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ELLIPTIC, PARABOLIC AND HYPERBOLIC ANALYTIC FUNCTION THEORY-0: GEOMETRY OF DOMAINS

VLADIMIR V. KISIL AND DEBAPRIYA BISWAS

ABSTRACT. This paper lays down a foundation for a systematic treatment of three main (elliptic, parabolic and hyperbolic) types of analytic function theory based on the representation theory of $SL_2(\mathbb{R})$ group. We describe here geometries of corresponding domains. The principal rôle is played by Clifford algebras of matching types.

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Most attractive feature of most exotic places is the presence
of a standard tourist accommodation

1. INTRODUCTION

Starting from the early age of mathematics as a science we repeatedly meet the division of various mathematical objects into three main classes. In very different areas (equations, quadratic forms, metrics, manifolds, operators, etc.) these classes preserved names obtained by the very first example—the classification of conic sections: elliptic, parabolic, hyperbolic. We will abbreviate this separation as *EPH-classification*. The *common origin* of this fundamental division can be seen from the simple picture of a coordinate line split by the zero into negative and positive half-axes:



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However connections between different objects admitting EPH-classification are not limited to this common source. There are many deep results linking, for example, ellipticity of quadratic forms, metrics and operators. On the other hand there are a lot of white spots and obscure gaps between some subjects as well.

For example, it is well known that elliptic operators are effectively treated through complex analysis, which can be naturally identified as the *elliptic analytic function theory* [9, 11, 14]. Thus there are natural questions about *hyperbolic* and *parabolic* analytic function theories, which will be of similar importance for corresponding types of operators. A search for hyperbolic function theory was initiated in the book [19] with some important advances achieved.

An alternative approach to analytic function theories based on the representation theory of semisimple Lie groups was developed in the series of papers [7, 8, 9, 10, 11, 12, 13, 14]. Particularly, a hyperbolic function theory was built in [8, 9, 11] along the same lines as the elliptic one—standard complex analysis.

This paper makes a further step forwards in this direction. We lay down foundations for all three (including parabolic!) EPH-types of analytic function theories. But the present step is rather modest: we just study geometries of corresponding domains.

Remark 1.1. Introducing parabolic objects on a common ground with elliptic and hyperbolic ones we should warn against two common prejudices suggested by picture (1.1):

- (1) The parabolic case is unimportant (has “zero measure”) in comparison to the elliptic and hyperbolic ones. As we shall see (e.g. Remark 2.18) some geometrical features are richer in parabolic case.
- (2) The parabolic case is a limiting situation or an intermediate position between the elliptic and hyperbolic: all properties of the former can be guessed or obtained as a limit or an average from the later two. Particularly this point of view is implicitly supposed in [19].

Although there are some confirmations of this (e.g. Figures 6(*E*)–(*H*)), we shall see (e.g. Remark 2.11) that some properties of the parabolic case cannot be straightforwardly guessed from a combination of elliptic and hyperbolic cases.

An amazing aspect of this topic is a transparent similarity between all three EPH cases which is combined with some non-trivial exceptions like *non-invariance* of the upper half plane in the hyperbolic case (Subsection 2.5) or *non-symmetric* length and orthogonality in the parabolic case (Lemma 2.10.p). The elliptic case seems to be free from any such irregularities only because it sets itself the standards to others.

This paper contains some results and many pictures but almost no proofs, which are not difficult anyway in most cases. There are only two notable exceptions (Lemmas ?? and 2.13) when proofs themselves bring additional insights into the subject.

2. ELLIPTIC, PARABOLIC AND HYPERBOLIC SPACES

2.1. $SL_2(\mathbb{R})$ group and Clifford Algebras. We use representations of the $SL_2(\mathbb{R})$ group in Clifford valued function spaces. There will be three different Clifford algebras $\mathcal{C}(e)$, $\mathcal{C}(p)$, $\mathcal{C}(h)$ corresponding to *elliptic*, *parabolic*, and *hyperbolic* cases respectively. The notation $\mathcal{C}(a)$ refers to *any* of these three algebras.

A Clifford algebra $\mathcal{C}\ell(a)$ as a 4-dimensional linear space is spanned by $1, e_1, e_2, e_1e_2$ with *non-commutative* multiplication defined by the identities:

$$(2.1) \quad e_1^2 = -1, \quad e_2^2 = \begin{cases} -1, & \text{for } \mathcal{C}\ell(e)\text{—elliptic case} \\ 0, & \text{for } \mathcal{C}\ell(p)\text{—parabolic case} \\ 1, & \text{for } \mathcal{C}\ell(h)\text{—hyperbolic case} \end{cases}, \quad e_1e_2 = -e_2e_1.$$

The two-dimensional subalgebra of $\mathcal{C}\ell(e)$ spanned by 1 and $i = e_2e_1 = -e_1e_2$ is *isomorphic* (and can actually replace in all calculations!) the field of complex numbers \mathbb{C} . For any $\mathcal{C}\ell(a)$ we identify \mathbb{R}^2 with the set of vectors $w = ue_1 + ve_2$, where $(u, v) \in \mathbb{R}^2$. In the elliptic case of $\mathcal{C}\ell(e)$ this maps

$$(2.2) \quad (u, v) \mapsto e_1(u + iv) = e_1z, \quad \text{with } z = u + iv \text{ a standard complex number.}$$

We denote \mathbb{R}^2 by $\mathbb{R}^e, \mathbb{R}^p$ or \mathbb{R}^h to highlight which of Clifford algebras is used in the present context. The notation \mathbb{R}^a assumes $\mathcal{C}\ell(a)$.

