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# GUPTA-BLEULER QUANTIZATION OF THE MAXWELL FIELD IN GLOBALLY HYPERBOLIC SPACE-TIMES

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ABSTRACT. We give a complete framework for the Gupta-Bleuler quantization of the free electromagnetic field on globally hyperbolic space-times. We describe one-particle structures that give rise to states satisfying the microlocal spectrum condition. The field algebras in the so-called Gupta-Bleuler representations satisfy the time-slice axiom, and the corresponding vacuum states satisfy the microlocal spectrum condition. We also give an explicit construction of ground states on ultrastatic space-times. Unlike previous constructions, our method does not require a spectral gap or the absence of zero modes. The only requirement, the absence of zero-resonance states, is shown to be stable under compact perturbations of topology and metric. Usual deformation arguments based on the time-slice axiom then lead to a construction of Gupta-Bleuler representations on a large class of globally hyperbolic space-times. As usual, the field algebra is represented on an indefinite inner product space, in which the physical states form a positive semi-definite subspace. Gauge transformations are incorporated in such a way that the field can be coupled perturbatively to a Dirac field. Our approach does not require any topological restrictions on the underlying space-time.

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## 1. INTRODUCTION

The classical approach to perturbative quantum electrodynamics begins with the canonical quantization scheme for the Dirac field and the electromagnetic field. While the free electromagnetic field may well be described by the quantized field algebra of the electromagnetic field strength, minimal coupling to a Dirac field requires the quantization of the vector potential. In a coupled theory, the gauge group will act on both the Dirac field and the vector potential. Therefore, the gauge group cannot be factored out before the coupling is introduced. It was realized by Gupta and Bleuler [23, 8] that in Minkowski space-time, the field algebra of the vector potential is most conveniently represented in a Poincaré covariant manner on an indefinite inner product space in which the physical states form a positive semi-definite subspace. Perturbative quantum electrodynamics can then be carried out consistently on the level of formal power series.

Conceptually, it is best to split the construction of the field operators in canonical quantization into two steps. Step one is to construct the so called field algebra, i.e. a  $*$ -algebra that satisfies the canonical commutation relations. Step two consists in finding representations of this algebra that are physically reasonable. In Minkowski space-time, there is usually a preferred Poincaré invariant ground state and therefore a physically preferred representation. In this situation, the construction is canonical and is often carried out in one step by employing a procedure which in physics is called frequency splitting.

In quantum field theory on curved space-time, one considers quantized fields on a classical curved space-time. In a space-time  $(M^n, g)$ , the classical Maxwell equations can be formulated with differential forms by

$$dF = 0, \quad \delta F = J,$$

where  $F \in \Omega^2(M)$  is the field strength, and  $J$  is the electromagnetic current. In the so-called potential method for finding cohomologically trivial solutions of these equations, one sets  $F = dA$  with a one-form  $A$ , the so-called *electromagnetic potential*. Then the equation  $dF = 0$  is automatically satisfied, so that Maxwell's equations reduce to

$$\delta dA = J. \tag{1.1}$$

The potential  $A$  is not uniquely determined. Namely, transforming  $A$  according to

$$A(x) \mapsto A(x) - d\Lambda(x) \tag{1.2}$$

with  $\Lambda \in \Omega^0(M)$  maps the solution space of (1.1) to itself and leaves  $F$  unchanged. The transformations (1.2) are the *classical gauge transformations* of electrodynamics.

The physical requirement of *gauge invariance* states that all observable quantities should be invariant under gauge transformations. In particular, the electromagnetic potential  $A$  is not gauge invariant. The field strength  $F = dA$  is, making the electromagnetic field an observable quantity. Another way of forming gauge invariant quantities is to integrate the electromagnetic potential along a closed curve, or more generally a cycle,

$$\int_{\gamma} A \quad \text{for a cycle } \gamma.$$

By Stokes' theorem, knowing  $\int_{\gamma} A$  for all homologically trivial cycles is equivalent to knowing the field strength  $F = dA$ . However, as the Aharonov-Bohm experiment

shows (see for example [36]), also homologically non-trivial cycles correspond to measurable quantities. Thus not the field strength alone, but the integrals of the electromagnetic potential along all cycles, should be regarded as the fundamental physical objects of electrodynamics.

This physical significance of the electromagnetic potential becomes clearer in quantum mechanics, where  $A$  is needed to describe the coupling of the electromagnetic field to the quantum mechanical particle. For example, in the Dirac equation the coupling is described by the term  $\gamma^j A_j(x)\psi(x)$ , which can be generated by the so-called *minimal coupling* procedure where one replaces the partial derivatives in the Dirac equation according to  $\partial_j \rightarrow \partial_j - iA_j(x)$ . Thus it is impossible to work with the field strength alone; one must consider the potential  $A(x)$  as being the basic object describing the electromagnetic field. In such a coupled situation, the gauge transformations (1.2) extend to transformations on the whole system, which typically describe local phase transformations of the wave functions

$$\psi(x) \mapsto e^{-i\Lambda(x)} \psi(x).$$

In geometric terms, minimal coupling is best understood as follows. The solutions of the Dirac equation are sections of a Dirac bundle that is twisted by a line bundle. The classical electromagnetic field vector potential should be regarded as a connection on this line bundle, and the Dirac equation is formed using the connection on the bundle. Once a local trivialization of the line bundle and the Dirac bundle is fixed, the connection determines a one-form, the vector potential. Gauge transformations then correspond to different choices of local trivializations.

As a consequence of the classical gauge freedom (1.2), the Cauchy problem for Maxwell's equations (1.1) is ill-posed. In order to circumvent this problem, one typically chooses a specific gauge. A common choice is the *Lorenz gauge*

$$\delta A = 0. \tag{1.3}$$

Then the Maxwell equations go over to the wave equation

$$\square A = 0.$$

When performing a gauge transformation (1.2), the gauge condition (1.3) becomes

$$\delta A = \square \Lambda, \tag{1.4}$$

and the field equations transform to

$$\square A = \square d\Lambda.$$

The goal of this paper is to quantize the electromagnetic field in a curved space-time in such a way that this field can readily be coupled to a quantized Dirac field. We restrict attention to the first step where the electromagnetic field is quantized. In a second step, the coupling to quantum particles could be described perturbatively. With this in mind, we only consider the free dynamics for the vector potential without source term. However, as the coupling to other particles and fields requires the electromagnetic potential, we want to construct field operators  $\hat{A}$  for the electromagnetic potential. We do not impose any cohomological restrictions on our space-time. When coupling to the Dirac field, we assume that the quantization starts in a fixed topological sector. This means that we assume that we have chosen a fixed line bundle and a fixed connection with respect to which we perturb. Thus the electromagnetic potential to be quantized will consist of globally defined one-forms. Passing from one

background connection to another can be implemented by an algebra automorphism that may however not be unitarily implementable in a given standard Fock representation (see e.g. [1] and [40]). Different background connections may therefore be thought of as choices of different classes of representations or superselection sectors.

The paper is organized as follows. After a brief mathematical introduction (Section 2), we define the field algebras, introduce gauge transformations and prove the time slice axiom (Section 3). In Section 4, we consider representations of the field algebras and explain the properties which we demand from a physically reasonable representation. More precisely, we define so-called *Gupta-Bleuler representations* as representations on an indefinite inner product space which satisfy a microlocal spectrum condition. Moreover, we demand that applying the observables to the vacuum should generate the positive semi-definite subspace of physical states. Furthermore, the gauge condition  $\delta\hat{A} = \square\Lambda$  should be satisfied for the expectation values of the physical states. In Section 5, we construct Gupta-Bleuler representations for ultrastatic manifolds. Here our main point is to treat the zero resonance states (Section 5.1) and the zero modes (Section 5.2). Using a glueing construction, these representations are then extended to general globally hyperbolic space-times (Section 6). All our constructions are gauge covariant, where we extend the classical gauge transformation law (1.2) to the field operators  $\hat{A}$  by

$$\hat{A}(x) \mapsto \hat{A}(x) - d\Lambda(x), \quad (1.5)$$

and  $\Lambda$  is again a real-valued function. This corresponds to the usual procedure in canonical quantization schemes (see for example [46, Section 8]) in which the gauge freedom described by so-called scalar photons is not quantized. In particular, the gauge transformations (1.5) leave the commutator relations of the field operators unchanged. An alternative procedure described in the literature is to fix the gauge with a *gauge parameter*. This leads to modifications of the commutator relations for non-observable quantities. In Appendix A we show that working with different gauge parameters gives an equivalent description of the physical system. Since the construction of one-particle states for massless spin one fields is interesting in its own right and independent of the Gupta-Bleuler framework, we show in Appendix B that it can also be used to construct states in the BRST framework as formulated in [18] and in curved space-times in [25]. In Appendix C it is shown that the presence of zero modes is intimately related to cohomology and may be presented independent of foliations.

**1.1. Discussion and Relation to Previous Work.** In curved space-time, it was first shown by Dimock [16] that the algebra of the free scalar field on a globally hyperbolic space-time can be constructed in a functorial manner. Thus the first step, the construction of the field algebra, can be carried out just as in Minkowski space-time. Dimock later used this procedure to quantize the electromagnetic field strength [17]. The canonical quantization of the electromagnetic vector potential in a curved background in the Gupta-Bleuler framework was first described by Furlani [21], who assumes the space-time to be ultrastatic with compact Cauchy surfaces. We note here, however, that in the presence of zero modes, the construction given in [21] contains gaps (in particular, Theorem III.1 does not hold if  $H^1(M) \neq \{0\}$ , essentially because when projecting out the zero modes, the locality of the commutation relations is lost). Another series of papers [19, 13, 14] deals with the uncoupled electromagnetic field in curved space-times. More specifically, in [19] the field algebra for the field strength is constructed under certain cohomological conditions, and some representations are

given. In [13] the field algebra for the field strength is constructed without cohomological conditions. In the paper [14] the field algebra smeared out with co-closed test functions is constructed; under the stated cohomological assumptions this algebra is equivalent to the algebra for the field strength. In the construction of physical representations, in previous works the term ‘‘Gupta-Bleuler quantization’’ is ambiguous. It is often referred to as a method to construct a state over the *algebra of observables* that satisfies the microlocal spectrum condition. In the present paper, we take the view that such a quantization procedure should result in a physical representation of the *whole field algebra* that has all the properties needed for coupling it to an electron field and for constructing quantum electrodynamics, at least on the level of formal power series in a localized coupling constant. The procedure we propose is as close as possible to the standard textbook Gupta-Bleuler construction in Minkowski space-time whilst at the same time keeping the language suitable for general globally hyperbolic space-times and independent of preferred foliations. Our method of construction of Gupta-Bleuler states in ultrastatic space-times does not require the presence of a spectral gap or the absence of zero modes. To our knowledge, the construction in this generality is new even on the level of one-particle structures. We therefore show in Appendix B that it also applies to the BRST framework as described in [24, 4, 25]. Finally, the paper [39] deals with the construction of the abstract field algebras and observable algebras without topological restrictions and in the presence of classical external sources, but physical representations of these algebras are not considered. Finally, a discussion of the quantization of the full bundle of connections modulo gauge transformations can be found in [6, 5].

## 2. MATHEMATICAL PRELIMINARIES

Let  $(M, g)$  be a globally hyperbolic Lorentzian space-time of dimension  $n \geq 2$ , i.e.  $M$  is an oriented, time-oriented Lorentzian manifold that admits a smooth global Cauchy surface  $\Sigma$  (see [7]). We assume that the metric has signature  $(+1, -1, \dots, -1)$ . Let  $\Omega^p(M) \subset C^\infty(M; \Lambda^p T^*M)$  be the space of smooth real-valued  $p$ -forms and  $\Omega_0^p(M) \subset \Omega^p(M)$  be the forms with compact support. As usual denote by  $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$  the exterior derivative and by  $\delta : \Omega^{p+1}(M) \rightarrow \Omega^p(M)$  its formal adjoint with respect to the inner product on the space of  $p$ -forms

$$\langle f, g \rangle = \int_M f \wedge *g,$$

where  $*$  is the Hodge star operator. Note that this inner product is indefinite if  $0 < p < n$ .

The wave operator  $\square_p : \Omega^p(M) \rightarrow \Omega^p(M)$  is defined by  $\square_p = d\delta + \delta d$ . It is formally self-adjoint with respect to the above inner product. The wave equation  $\square_p A = 0$  for  $p$ -forms  $A \in \Omega^p(M)$  is a normally hyperbolic differential equation. It is well-known that the Cauchy problem for this equation can be solved uniquely, and moreover there exist unique advanced and retarded fundamental solutions  $G_\pm^p : \Omega_0^p(M) \rightarrow \Omega^p(M)$  such that

- (1)  $G_\pm^p$  is continuous with respect to the usual locally convex topologies on  $\Omega_0^p(M)$  and  $\Omega^p(M)$ , respectively.
- (2)  $\square_p G_\pm^p f = G_\pm^p \square_p f = f$  for all  $f \in \Omega_0^p(M)$ ,
- (3)  $\text{supp } G_\pm^p f \subset J^\pm(\text{supp } f)$ , where  $J^\pm(\text{supp } f)$  denotes the causal future respectively past of  $\text{supp } f$ .

(we refer the reader to the monograph [3] for a detailed general proof in the context of operators on vector bundles). The map  $G^p$  is then defined to be the difference of retarded and advanced fundamental solutions  $G^p := G_+^p - G_-^p$ . Note that  $G^p$  maps onto the space of smooth solutions of the equation  $\square_p A = 0$  with spatially compact support, i.e. solutions whose support have compact intersection with  $\Sigma$ . The function  $G^p(f)$  can be viewed as a distribution and may be paired with a test function  $g \in \Omega_0^p(M)$ , using the inner product  $\langle \cdot, \cdot \rangle$ . We will denote  $G^p(f)(g) = \langle G^p(f), g \rangle$  by  $G^p(f, g)$ . The bilinear form  $G^p(\cdot, \cdot)$  defines a distribution on  $M \times M$  with values in  $\Lambda^p T^*M \boxtimes \Lambda^p T^*M$ . It is straightforward to verify that  $G^p(f, g) = -G^p(g, f)$ .

