of $H_{k}$ is a segment of a line which is parallel to an $x_{l}$-axis. Let $P=\left(y_{1}, \ldots, y_{n}\right)$ and $P^{\prime}=\left(y_{1}^{\prime}, \ldots, y_{n}^{\prime}\right)$ denote the points of intersection of this line parallel to the $x_{l}$-axis with $\mathcal{L}(m, p)$ and $\mathcal{L}\left(m, p^{\prime}\right)$ respectively. The fact that $P$ and $P^{\prime}$ respectively belong to $\mathcal{L}(m, p)$ and $\mathcal{L}\left(m, p^{\prime}\right)$ is described by the equations

$$
\begin{equation*}
m \cdot y+p=0, \quad m \cdot y^{\prime}+p^{\prime}=0 \tag{22}
\end{equation*}
$$

The fact that $P$ and $P^{\prime}$ both belong to a line parallel to the $x_{l}$-axis implies that $y_{i}=y_{i}^{\prime} \forall i \neq l$. Hence, by substracting the second equation in (22) to the first one, we have that

$$
\begin{equation*}
\left|y_{l}-y_{l}^{\prime}\right|-\frac{2 \epsilon}{\left|m_{l}\right|} \geq \frac{\left|p-p^{\prime}\right|}{\left|m_{l}\right|}-\frac{2 \epsilon}{\left|m_{l}\right|}>\frac{1}{R^{k j_{l}}}-\frac{1}{2 R^{k j_{l}}}=\frac{1}{2} R^{-k j_{l}} . \tag{23}
\end{equation*}
$$

Since the length size of $H_{k}$ in the $x_{l}$-direction is $R^{-k j_{l}}$, the inequality (23) implies that there are not more than two thickenings intersecting the same edge of $H_{k}$. Thus the number of thickenings $\Delta(m, p)$ that intersect $H_{k}$ is at most twice the number of edges of $H_{k}$, and this is $2^{n} n$.

Fact 3. Given a thickening $\Delta(m, p)$, we give an upper bound for the number of hyperrectangles $H_{k+1} \subset H_{k}$ that intersect $\Delta(m, p)$. Fix $m \in \mathcal{M}_{k}$ and $p \in \mathbb{Z}$. Denote by $l$ the index such that $M_{m}=\left|m_{l}\right|^{1 / j l}$. Consider the projection of $\Delta(m, p) \cap H_{k}$ onto one of the faces of $H_{k}$ parallel to the plane given by the $x_{l}$-axis and an $x_{i}$-axis. We split this projected of $\Delta(m, p) \cap H_{k}$ into right triangles with perpendicular sides of length $2 \epsilon /\left|m_{l}\right|$ and $2 \epsilon /\left|m_{i}\right|$ respectively. From this splitting and the inequality

$$
\frac{2 \epsilon}{\left|m_{l}\right|}<\frac{1}{2 R^{j_{l}(k+1)}}
$$

we deduce that $\Delta(m, p)$ intersects at most $2\left[R^{1-j_{\min }}\right]$ hyperrectangles $H_{k+1} \subset$ $H_{k}$.

Conclusion. There are at most $\left[2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{1-j_{\text {min }}}\right]$ hyperrectangles $H_{k+1} \subset H_{k}$ that intersect some $\Delta(m, p)$ with $m \in \mathcal{M}_{k}, p \in \mathbb{Z}$. Hence

$$
\# \mathcal{G}^{(0)}\left(H_{k}\right) \geq R-\sum_{i=1}^{n} R^{j_{i}}-\left[2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{1-j_{\min }}\right]
$$

4.2. The subcollections $\mathcal{G}_{k}^{(i)}$. For each $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, consider the sets

$$
\begin{equation*}
\Gamma_{i}(q, p):=\left\{x \in \mathbb{R}^{n}: q\left|q x_{i}-p\right|<\epsilon\right\} \quad(1 \leq i \leq n) \tag{24}
\end{equation*}
$$

Geometrically, each $\Gamma_{i}(q, p)$ is a thickening of a hyperplane described by the equation $x_{i}=p / q$ with width $2 \epsilon / q^{2}$ in the $x_{i}$-coordinate direction.