The $SL_2(\mathbb{R})$ group [5, 18, 22] consists of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{with } a, b, c, d \in \mathbb{R} \text{ and the determinant } ad - bc = 1.$$

An isomorphic realisation of $SL_2(\mathbb{R})$ with the same multiplication is obtained if we replace a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix}$ within any $\mathcal{C}\ell(a)$. The advantage of the later form is that we can define the *Möbius transformation* of $\mathbb{R}^a \rightarrow \mathbb{R}^a$ for all three algebras $\mathcal{C}\ell(a)$ by the same expression:

$$(2.3) \quad \begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : ue_1 + ve_2 \mapsto \frac{a(ue_1 + ve_2) - be_1}{ce_1(ue_1 + ve_2) + d},$$

where the expression $\frac{a}{b}$ in a non-commutative algebra is always understood as ab^{-1} , see [3, 4]. Therefore $\frac{ac}{bc} = \frac{a}{b}$ but $\frac{ca}{cb} \neq \frac{a}{b}$ in general.

Again in the elliptic case the transformation (2.3) is equivalent to

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : e_1z \mapsto \frac{e_1(a(u + e_2e_1v) - b)}{-c(u + e_2e_1v) + d} = e_1 \frac{az - b}{-cz + d}, \quad \text{where } z = u + iv,$$

which is the standard form of a Möbius transformation. One can straightforwardly verify that the map (2.3) is a left action of $SL_2(\mathbb{R})$ on \mathbb{R}^a , i.e. $g_1(g_2w) = (g_1g_2)w$.

To study the finer structure of Möbius transformations it is useful to decompose an element g of $SL_2(\mathbb{R})$ into the product $g = g_ag_ng_k$:

$$(2.4) \quad \begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & \chi e_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & e_1 \sin \phi \\ e_1 \sin \phi & \cos \phi \end{pmatrix},$$

where the values of parameters are as follows:

$$(2.5) \quad \alpha = \sqrt{c^2 + d^2}, \quad \chi = \frac{d - a(c^2 + d^2)}{c} = \frac{b(c^2 + d^2) - c}{d}, \quad \phi = \tan^{-1} \frac{c}{d}.$$

Consequently $\cos \phi = \frac{d}{\sqrt{c^2 + d^2}}$ and $\sin \phi = \frac{c}{\sqrt{c^2 + d^2}}$. The product (2.4) gives a realisation of the *Iwasawa decomposition* [18, § III.1] in the form $SL_2(\mathbb{R}) = ANK$, where K is the maximal compact group, N is nilpotent and A normalises N .

2.2. Actions of Subgroups and Invariance of Sections. In all three EPH cases subgroups the A and N act through Möbius transformation uniformly:

Lemma 2.1. *For any type of the Clifford algebra $\mathcal{C}\ell(a)$:*

- (1) *The subgroup N defines shifts $ue_1 + ve_2 \mapsto (u + \chi)e_1 + ue_2$ along the “real” axis U by χ .*

The vector field of the derived representation is $dN_a(u, v) = (1, 0)$.

- (2) The subgroup A defines dilations $ue_1 + ve_2 \mapsto \alpha^2(ue_1 + ve_2)$ by the factor α^2 which fixes origin $(0, 0)$.

The vector field of the derived representation is $dA_a(u, v) = (2u, 2v)$.

Orbits and vector fields corresponding to the *derived representation* [6, § 6.3], [18, Chap. VI] of the Lie algebra \mathfrak{sl}_2 for subgroups A and N are shown in Figure 1. Thin transverse lines join points of orbits corresponding to the same values of the parameter.

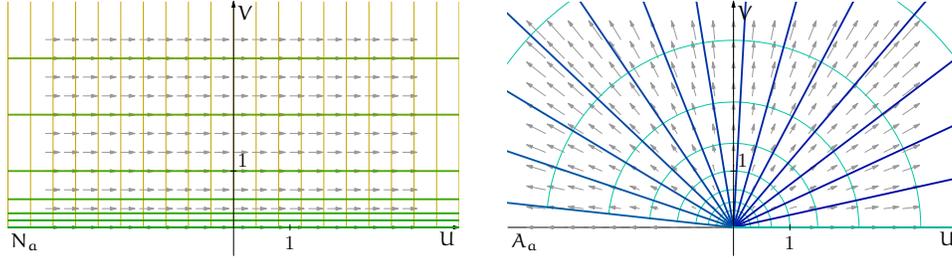


FIGURE 1. Actions of the subgroups A and N by Möbius transformations

By contrast the actions of the subgroup K is significantly different between the EPH cases and correlates with names chosen for $\mathcal{C}(e)$, $\mathcal{C}(p)$, $\mathcal{C}(h)$:

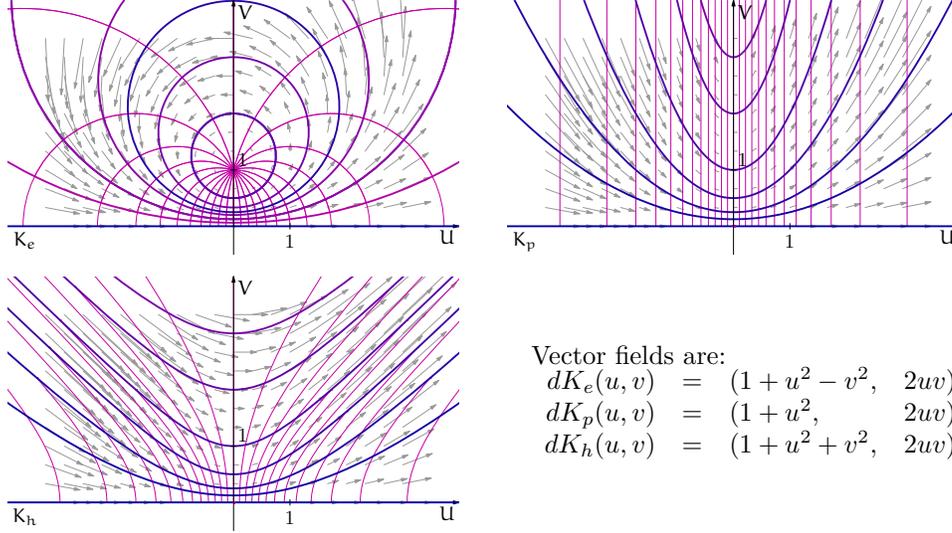


FIGURE 2. Action of the K subgroup. The corresponding orbits are circles, parabolas and hyperbolas.