Throughout the paper, we regard the space of  $p$ -forms as a subset of the set of distributional  $p$ -forms by using the inner product. That is if  $A \in \Omega^p(M)$  we may pair  $A$  with a test function  $f \in \Omega_0^p(M)$

$$A(f) := \int_M A \wedge *f.$$

For example if  $A$  is a one-form which in local coordinates is given by  $A = \sum_{i=1}^n A_i(x) dx^i$ , then the corresponding integral in local coordinates is

$$A(f) = \int_M \left( \sum_{i,k=1}^n g^{ik}(x) A_i(x) f_k(x) \right) \sqrt{|g|} dx,$$

where  $f = \sum_{i=1}^n f_i(x) dx^i$ . In physics this is often referred to as the field "smeared out" with a test function  $f$ . Of course the equation  $dA = 0$  is then equivalent to  $A(\delta f) = 0$  for all test functions  $f \in \Omega_0^{p+1}(M)$ . Similarly, the wave equation  $\square_p A = 0$  is equivalent to  $A(\square_p f) = 0$  for all  $f \in \Omega_0^p(M)$ . When dealing with quantum fields, we shall always take this "dual" point of view. Note that any cycle  $\gamma$  can be thought of as a co-closed distributional current. This means that knowing  $\int_\gamma A$  for all cycles is equivalent to knowing  $A(f)$  for all  $f \in \Omega_0^1(M)$  with  $\delta f = 0$ .

### 3. THE FIELD ALGEBRAS AND THE GAUGE IDEALS

In this section, we construct the field algebra of the quantized Maxwell field. Since we will be dealing mostly with one-forms, we shall often omit the subscript  $p$  in the case  $p = 1$  and simply write  $G_\pm$  for the advanced and retarded fundamental solutions and set  $G = G_+ - G_-$ . The *field algebra*  $\mathcal{F}$  is defined to be the unital  $*$ -algebra generated by symbols  $A(f)$  for  $f \in \Omega_0^1(M)$  together with the relations

$$f \mapsto A(f) \text{ is linear,} \tag{3.1}$$

$$A(f)A(g) - A(g)A(f) = -iG(f, g), \tag{3.2}$$

$$A(\square f) = 0, \quad \text{for all } f \in \Omega_0^1(M) \text{ with } \delta f = 0, \tag{3.3}$$

$$(A(f))^* = A(f). \tag{3.4}$$

For every open subset  $\mathcal{O} \subset M$ , we define the *local field algebra*  $\mathcal{F}(\mathcal{O}) \subset \mathcal{F}$  to be the sub-algebra generated by the  $A(f)$  with  $\text{supp}(f) \subset \mathcal{O}$ . Inside  $\mathcal{F}$ , the *algebra of observables*  $\mathcal{A}$  is defined as the unital subalgebra generated by  $A(f)$  with  $\delta f = 0$ . The local algebras of observables  $\mathcal{A}(\mathcal{O})$  are given by  $\mathcal{A}(\mathcal{O}) = \mathcal{A} \cap \mathcal{F}(\mathcal{O})$ .

**Remark 3.1. (Choice of field algebra)** We point out that the relation (3.3) is weaker than the more common requirement  $A(\delta f) = 0$  for all  $f \in \Omega_0^1(M)$ . In particular, we do not impose that the field operators satisfy the homogeneous Maxwell

equations. This is a well-known feature of the Gupta-Bleuler formalism. A general no-go theorem by Strocchi (see [41, 42, 43]) states that under the assumption of locality of the field, there is no non-trivial Lorentz covariant quantization of the electromagnetic potential on four dimensional Minkowski space in which the homogeneous Maxwell equations are satisfied as an operator identity. As remarked in [42, page 2199], the Gupta-Bleuler formalism circumvents this by relaxing the homogeneous Maxwell equation to a weak condition on the physical subspace. Our choice of field algebra together with the gauge ideal below (with  $\Lambda \equiv 0$  to obtain the Lorenz gauge) reflects the construction in the original article [8] as well as the presentation in the standard textbooks on quantum electrodynamics.  $\diamond$

The physical interpretation of the algebra  $\mathcal{A}(\mathcal{O})$  is that it consists of all the physical quantities that can be measured in the space-time region  $\mathcal{O}$ . In particular, if  $g \in \Omega_0^2(\mathcal{O})$ , then  $A(\delta g)$  is an observable. Since  $A(\delta g) = dA(g)$ , this observable corresponds to the field strength operator smeared out with the test function  $g$ . However, as explained at the end of the previous section, the algebra of observables may also contain observables that correspond to smeared out measurements of  $A$  along homologically non-trivial cycles. Thus it may be strictly larger than the algebra generated by  $dA(g)$ .

**Definition 3.2.** Let  $\Lambda \in C^\infty(M)$ . The **gauge ideal**  $\mathcal{I}_\Lambda \subset \mathcal{F}$  is the two-sided ideal generated by

$$\{A(\square f) - \Lambda(\delta \square f) \mid f \in \Omega_0^1(M)\}.$$

**Lemma 3.3.**  $\mathcal{A} \cap \mathcal{I}_\Lambda = \{0\}$ .

*Proof.* We first observe that

$$K := \square \ker \delta|_{\Omega_0^1(M)} = \square(\Omega_0^1(M)) \cap \ker \delta|_{\Omega_0^1(M)}. \quad (3.5)$$

Indeed, if  $f$  lies in  $\square \ker \delta|_{\Omega_0^1(M)}$ , then it is obviously both in the image of  $\square$  and co-closed. Conversely, if  $f$  lies in the intersection on the right, then  $f = \square g$  and  $0 = \delta f$  and thus  $\square \delta g = 0$ . Since  $\delta g$  has compact support and solves the wave equation, it follows that  $\delta g = 0$ , proving (3.5). We introduce the vector spaces

$$W = \Omega_0^1(M)/K \quad \text{and} \quad U = \square(\Omega_0^1(M))/K, \quad V = \ker \delta|_{\Omega_0^1(M)}/K.$$

Then

$$U \cap V = \{0\}. \quad (3.6)$$

We introduce  $W_U^*$  and  $W_V^*$  as the subspaces of the dual space  $W^*$  consisting of elements that vanish on  $U$  and  $V$ , respectively.

The algebra  $\mathcal{F}$  has a natural filtration  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$ , where  $\mathcal{F}_n$  is the span of products of the form  $A(f_1) \cdots A(f_m)$  with  $m \leq n$ . Moreover, the linear map  $\sigma_n : \mathcal{F}_n \rightarrow \otimes_s^n W$  with kernel  $\mathcal{F}_{n-1}$  mapping  $A(f_1) \cdots A(f_n)$  to  $f_1 \otimes_s \cdots \otimes_s f_n$  is well-defined, where  $\otimes_s$  denotes the symmetric tensor product.

Let  $\mathfrak{f} \in \mathcal{A} \cap \mathcal{I}_\Lambda$ . We want to show that  $\mathfrak{f}$  vanishes. Thus assume by contradiction that  $\mathfrak{f} \neq 0$ . Then there is a minimal  $n$  such that  $\mathfrak{f} \in \mathcal{F}_n$  and  $F := \sigma_n(\mathfrak{f}) \neq 0$ . Since  $\mathfrak{f} \in \mathcal{A}$  is an observable, we know that  $F \in \otimes_s^n V$ , and thus

$$\iota_\lambda F = 0 \quad \text{for all } \lambda \in W_V^*, \quad (3.7)$$

where  $\iota_\lambda$  denotes the contraction with  $\lambda$ . On the other hand, as  $\mathfrak{f} \in \mathcal{I}_\Lambda$ , we know that

$$(\lambda_1 \otimes_s \cdots \otimes_s \lambda_n)F = 0 \quad \text{for all } \lambda_i \in W_U^*. \quad (3.8)$$

Combining (3.7) and (3.8), we obtain by linearity that

$$(\lambda_1 \otimes_s \cdots \otimes_s \lambda_n)F = 0 \quad \text{for all } \lambda_i \in W_U^* + W_V^*. \quad (3.9)$$

In view of (3.6), the set  $W_U^* + W_V^*$  is a point-separating subspace of  $W^*$ . Hence (3.9) implies that  $F = 0$ , a contradiction.  $\square$

This lemma gives a canonical injective map  $\mathcal{A} \rightarrow \mathcal{F}/\mathcal{I}_\Lambda$ .

**Remark 3.4. (gauge transformations)** The analog of classical gauge transformations can be realized by the algebra homomorphism

$$\mathfrak{G}_\Lambda : A(f) \mapsto A(f) - (d\Lambda)(f). \quad (3.10)$$

This algebra homomorphism leaves the algebra of observables  $\mathcal{A}$  invariant. Moreover, it transforms the gauge ideals by

$$\mathfrak{G}_{\Lambda'}\mathcal{I}_\Lambda = \mathcal{I}_{\Lambda+\Lambda'}. \quad (3.11)$$

Thus the gauge freedom is described in the algebraic formulation by the freedom in choosing a gauge ideal.  $\diamond$

In classical gauge theories, the time evolution is uniquely defined only after a gauge-fixing procedure. In the same way, in our framework the time slice axiom holds only after dividing out the gauge ideal:

**Proposition 3.5. (time slice axiom)** *Let  $U$  be an open neighborhood of a Cauchy surface in  $M$  and  $\Lambda \in C^\infty(M)$ . Then*

$$\mathcal{F}(U)/\mathcal{I}_\Lambda = \mathcal{F}/\mathcal{I}_\Lambda \quad \text{and} \quad \mathcal{A}(U) = \mathcal{A}.$$

*Proof.* Since the above gauge transformations leave  $\mathcal{F}(U)$  invariant, we can arrange in view of (3.10) and (3.11) that  $\Lambda = 0$ . Let  $f \in \Omega_0^1(M)$ . By Lemma 3.6 below there exist forms  $h \in \Omega_0^1(M)$  and  $g \in \Omega_0^1(U)$  with  $\square h = f - g$ . We conclude that  $A(f - g) = A(\square h) \in \mathcal{I}_\Lambda$ .  $\square$

**Lemma 3.6.** *Suppose that  $U$  is an open neighborhood of a Cauchy surface  $\Sigma$  in  $M$ . Then for every  $f \in \Omega_0^\bullet(M)$  there exists  $h \in \Omega_0^\bullet(M)$  such that the form*

$$f - \square h$$

*is compactly supported in  $U$ . If  $df = 0$ , then  $h$  can be chosen to be closed. If  $\delta f = 0$ , then  $h$  can be chosen to be co-closed.*

*Proof.* Let  $\eta_+ \in C^\infty(M)$  and  $\eta_-$  be non-negative smooth functions such that

- $\eta_+(x)^2 + \eta_-(x)^2 = 1$  for all  $x \in M$ .
- $\eta_+$  has future compact support and  $\eta_-$  has past compact support.
- $\text{supp}(d\eta_\pm) \subset U$ .

Now one checks by direct computation that  $h := \eta_+^2 G_-(f) + \eta_-^2 G_+(f)$  has the required properties. If  $f$  is closed, we may take

$$h := d(\eta_+ G_-(\eta_+ G_-(\delta f)) + \eta_- G_+(\eta_- G_+(\delta f))).$$

Again one checks that  $f - \square h$  has compact support in  $U$ . Moreover, by construction,  $h$  is closed. A straightforward modification of this argument shows that  $h$  can be chosen to be co-closed if  $f$  is co-closed.  $\square$

4. REPRESENTATIONS OF  $\mathcal{F}$ 

In Minkowski space, there is a unique vacuum state determined by Poincaré invariance and the spectrum condition. In general curved space-times, the lack of such a distinguished vacuum state has led to alternative selection criteria for physical states. The spectrum condition is then replaced by microlocal versions [37]. Before introducing representations, we therefore recall some basic notions of microlocal analysis.

**4.1. Polarization Sets and Wavefront Sets of Bundle-Valued Distributions.**

We denote by  $\psi\text{DO}^m(M, E)$  the set of properly supported pseudo-differential operators acting on sections of a vector bundle  $E \rightarrow M$ . More precisely, we work with polyhomogeneous symbols, i.e. symbols in the Hörmander classes  $S_{\text{phg}}^m$  defined in [28, Chapter 18]. The principal symbol  $\sigma_A$  of a pseudo-differential operator  $A \in \psi\text{DO}^m(M, E)$  is then a positive homogeneous section of degree  $m$  in  $C^\infty(\dot{T}^*M, \pi^* \text{End}(E))$  (where  $\dot{T}^*M$  denotes the cotangent space with its zero section removed, and  $\pi : \dot{T}^*M \rightarrow M$  is the canonical projection). Following [15], we define:

**Definition 4.1.** *Let  $u \in \mathcal{D}'(M; E)$  be a distribution with values in  $E$ . Then the polarization set  $\text{WF}_{\text{pol}}(u)$  is defined by*

$$\text{WF}_{\text{pol}}(u) = \bigcap_{\substack{A \in \psi\text{DO}^0(M; E), \\ Au \in C^\infty(M; E)}} \mathcal{N}_A,$$

where

$$\mathcal{N}_A := \left\{ (x, \xi; v) \in \dot{T}^*M \times E \mid v \in E_x \text{ and } \sigma_A(x, \xi)v = 0 \right\}.$$

Moreover, the wave front set can be defined by

$$\text{WF}(u) = \pi \left( \text{WF}_{\text{pol}}(u) \setminus \dot{T}^*M \times \{0\} \right),$$

where  $\pi : \dot{T}^*M \times E \rightarrow \dot{T}^*M$  is the natural projection.