We construct a tower of subcollections

$$
\mathcal{G}_{k}^{(n)} \subset \mathcal{G}_{k}^{(n-1)} \subset \ldots \subset \mathcal{G}_{k}^{(1)} \subset \mathcal{G}_{k}^{(0)}
$$

where each $\mathcal{G}_{k}^{(i)}$ consists of hyperrectangles in $\mathcal{G}_{k}^{(i-1)}$ which points avoid each thickening $\Gamma_{i}(q, p)$ for $q \in \mathcal{Q}_{k-1}^{(i)}$. More precisely, for $1 \leq i \leq n$, we form $\mathcal{G}_{k}^{(i)}$ by letting

$$
\mathcal{G}^{(i)}\left(H_{k}\right):=\left\{H_{k+1} \in \mathcal{G}^{(i-1)}\left(H_{k}\right): H_{k+1} \cap \Gamma_{i}(q, p)=\emptyset \forall q \in \mathcal{Q}_{k}^{(i)}\right\}
$$

and

$$
\mathcal{G}_{k+1}^{(i)}:=\bigcup_{H_{k} \in \mathcal{G}_{k}^{(i-1)}} \mathcal{G}^{(i)}\left(H_{k}\right)
$$

Clearly the hyperrectangles in $\mathcal{G}_{k+1}^{(i)}$ satisfy the conditions $(0),(1), \ldots,(i)$, so the collection $\mathcal{G}_{k}^{(n)}$ satisfies the $n$ conditions ( 0 ), $\ldots,(n)$.

Next, for each $1 \leq i \leq n$ and $H_{k} \in \mathcal{G}_{k}^{(i-1)}$, we give a lower bound of $\# \mathcal{G}^{(i)}\left(H_{k}\right)$. Suppose that there are two pairs $(q, p)$ and $\left(q^{\prime}, p^{\prime}\right)$ in $\mathcal{Q}_{k}^{(i)} \times \mathbb{Z}$ such that

$$
H_{k} \cap \Gamma_{i}(q, p) \neq \emptyset, \quad H_{k} \cap \Gamma_{i}\left(q^{\prime}, p^{\prime}\right) \neq \emptyset .
$$

In other words, suppose there exist $\eta, \eta^{\prime}$ in $H_{k}$ such that

$$
\begin{equation*}
q\left|q \eta_{i}-p\right|<\epsilon, \quad q^{\prime}\left|q^{\prime} \eta_{i}^{\prime}-p^{\prime}\right|<\epsilon . \tag{25}
\end{equation*}
$$

Then, by (19) and (20), we have

$$
\begin{equation*}
\left|\frac{p}{q}-\frac{p^{\prime}}{q^{\prime}}\right|-\frac{\epsilon}{q^{2}}-\frac{\epsilon}{q^{\prime 2}} \geq \frac{1}{q q^{\prime}}-\frac{\epsilon}{q^{2}}-\frac{\epsilon}{q^{\prime 2}}>\frac{1}{R^{k j_{i}}}-\frac{1}{2 R^{k j_{i}}}=\frac{1}{2} R^{-k j_{i}} . \tag{26}
\end{equation*}
$$

Since the length sides of $H_{k}$ in the $x_{i}$-direction is $R^{-k j_{i}}$, the inequality (26) implies that at most two thickenings of the form (24) can intersect $H_{k}$.

Now, from (19) and (20), it follows that if $\eta \in \Gamma_{i}(q, p)$, then

$$
\left|\eta_{i}-\frac{p}{q}\right|<\frac{\epsilon}{q^{2}}<\frac{1}{2} R^{-k j_{i}}
$$

which implies that each thickening $\Gamma_{i}(q, p)$ intersects at most

$$
2\left[R^{j_{1}}\right] \times \ldots \times\left[\widehat{R^{j_{i}}}\right] \times \ldots \times\left[R^{j_{n}}\right] \leq 2\left[R^{1-j_{i}}\right]
$$

hyperrectangles $H_{k+1} \subset H_{k}$.