Lemma 2.2. *The actions of the subgroup K in three cases are as follows:*

- (e) For $\mathcal{C}(e)$ the orbits of K are circles. A circle with centre at $(0, (v + v^{-1})/2)$ passing through two points $(0, v)$ and $(0, v^{-1})$.
The vector field of the derived representation is $dK_e(u, v) = (u^2 - v^2 + 1, 2uv)$.
- (p) For $\mathcal{C}(p)$ the orbits of K are parabolas with the vertical axis V . A parabola passing through $(0, v/2)$ has its horizontal directrix passing through $(0, (v - v^{-1})/2)$ and focus at $(0, (v + v^{-1})/2)$.
The vector field of the derived representation is $dK_p(u, v) = (u^2 + 1, 2uv)$.

- (h) For $\mathcal{C}\ell(h)$ the orbits of K are hyperbolas with asymptotes parallel to lines $u = \pm v$. A hyperbola passing through the point $(0, v)$ has the focal distance between foci $2p$, where $p = \frac{v^2+1}{\sqrt{2v}}$ and the upper focus is located at $(0, f)$ with:

$$f = \begin{cases} p - \sqrt{\frac{p^2}{2} - 1}, & \text{for } 0 < v < 1; \text{ and} \\ p + \sqrt{\frac{p^2}{2} - 1}, & \text{for } v \geq 1. \end{cases}$$

The vector field of the derived representation is $dK_h(u, v) = (u^2 + v^2 + 1, 2uv)$.

Orbits and the corresponding derived actions of the subgroup K are shown in Figure 2.

- Remark 2.3.* (1) The values of all three vector fields dK_e , dK_p and dK_h coincide on the “real” U -axis $v = 0$, i.e. they are three different extensions into the domain of the same boundary condition.
 (2) The hyperbola passing through the point $(0, 1)$ has the shortest focal length $\sqrt{2}$ among all other hyperbolic orbits; two hyperbolas passing through $(0, v)$ and $(0, v^{-1})$ have the same focal length and are related to each other as explained in Remark 2.15.1.

Definition 2.4. We use the word *cycle* to denote straight lines **and one of the following**

- (e) Circles in the elliptic case;
- (p) Parabolas with a vertical axis of symmetry in the parabolic case;
- (h) Equilateral hyperbolas with a vertical axis of symmetry in the hyperbolic case.

Moreover the words *parabola* and *hyperbola* in this paper always assume only ones of the above described types.

Centre of a cycle is its geometrical centre for a circle or a hyperbola and its focus for a parabola. Centres of straight lines are at infinity.

Using the Lemmas 2.1 and 2.2 we can give an easy proof of invariance for corresponding cycles.

Lemma 2.5. *Möbius transformations preserve the cycles in the upper half plane, i.e.:*

- (e) For $\mathcal{C}\ell(e)$ Möbius transformations map circles to circles.
- (p) For $\mathcal{C}\ell(p)$ Möbius transformations map parabolas to parabolas.
- (h) For $\mathcal{C}\ell(h)$ Möbius transformations map hyperbolas to hyperbolas.

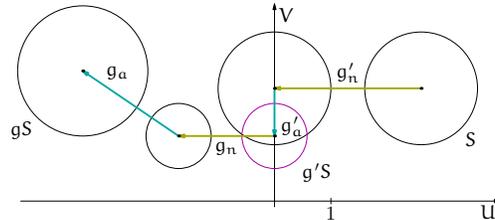


FIGURE 3. The decomposition of an arbitrary Möbius transformation g into a product $g = g_a g_n g_k g'_a g'_n$.

Proof. Our first observation is that the subgroups A and N obviously preserve all circles, parabolas, hyperbolas and straight lines in all $\mathcal{C}\ell(a)$. Thus we use subgroups

A and N to fit a given cycle exactly on a particular orbit of subgroup K shown on Figure 2 of the corresponding type.

To this end for an arbitrary cycle S we can find $g'_n \in N$ which puts centre of S on the V -axis, see Figure 3. Then there is a unique $g'_a \in A$ which scales it exactly to an orbit of K , e.g. for a circle passing through point $(0, v_1)$ and $(0, v_2)$ the scaling factor is $\frac{1}{\sqrt{v_1 v_2}}$ accordingly to Lemma 2.2.e. Let $g' = g'_a g'_n$, then for any element $g \in SL_2(\mathbb{R})$ using the Iwasawa decomposition of $gg'^{-1} = g_a g_n g_k$ we get the presentation $g = g_a g_n g_k g'_a g'_n$ with $g_a, g'_a \in A$, $g_n, g'_n \in N$ and $g_k \in K$.

Then the image $g'S$ of the cycle S under $g' = g'_a g'_n$ is a cycle itself in the obvious way, then $g_k(g'S)$ is again a cycle since $g'S$ was arranged to coincide with a K -orbit, and finally $gS = g_a g_n(g_k(g'S))$ is a cycle due to the obvious action of $g_a g_n$, see Figure 3 for an illustration. \square

2.3. Lengths and Orthogonality. The invariance of cycles (see Lemma 2.5) suggests using them in the rôle of circles in each of the EPH cases and play *the standard mathematical game*: turn some properties of classical objects into definitions of new ones.