**4.2. Gupta-Bleuler Representations.** As is usual in the Gupta-Bleuler formalism, the representation of the field algebra will not be on a Hilbert space, but rather on a space equipped with an indefinite inner product. Thus we let  $(\mathfrak{K}, \langle \cdot, \cdot \rangle)$  be a locally convex topological vector space endowed with an indefinite inner product.

For a given real-valued function  $\Lambda \in C^\infty(M)$ , we let  $\pi$  be a representation of  $\mathcal{F}$  on  $\mathfrak{K}$  and  $\Omega \in \mathfrak{K}$  such that the following hold:

- (a)  $\overline{\pi(\mathcal{F})\Omega} = \mathfrak{K}$ , (cyclicity)
- (b)  $\pi(\mathcal{A})\Omega$  is a positive semi-definite subspace  $\mathfrak{H}_0 \subset \mathfrak{K}$  and  $\langle \Omega, \Omega \rangle = 1$ .
- (c)  $\pi(\mathcal{I}_\Lambda) = 0$ ,
- (d) For any  $n \in \mathbb{N}$ , the space  $\mathfrak{K}_n$  defined by

$$\mathfrak{K}_n = \overline{\pi(\mathcal{F}_n)\Omega} \subset \mathfrak{K}$$

is a Krein space (endowed with the inner product  $\langle \cdot, \cdot \rangle$  and the locally convex topology induced by  $\mathfrak{K}$ ). As before,  $\mathcal{F}_n$  is the span of products of the form  $A(f_1) \cdots A(f_m)$  with  $m \leq n$ .

- (e) microlocal spectrum condition:

$$\text{WF} \left( \underbrace{\pi(A(\cdot) \cdots A(\cdot))}_{m \text{ factors}} \Omega \right) \subseteq \Gamma_m^+ \quad \text{for all } m, \quad (4.1)$$

where  $\pi(A(\cdot) \cdots A(\cdot))\Omega$  is a Krein-space-valued distribution. The sets  $\Gamma_m^+$  are defined below.

(f) Gupta-Bleuler condition:

$$\langle \phi, \pi(A(df_1) - \Lambda(\square f_1)) \cdots \pi(A(df_n) - \Lambda(\square f_n)) \phi \rangle = 0$$

for all  $\phi \in \mathfrak{H}_0$  and  $n \geq 1$ .

Then  $(\pi, \mathfrak{K}, \Omega)$  is called a **Gupta-Bleuler representation in the  $\Lambda$ -gauge**. The distribution in (4.1) can be expressed in terms the  $n$ -point distributions defined by

$$\omega_n(f_1, \dots, f_n) = \langle \Omega, \pi(A(f_1) \cdots A(f_n)) \Omega \rangle.$$

Note that  $\mathfrak{H}_0$  is not a Hilbert space, because its inner product is only positive semi-definite. Dividing out the null subspace

$$\mathcal{N} = \{\psi \in \mathfrak{H}_0 \mid \langle \psi, \psi \rangle = 0\}$$

and forming the completion, one gets a Hilbert space, which in the usual Gupta-Bleuler formalism is interpreted as the physical Hilbert space. Note that as a consequence of the commutation relations the  $A(df)$  generate a commutative  $*$ -algebra that commutes with the observable algebra  $\mathcal{A}$ .

Here the sets  $\Gamma_m$  are defined as follows. We denote the closed light cone and its boundary by

$$\begin{aligned} J^+ &= \{(x, \xi) \mid g_x(\xi, \xi) \geq 0 \text{ and } \xi_0 \geq 0\} \\ L^+ &= \{(x, \xi) \mid g_x(\xi, \xi) = 0 \text{ and } \xi_0 \geq 0\}. \end{aligned}$$

Let  $\mathcal{G}_k$  be the set of all finite graphs with vertices  $\{1, \dots, k\}$  such that for every element  $G \in \mathcal{G}_k$  all edges occur in both admissible directions. We write  $s(e)$  and  $r(e)$  for the source and the target of an edge respectively. Following [10], we define an immersion of a graph  $G \in \mathcal{G}_k$  into the space-time  $M$  as an assignment of the vertices  $\nu$  of  $G$  to points  $x(\nu)$  in  $M$ , and of edges  $e$  of  $G$  to piecewise smooth curves  $\gamma(e)$  in  $M$  with source  $s(\gamma(e)) = x(s(e))$  and range  $r(\gamma(e)) = x(r(e))$ , together with a covariantly constant causal co-vector field  $\xi_e$  on  $\gamma$  such that

- (1) If  $e^{-1}$  denotes the edge with opposite direction as  $e$ , then the corresponding curve  $\gamma(e^{-1})$  is the inverse of  $\gamma(e)$ .
- (2) For every edge  $e$  the co-vector field  $\xi_e$  is directed towards the future whenever  $s(e) < r(e)$ .
- (3)  $\xi_{e^{-1}} = -\xi_e$ .

We set

$$\begin{aligned} \Gamma_m &:= \left\{ (x_1, \xi_1; \dots; x_m, \xi_m) \in T^*M^m \setminus 0 \mid \text{there exists } G \in \mathcal{G}_m \right. \\ &\quad \text{and an immersion } (x, \gamma, \xi) \text{ of } G \text{ in } M \text{ such that} \\ &\quad \left. x_i = x(i) \text{ for all } i = 1, \dots, m \text{ and } \xi_i = - \sum_{e, s(e)=i} \xi_e(x_i) \right\}. \end{aligned}$$

The set  $\Gamma_m^+$  is defined as

$$\Gamma_m^+ = \Gamma_m + (J^+)^m.$$

The microlocal spectrum condition for  $n$ -point distributions

$$\text{WF}(\omega_m) \subset \Gamma_m$$

was introduced for scalar fields by Brunetti, Fredenhagen and Köhler in [10], who also showed that for quasi-free representations, it suffices to verify their microlocal spectrum condition for the two-point functions (see also [38]). Quasi-free representations are those which satisfy the Wick rule

$$\omega_m(f_1, \dots, f_m) = \sum_P \prod_r \omega_2(f_{(r,1)}, f_{(r,2)}),$$

where  $P$  denotes a partition of the set  $\{1, \dots, m\}$  into subsets which are pairings of points labeled by  $r$ .

The microlocal spectrum condition for quasi-free states of the Klein-Gordon field was shown in [37] to be equivalent to the well-known Hadamard condition. Moreover, the microlocal spectrum condition is a sufficient condition for the construction of Wick polynomials (see [10, 26]) and interacting fields (see [9, 27]) in general globally hyperbolic space-times. For this reason, the microlocal spectrum condition is generally recognized to be a useful substitute for the spectrum condition in Minkowski space valid in general globally hyperbolic space-times.

For scalar fields one can impose the microlocal spectrum condition for Hilbert space valued distributions

$$\text{WF}(\underbrace{\phi(\cdot) \cdots \phi(\cdot)}_{m \text{ factors}} \Omega) \subset \Gamma_m^+.$$

Note that if  $(x_1, \xi_1; \dots; x_n, \xi_n) \in \Gamma_m^+$  and  $(x_n, -\xi_n; \dots; x_1, -\xi_1) \in \Gamma_m^+$  then we have  $(x_1, \xi_1; \dots; x_n, \xi_n) \in \Gamma_m$ . Therefore the above microlocal spectrum condition implies the microlocal spectrum condition for the corresponding  $m$ -point distribution. Following [10] one can show that the microlocal spectrum condition for the Hilbert space valued distribution as satisfied for Wightman fields in Minkowski space-time and also for quasifree Hadamard states of scalar fields in globally hyperbolic space-times. For quasifree states of scalar fields the microlocal spectrum condition for states is therefore equivalent to the one for Hilbert space valued distributions. This equivalence is based however on the Cauchy-Schwartz inequality which fails in Krein-spaces. We therefore require the microlocal spectrum condition to hold for the Krein space valued distributions as above. For Fock representations as described in Section 4.3 the microlocal spectrum condition is equivalent to the requirement

$$\text{WF}(\pi(A(\cdot)) \Omega) \subset J^+$$

(where  $\pi(A(\cdot)) \Omega$  is again understood as a Krein-space-valued distribution), which is not difficult to show using the commutation relations and Wick's theorem.

**Remark 4.2.** Requiring the microlocal spectrum condition on the level of observables only results in a slightly weaker condition that manifests itself in a condition on the polarization set. Namely, assume that  $u \in \mathcal{D}'(M, \Lambda^1(M))$  satisfies the condition

$$\text{WF}(du) \subseteq \{(x, \xi) \in \dot{T}^*M \mid \xi \in J^+\}. \quad (4.2)$$

Then

$$\text{WF}_{\text{pol}}(u) \subseteq \{(x, \xi; v) \in \dot{T}^*M \times T^*M \mid \xi \in J^+ \text{ or } \xi \sim v\}, \quad (4.3)$$

where  $\xi \sim v$  denotes linear dependence of  $\xi$  and  $v$ . Note that

$$\text{WF}_{\text{pol}}(u) \subseteq \text{WF}_{\text{pol}}(du) + \mathcal{N}_d$$

with

$$\mathcal{N}_d = \{(x, \xi; v) \mid \sigma_d(x, \xi) \cdot v = 0\} = \{(x, \xi; v) \mid v \sim \xi\}.$$

Microlocally, the set  $\mathcal{N}_d$  corresponds to the so-called longitudinal photons. Our condition (e) is stronger in that it imposes the microlocal spectrum condition in all directions, including those corresponding to longitudinal photons. We point out that the inverse implication (4.3) $\Rightarrow$ (4.2) is in general false, because the set  $\text{WF}_{\text{pol}}(u)$  only detects the highest order of the singularities.  $\diamond$

**Remark 4.3.** (*gauge invariance*) Note that property (c) implements the field equation. The property (f), on the other hand, realizes the gauge condition (1.4), but only if we take the inner product with a vector in  $\mathfrak{H}_0$ .  $\diamond$

**4.3. Fock Representations.** In order to construct Gupta-Bleuler-Fock representations of the field algebra  $\mathcal{F}$ , one can proceed as follows. Let  $\kappa : \Omega_0^1(M) \rightarrow \mathcal{K}$  be a real-linear continuous map into a complex Krein space  $\mathcal{K}$  with the following properties:

- (i)  $\kappa(\square f) = 0$  for all  $f \in \Omega_0^1(M)$
- (ii)  $\langle \kappa(f), \kappa(f) \rangle \geq 0$  if  $\delta f = 0$ .
- (iii)  $\text{Im} \langle \kappa(f), \kappa(g) \rangle = G(f, g)$
- (iv) microlocal spectrum condition:

$$\text{WF}(\kappa) \subseteq \{(x, \xi) \in \dot{T}^*M \mid \xi \in J^+\}.$$

- (v)  $\text{span}_{\mathbb{C}} \text{Rg}(\kappa)$  is dense in  $\mathcal{K}$ .
- (vi)  $\langle \kappa(df), \kappa(g) \rangle = 0$  for all  $f \in C_0^\infty(M)$ ,  $g \in \Omega_0^1(M)$  with either  $\delta g = 0$  or with  $g = dh$  for some  $h \in C_0^\infty(M)$ .

We introduce the Bosonic Fock space by

$$\mathfrak{K} = \bigoplus_{N=0}^{\infty} \hat{\otimes}_s^N \mathcal{K}, \quad (4.4)$$

where  $\hat{\otimes}$  denotes the completed symmetric tensor products of Krein spaces. Note that  $\mathfrak{K}$  is an indefinite inner product space but does not have a canonical completion to a Krein space. For  $\psi \in \mathcal{K}$ , we let  $a(\psi)$  be the annihilation operator and  $a^*(\psi)$  be the creation operator, defined as usual by

$$\begin{aligned} a^*(\psi) \phi_1 \otimes_s \dots \otimes_s \phi_N &= \psi \otimes_s (\phi_1 \otimes_s \dots \otimes_s \phi_N) \\ a(\psi) \phi_1 \otimes_s \dots \otimes_s \phi_N &= \langle \psi, \phi_1 \rangle \phi_2 \otimes_s \dots \otimes_s \phi_N. \end{aligned} \quad (4.5)$$

By construction, we have the canonical commutation relations

$$[a(\psi), a^*(\phi)] = \langle \psi, \phi \rangle. \quad (4.6)$$

For each  $f \in \Omega_0^1(M)$  and a given  $\Lambda \in C^\infty(M)$ , we let  $\hat{A}(f)$  be the following endomorphism of  $\mathfrak{K}$ :

$$\hat{A}(f) = \frac{1}{\sqrt{2}} \left( a(\kappa(f)) + a^*(\kappa(f)) \right) + d\Lambda(f). \quad (4.7)$$

Then the mapping

$$\pi : A(f) \mapsto \hat{A}(f)$$

extends to a  $*$ -representation  $\pi$  of the field algebra  $\mathcal{F}$  by operators that are symmetric with respect to the indefinite inner product on  $\mathfrak{K}$ .

**Theorem 4.4.** *The representation  $\pi$  is a Gupta-Bleuler representation.*

*Proof.* We need to check the properties (a)-(f) of a Gupta-Bleuler representation.