Therefore, there are at most $4\left[R^{1-j_{\text {min }}}\right]$ hyperrectangles $H_{k+1} \subset H_{k}$ that do not satisfy condition (i). Hence
(27) $\# \mathcal{G}^{(i)}\left(H_{k}\right) \geq R-\sum_{i=1}^{n} R^{j_{i}}-\left[2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{1-j_{\text {min }}}\right]-4 i\left[R^{1-j_{\text {min }}}\right]$.
4.3. The right subcollection $\mathcal{F}_{k}$. We choose a subcollection of $\mathcal{G}_{k}^{(n)}$ that we can exactly quantify in the following way. Let $\mathcal{F}_{0}:=\mathcal{G}_{0}^{(0)}$. Choose $R$ sufficiently large so that $\left[R-\sum_{i=1}^{n} R^{j_{i}}-2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) \cdot R^{1-j_{\text {min }}}-\right.$ $\left.4 n R^{1-j_{\min }}\right]>1$. For $k \geq 0$, for each $H_{k} \in \mathcal{F}_{k}$, we choose exactly [ $R-$ $\sum_{i=1}^{n} R^{j_{i}}-2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{1-j_{\text {min }}}-4 n R^{\left.1-j_{\text {min }}\right]}$ hyperrectangles from the collection $\mathcal{G}^{(n)}\left(H_{k}\right)$ and denote this collection by $\mathcal{F}\left(H_{k}\right)$. Trivially,

$$
\begin{equation*}
\# \mathcal{F}\left(H_{k}\right)=\left[R-\sum_{i=1}^{n} R^{j_{i}}-2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{1-j_{\min }}-4 n R^{1-j_{\min }}\right]>1 \tag{28}
\end{equation*}
$$

so each hyperrectangle $H_{k} \in \mathcal{F}_{k}$ gives rise to exactly the same number of hyperrectangles $H_{k+1}$ in $\mathcal{F}\left(H_{k}\right)$. Finally, define

$$
\mathcal{F}_{k+1}:=\bigcup_{H_{k} \in \mathcal{F}_{k}} \mathcal{F}\left(H_{k}\right) .
$$

This completes the construction of the Cantor-type set

$$
K(R):=\bigcap_{k=0}^{\infty} \mathcal{F}_{k} .
$$

By construction, we have $K(R) \subset \mathcal{V} \cap \operatorname{Bad}(1,0 \ldots, 0) \cap \ldots \cap \operatorname{Bad}(0, \ldots, 0,1)$. Moreover, in view of (28), we have

$$
\begin{align*}
\# \mathcal{F}_{k+1} & =\# \mathcal{F}_{k} \# \mathcal{F}\left(H_{k}\right)  \tag{29}\\
& =\left[R-\sum_{i=1}^{n} R^{j_{i}}-2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{1-j_{\min }}-4 n R^{1-j_{\min }}\right]^{k+1} . \tag{30}
\end{align*}
$$

## 5. The measure $\mu$ on $K(R)$

We now describe a probablity measure $\mu$ supported on the Cantor-type set $K(R)$ constructed in the previous section. The measure we define is analogous to the probability measure used in [21] and [2] on a Cantor-type set of $\mathbb{R}^{2}$. For any hyperrectangle $H_{k} \in \mathcal{F}_{k}$ we attach a weight $\mu\left(H_{k}\right)$ which is defined recursively as follows: for $k=0$,

$$
\mu\left(H_{0}\right)=\frac{1}{\# \mathcal{F}_{0}}=1
$$

and for $k \geq 1$,

$$
\mu\left(H_{k}\right)=\frac{1}{\# \mathcal{F}\left(H_{k-1}\right)} \mu\left(H_{k-1}\right) \quad\left(H_{k} \in \mathcal{F}\left(H_{k-1}\right)\right)
$$

This procedure defines inductively a mass on any hyperrectangle used in the construction of $K(R)$. Moreover, $\mu$ can be further extended to all Borel subsets $X$ of $\mathbb{R}^{n}$, so that $\mu$ actually defines a measure supported on $K(R)$, by letting

$$
\mu(X)=\inf \sum_{H \in \mathcal{C}} \mu(H)
$$

where the infimum is taken over all coverings $\mathcal{C}$ of $X$ by rectangles $H \in$ $\left\{\mathcal{F}_{k}: k \geq 0\right\}$. For further details, see [9], Proposition 1.7.

Notice that, in view of (29), we have

$$
\mu\left(H_{k}\right)=\frac{1}{\# \mathcal{F}_{k}} \quad(k \geq 0)
$$

A classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle (see [9] p. 55).