Definition 2.6. The length $l_a(\overrightarrow{AB})$ of a vector \overrightarrow{AB} in \mathbb{R}^a is defined as a real valued function, such that for a fixed A the level curves of $l_a(\overrightarrow{AB})$ are of corresponding shapes: circles, parabolas or hyperbolas.

Lemma 2.7. *The following are lengths in the sense of Definition 2.6:*

- (e) *In the elliptic case: the Euclidean metric $l_e(ue_1 + ve_2) = u^2 + v^2$.*
- (p) *In the parabolic case: a monotonic function of focal length:*

$$(2.6) \quad l_p(ue_1 + ve_2) = \sqrt{u^2 + v^2} - v$$

for a parabola with focus A passing through B . Note, that $l(\overrightarrow{AB}) \neq l(\overrightarrow{BA})!$

- (h) *In the hyperbolic case: the Minkowski metric $l_h(ue_1 + ve_2) = u^2 - v^2$.*

Remark 2.8. In the elliptic and hyperbolic cases the above lengths are conveniently defined by the Clifford algebra multiplication

$$l_{e,h}(ue_1 + ve_2) = -(ue_1 + ve_2)^2.$$

It will be interesting to find some sort of such relation for the parabolic length (2.6) as well.

Definition 2.9. We say that a vector \overrightarrow{AB} is *s-orthogonal* to a vector \overrightarrow{CD} and denote it $\overrightarrow{AB} \dashv \overrightarrow{CD}$ (for the reasons clear from Lemma 2.10.p) if the function $l(\overrightarrow{AB} + \epsilon \overrightarrow{CD})$ of a variable ϵ has a local extremum at $\epsilon = 0$ (i.e. orthogonality provides the shortest length).

Again s-orthogonality turns out to be the usual orthogonality in the elliptic case. For the two other cases the description is given as follows:

Lemma 2.10. *A vector $ue_1 + ve_2$ is s-orthogonal to a vector $u'e_1 + v'e_2$ if in terms of Euclidean geometry:*

- (e) *In the elliptic case: two vectors form a right angle, or analytically $uu' + vv' = 0$.*
- (p) *In the parabolic case: the vector $ue_1 + ve_2$ bisects the angle between $u'e_1 + v'e_2$ and the vertical directions or analytically:*

$$(2.7) \quad u'u - v'l_p(ue_1 + ve_2) = u'u - v'(\sqrt{u^2 + v^2} - v) = 0.$$

*Note that $\overrightarrow{AB} \dashv \overrightarrow{CD}$ does **not** necessarily imply $\overrightarrow{CD} \dashv \overrightarrow{AB}!$*

- (h) *In the hyperbolic case the angles between two vectors are bisected by lines parallel to $u = \pm v$, or analytically $u'u - v'v = 0$.*

Remark 2.11. If one tries to devise a parabolic length as a limit or an intermediate case between the elliptic $l_e = u^2 + v^2$ and hyperbolic $l_p = u^2 - v^2$ lengths then the only possible guess is $l'_p = u^2$, which is too trivial for an interesting geometry.

Similarly the only orthogonality conditions bridging elliptic $u_1u_2 + v_1v_2 = 0$ and hyperbolic $u_1u_2 - v_1v_2 = 0$ seems to be $u_1u_2 = 0$ which is again too trivial. This support our Remark 1.1.2.

2.4. Zero Radius Cycles, Invariant Measure and Compactification. Of course, Möbius transformations may not preserve centres of cycles. However this happens in a trivial way for “zero radius” cycles, as follows.

Lemma 2.12. *A zero-radius cycle with centre at (u_0, v_0) defined by the equation $l_a(u - u_0, v - v_0) = 0$ is:*

- (e) *a single point (u_0, v_0) in the elliptic case;*
- (p) *the vertical upward directed ray with origin at (u_0, v_0) in the parabolic case;*
- (h) *the light cone with origin at (u_0, v_0) defined by the equation*

$$(2.8) \quad (u - u_0)^2 - (v - v_0)^2 = 0$$

in the hyperbolic case.

In the elliptic and hyperbolic cases it is often useful [3, 4] to consider zero radius cycles instead of corresponding points which is known as Fillmore-Springer-Cnops construction. The same advantages are expected in the parabolic case as well. Among many useful applications the embedding of \mathbb{R}^a into a bigger space of spheres produces the invariant measure in an elegant way [3, 4]. We give another proof based on the Iwasawa decomposition.

Lemma 2.13. *A Möbius invariant measure on \mathbb{R}^a is given by $\frac{du dv}{v^2}$.*

Proof. Let $f(u, v) du dv$ be an invariant measure. Then considering shifts generated by the subgroup N (see Figure 1) we conclude that $f(u, v)$ is independent of u , thus we denote it by $f(v)$. The dilations generated by the subgroup A (see Figure 1) put the restriction $f(v) = cv^{-2}$ which is obviously compatible with any K action since ∂_v components of all vector fields dK_a are the same. \square

Another important rôle of the zero radius cycles is the proper compactification of \mathbb{R}^a . Indeed the initial space \mathbb{R}^a is not a closed set under a generic Möbius transformations. In the elliptic case the problem is solved by the compactification of \mathbb{R}^e with a point ∞ at infinity. However in the parabolic and hyperbolic cases the singularity of the Möbius transform is not localised in a single point—the denominator vanish for the whole zero radius cycle. Thus in each EPH case the correct compactification is made by a *zero radius cycle at infinity*. Of course in the elliptic case this is still a point, but for the two other cases the result is significantly different.