- (a) Cyclicity: The Fock space is the direct sum of finite particle subspaces. Suppose that  $N \geq 1$  and let  $P_N$  be the canonical projection onto the  $N$ -particle subspace  $\hat{\otimes}_s^N \mathcal{K}$ . Since

$$P_N \hat{A}(f_1) \cdots \hat{A}(f_N) \Omega = \left( \frac{1}{\sqrt{2}} \right)^N \kappa(f_1) \otimes_s \cdots \otimes_s \kappa(f_N)$$

and the complex span of the range of  $\kappa$  is dense in  $\mathcal{K}$ , we know that the complex span of  $\{P_N \hat{A}(f_1) \cdots \hat{A}(f_N) \Omega\}$  is dense in  $\hat{\otimes}_s^N \mathcal{K}$ . Therefore, if any element in  $\bigoplus_{k=0}^{N-1} \hat{\otimes}_s^k \mathcal{K}$  can be approximated by elements in  $\pi(\mathcal{F})\Omega$ , so can be any element in  $\bigoplus_{k=0}^N \hat{\otimes}_s^k \mathcal{K}$ . By induction in  $N$ , we conclude that  $\pi(\mathcal{F})\Omega$  is dense in  $\mathfrak{K}$ .

- (b) is a direct consequence of (ii).  
 (c) follows from  $\kappa(\square(f)) = 0$  and the definition of  $\hat{A}$  by direct computation.  
 (d) is clear by construction, because the finite tensor product of Krein spaces is a Krein space.  
 (e) The microlocal spectrum condition can be proved exactly as in [44, Propositions 2.2 and 6.1] and [10, Proposition 4.3].  
 (f) The space  $\mathfrak{H}_0$  is generated by vectors of the form

$$\phi = \hat{A}(f_1) \cdots \hat{A}(f_n) \Omega \quad \text{with } \delta f_i = 0.$$

Using (4.7), we obtain

$$\pi(A(df) - \Lambda(\square f)) = \frac{1}{\sqrt{2}} \left( a(\kappa(df)) + a^*(\kappa(df)) \right).$$

As a consequence of the commutation relations (4.6) and the property (vi), the operators  $a(\kappa(df))$  and  $a^*(\kappa(df))$  commute with all the  $\hat{A}(f_k)$  and with each other. We conclude that  $a(\kappa(df))\phi = 0$  and

$$\begin{aligned} & \langle \phi, \pi(A(df_1) - \Lambda(\square f_1)) \cdots \pi(A(df_n) - \Lambda(\square f_n)) \phi \rangle \\ &= \left( \frac{1}{\sqrt{2}} \right)^n \left\langle \phi, \left( a(\kappa(df_1)) + a^*(\kappa(df_1)) \right) \cdots \left( a(\kappa(df_n)) + a^*(\kappa(df_n)) \right) \phi \right\rangle = 0. \quad \square \end{aligned}$$

**4.4. Generalized Fock Representations.** In order to quantize the zero modes, we need to generalize the previous construction as follows. Let  $\kappa : \Omega_0^1(M) \rightarrow \mathcal{K}$  be a real-linear continuous map into a complex Krein space  $\mathcal{K}$  with the following properties:

- (i)  $\kappa(\square f) = 0$  for all  $f \in \Omega_0^1(M)$
- (ii)  $\langle \kappa(f), \kappa(f) \rangle \geq 0$  if  $\delta f = 0$ .
- (iii) There is a bilinear form  $G_Z$  on  $\Omega_0^1(M) \times \Omega_0^1(M)$  with smooth integral kernel and the following properties:

$$\text{Im} \langle \kappa(f), \kappa(g) \rangle + G_Z(f, g) = G(f, g) \tag{4.8}$$

$$\text{The vector space } Z := \Omega_0^1(M) / \{f \mid G_Z(f, \cdot) = 0\} \text{ is finite dimensional.} \tag{4.9}$$

- (iv) microlocal spectrum condition:

$$\text{WF}(\kappa) \subseteq \{(x, \xi) \in T^*M \mid \xi \in J^+\}.$$

- (v)  $\text{span}_{\mathbb{C}} \text{Rg}(\kappa)$  is dense in  $\mathcal{K}$ .
- (vi)  $\langle \kappa(df), \kappa(g) \rangle = 0$  for all  $f \in C_0^\infty(M)$ ,  $g \in \Omega_0^1(M)$  with either  $\delta g = 0$  or with  $g = dh$  for some  $h \in C_0^\infty(M)$ .

We introduce  $\mathfrak{K}$  and  $a$  as in the previous section (see (4.4) and (4.5)). Let  $\nu : \Omega_0^1(M) \rightarrow Z$  be the quotient map, and  $\tilde{G}_Z$  the induced symplectic form on  $Z$ .

We choose a complex structure  $\mathfrak{J}$  on  $Z$  such that  $K(\cdot, \cdot) := -\tilde{G}_Z(\cdot, \mathfrak{J}\cdot)$  is a real inner product. This complex structure then induces a canonical splitting  $Z = Y \oplus \tilde{Y}$  into two  $K$ -orthogonal Lagrangian subspaces  $Y$  and  $\tilde{Y}$  such that the symplectic form is given by  $\tilde{G}_Z((x_1, x_2), (y_1, y_2)) = K(x_1, y_2) - K(x_2, y_1)$ . Let  $\text{pr}_1$  and  $\text{pr}_2$  be the canonical projections and let  $\nu_i := \text{pr}_i \circ \nu$ . On the Schwartz space  $\mathcal{S}(Y, \mathbb{C})$ , we define  $\hat{A}_{\mathfrak{J}}(f) \in \text{End}(\mathcal{S}(Y, \mathbb{C}))$  by

$$(\hat{A}_{\mathfrak{J}}(f)\phi)(x) = K(\nu_1(f), x) \phi(x) + i(D_{\mathfrak{J}\nu_2(f)}\phi)(x)$$

(where  $D_{\mathfrak{J}\nu_2(f)}$  denotes the derivative in the direction  $\mathfrak{J}\nu_2(f) \in Y$ ). A short computation using the identity

$$\begin{aligned} & K(\nu_1(f), x) (D_{\mathfrak{J}\nu_2(g)}\phi)(x) - D_{\mathfrak{J}\nu_2(g)}(K(\nu_1(f), x) \phi(x)) \\ &= - (D_{\mathfrak{J}\nu_2(g)}K(\nu_1(f), x)) \phi(x) = -K(\nu_1(f), \nu_2(g)) \phi(x) \end{aligned}$$

shows that  $\hat{A}_{\mathfrak{J}}$  satisfies the canonical commutation relations

$$[\hat{A}_{\mathfrak{J}}(f), \hat{A}_{\mathfrak{J}}(g)] = -iG_Z(f, g).$$

We now define  $\hat{A}(f)$  on  $\mathfrak{K} \otimes \mathcal{S}(Y, \mathbb{C})$  by

$$\hat{A}(f) = \frac{1}{\sqrt{2}} \left( a(\kappa(f)) + a^*(\kappa(f)) \right) \otimes \mathbf{1} + \mathbf{1} \otimes A_{\mathfrak{J}}(f) + d\Lambda(f).$$

Since by assumption  $G_Z$  has a smooth integral kernel, the wave front set of the distribution  $\mathbf{1} \otimes A_{\mathfrak{J}}(f)$  is empty. A straightforward modification of the proof of Theorem 4.4 leads to the following result.

**Theorem 4.5.** *Under the assumptions (i)–(vi) stated above, the mapping  $\pi : f \mapsto \hat{A}(f)$  defines a Gupta-Bleuler representation.*

## 5. CONSTRUCTIONS FOR ULTRASTATIC MANIFOLDS

In this section we assume that the manifold  $M$  is ultrastatic, i.e. that it is of the form  $M = \mathbb{R} \times \Sigma$  with metric of product type  $g = dt^2 - h$ , where  $h$  is a complete Riemannian metric on  $\Sigma$ . Then  $M$  is globally hyperbolic and each  $\Sigma_t := \{t\} \times \Sigma$  is a Cauchy surface. A one-form  $f \in \Omega^1(M)$  can be decomposed as

$$f = f_0 dt + f_{\Sigma},$$

where  $f_0 \in C^\infty(M)$  and  $f_{\Sigma} \in C^\infty(M) \otimes_{\pi} \Omega^1(\Sigma)$ . Here  $\otimes_{\pi}$  denotes the projective tensor product of two locally convex spaces. We can think of  $f_{\Sigma}$  as a family of one-forms  $f_{\Sigma}(t)$  on  $\Sigma$  that depends smoothly on the parameter  $t$ . Let

$$\Psi^f := \begin{pmatrix} f \\ \dot{f} \end{pmatrix},$$

where  $\dot{f} := \frac{df}{dt}$ . The restriction  $\Psi_t^f$  of  $\Psi^f$  to the hypersurface  $\Sigma_t$  will be viewed as an element in  $(C^\infty(\Sigma) \oplus \Omega^1(\Sigma))^2$ . We say that a one-form  $f \in \Omega^1(M)$  has *spatially compact support* if its restriction to any Cauchy surface has compact support. In particular,  $\Psi_t^f \in (C_0^\infty(\Sigma) \oplus \Omega_0^1(\Sigma))^2$  for all  $t \in \mathbb{R}$ . The set of spatially compact one-forms is denoted by  $\Omega_{\text{sc}}^1(M)$ .

In view of the unique solvability of the Cauchy problem, the set of smooth solutions  $\Omega_{\text{sc}}^1(M) \cap \ker(\square)$  of the wave equation with spatially compact support can be identified with the space of initial data with compact support on  $\Sigma_0$ . Thus, the map  $f \mapsto \Psi_0^f$  defines an isomorphism between  $\Omega_{\text{sc}}^1(M) \cap \ker(\square)$  and  $(C_0^\infty(\Sigma) \oplus \Omega_0^1(\Sigma))^2$ . Since  $G$  maps  $\Omega_0^1(M)$  onto  $\Omega_{\text{sc}}^1(M) \cap \ker(\square)$ , the assignment

$$f \mapsto \Psi_0^{G(f)}$$

defines a surjective map to the Cauchy data space  $(C_0^\infty(\Sigma) \oplus \Omega_0^1(\Sigma))^2$ . There exists a natural symplectic form  $\sigma$  on the Cauchy data space defined by

$$\sigma \left( \begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix}, \begin{pmatrix} \dot{g} \\ \dot{g} \end{pmatrix} \right) = - \int_{\Sigma} (\dot{f}_0 \dot{g}_0 - \dot{f}_0 \dot{g}_0) d\mu_{\Sigma} + \int_{\Sigma} (\langle \dot{f}_{\Sigma}, \dot{g}_{\Sigma} \rangle - \langle \dot{f}_{\Sigma}, \dot{g}_{\Sigma} \rangle) d\mu_{\Sigma}, \quad (5.1)$$

where  $\langle \cdot, \cdot \rangle$  is the fibrewise inner product on forms on  $\Sigma$  induced by the Riemannian metric  $h$ , and  $\mu_{\Sigma}$  is the Riemannian measure on  $\Sigma$ . An elementary computation using Stokes' formula shows that (see for example [3, eq. (4.6)])

$$G(f, g) = \sigma(\Psi^{Gf}, \Psi^{Gg}).$$

**5.1. Absence of Zero Resonance States.** Usually, the construction of ground states on ultrastatic space-times assumes the existence of a spectral gap. In what follows, we shall generalize this construction significantly assuming a weaker condition, which we now formulate. Let  $\Omega_{(2)}^p(\Sigma)$  be the space of square-integrable  $p$ -forms on  $\Sigma$ . Since  $\Sigma$  is assumed to be complete, the Hodge Laplacian  $\Delta$  with domain of definition  $\Omega_0^p(\Sigma)$  is an essentially self-adjoint operator on  $\Omega_{(2)}^p(\Sigma)$ . We denote the self-adjoint extension again by  $\Delta$  with domain  $\mathcal{D}(\Delta)$ . Let  $dE_z$  be the spectral measure of  $\Delta$ . Moreover, let  $\Omega_{(2)}^{p\perp}(\Sigma) \subset \Omega_{(2)}^p(\Sigma)$  be the orthogonal complement of  $\ker \Delta$ , and  $\Omega^{p\perp}(\Sigma) \subset \Omega_{(2)}^{p\perp}(\Sigma) \cap \Omega^p(\Sigma)$  be the projection of  $\Omega_0^p(\Sigma)$  onto the orthogonal complement of  $\ker \Delta$ . Of course,  $\Delta$  leaves  $\Omega_{(2)}^{p\perp}(\Sigma)$  invariant. For simplicity, we denote its restriction to  $\Omega_{(2)}^{p\perp}(\Sigma)$  again by  $\Delta$ . Our constructions rely on the following condition:

$$(A) \quad \begin{aligned} &\text{The kernel of } \Delta \text{ is finite dimensional,} \\ &\text{and the domain of the operator } \Delta^{-\frac{1}{4}} \text{ contains } \Omega_{(2)}^{p\perp}(\Sigma). \end{aligned} \quad (5.2)$$

It is remarkable that for a large class of manifolds, this condition can be guaranteed under topological conditions on the boundary at infinity. In fact, this condition is closely related to the absence of zero resonance states. Namely, assume that the resolvent family  $(\Delta - \lambda^2)^{-1}$  of the Laplacian on differential forms admits a meromorphic continuation in the following sense. For suitably weighted  $L^2$ -spaces

$$\mathcal{H}_1 := L^2(\Sigma, \rho^{-1} d\mu_{\Sigma}) \subset \Omega_{(2)}^{\bullet}(\Sigma) \subset \mathcal{H}_1^* = \mathcal{H}_{-1} := L^2(\Sigma, \rho d\mu_{\Sigma})$$

with a positive weight function  $\rho \in C^\infty(\Sigma)$  that vanishes at infinity, we assume that the family of operators

$$(\Delta - \lambda^2)^{-1} : \mathcal{H}_1 \rightarrow \mathcal{H}_{-1}$$

has a meromorphic extension to a neighborhood of  $\lambda = 0$ , with the property that the negative Laurent coefficient are operators of finite rank. This assumption is well-known to be satisfied for odd-dimensional manifolds which are isometric to  $\mathbb{R}^{2n+1}$  outside compact sets (see for example [32]). Moreover, meromorphic continuations have been established for manifolds with cylindrical ends [31] in the context of the Atiyah-Patodi-Singer index theorem. It follows from standard glueing constructions

and the meromorphic Fredholm theorem that the meromorphic properties of the resolvent are stable under compactly supported metric and topological perturbations and therefore only depend on the structure near infinity.

