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Spectrum as the Support of Functional Calculus

Vladimir V. Kisil

ABSTRACT. We investigate the new definition of analytic functional calculus in the terms of representation theory of $SL_2(\mathbb{R})$. We avoid any usage of its algebraic homomorphism property and replace it by the demand to be an intertwining operator. The related notion of spectrum and spectral mapping theorem are given. The construction is illustrated by a simple example of calculus and spectrum of non-normal $n \times n$ matrix.

CONTENTS

0. Introduction	1
1. Another Approach to Analytic Functional Calculus	3
2. Background in Complex Analysis from $SL_2(\mathbb{R})$ Group	4
3. Representations of $SL_2(\mathbb{R})$ in Banach Algebras	5
4. Jet Bundles and Prolongations of ρ_1	6
5. Spectrum and the Jordan Normal Form of a Matrix	7
6. Spectral Mapping Theorem	9
References	9

0. Introduction

United in the trinity functional calculus, spectrum, and spectral mapping theorem play the exceptional rôle in functional analysis and could not be substituted by anything else. All traditional definitions of functional calculus are covered by the following rigid template based on *algebra homomorphism* property:

DEFINITION 0.1. An *functional calculus* for an element $a \in \mathfrak{A}$ is a continuous linear mapping $\Phi : \mathcal{A} \rightarrow \mathfrak{A}$ such that

- i. Φ is a unital *algebra homomorphism*

$$\Phi(f \cdot g) = \Phi(f) \cdot \Phi(g).$$

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On leave from Odessa University.

- ii. There is an initialisation condition: $\Phi[v_0] = a$ for a fixed function v_0 , e.g. $v_0(z) = z$.

Most typical definition of the spectrum is seemingly independent and uses the important notion of resolvent:

DEFINITION 0.2. A *resolvent* of element $a \in \mathfrak{A}$ is the function $R(\lambda) = (a - \lambda e)^{-1}$, which is the image under Φ of the Cauchy kernel $(z - \lambda)^{-1}$.

A *spectrum* of $a \in \mathfrak{A}$ is the set \mathbf{spa} of singular points of its resolvent $R(\lambda)$.

Then the following important theorem links spectrum and functional calculus together.

THEOREM 0.3 (Spectral Mapping). *For a function f suitable for the functional calculus:*

$$(0.1) \quad f(\mathbf{spa}) = \mathbf{sp}f(a).$$

However the power of the classic spectral theory rapidly decreases if we move beyond the study of one normal operator (e.g. for quasinilpotent ones) and is virtually nil if we consider several non-commuting ones. Sometimes these severe limitations are seen to be irresistible and alternative constructions, i.e. model theory [10], were developed.

Yet the spectral theory can be revived from a fresh start. While three components—functional calculus, spectrum, and spectral mapping theorem—are highly interdependent in various ways we will nevertheless arrange them as follows:

- i. Functional calculus is an *original* notion defined in some independent terms;
- ii. Spectrum (or spectral decomposition) is derived from previously defined functional calculus as its *support* (in some appropriate sense);
- iii. Spectral mapping theorem then should drop out naturally in the form (0.1) or some its variation.

Thus the entire scheme depends from the notion of the functional calculus and our ability to escape limitations of Definition 0.1. The first known to the present author definition of functional calculus not linked to algebra homomorphism property was the Weyl functional calculus defined by an integral formula [1]. Then its intertwining property with affine transformations of Euclidean space was proved as a theorem. However it seems to be the only “non-homomorphism” calculus for decades.

The different approach to whole range of calculi was given in [4] and developed in [7] in terms of *intertwining operators* for group representations. It was initially targeted for several non-commuting operators because no non-trivial algebra homomorphism with a commutative algebra of function is possible in this case. However it emerged later that the new definition is a useful replacement for classical one across all range of problems.

In the present note we will support the last claim by consideration of the simple known problem: characterisation a $n \times n$ matrix up to similarity. Even that “freshman” question could be only sorted out by the classical spectral theory for a small set of diagonalisable matrices. Our solution in terms of new spectrum will be full and thus unavoidably coincides with one given by the Jordan normal form of matrix. Other more difficult questions are the subject of ongoing research.