It is common to identify the compactification \mathbb{R}^e of the space \mathbb{R}^e by a point ∞ with a Riemann sphere. This model can be visualised by the stereographic projection [2, § 18.1.4]. The projection from the centre of a sphere provides a model for the compactification $\widehat{\mathbb{R}}^p$ in the parabolic case. The space $\widehat{\mathbb{R}}^p$ is represented by a sphere where all pairs of opposite points are identified. The “half of the equator” in this model represents the parabolic zero radius cycle (see Lemma 2.12.p) at infinity. More informative models are provided by the Fillmore-Springer-Cnops construction, which represent Möbius transformations through orthogonal rotations in the bigger space of spheres [3, 4].

The hyperbolic case produces its own caveats. A compactification of the hyperbolic space \mathbb{R}^h by a light cone at infinity will produce a closed Möbius invariant object. However it will not be satisfactory for some other reasons explained in the next Subsection.

2.5. (Non)-Invariance of The Upper Half Plane. The important difference between the hyperbolic case and the two others is that

Lemma 2.14. *In the elliptic and parabolic cases the upper halfplane in \mathbb{R}^a is preserved by Möbius transformations from $SL_2(\mathbb{R})$. However in the hyperbolic case any point (u, v) with $v > 0$ can be mapped into an arbitrary point (u', v') with $v' \neq 0$.*

The lack of invariance in the hyperbolic case has many important consequences in seemingly different areas, for example:

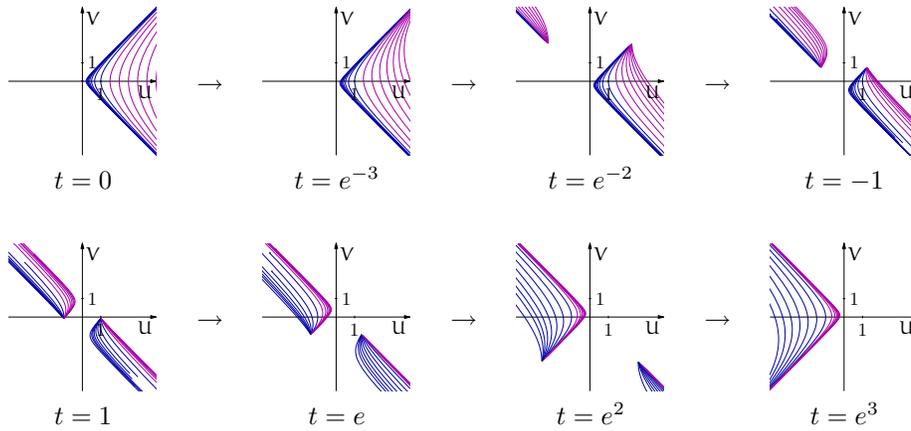


FIGURE 4. Eight frames from a continuous transformation from future to the past parts of the light cone.

Geometry: \mathbb{R}^h is not split by the real axis into two disjoint pieces: there is a continuous path (through the light cone at infinity) from the upper half plane to lower which does not cross the real axis (see the sin-like joined two pieces of the hyperbola in Figure 5(a)).

Physics: There is no Möbius invariant way to separate “past” and “future” parts of the light cone [21], i.e. there is a continuous family of Möbius transformations reversing the arrow of time. For example, The family of matrices $\begin{pmatrix} 1 & -te_2 \\ te_2 & 1 \end{pmatrix}$, $t \in [0, \infty)$ provide the transformations and Figure 4 presents images for eight values of t .

Analysis: There is no a possibility to split $L_2(\mathbb{R})$ space of function into a direct sum of the Hardy space of functions having an analytic extension into the upper half plane and its non-trivial complement, i.e. any function from $L_2(\mathbb{R})$ has an “analytic extension” into the upper half plane, see [11].

All the above problems can be resolved in the following way [11, § A.3]. We take two copies \mathbb{R}_+^h and \mathbb{R}_-^h of \mathbb{R}^h , depicted by squares $ACA'C''$ and $A'C'A''C''$ in Figure 5 correspondingly. The boundaries of these squares are light cones at infinity and we glue \mathbb{R}_+^h and \mathbb{R}_-^h in such a way that the construction is invariant under the natural action of the Möbius transformation. That is achieved if the same letters A, B, C, D, E in Figure 5 are identified regardless the number of attached primes. This aggregate denoted by $\tilde{\mathbb{R}}^h$ is a two-fold cover of \mathbb{R}^h . The hyperbolic “upper” half

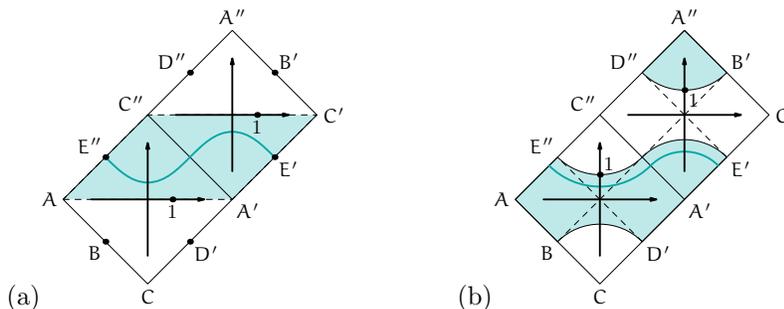


FIGURE 5. Hyperbolic objects in the double cover of \mathbb{R}^h :
 (a) the “upper” half plane; (b) the unit circle.

plane in $\tilde{\mathbb{R}}^h$ consists of the upper halfplane in \mathbb{R}_+^h and the lower one in \mathbb{R}_-^h . A similar conformally invariant two-fold cover of the Minkowski space-time was constructed in [21, § III.4] in connection with the red shift problem in extragalactic astronomy.

Remark 2.15. (1) The hyperbolic orbit of the K subgroup in the $\tilde{\mathbb{R}}^h$ consists of two branches of the hyperbola passing through $(0, v)$ in \mathbb{R}_+^h and $(0, -v^{-1})$ in \mathbb{R}_-^h , see Figure 5. As explained in Remark 2.3.2 they both have the same focal length.