Under these assumptions, there exist finite-rank operators  $A, B : \mathcal{H}_1 \rightarrow \mathcal{H}_{-1}$  such that for any  $f \in \Omega_0^\bullet$ , the measure  $d\langle f, E_z f \rangle$  has the representation

$$d\langle f, E_z f \rangle = \left( \langle f, Af \rangle \delta(z) + \langle f, Bf \rangle \frac{\Theta(z)}{\sqrt{z}} + \langle f, C(\sqrt{z})f \rangle \Theta(z) \right) dz, \quad (5.3)$$

where  $\Theta$  denotes the Heaviside step function, and  $C$  is a holomorphic family of operators with values in the bounded operators  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_{-1})$ . The operator  $A$  is in fact the orthogonal projection onto  $\ker \Delta$ . Since (5.3) remains true for vectors in  $\ker \Delta$  if we set  $B = 0$ ,  $C = 0$  and  $A = 1$ , we can extend this formula by linearity to the domain  $\Omega_{(2)}^{\bullet\perp}(\Sigma)$ . Using the spectral theorem, one easily sees that the above condition (A) is equivalent to the vanishing of the operator  $B$ .

Vectors in the range of  $B$  are commonly referred to as *zero resonant states*. The topological significance of these states was first pointed out by Atiyah, Patodi and Singer [2] and elaborated in [34, 31] in the case of manifolds with cylindrical ends (see also the introduction in [35]). For another class of manifolds, referred to as non-parabolic at infinity, it was pointed out by Carron [11] that the existence of certain non-square-integrable harmonic forms depends only on the geometry near infinity, the obstruction being an index of a certain Dirac operator. In many situations, it can be shown that these non-square-integrable harmonic forms correspond to zero resonant states [12, 45].

In order to illustrate that assumption (A) is stable and holds for a large class of manifolds, we now work out the above connections in the case of odd-dimensional manifolds which are isometric to  $\mathbb{R}^{2n+1}$  outside compact sets. This covers the physically interesting case of three space dimensions. Our results could be extended to even dimensions by a straightforward analysis of the logarithmic terms that are known to be present in the corresponding expansion (5.3).

**Proposition 5.1.** *Let  $(\Sigma^{2n+1}, g)$  with  $n \geq 1$  be a complete Riemannian manifold which is asymptotically Euclidean in the sense that there exist compact subsets  $K_1 \subset \Sigma$  and  $K_2 \subset \mathbb{R}^{2n+1}$  such that  $\Sigma \setminus K_1$  is isometric to  $\mathbb{R}^{2n+1} \setminus K_2$ . Then the operator  $B$  in (5.3) vanishes and condition (A) in (5.2) is satisfied.*

*Proof.* By the above, it suffices to show that  $B = 0$ . Following [11], we introduce the Sobolev space  $W(\Lambda^\bullet T^* \Sigma)$  as the completion of  $\Omega_0^\bullet(\Sigma)$  with respect to the quadratic form

$$\int_U |\alpha|^2 d\mu_\Sigma + \int_\Sigma (|d\alpha|^2 + |\delta\alpha|^2) d\mu_\Sigma, \quad (5.4)$$

where  $U$  is a non-empty relatively compact open subset of  $\Sigma$ . Note that for  $\Sigma = \mathbb{R}^{2n+1}$ , the space  $\{\alpha \in W(\Lambda^\bullet T^* \Sigma) \mid d\alpha + \delta\alpha = 0\}$  is zero provided that  $n \geq 1$ . As shown in [11, Theorem 0.6], the number

$$\dim \frac{\{\alpha \in W(\Lambda^\bullet T^* \Sigma) \mid d\alpha + \delta\alpha = 0\}}{\ker(\Delta)}$$

depends only on the geometry of  $\Sigma$  near infinity. Therefore, it is enough to show that the range of  $B$  is contained in  $W(\Lambda^\bullet T^* \Sigma)$ .

To this end, we must show that for every zero-resonance state  $u \in \text{rg } B$ , there is a sequence  $u_n \in \Omega_0^\bullet(\Sigma)$  which converges to  $u$  in  $W(\Lambda^\bullet T^* \Sigma)$ . If  $\chi \in C_0^\infty(\mathbb{R})$  is an

even real-valued function with  $\int_{\mathbb{R}} \chi(x) dx = 0$ , then its Fourier transform  $\hat{\chi}(\lambda) \in \mathcal{S}(\mathbb{R})$  is a Schwartz function that vanishes at  $\lambda = 0$ . We choose the function  $\chi$  with the additional property that

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\hat{\chi}(\sqrt{z})}{\sqrt{z}} dz = 1.$$

Moreover, for any  $\varepsilon > 0$  we define

$$\chi_\varepsilon(x) = \chi(\varepsilon x) \quad \text{and thus} \quad \hat{\chi}_\varepsilon(\lambda) = \frac{1}{\varepsilon} \hat{\chi}\left(\frac{\lambda}{\varepsilon}\right).$$

Since  $B$  has finite rank, there exists a compactly supported section  $v \in \Omega_0^\bullet(\Sigma)$  such that  $u = Bv$ . By finite propagation speed of the operator  $\cos(t\Delta^{\frac{1}{2}})$ , the section  $u_\varepsilon := \sqrt{\varepsilon} \hat{\chi}_\varepsilon(\Delta^{\frac{1}{2}})(v)$  is again compactly supported. We want to show that

$$\lim_{\varepsilon \searrow 0} u_\varepsilon = u \quad \text{with convergence in } W(\Lambda^\bullet T^* \Sigma).$$

First, it follows from (5.3) that  $u_\varepsilon$  converges in  $\mathcal{H}_{-1}$  to  $u$ . This implies in particular that

$$\lim_{\varepsilon \searrow 0} \int_U |u_\varepsilon - u|^2 d\mu_\Sigma = 0$$

(where  $U$  is again the relatively compact set in (5.4)). Next, again using (5.3),

$$\int_\Sigma (|du_\varepsilon|^2 + |\delta u_\varepsilon|^2) d\mu_\Sigma = \langle u_\varepsilon, \Delta u_\varepsilon \rangle = \varepsilon \int_0^\infty z |\hat{\chi}_\varepsilon(\sqrt{z})|^2 d\langle v, E_z v \rangle \xrightarrow{\varepsilon \searrow 0} 0,$$

showing that  $u_\varepsilon$  converges in  $W$ . Since it converges in  $\mathcal{H}_{-1}$  to  $u$ , the limit in  $W$  is again  $u$ .  $\square$

**5.2. Construction of  $\kappa$ .** We assume throughout this section that condition (A) in (5.2) holds. We choose the Krein space  $\mathcal{K}$  as

$$\mathcal{K} = (-\Omega_{(2)}^{0\perp}(\Sigma) \oplus \Omega_{(2)}^{1\perp}(\Sigma)) \otimes_{\mathbb{R}} \mathbb{C}.$$

Our assumptions imply that the operator  $\Delta^s$  has a trivial kernel on  $\Omega_{(2)}^{p\perp}(\Sigma)$  and is densely defined for all  $s \in \mathbb{R}$ . We introduce the spaces

$$\mathcal{H}^s = \overline{\mathcal{D}(\Delta^{\frac{s}{2}})}^{\|\cdot\|^s},$$

where the bar denotes the completion with respect to the norm  $\|\phi\|_s := \|\Delta^{\frac{s}{2}}\phi\|$  of the subspaces  $\mathcal{D}(\Delta^{\frac{s}{2}}) \subset \Omega_{(2)}^{p\perp}(\Sigma)$ . It follows from the spectral calculus that  $\mathcal{H}^s$  is the topological dual of  $\mathcal{H}^{-s}$ . Moreover, it is obvious that

$$\mathcal{D}(\Delta^s) \subset \mathcal{H}^{2s},$$

with continuous inclusion. Next, the following map is continuous:

$$\Delta^t : \mathcal{H}^s \rightarrow \mathcal{H}^{s-2t} \quad \text{for all } t \in \mathbb{R}.$$

Furthermore, using that  $\Delta$  commutes with all projections onto the form degree and  $\Delta = (d_\Sigma + \delta_\Sigma)^2$ , we also have the continuous mappings

$$d_\Sigma, \delta_\Sigma : \mathcal{H}^s \rightarrow \mathcal{H}^{s-1},$$

which commute with  $\Delta^t$  in the sense that

$$\Delta^t d_\Sigma = d_\Sigma \Delta^t \quad \text{and} \quad \Delta^t \delta_\Sigma = \delta_\Sigma \Delta^t$$

as continuous operators from  $\mathcal{D}(\Delta^s)$  to  $\mathcal{D}(\Delta^{s-2t-1})$ . Finally, the adjoints with respect to the dual pairings  $\mathcal{H}^s$  and  $\mathcal{H}^{-s}$  are computed as usual, i.e.

$$(\Delta^t)^* = \Delta^t, \quad d_\Sigma^* = \delta_\Sigma, \quad \delta_\Sigma^* = d_\Sigma.$$

In the following computations, by  $\langle \cdot, \cdot \rangle$  we denote the dual pairing between the spaces  $\mathcal{H}^s$  and  $\mathcal{H}^{-s}$ . We define the maps  $\tau$  and  $\kappa$  by

$$\begin{aligned} \tau : (\Omega_{(2)}^{0\perp}(\Sigma) \oplus \Omega_{(2)}^{1\perp}(\Sigma))^2 &\rightarrow \mathcal{K}, \\ \begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix} &\mapsto \left( \Delta^{\frac{1}{4}} \mathfrak{f}_0 + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_0 \right) \oplus \left( \Delta^{\frac{1}{4}} \mathfrak{f}_\Sigma + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_\Sigma \right) \end{aligned} \quad (5.5)$$

$$\kappa : \Omega_0^1(M) \rightarrow \mathcal{K}, \quad f \mapsto \tau(P_\perp \Psi_0^{Gf}), \quad (5.6)$$

where  $P_\perp$  is the orthogonal projection onto  $\Omega^{\bullet\perp}(\Sigma)$ .

**Proposition 5.2.** *The mapping  $\kappa$  has the following properties:*

- (i)  $\kappa(\square f) = 0$  for all  $f \in \Omega_0^1(M)$ .
- (ii)  $\langle \kappa(f), \kappa(f) \rangle \geq 0$  if  $\delta f = 0$ .
- (iii)  $\text{Im} \langle \kappa f, \kappa g \rangle = G(f, g)$  for all  $f, g \in \Omega_0^1(M)$  with  $\Psi_0^{Gf} \perp \ker \Delta$ .
- (v)  $\text{span}_{\mathbb{C}} \text{Rg}(\kappa)$  is dense in  $\mathcal{K}$ .
- (vi)  $\langle \kappa(df), \kappa(g) \rangle = 0$  for all  $f \in C_0^\infty(M)$ ,  $g \in \Omega_0^1(M)$  with either  $\delta g = 0$  or with  $g = dh$  for some  $h \in C_0^\infty(M)$ .

*Proof.* The properties (i) and (v) hold by construction. For both one-forms and functions we set

$$\begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix} (t) = P_\perp \Psi_t^{Gf},$$

and similarly for  $g$ . Moreover, we denote the negative of the  $L^2$  inner product on  $\Sigma_0$  by  $\langle \cdot, \cdot \rangle_\Sigma$ , i.e.  $\langle \mathfrak{f}, \mathfrak{g} \rangle_\Sigma$  is the  $L^2$ -inner product of the restrictions of  $\mathfrak{f}$  and  $\mathfrak{g}$  to  $\Sigma_0$ . The convention is chosen so that this inner product coincides with the Krein space inner product and the notation makes it clear that integration is over  $\Sigma_0$  and not over the entire space  $M$ . Then the computation

$$\begin{aligned} &\text{Im} \left\langle \tau \left( \begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix} \right), \tau \left( \begin{pmatrix} \mathfrak{g} \\ \dot{\mathfrak{g}} \end{pmatrix} \right) \right\rangle \\ &= -\text{Im} \left\langle \Delta^{\frac{1}{4}} \mathfrak{f}_0 + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_0, \Delta^{\frac{1}{4}} \mathfrak{g}_0 + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_0 \right\rangle_\Sigma \\ &\quad + \text{Im} \left\langle \Delta^{\frac{1}{4}} \mathfrak{f}_\Sigma + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_\Sigma, \Delta^{\frac{1}{4}} \mathfrak{g}_\Sigma + i \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_\Sigma \right\rangle_\Sigma \\ &= -\langle \Delta^{\frac{1}{4}} \mathfrak{f}_0, \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_0 \rangle_\Sigma + \langle \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_0, \Delta^{\frac{1}{4}} \mathfrak{g}_0 \rangle_\Sigma \\ &\quad + \langle \Delta^{\frac{1}{4}} \mathfrak{f}_\Sigma, \Delta^{-\frac{1}{4}} \dot{\mathfrak{g}}_\Sigma \rangle_\Sigma - \langle \Delta^{-\frac{1}{4}} \dot{\mathfrak{f}}_\Sigma, \Delta^{\frac{1}{4}} \mathfrak{g}_\Sigma \rangle_\Sigma \\ &= -\langle \mathfrak{f}_0, \dot{\mathfrak{g}}_0 \rangle_\Sigma + \langle \dot{\mathfrak{f}}_0, \mathfrak{g}_0 \rangle_\Sigma + \langle \mathfrak{f}_\Sigma, \dot{\mathfrak{g}}_\Sigma \rangle_\Sigma - \langle \dot{\mathfrak{f}}_\Sigma, \mathfrak{g}_\Sigma \rangle_\Sigma = \sigma \left( \begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix}, \begin{pmatrix} \mathfrak{g} \\ \dot{\mathfrak{g}} \end{pmatrix} \right) \end{aligned}$$

together with (5.1) yields (iii). To prove (ii), we first note that

$$\begin{aligned} &\left\langle \tau \left( \begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix} \right), \tau \left( \begin{pmatrix} \mathfrak{f} \\ \dot{\mathfrak{f}} \end{pmatrix} \right) \right\rangle = -\langle \Delta^{\frac{1}{2}} \mathfrak{f}_0, \mathfrak{f}_0 \rangle_\Sigma - \langle \Delta^{-\frac{1}{2}} \dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}_0 \rangle_\Sigma \\ &\quad + \langle \Delta^{\frac{1}{2}} \mathfrak{f}_\Sigma, \mathfrak{f}_\Sigma \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} \dot{\mathfrak{f}}_\Sigma, \dot{\mathfrak{f}}_\Sigma \rangle_\Sigma. \end{aligned} \quad (5.7)$$