Lemma 5.1 (mass distribution principle). Let $\delta$ be a probability measure supported on a subset $X$ of $\mathbb{R}^{n}$. Suppose there are positive constants $c, s$ and $l_{0}$ such that

$$
\begin{equation*}
\delta(S) \leq c l^{s} \tag{31}
\end{equation*}
$$

for any hypercube $S \subset \mathbb{R}^{n}$ with side length $l \leq l_{0}$. Then $\operatorname{dim}(X) \geq s$.
The goal in the next section is to prove that there exist constants $c$ and $l_{0}$ satisfying (31) with $\delta=\mu, X=K(R)$ and $s=n-\lambda(R)$, where $\lambda(R) \rightarrow 0$ as $R \rightarrow \infty$. Then from the mass distribution principle it will follow that $\operatorname{dim}(K(R))=n$.

## 6. A LOWER BOUND FOR $\operatorname{dim}(K(R))$

Recall that

$$
j_{\min }=\min _{1 \leq i \leq n}\left(j_{i}\right)
$$

Let $k_{0}$ be a positive integer such that

$$
\begin{equation*}
R^{-k j_{i}}<R^{-(k+1) j_{\min }} \quad \forall j_{i} \neq j_{\min } \text { and } k \geq k_{0} \tag{32}
\end{equation*}
$$

Consider an arbitrary hypercube $S$ of side length $l \leq l_{0}$ where $l_{0}$ satisfies

$$
\begin{equation*}
l_{0}<R^{-\left(k_{0}+1\right) j_{\min }} \tag{33}
\end{equation*}
$$

together with a second inequality to be determined later. We can choose $k>k_{0}$ so that

$$
\begin{equation*}
R^{-(k+1) j_{\min }}<l<R^{-k j_{\min }} \tag{34}
\end{equation*}
$$

From the inequality (32) it follows that

$$
\begin{equation*}
l>R^{-k j_{i}} \quad \forall j_{i} \neq j_{\min } \tag{35}
\end{equation*}
$$

Then it is easy to see that $S$ intersects at most $2^{n} l^{n-1} \prod_{j_{i} \neq j_{\text {min }}} R^{k j_{i}}$ hyperrectangles $H_{k} \in \mathcal{F}_{k}$, so

$$
\mu(S) \leq 2^{n} l^{n-1} \prod_{j_{i} \neq j_{\min }} R^{k j_{i}} \mu\left(H_{k}\right)=2^{n} l^{n-1} R^{k-k j_{\min }} \frac{1}{\# \mathcal{F}_{k}}
$$

Since $R^{(k+1) j_{\text {min }}}>l^{-1}$ (see (34)), we have that

$$
\mu(S) \leq 2^{n} l^{n} R^{j_{\min }} R^{k} \frac{1}{\# \mathcal{F}_{k}}
$$

Remember that we mentioned in Section 3 that later we would choose the parameter $R$ big enough so that it satisfies various conditions. We choose $R$ so that

$$
R^{-1} \sum_{i=1}^{n} R^{j_{i}}-2^{n+2} 3^{n} n\left(1+\log _{2}(R)\right) R^{-j_{\min }}-4 n R^{-j_{\min }}-R^{-1} \leq 2^{-1}
$$

Then, by (29) we have that

$$
\mu(S) \leq 2^{n} l^{n} R^{j_{\min }} 2^{k}
$$

We choose

$$
k \geq \log (R) \quad \text { and } \quad \lambda(R)=\frac{1+\log (2)}{j_{\min } \log (R)},
$$

so

$$
\mu(S) \leq 2^{n} l^{n} R^{k j_{\min } \lambda(R)}
$$

Since $R^{k j_{\text {min }}}<l^{-1}$ (see (34)), it follows that

$$
\mu(S) \leq 2^{n} l^{n-\lambda(R)}
$$

Finally, by applying the mass distribution principle we obtain

$$
\operatorname{dim} K(R) \geq n-\lambda(R) \rightarrow n \quad \text { as } R \rightarrow \infty
$$

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[^1]:    ${ }^{1}$ We recall that winning sets in $\mathbb{R}^{n}$ have maximal Hausdorff dimension, and that countable intersections of winning sets are again winning. We refer the reader to [22] for all necessary definitions and results on winning sets.