(2) The “upper” halfplane is bounded by two disjoint “real” axes denoted by AA' and $C'C''$ on Figure 5.

For the hyperbolic Cayley transform in the next subsection we need the conformal version of the hyperbolic unit disk. We define it in $\tilde{\mathbb{R}}^{1,1}$ as follows:

$$\begin{aligned} \tilde{\mathbb{D}} = & \{(ue_1 + ve_2) \mid l_h(ue_1 + ve_2) < -1, u \in \mathbb{R}_+^{1,1}\} \\ & \cup \{(ue_1 + ve_2) \mid l_h(ue_1 + ve_2) > -1, u \in \mathbb{R}_-^{1,1}\}. \end{aligned}$$

It can be shown that $\tilde{\mathbb{D}}$ is conformally invariant and has a boundary $\tilde{\mathbb{T}}$ —the two copies of the unit circles in $\mathbb{R}_+^{1,1}$ and $\mathbb{R}_-^{1,1}$. We call $\tilde{\mathbb{T}}$ the (*conformal*) *unit circle* in $\mathbb{R}^{1,1}$. Figure 5 illustrates¹ the geometry of the “upper” half plane as well as the conformal unit disk in $\tilde{\mathbb{R}}^{1,1}$ conformally equivalent to it.

2.6. The Cayley Transform and Unit “Circles”. The upper half plane is the universal starting point for an analytic function theory of any EPH type. However universal models are rarely best suited to particular circumstances. For many reasons it is more convenient to consider analytic functions on the unit disk rather than on the upper half plane, although both theories are completely isomorphic, of course. This isomorphism is delivered by the *Cayley transform*.

Let $\sigma = e_2^2$, i.e. $-1, 0$, or 1 as in (2.1). Then the first possibility to define the Cayley transform is given by the matrix $C = \begin{pmatrix} 1 & -e_2 \\ \sigma e_2 & 1 \end{pmatrix}$ with determinant 1. It can be applied as the Möbius transformation

$$(2.9) \quad \begin{pmatrix} 1 & -e_2 \\ \sigma e_2 & 1 \end{pmatrix} : w = (ue_1 + ve_2) \mapsto Cw = \frac{(ue_1 + ve_2) - e_2}{\sigma e_2(ue_1 + ve_2) + 1}$$

to a point $(ue_1 + ve_2) \in \mathbb{R}^a$. Alternatively it acts by conjugation $g_C = CgC^{-1}$ on an element $g \in SL_2(\mathbb{R})$:

$$(2.10) \quad g_C = \frac{1}{2} \begin{pmatrix} 1 & -e_2 \\ \sigma e_2 & 1 \end{pmatrix} \begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} \begin{pmatrix} 1 & e_2 \\ -\sigma e_2 & 1 \end{pmatrix}$$

¹Note that similar figures in papers [11, 15] have letters D' and E' misplaced.

The connection between the two forms (2.9) and (2.10) of the Cayley transform is given by $g_C C w = C(gw)$, i.e. C intertwines the actions of g and g_C .

The Cayley transform $(u'e_1 + v'e_2) = C(ue_1 + ve_2)$ in the elliptic case is very important [18, § IX.3], [22, Ch. 8, (1.12)]. The transformation $g \mapsto g_C$ (2.10) is an isomorphism of the groups $SL_2(\mathbb{R})$ and $SU(1, 1)$, namely in $\mathcal{C}(e)$ we have

$$(2.11) \quad g_C = \begin{pmatrix} f & h \\ -h & f \end{pmatrix}, \text{ with } f = (a+d) - (c-b)e_2e_1 \text{ and } h = (a-d)e_2 - (b+c)e_1.$$

Under the map $\mathbb{R}^e \rightarrow \mathbb{C}$ (2.2) this matrix becomes $\begin{pmatrix} \alpha & \beta \\ \bar{\alpha} & \bar{\beta} \end{pmatrix}$, i.e. the standard form of elements of $SU(1, 1)$ [18, § IX.1], [22, Ch. 8, (1.11)].

The images of elliptic actions of subgroups A , N , K are given in Figure 6(E). The types of orbits can be easily distinguished by the number of *fixed points on the boundary*: two, one and zero correspondingly. In some sense the Cayley transform swaps complexities: in contrast to on the upper half plane the K -action is now simple but A and N are not. The simplicity of K orbits is explained by the diagonalisation of matrices:

$$(2.12) \quad \frac{1}{2} \begin{pmatrix} 1 & -e_2 \\ -e_2 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & -e_1 \sin \phi \\ -e_1 \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 & e_2 \\ e_2 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix},$$

where $i = e_1e_2$ behaves as the complex imaginary unit, i.e. $i^2 = -1$.

A hyperbolic version of the Cayley transform was used in [11]. The above formula (2.10) in \mathbb{R}^h becomes as follows:

$$(2.13) \quad g_C = \begin{pmatrix} f & h \\ h & f \end{pmatrix}, \text{ with } h = a + d - (b+c)e_2e_1 \text{ and } f = (a-d)e_2 + (c-b)e_1,$$

with some subtle differences in comparison with (2.11). The corresponding A , N and K orbits are given on Figure 6(H). However there is an important difference between the elliptic and hyperbolic cases similar to the one discussed in subsection 2.5.

Lemma 2.16. (1) *In the elliptic case the “real axis” U is transformed to the unit circle and the upper half plane—to the unit disk:*

$$(2.14) \quad \{(u, v) \mid v = 0\} \rightarrow \{(u', v') \mid l_e(u'e_1 + v'e_2) = u'^2 + v'^2 = 1\}$$

$$(2.15) \quad \{(u, v) \mid v > 0\} \rightarrow \{(u', v') \mid l_e(u'e_2 + v'e_2) = u'^2 + v'^2 < 1\}.$$

On both sets $SL_2(\mathbb{R})$ acts transitively and the unit circle is generated, for example, by the point $(1, 0)$ and the unit disk is generated by $(0, 0)$.