Since  $Gf$  solves the wave equation,

$$0 = \square f = (-\ddot{f}_0 - \Delta f_0) dt + (-\ddot{f}_\Sigma - \Delta f_\Sigma) \quad (5.8)$$

Moreover, if  $\delta f = 0$ , we have

$$0 = \delta f = -\dot{f}_0 - \delta_\Sigma f_\Sigma. \quad (5.9)$$

The last equality in (5.9) can be verified with the computation

$$\begin{aligned} \langle \delta f, \varphi \rangle &= \langle f, d\varphi \rangle = \langle f_0 dt + f_\Sigma, \dot{\varphi} dt + d_\Sigma \varphi \rangle \\ &= \langle f_0, \dot{\varphi} \rangle - \langle f_\Sigma, d_\Sigma \varphi \rangle \stackrel{(*)}{=} -\langle \dot{f}_0, \varphi \rangle - \langle \delta_\Sigma f_\Sigma, \varphi \rangle = -\langle (\dot{f}_0 + \delta_\Sigma f_\Sigma), \varphi \rangle, \end{aligned}$$

where in (\*) we integrated by parts. Then

$$\begin{aligned} &-\langle \Delta^{-\frac{1}{2}} \dot{f}_0, \dot{f}_0 \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} f_\Sigma, f_\Sigma \rangle_\Sigma \\ &= -\langle \Delta^{-\frac{1}{2}} (d_\Sigma \delta_\Sigma) f_\Sigma, f_\Sigma \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} (d_\Sigma \delta_\Sigma + \delta_\Sigma d_\Sigma) f_\Sigma, f_\Sigma \rangle_\Sigma \\ &= \langle \Delta^{-\frac{1}{2}} \delta_\Sigma d_\Sigma f_\Sigma, f_\Sigma \rangle_\Sigma = \langle \Delta^{-\frac{1}{2}} d_\Sigma f_\Sigma, d_\Sigma f_\Sigma \rangle_\Sigma \geq 0. \end{aligned}$$

Moreover, differentiating (5.9) with respect to  $t$  and using (5.8) gives

$$\delta_\Sigma \dot{f}_\Sigma = \Delta f_0.$$

Hence

$$\begin{aligned} &-\langle \Delta^{\frac{1}{2}} \dot{f}_0, \dot{f}_0 \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} \dot{f}_\Sigma, \dot{f}_\Sigma \rangle_\Sigma \\ &= -\langle \Delta^{-\frac{3}{2}} \Delta f_0, \Delta f_0 \rangle_\Sigma + \langle \Delta^{-\frac{3}{2}} \Delta \dot{f}_\Sigma, \dot{f}_\Sigma \rangle_\Sigma \\ &= -\langle \Delta^{-\frac{3}{2}} \delta_\Sigma \dot{f}_\Sigma, \delta_\Sigma \dot{f}_\Sigma \rangle_\Sigma + \langle \Delta^{-\frac{3}{2}} (d_\Sigma \delta_\Sigma + \delta_\Sigma d_\Sigma) \dot{f}_\Sigma, \dot{f}_\Sigma \rangle_\Sigma \\ &= \langle \Delta^{-\frac{3}{2}} \delta_\Sigma d_\Sigma \dot{f}_\Sigma, \dot{f}_\Sigma \rangle_\Sigma = \langle \Delta^{-\frac{3}{2}} d_\Sigma \dot{f}_\Sigma, d_\Sigma \dot{f}_\Sigma \rangle_\Sigma \geq 0. \end{aligned}$$

This shows (ii).

In order to prove (vi) in case  $\delta g = 0$ , we polarize (5.7) to obtain

$$\begin{aligned} \langle \kappa(df), \kappa(g) \rangle &= -\langle \Delta^{\frac{1}{2}} \dot{f}, \mathfrak{g}_0 \rangle_\Sigma - \langle \Delta^{-\frac{1}{2}} \ddot{f}, \dot{\mathfrak{g}}_0 \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} d_\Sigma f, \mathfrak{g}_\Sigma \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{f}, \dot{\mathfrak{g}}_\Sigma \rangle_\Sigma \\ &= -\langle \Delta^{\frac{1}{2}} \dot{f}, \mathfrak{g}_0 \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} \dot{f}, \dot{\mathfrak{g}}_0 \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} d_\Sigma f, \mathfrak{g}_\Sigma \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{f}, \dot{\mathfrak{g}}_\Sigma \rangle_\Sigma \\ &= -\langle \Delta^{\frac{1}{2}} \dot{f}, \mathfrak{g}_0 \rangle_\Sigma - \langle \Delta^{\frac{1}{2}} \dot{f}, \delta_\Sigma \mathfrak{g}_\Sigma \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} d_\Sigma f, \mathfrak{g}_\Sigma \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{f}, \dot{\mathfrak{g}}_\Sigma \rangle_\Sigma \\ &= 0, \end{aligned}$$

where we have used the wave equations  $\square f = 0$  and  $\square g = 0$  together with the identity  $0 = \delta g = -\dot{\mathfrak{g}}_0 - \delta_\Sigma \mathfrak{g}_\Sigma$ .

Similarly, if  $g \in C_0^\infty(M)$  we have

$$\begin{aligned} \langle \kappa(df), \kappa(dg) \rangle &= -\langle \Delta^{\frac{1}{2}} \dot{f}, \dot{\mathfrak{g}} \rangle_\Sigma - \langle \Delta^{-\frac{1}{2}} \ddot{f}, \ddot{\mathfrak{g}} \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} d_\Sigma f, d_\Sigma \mathfrak{g} \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{f}, d_\Sigma \dot{\mathfrak{g}} \rangle_\Sigma \\ &= -\langle \Delta^{\frac{1}{2}} \dot{f}, \dot{\mathfrak{g}} \rangle_\Sigma - \langle \Delta^{\frac{3}{2}} \dot{f}, \mathfrak{g} \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} d_\Sigma f, d_\Sigma \mathfrak{g} \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} d_\Sigma \dot{f}, d_\Sigma \dot{\mathfrak{g}} \rangle_\Sigma \\ &= -\langle \Delta^{\frac{1}{2}} \dot{f}, \dot{\mathfrak{g}} \rangle_\Sigma - \langle \Delta^{\frac{3}{2}} \dot{f}, \mathfrak{g} \rangle_\Sigma + \langle \Delta^{\frac{3}{2}} \dot{f}, \mathfrak{g} \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} \dot{f}, \dot{\mathfrak{g}} \rangle_\Sigma = 0. \end{aligned}$$

This concludes the proof.  $\square$

In view of (4.8), we define

$$G_Z(f, g) = G(f, g) - \text{Im}\langle \kappa(f), \kappa(g) \rangle. \quad (5.10)$$

Proposition 5.2 (iii) has the following implication.

**Corollary 5.3.** *The symplectic vector space  $Z$  defined by (4.9) is canonically isomorphic to  $\left(\ker \Delta \cap \left(\Omega_{(2)}^0(\Sigma) \oplus \Omega_{(2)}^1(\Sigma)\right)\right)^2$  with the restriction of the standard symplectic form defined in (5.1). With the complex structure*

$$\mathfrak{J} \left( \begin{pmatrix} (f_0, f_\Sigma) \\ (g_0, g_\Sigma) \end{pmatrix} \right) = \begin{pmatrix} (g_0, -g_\Sigma) \\ (-f_0, f_\Sigma) \end{pmatrix}$$

the associated inner product is the usual  $L^2$ -inner product.

Summarizing the results of this section, we come to the following conclusion.

**Theorem 5.4.** *Let  $(\Sigma, g)$  be a Riemannian manifold satisfying assumption (A) in (5.2). The above mapping  $\kappa$  together with the form  $G_Z$  given by (5.10) and the complex structure  $\mathfrak{J}$  given above defines a generalized Fock representation (see Section 4.4) in the ultrastatic space-time  $(\mathbb{R} \times \Sigma, dt^2 - g)$ , thereby giving rise to a Gupta-Bleuler representation of  $\mathcal{F}$  (see Section 4.2).*

**5.3. Positivity of the Energy and the Microlocal Spectrum Condition.** It is obvious from Definition 5.5 that  $\tau$  is injective and that its image is dense in  $\mathcal{K}$ . We introduce the operator  $H$  on  $\mathfrak{K}$  by

$$H = \begin{pmatrix} \Delta^{\frac{1}{2}} & 0 \\ 0 & \Delta^{\frac{1}{2}} \end{pmatrix} \quad (5.11)$$

(acting in the first component on functions and on the second component on one-forms). Then

$$\begin{aligned} H \tau \begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix} &= \left( \Delta^{-\frac{1}{4}} \Delta \dot{f}_0 + i \Delta^{\frac{1}{4}} \dot{f}_0 \right) \oplus \left( \Delta^{-\frac{1}{4}} \Delta \dot{f}_\Sigma + i \Delta^{\frac{1}{4}} \dot{f}_\Sigma \right) \\ &= i \left( \Delta^{\frac{1}{4}} \dot{f}_0 + i \Delta^{-\frac{1}{4}} (-\Delta) \dot{f}_0 \right) \oplus \left( \Delta^{\frac{1}{4}} \dot{f}_\Sigma + i \Delta^{-\frac{1}{4}} (-\Delta) \dot{f}_\Sigma \right) \\ &= i \tau \begin{pmatrix} 0 & 1 \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix} \stackrel{(5.8)}{=} i \partial_t \tau \begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix}. \end{aligned}$$

Assume that  $\delta f = 0$ . Then, similar as above,

$$\begin{aligned} &\left\langle \tau \begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix}, H \tau \begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix} \right\rangle \\ &= -\langle \Delta \dot{f}_0, \dot{f}_0 \rangle_\Sigma - \langle \dot{f}_0, \dot{f}_0 \rangle_\Sigma + \langle \Delta \dot{f}_\Sigma, \dot{f}_\Sigma \rangle_\Sigma + \langle \dot{f}_\Sigma, \dot{f}_\Sigma \rangle_\Sigma \\ &= \langle d_\Sigma \dot{f}_\Sigma, d_\Sigma \dot{f}_\Sigma \rangle_\Sigma + \langle \Delta^{-1} d_\Sigma \dot{f}_\Sigma, d_\Sigma \dot{f}_\Sigma \rangle_\Sigma \geq 0. \end{aligned}$$

**Proposition 5.5.** *The mapping  $\kappa$  satisfies the*

(iv) *microlocal spectrum condition:*

$$\text{WF}(\kappa(\cdot)) \subset J^+$$

*Proof.* Let  $T_t$  be the operator that shifts functions and distributions by  $t \in \mathbb{R}$  in time. By construction of  $H$ , we know that

$$\kappa(T_{-t}f) = e^{iHt} \kappa(f).$$

For a given point  $x_0 \in \Sigma$ , we choose a chart  $x$  and a bundle chart. Let  $\chi \in \Omega_0^1(\Sigma)$  be any smooth form supported in our chart with  $\chi(x_0) \neq 0$ . The following computation will be carried out in local coordinates. Let  $u_\xi \in \mathcal{D}'(\mathbb{R})$  be the family of distributions

$$u_\xi(g) = \kappa(\chi e^{-i\xi x} \otimes g).$$

This family is polynomially bounded in  $\xi$  and, by construction,

$$T_t u_\xi = e^{iHt} u_\xi.$$

As a consequence,

$$u_\xi(\chi * g) = \sqrt{2\pi} \int_0^\infty \hat{\chi}(\lambda) dE_\lambda u_\xi(g),$$

where  $dE_\lambda$  is the spectral measure of the generator (5.11), considered as a self-adjoint operator on the Hilbert space  $L^2(\Sigma) \otimes \mathbb{C}^2$ .

Choosing a test function  $\eta \in C_0^\infty(\mathbb{R})$ , we have

$$\begin{aligned} \kappa\left(\chi(x) e^{-i\xi x} \otimes (\eta * \eta)(t) e^{-i\xi_0 t}\right) &= u_\xi\left((\eta * \eta) e^{-i\xi_0 \cdot}\right) \\ &= u_\xi\left((\eta e^{-i\xi_0 \cdot}) * (\eta e^{-i\xi_0 \cdot})\right) = \sqrt{2\pi} \int_0^\infty \hat{\eta}(\lambda - \xi_0) dE_\lambda u_\xi(\eta e^{-i\xi_0 \cdot}). \end{aligned}$$

Taking the Hilbert space norm, one sees that  $\|\eta e^{-i\xi_0 \cdot}\|$  is polynomially bounded in  $(\xi, \xi_0)$ , whereas the operator norm of the spectral integral decays rapidly in  $\xi_0$ . We thus obtain rapid decay in  $(\xi, \xi_0)$  in a conic neighborhood of any direction  $(\tilde{\xi}, \tilde{\xi}_0)$  with  $\tilde{\xi}_0 < 0$ .  $\square$

We remark that this result could also be inferred somewhat less directly from [44, Theorem 2.8].

## 6. CONSTRUCTION OF GUPTA-BLEULER REPRESENTATIONS

In this section, we use a variation of a deformation argument by [20] to show that Gupta-Bleuler representations and states exist for a large class of globally hyperbolic space-times. Let  $(M, g)$  be a globally hyperbolic space-time. According to [7], the manifold admits a smooth foliation  $(\Sigma_t)_{t \in \mathbb{R}}$  by Cauchy hypersurfaces. Assume there exists a metric  $g$  on  $\Sigma_0$  such that  $(\Sigma_0, g)$  satisfies condition (A) in (5.2). Then, using the constructions in [33], there is a globally hyperbolic space-time  $(M, \tilde{g})$  which is future-isometric to  $(M, g)$  and past isometric to the ultrastatic space-time  $(\mathbb{R} \times \Sigma_0, dt^2 - g)$ . On  $\mathbb{R} \times \Sigma$ , we choose  $\kappa$  as in Section 5.

By propagation of singularities (see [28, Theorem 23.2.9]) and Lemma 3.6 we have the following.

**Lemma 6.1.** *Suppose that  $U$  is an open neighborhood of a Cauchy surface. Assume that  $\kappa : \Omega_0^1(U) \rightarrow \mathcal{K}$  satisfies properties in Section 4.4 in  $U$ , i.e.*

- (i)'  $\kappa(\square f) = 0$  for all  $f \in \Omega_0^1(U)$ .
- (ii)'  $\langle \kappa(f), \kappa(f) \rangle \geq 0$  if  $f \in \Omega_0^1(U)$  with  $\delta f = 0$ .

(iii)' *There is a bilinear form  $G_Z$  on  $\Omega_0^1(U) \times \Omega_0^1(U)$  with smooth integral kernel and the following properties:*

$$\operatorname{Im}\langle \kappa(f), \kappa(g) \rangle + G_Z(f, g) = G(f, g)$$

*The vector space  $Z_U := \Omega_0^1(U) / \{f \mid G_Z(f, \cdot) = 0\}$  is finite dimensional*

$$G_Z(f, \delta g) = 0 \quad \text{for all } f \in \Omega_0^1(U) \text{ and } g \in \Omega_0^2(U)$$

(iv)'  $\operatorname{WF}(\kappa) \cap U \subseteq \{(x, \xi) \in \dot{T}^*U \mid \xi \in J^+\}$ .

(v)'  $\operatorname{span}_{\mathbb{C}} \kappa(\Omega_0^1(U))$  is dense in  $\mathcal{K}$ .

(vi)'  $\langle \kappa(df), \kappa(g) \rangle = 0$  for all  $f \in C_0^\infty(U)$ ,  $g \in \Omega_0^1(U)$  with  $\delta g = 0$  or with  $g = dh$  for some  $h \in C_0^\infty(M)$ .

*Then there exists a unique extension  $\tilde{\kappa} : \Omega_0^1(M) \rightarrow \mathcal{K}$  that satisfies (i) everywhere. This unique extension then satisfies (ii), (iii), (iv), (v) and (vi) everywhere.*

One can now construct a Gupta-Bleuler representation on  $(\tilde{M}, \tilde{g})$  by constructing the generalized Fock representation on the ultrastatic space-time  $(\mathbb{R} \times \Sigma_0, dt^2 - g)$ . Since there exists a neighborhood  $\tilde{U}_1$  of a Cauchy surface in  $(\tilde{M}, \tilde{g})$  that is isometric to a neighborhood  $U_{\text{us}}$  of a Cauchy surface in  $(\mathbb{R} \times \Sigma_0, dt^2 - g)$  one can use the above Lemma to migrate this Gupta-Bleuler representation to  $(\tilde{M}, \tilde{g})$  by first restricting  $\kappa$  to  $U_{\text{us}}$  and then extending from  $\tilde{U}$  to  $\tilde{M}$ . Similarly, since there exists a neighborhood  $U$  of a Cauchy surface in  $(M, g)$  that is isometric to a neighborhood  $\tilde{U}_2$  of a Cauchy surface in  $(\tilde{M}, \tilde{g})$  one constructs a Gupta-Bleuler representation on  $(M, g)$  by restricting  $\kappa$  to  $\tilde{U}_2$  and then extending it to  $M$ .

If a globally hyperbolic space-time has symmetries, one would want to construct Gupta-Bleuler representations in which these symmetries can be implemented unitarily. Unfortunately, the above method of construction does not respect symmetries. This is of course analogous to the problem of the construction of invariant Hadamard states on space-times with non-trivial symmetry group.

## APPENDIX A. THE GAUGE PARAMETER AND OTHER COMMUTATOR RELATIONS

A common procedure in physics is to add a gauge fixing term to the classical Lagrangian of the electromagnetic field. This leads to the modified wave equation

$$\square_\xi A = 0 \quad \text{where} \quad \square_\xi = \delta d + \frac{1}{\xi} d\delta,$$

which involves the *gauge parameter*  $\xi \in (0, \infty)$ . Choosing  $\xi = 1$ , the so-called *Feynman gauge*, gives again the ordinary wave equation  $\square A = 0$ . Another common gauge is the *Landau gauge*  $\xi \searrow 0$  (where the limit is taken after computing expectation values).

Working with a gauge-parameter  $\xi \neq 1$  has the unpleasant consequence that the modified wave operator  $\square_\xi$  is not normally hyperbolic. But, in a chosen foliation, the modified wave equation can be written as a symmetric hyperbolic system (see for example [29]), showing that the Cauchy problem is well-posed, and that the propagation speed is finite. Moreover, the formulation as a symmetric hyperbolic system depends on the choice of the foliation.

We now show that working with different gauge parameters gives an equivalent description of the physical system. To this end, we will construct a bijection of the corresponding field algebras. In preparation, we relate the solutions of the modified

wave equation to the solutions of the ordinary wave equation: We choose an operator  $R : C_{\text{sc}}^\infty(M) \rightarrow C_{\text{sc}}^\infty(M)$  such that

$$\square Rf = f \quad \text{for all } f \in C_{\text{sc}}^\infty(M) \cap \ker(\square_\xi).$$

One method for constructing the operator  $R$  is to solve the Cauchy problem  $\square\phi = f$  for vanishing initial data on a Cauchy surface. In Minkowski space, a particular choice of the operator  $R$  is discussed in [30] and [22, Exercise 7.3]. There are of course many other choices. Suppose that  $\square_\xi\psi = 0$ . Then, of course  $\square\delta\psi = 0$ , and the calculation

$$\begin{aligned} \square(\mathbf{1} + (\xi^{-1} - 1)dR\delta)\psi &= (d\delta + \delta d + (\xi^{-1} - 1)d\delta dR\delta)\psi \\ &= (d\delta + \delta d + (\xi^{-1} - 1)d\square R\delta)\psi = (d\delta + \delta d + (\xi^{-1} - 1)d\delta)\psi \\ &= (\delta d + \frac{1}{\xi}d\delta)\psi = \square_\xi\psi = 0 \end{aligned}$$

shows that the following operator maps solutions of the corresponding wave equations into each other,

$$\mathfrak{J}_R := \mathbf{1} + (\xi^{-1} - 1)dR\delta : \Omega_{\text{sc}}^1(M) \cap \ker(\square_\xi) \rightarrow \Omega_{\text{sc}}^1(M) \cap \ker(\square).$$

One easily checks that  $\mathfrak{J}_R$  is invertible with explicit inverse given by

$$\mathfrak{J}_R^{-1} = \mathbf{1} + (1 - \xi)dR\delta.$$

We thus obtain a one-to-one correspondence between solutions of the ordinary and modified wave equations.

In order to extend this correspondence to the field operators, we use the following dual formulation. Assume that  $L$  is a given map  $L : C_0^\infty(M) \rightarrow C_0^\infty(M)$  such that

$$f - L\square f \in \text{Im}(\square) \quad \text{for all } f \in C_0^\infty(M).$$

Again, there are many possibilities to choose  $L$ . A particular choice is

$$L = \eta_- G_+^0 + \eta_+ G_-^0,$$

where  $\eta_+$  and  $\eta_-$  are smooth functions with  $\eta_+ + \eta_- = 1$ , having past and future compact support, respectively (as before,  $G_\pm^0$  denote the causal fundamental solutions for the scalar wave operator). Then the computation

$$(1 + (\xi^{-1} - 1)dL\delta)\square f = \square_\xi f + (\xi^{-1} - 1)d(L\square - 1)\delta f$$

shows that the map

$$\mathfrak{J}_L := 1 + (\xi^{-1} - 1)dL\delta$$

has the property

$$\mathfrak{J}_L\square f \in \text{Im}(\square_\xi) \quad \text{for all } f \in \Omega_0^1(M).$$

This allows us to identify the field algebras of the modified and the ordinary wave operators via the relation

$$\tilde{A}(f) = A(\mathfrak{J}_L f).$$

By the above,  $\tilde{A}$  satisfies the equation

$$\square_\xi \tilde{A} = 0$$

as an operator-valued distribution. The commutation relations become

$$\tilde{A}(f)\tilde{A}(g) - \tilde{A}(g)\tilde{A}(f) = -iG(\mathfrak{J}_L f, \mathfrak{J}_L g).$$

## APPENDIX B. THE CONSTRUCTION OF STATES IN THE BRST FRAMEWORK

The purpose of this appendix is to show that a small variation of our construction of Gupta-Bleuler states can also be used to construct states in the BRST framework. For details of BRST quantization we would like to refer the reader to [25]. Here we define only a minimal set of objects needed for the construction.

As before, let  $G^1$  be the difference of advanced and retarded Green's operators on one-forms and let  $G^0$  be the corresponding operator on functions. We think of  $G^1$  and  $G^0$  as bidistributions. Following [25, Section 4.2], one wants to find bi-distributions  $\omega^1 \in (\Omega_0^1(M) \otimes_\pi \Omega_0^1(M))'$  and  $\omega^0 \in (C_0^\infty(M) \otimes_\pi C_0^\infty(M))'$  such that the following list of compatibility conditions is satisfied:

$$\begin{aligned} \omega^1(\square f, g) &= \omega^1(f, \square g) = 0 & \forall f, g \in \Omega_0^1(M) \\ \omega^0(\square f, g) &= \omega^0(f, \square g) = 0 & \forall f, g \in C_0^\infty(M) \\ \omega^1(f, g) - \omega^1(g, f) &= -i G^1(f, g) & \forall f, g \in \Omega_0^1(M) \\ \omega^0(f, g) - \omega^0(g, f) &= -i G^0(f, g) & \forall f, g \in C_0^\infty(M) \\ \omega^0(\delta f, g) &= -\omega^1(f, dg) & \forall f \in \Omega_0^1(M), g \in C_0^\infty(M) \\ \omega^1(f, f) &\geq 0 & \forall f \in \Omega_0^1(M) \text{ with } \delta f = 0 \\ \omega^0(f, f) &\geq 0 & \forall f \in C_0^\infty(M). \end{aligned}$$

Here we write  $\omega(f, g)$  for  $\omega(f \otimes g)$ , understanding the bidistributions as bilinear forms. Moreover, one needs to impose the microlocal spectrum condition.

We will show that one can always construct  $\omega^0$  and  $\omega^1$  for ultrastatic space-times  $M = \mathbb{R} \times \Sigma$  under the following assumption on  $\Sigma$ .

**Assumption B.1.** *Condition (A) of (5.2) holds and  $\text{Vol}(\Sigma) = \infty$ .*

This assumption implies that the space of square integrable harmonic functions has dimension 0 but does not rule out the existence of non-trivial square integrable harmonic one-forms. In particular, the assumption is satisfied on compactly supported topological and metric perturbations of  $\mathbb{R}^{2n+1}$ . The two point function for one-forms  $\omega^1$  will simply be given by

$$\omega^1(f, g) = \langle \Omega, \hat{A}(f) \hat{A}(g) \Omega \rangle,$$

where  $\hat{A}$  is the Gupta-Bleuler representation constructed in Section 5 and the complex structure is chosen (for the sake of concreteness) as in Corollary 5.3 to be the one associated with the  $L^2$ -inner product on the space of square integrable harmonic forms.

The state  $\omega^0$  is obtained if the same construction is carried out on the level of zero forms. Define the maps  $\tau^0$  and  $\kappa^0$  by

$$\begin{aligned} \tau^0 : (\Omega_{(2)}^{0\perp}(\Sigma))^2 &\rightarrow \mathcal{K}, \quad \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \mapsto \left( \Delta^{\frac{1}{4}} \mathfrak{f} + i \Delta^{-\frac{1}{4}} \mathfrak{g} \right) \\ \kappa^0 : C_0^\infty(M) &\rightarrow L^2(\Sigma, \mathbb{C}), \quad f \mapsto \tau(\Psi_0^{G^0} f), \end{aligned}$$

where as before we define  $\mathfrak{f}$  and  $\mathfrak{g}$  as  $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} (t) = \Psi_t^{G^f}$ . Note that, since square integrable harmonic functions vanish, these do not have to be projected out as it was necessary for forms. In the same way as in Proposition 5.2, one shows easily that the mapping  $\kappa^0$  has the following properties:

- (i)  $\kappa^0(\square f) = 0$  for all  $f \in C_0^\infty(M)$ .
- (ii)  $\langle \kappa^0(f), \kappa^0(f) \rangle \geq 0$  for all  $f \in C_0^\infty(M)$ .
- (iii)  $\text{Im}(\langle \kappa^0(f), \kappa^0(g) \rangle) = G^0(f, g)$  for all  $f, g \in C_0^\infty(M)$ .
- (v)  $\text{span}_{\mathbb{C}} \text{Rg}(\kappa^0)$  is dense in  $L^2(\Sigma, \mathbb{C})$ .