(2) *In the hyperbolic case the “real axis” U is transformed to the hyperbolic unit circle:*

$$(2.16) \quad \{(u, v) \mid v = 0\} \rightarrow \{(u', v') \mid l_h(u'e_1 + v'e_2) = u'^2 - v'^2 = -1\}$$

On the hyperbolic unit circle $SL_2(\mathbb{R})$ acts transitively and it is generated, for example, by point $(0, 1)$.

$SL_2(\mathbb{R})$ acts also transitively on the whole complement $\{(u', v') \mid l_e(u'e_2 + v'e_2) \neq -1\}$ to the unit circle, i.e. on its “inner” and “outer” parts together.

The last feature of the hyperbolic Cayley transform can be treated in a way described in the end of subsection 2.5, see also Figure 5(b). With such an arrangement the hyperbolic Cayley transform maps the “upper half” plane from Figure 5(a) onto the “inner part” of the unit disk from Figure 5(b).

One may wish that the hyperbolic Cayley transform diagonalises the action of A subgroup in a fashion similar to (2.12) for K . That is achieved [11, Ex. 3.1(b)]

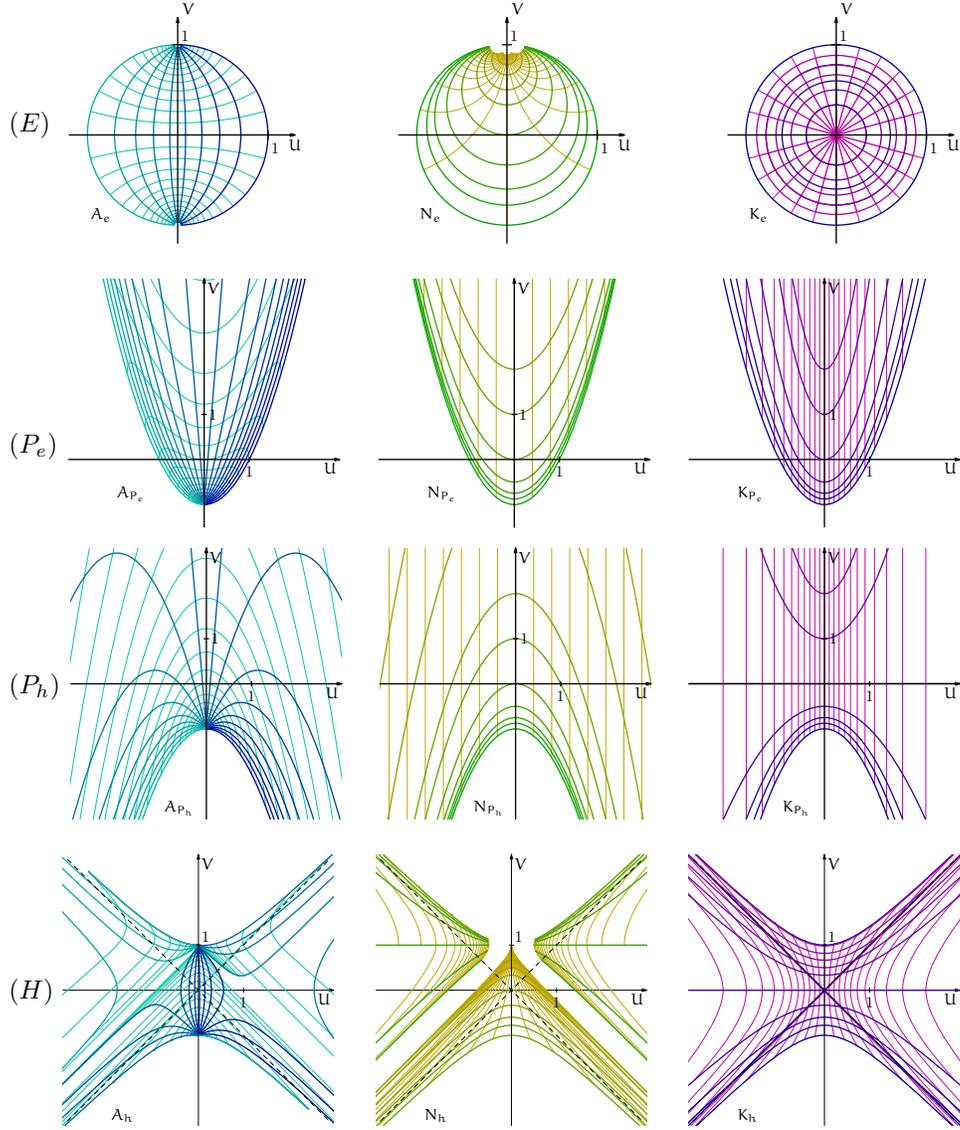


FIGURE 6. The images of unit disks with orbits of subgroups A , N and K correspondingly:
 (E): The elliptic unit disk;
 (Pe): The first version of parabolic unit disk with an *elliptic type* of Cayley transform (the second—pure *parabolic type* (P_p) transform—is the shift down by 1 of Figures 1 and 2(K_p)).
 (Ph): The third version of parabolic unit disk with a *hyperbolic type* of Cayley transform.
 (H): The hyperbolic unit disk.

if transformation (2.10) is preceded by conjugation with the matrix $\begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix}$, or in total the Cayley transform is given by the matrix:

$$(2.17) \quad C_1 = \begin{pmatrix} 1 - e_2 e_1 & e_1 - e_2 \\ e_1 + \sigma e_2 & 1 + \sigma e_2 e_1 \end{pmatrix} = \begin{pmatrix} 1 & -e_2 \\ \sigma e_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix}.$$

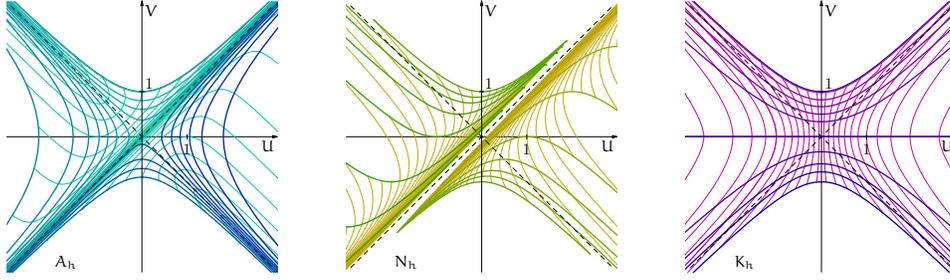


FIGURE 7. The hyperbolic unit disk with alternative orbits

This gives the transformation, cf. [11, (3.6–3.7)]

$$g_{C_1} = \begin{pmatrix} f & h \\ h & f \end{pmatrix}, \text{ with } h = a(1 + e_2 e_1) + d(1 - e_2 e_1) \text{ and } f = b(e_2 - e_1) + c(e_2 + e_1),$$

which obviously keep diagonal form of matrices $\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \in A$. Orbits of subgroup A , N and K for this transformation are shown on Figure 7. The subgroup A acts by hyperbolic rotations of \mathbb{R}^h now.

Note that the alternative Cayley transformation (2.17) diagonalises the subgroup K in the elliptic case ($\sigma = -1$) as well, thus it is also convenient as an alternative form of the Cayley transform in both the elliptic and hyperbolic cases. Images of subgroups action in the elliptic case are obtained from Figure 6(E) by rotation by 90° .

Now we turn to the parabolic case, which benefits from a bigger variety of choices. The first natural attempt to define a Cayley transform can be taken from the same formula (2.9) with the parabolic value $\sigma = 0$. The corresponding transformation defined by the matrix $\begin{pmatrix} 1 & -e_2 \\ 0 & 1 \end{pmatrix}$ turns to be a shift by one unit down. We will denote it by P_p for reasons which will become clear shortly.

While being trivial this transform still possesses some properties of the elliptic case. For example, the K -orbits in the elliptic case (Figure 6(K_e)) and the A -orbits in the hyperbolic case (Figure 7(A_h)) are concentric. The same happens for N -orbits in the parabolic case (Figure 1(N_a))—they all are parabolas (straight lines) with focus at infinity.

However in the parabolic case it worth considering also both the elliptic $\begin{pmatrix} 1 & e_2 \\ e_2 & 1 \end{pmatrix}$ and hyperbolic $\begin{pmatrix} 1 & e_2 \\ -e_2 & 1 \end{pmatrix}$ transformations (2.9). They are presented on Figure 6, rows (P_e) and (P_h) correspondingly. The missing row (P_p) is formed by the parabolic transformation discussed in the previous paragraph and illustrated by Figures 1(A_a), 1(N_a) and 2(K_p) with the upper half plane shifted down by one unit. Consideration of Figure 6 by columns from top to bottom gives an impressive mixture of many common properties (e.g. the number of fixed point on the boundary for each subgroup) with several gradual mutations.

Some properties of parabolic unit disks are as follows:

- Lemma 2.17.** (1) All Cayley transforms P_e , P_p and P_h act on the axes V as the shift down by 1.
 (2) The parabolic unit disk at P_e is given by the inequality $l_p(ue_1 + ve_2) \leq 1$ with boundary given by the parabolic unit circle $l_p(ue_1 + ve_2) = 1$ in sharp resemblance to (2.14) and (2.16).

- (3) The parabolic unit disk at P_h is given by the inequality $l_p(-(u+2)e_1 - ve_2) \leq 1$ with boundary given by the parabolic unit circle $l_p(-(u+2)e_1 + ve_2) = 1$.
- (4) N -orbits in both transforms P_e and P_h are parabolas with focal length $1/2$.
- (5) A -orbits in transforms P_e and P_h are segments of parabolas with focal length $1/2$ passing through $(0, -1)$. Their vertices belongs to two parabolas $v = -x^2 - 1$ and $v = x^2 - 1$ correspondingly, which are boundaries of parabolic circles in P_h and P_e (note the swap!).
- (6) K -orbits in transform P_e are parabolas with focal length less than $1/2$ and in transform P_h —with inverse of focal length bigger than -2 .

Of course property 2.17.2 makes transformations P_e very appealing as a “right” parabolic version of the Cayley transform. However it seems that all three transformations $P_{e,p,h}$ have their own merits which may be decisive in particular circumstances.

Remark 2.18. We see that the varieties of possible Cayley transforms in the parabolic case is bigger than in the two other cases. It is interesting that this parabolic richness is a consequence of the parabolic degeneracy of the generator $e_2^2 = 0$. Indeed for both the elliptic and the hyperbolic sign in $e_2^2 = \pm 1$ only one matrix (2.9) out of two possible $\begin{pmatrix} 1 & -e_2 \\ \pm e_2 & 1 \end{pmatrix}$ has a non-zero determinant. And only for the degenerate parabolic value $e_2^2 = 0$ both these matrices are non-degenerate!

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The extensive graphics in this paper were produced with the help of GiNaC [1] computer algebra system. Since this tool is of separate interest we explain its usage by examples from this article in the separate paper [16]. The `noweb` [20] wrapper for C++ source code is included in the `arXiv.org` files of both these papers [16, 17].

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