Moreover, by direct inspection, one can see that

$$\langle \kappa^0(\delta f), \kappa^0(g) \rangle = -\langle \kappa(f), \kappa(dg) \rangle$$

for all  $f \in \Omega_0^1(M)$  with  $\Psi_0^{Gf} \perp \ker \Delta$  and all  $g \in C_0^\infty(M)$ . Indeed,

$$\begin{aligned} \langle \kappa(f), \kappa(dg) \rangle &= -\langle \Delta^{\frac{1}{2}} \dot{f}_0, \dot{g} \rangle_\Sigma - \langle \Delta^{-\frac{1}{2}} \dot{f}_0, \ddot{g} \rangle_\Sigma + \langle \Delta^{\frac{1}{2}} \dot{f}_\Sigma, d_\Sigma \dot{g} \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} \dot{f}_\Sigma, d_\Sigma \dot{g} \rangle_\Sigma \\ &= \langle \Delta^{\frac{1}{2}} (\dot{f}_0 + \delta_\Sigma \dot{f}_\Sigma), \dot{g} \rangle_\Sigma + \langle \Delta^{-\frac{1}{2}} (\ddot{f}_0 + \delta_\Sigma \dot{f}_\Sigma), \dot{g} \rangle_\Sigma = -\langle \kappa^0(\delta f), \kappa^0(g) \rangle, \end{aligned}$$

where we used that  $\ddot{f} = -\Delta \dot{f}$  and  $\ddot{g} = -\Delta \dot{g}$ . This is consistent with the relation  $G(f, dg) = -G^0(\delta f, g)$ , which can be verified by a similar computation. If

$\begin{pmatrix} \dot{f} \\ \dot{f} \end{pmatrix} \in \ker \Delta$ , then our assumptions imply that  $\dot{f}_0 = \dot{f}_\Sigma = 0$ . Since in this case,  $\dot{f}_\Sigma$  and  $\dot{f}_\Sigma$  are both orthogonal to  $dC_0^\infty(\Sigma)$ , we infer that  $\omega^1(f, dg) = 0$ . The same argument as in the proof of 5.5 shows that  $\omega^0$  satisfies the micro local spectrum condition and we therefore conclude that the two point function  $\omega^0$  defined by

$$\omega^0(f, g) := \langle \kappa^0(f), \kappa^0(g) \rangle$$

has all the required properties. By the same argument as in Section 6, it suffices to verify these properties in a neighborhood of a Cauchy surface. Therefore, the construction extends to all globally hyperbolic manifolds that admit a Cauchy surface satisfying Assumption B.1.

### APPENDIX C. COHOMOLOGICAL INTERPRETATION OF THE SPACE $Z$

We now show that in many situations, the space  $Z$  has a cohomological interpretation. We begin with the case of an ultrastatic manifold  $M = \mathbb{R} \times \Sigma$ . Then, under the above Assumption B.1, the construction in Section 5 can be applied to obtain a generalized Fock representation. In this case, we know that  $H_{(2)}^0 = \{0\}$ , and moreover by Corollary 5.3 the space  $Z$  is naturally isomorphic to  $H_{(2)}^1(\Sigma) \oplus H_{(2)}^1(\Sigma)$  with standard complex structure. Thus, we obtain the decomposition  $Z = Y \oplus \tilde{Y}$  where  $Y$  is isomorphic to the first direct summand  $H_{(2)}^1(\Sigma)$ . Here  $H_{(2)}^p(\Sigma)$  denotes the reduced  $L^2$ -cohomology spaces  $\Omega_{(2)}^p(\Sigma) \cap \ker \Delta$ . For any  $y \in Y$ , there exists a unique form  $F_y \in \Omega^1(M)$  such that

$$\langle F_y, f \rangle = G_Z(\mathfrak{I}y, f), \quad \forall f \in \Omega_0^1(M).$$

This form is precisely the pull back of the corresponding  $L^2$ -harmonic form on  $\Sigma$  to  $\mathbb{R} \times \Sigma$ . Therefore, it not only solves the wave equation but is also closed. Depending on the interpretation of the reduced  $L^2$ -cohomology groups on  $\Sigma$ , this closed form has cohomological significance. For example, if  $\Sigma$  is a compactly supported perturbation of  $\mathbb{R}^3$ , the space  $H_{(2)}^1(\Sigma)$  is naturally isomorphic to  $H_0^1(\Sigma, \mathbb{R})$  so that each  $y \neq 0$  defines a non-trivial element in  $H_0^p(\Sigma)$ .

For general globally hyperbolic  $M$  and a given generalized Fock representation, the additional requirement that the complex structure is chosen such that

$$G_Z(y, \delta f) = 0 \quad \forall y \in Y, f \in \Omega_0^2(M)$$

seems reasonable and makes sense independent of a choice of foliation by Cauchy surfaces. For globally hyperbolic manifolds that admit a Cauchy surface satisfying Assumption B.1, the deformation argument as in Section 6 can be applied to show that such complex structures exist. Again, since  $G_Z$  was assumed to have smooth kernel, for any given  $y$  there exists a unique smooth closed one-form  $F_y$  such that

$$\langle F_y, f \rangle = G_Z(\mathfrak{J}y, f) \quad \forall f \in \Omega_0^1(M).$$

Under the above assumptions, this form will be closed and will therefore define a cohomology class.

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#### REFERENCES

- [1] A. Ashtekar and A. Sen, *On the role of space-time topology in quantum phenomena: superselection of charge and emergence of nontrivial vacua*, J. Math. Phys. **21** (1980), no. 3, 526–533.
- [2] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
- [3] C. Bär, N. Ginoux, and F. Pfäffle, *Wave Equations on Lorentzian Manifolds and Quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2007.
- [4] G. Barnich, F. Brandt, and M. Henneaux, *Local BRST cohomology in gauge theories*, arXiv:hep-th/0002245, Physics Reports **338** (2000), 439–569.
- [5] M. Benini, C. Dappiaggi, T.-P. Hack, and A. Schenkel, *A  $C^*$ -algebra for quantized principal  $U(1)$ -connections on globally hyperbolic lorentzian manifolds*, arXiv:1307.3052 [math-ph] (2013).
- [6] M. Benini, C. Dappiaggi, and A. Schenkel, *Quantized abelian principal connections on lorentzian manifolds*, arXiv:1303.2515 [math-ph] (2013).
- [7] A.N. Bernal and M. Sánchez, *On smooth Cauchy hypersurfaces and Geroch’s splitting theorem*, arXiv:gr-qc/0306108, Comm. Math. Phys. **243** (2003), no. 3, 461–470.
- [8] K. Bleuler, *Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen*, Helvetica Phys. Acta **23** (1950), 567–586.
- [9] R. Brunetti and K. Fredenhagen, *Interacting quantum fields in curved space: renormalizability of  $\phi^4$* , arXiv:gr-qc/9701048, Operator Algebras and Quantum Field Theory (Rome, 1996), Int. Press, Cambridge, MA, 1997, pp. 546–563.
- [10] R. Brunetti, K. Fredenhagen, and M. Köhler, *The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes*, arXiv:gr-qc/9510056, Comm. Math. Phys. **180** (1996), no. 3, 633–652.
- [11] G. Carron, *Théorèmes de l’indice sur les variétés non-compactes*, J. Reine Angew. Math. **541** (2001), 81–115.
- [12] ———, *Le saut en zéro de la fonction de décalage spectral*, J. Funct. Anal. **212** (2004), no. 1, 222–260.
- [13] C. Dappiaggi and B. Lang, *Quantization of Maxwell’s Equations on Curved Backgrounds and General Local Covariance*, arXiv:1104.1374v2 [gr-qc], Lett. Math. Phys. **101** (2012), no. 3, 265–287.
- [14] C. Dappiaggi and D. Siemssen, *Hadamard states for the vector potential on asymptotically flat spacetimes*, arXiv:1106.5575 [gr-qc] (2011).
- [15] N. Dencker, *On the propagation of polarization sets for systems of real principal type*, J. Funct. Anal. **46** (1982), no. 3, 351–372.
- [16] J. Dimock, *Algebras of local observables on a manifold*, Comm. Math. Phys. **77** (1980), no. 3, 219–228.
- [17] ———, *Quantized electromagnetic field on a manifold*, Rev. Math. Phys. **4** (1992), no. 2, 223–233.

- [18] M. Dütsch and K. Fredenhagen, *A local (perturbative) construction of observables in gauge theories: the example of QED*, arXiv:hep-th/9807078, *Comm. Math. Phys.* **203** (1999), no. 1, 71–105.
- [19] C.J. Fewster and M.J. Pfenning, *A quantum weak energy inequality for spin-one fields in curved space-time*, arXiv:gr-qc/0303106, *J. Math. Phys.* **44** (2003), no. 10, 4480–4513.
- [20] S.A. Fulling, F.J. Narcowich, and R.M. Wald, *Singularity structure of the two-point function in quantum field theory in curved spacetime. II*, *Ann. Physics* **136** (1981), no. 2, 243–272.
- [21] E.P. Furlani, *Quantization of the electromagnetic field on static space-times*, *J. Math. Phys.* **36** (1995), no. 3, 1063–1079.
- [22] W. Greiner and J. Reinhardt, *Field quantization*, Springer-Verlag, Berlin, 1996.
- [23] S.N. Gupta, *Theory of longitudinal photons in quantum electrodynamics*, *Proc. Phys. Soc. Sect. A.* **63** (1950), 681–691.
- [24] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton, NJ, 1992.
- [25] S. Hollands, *Renormalized quantum Yang-Mills fields in curved spacetime*, arXiv:0705.3340 [gr-qc], *Rev. Math. Phys.* **20** (2008), no. 9, 1033–1172.
- [26] S. Hollands and R.M. Wald, *Local Wick polynomials and time ordered products of quantum fields in curved spacetime*, arXiv:gr-qc/0103074, *Comm. Math. Phys.* **223** (2001), no. 2, 289–326.
- [27] ———, *Existence of local covariant time ordered products of quantum field in curved spacetime*, arXiv:gr-qc/0111108, *Comm. Math. Phys.* **231** (2002), no. 2, 309–345.
- [28] L. Hörmander, *The Analysis of Linear Partial Differential Operators. III*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 274, Springer-Verlag, Berlin, 1985.
- [29] F. John, *Partial Differential Equations*, fourth ed., Applied Mathematical Sciences, vol. 1, Springer-Verlag, New York, 1991.
- [30] B. Lautrup, *Canonical quantum electrodynamics in covariant gauges*, *Mat. Fys. Medd. Dan. Vid. Selsk.* **35** (1966), 11.
- [31] R.B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, Research Notes in Mathematics, vol. 4, A K Peters Ltd., Wellesley, MA, 1993.
- [32] ———, *Geometric Scattering Theory*, Stanford Lectures, Cambridge University Press, Cambridge, 1995.
- [33] O. Müller, *Asymptotic flexibility of globally hyperbolic manifolds*, arXiv:1110.1037 [math.DG], *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 7-8, 421–423.
- [34] W. Müller, *Eta invariants and manifolds with boundary*, *J. Differential Geom.* **40** (1994), no. 2, 311–377.
- [35] W. Müller and A. Strohmaier, *Scattering at low energies on manifolds with cylindrical ends and stable systoles*, arXiv:0907.3517 [math.AP], *Geom. Funct. Anal.* **20** (2010), no. 3, 741–778.
- [36] M.E. Peskin and A. Tonomura, *The Aharonov-Bohm Effect*, Lecture Notes in Physics, vol. 340, Springer Verlag, Berlin, 1989.
- [37] M.J. Radzikowski, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, *Comm. Math. Phys.* **179** (1996), no. 3, 529–553.
- [38] K. Sanders, *Equivalence of the (generalised) Hadamard and microlocal spectrum condition for (generalised) free fields in curved spacetime*, arXiv:0903.1021 [math-ph], *Comm. Math. Phys.* **295** (2010), no. 2, 485–501.
- [39] K. Sanders, C. Dappiaggi, and T.-H. Hack, *Electromagnetism, local covariance, the Aharonov-Bohm effect and Gauss’ law*, arXiv:1211.6420 [math-ph] (2012).
- [40] R.F. Streater, *Spontaneous breakdown of symmetry in axiomatic theory*, *Proc. Roy. Soc. Ser. A* **287** (1965), 510–518.
- [41] F. Strocchi, *Gauge problem in quantum field theory*, *Phys. Rev. (2)* **162** (1967), 1429–1438.
- [42] F. Strocchi and A.S. Wightman, *Proof of the charge superselection rule in local relativistic quantum field theory*, *J. Math. Phys.* **15** (1974), 2198–2224.
- [43] ———, *Erratum: “Proof of the charge superselection rule in local relativistic quantum field theory”* (*J. Math. Phys.* **15** (1974), 2198–2224), *J. Math. Phys.* **17** (1976), no. 10, 1930–1931.
- [44] A. Strohmaier, R. Verch, and M. Wollenberg, *Microlocal analysis of quantum fields on curved space-times: analytic wave front sets and Reeh-Schlieder theorems*, arXiv:math-ph/0202003, *J. Math. Phys.* **43** (2002), no. 11, 5514–5530.

- [45] X.P. Wang, *Asymptotic expansion in time of the Schrödinger group on conical manifolds*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 6, 1903–1945.
- [46] S. Weinberg, *The Quantum Theory of Fields. Vol. I*, Cambridge University Press, Cambridge, 1996, Foundations, Corrected reprint of the 1995 original.

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