1. Another Approach to Analytic Functional Calculus

Anything called “*functional calculus*” uses properties of *functions* to model properties of *operators*. Thus changing our viewpoint on functions we could get another approach to operators. We start from the following observation reflected in the almost any textbook on complex analysis:

PROPOSITION 1.1. Analytic function theory *in the unit disk* \mathbb{D} is a manifestation of the mock discrete series representation ρ_1 of $SL_2(\mathbb{R})$:

$$(1.1) \quad \rho_1(g) : f(z) \mapsto \frac{1}{\alpha - \beta z} f\left(\frac{\bar{\alpha}z - \bar{\beta}}{\alpha - \beta z}\right), \quad \text{where } \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} \in SL_2(\mathbb{R}).$$

The representation (1.1) is unitary irreducible when acts on the Hardy space H_2 . Consequently we have one more reason to abolish the template definition 0.1: H_2 is *not* an algebra. Instead we replace the *homomorphism property* by a *symmetric covariance*:

DEFINITION 1.2. An *analytic functional calculus* for an element $a \in \mathfrak{A}$ and an \mathfrak{A} -module M is a *continuous linear* mapping $\Phi : A(\mathbb{D}) \rightarrow A(\mathbb{D}, M)$ such that

- i. Φ is an *intertwining operator*

$$\Phi \rho_1 = \rho_a \Phi$$

between two representations of the $SL_2(\mathbb{R})$ group ρ_1 (1.1) and ρ_a defined bellow in (3.3).

- ii. There is an initialisation condition: $\Phi[v_0] = m$ for $v_0(z) \equiv 1$ and $m \in M$, where M is a left \mathfrak{A} -module.

Note that our functional calculus released from the homomorphism condition can take value in any left \mathfrak{A} -module M , which however could be \mathfrak{A} itself if suitable. This add much flexibility to our construction.

The earliest functional calculus, which is *not* an algebraic homomorphism, was the Weyl functional calculus and was defined just by an integral formula as an operator valued distribution [1]. In that paper (joint) spectrum was defined as support of the Weyl calculus, i.e. as the set of point where this operator valued distribution does not vanish. We also define the spectrum as a support of functional calculus, but due to our Definition 1.2 it will means the set of non-vanishing intertwining operators with primary subrepresentations.

DEFINITION 1.3. A corresponding *spectrum* of $a \in \mathfrak{A}$ is the *support* of the functional calculus Φ , i.e. the collection of intertwining operators of ρ_a with *prime representations* [3, § 8.3].

More variations of functional calculi are obtained from other groups and their representations [4, 7].

2. Background in Complex Analysis from $SL_2(\mathbb{R})$ Group

To understand the functional calculus from Definition 1.2 we need first to realise the function theory from Proposition 1.1, see [5, 6, 8, 9] for more details.

Elements of $SL_2(\mathbb{R})$ could be represented by 2×2 -matrices with complex entries such that:

$$g = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1.$$

There are other realisations of $SL_2(\mathbb{R})$ which may be more suitable under other circumstances, e.g. in the upper half-plane.

We may identify the unit disk \mathbb{D} with the left coset $\mathbb{T} \backslash SL_2(\mathbb{R})$ for the unit circle \mathbb{T} through the important decomposition $SL_2(\mathbb{R}) \sim \mathbb{T} \times \mathbb{D}$ with $K = \mathbb{T}$ —the only compact subgroup of $SL_2(\mathbb{R})$:

$$(2.1) \quad \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \frac{1}{\sqrt{1-|u|^2}} \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix},$$

where

$$\omega = \arg \alpha, \quad u = \bar{\beta}\alpha^{-1}, \quad |u| < 1.$$

Each element $g \in SL_2(\mathbb{R})$ acts by the linear-fractional transformation (the Möbius map) on \mathbb{D} and $\mathbb{T} \subset H_2(\mathbb{T})$ as follows:

$$(2.2) \quad g^{-1} : z \mapsto \frac{\bar{\alpha}z - \bar{\beta}}{\alpha - \beta z}, \quad \text{where} \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}.$$

In the decomposition (2.1) the first matrix on the right hand side acts by transformation (2.2) as an orthogonal rotation of \mathbb{T} and \mathbb{D} ; and the second one—by transitive family of maps of the unit disk onto itself.

The standard linearisation procedure [3, § 7.1] leads from Möbius transformations (2.2) to the unitary representation ρ_1 irreducible on the *Hardy space*:

$$(2.3) \quad \rho_1(g) : f(z) \mapsto \frac{1}{\alpha - \beta z} f\left(\frac{\bar{\alpha}z - \bar{\beta}}{\alpha - \beta z}\right) \quad \text{where} \quad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}.$$

Möbius transformations provide a natural family of intertwining operators for ρ_1 coming from inner automorphisms of $SL_2(\mathbb{R})$ (will be used later).

We choose [7, 8] K -invariant function $v_0(z) \equiv 1$ to be a *vacuum vector*. Thus the associated *coherent states*

$$v(g, z) = \rho_1(g)v_0(z) = (u - z)^{-1}$$

are completely determined by the point on the unit disk $u = \bar{\beta}\alpha^{-1}$. The family of coherent states considered as a function of both u and z is obviously the *Cauchy kernel* [5]. The *wavelet transform* [5, 7] $\mathcal{W} : L_2(\mathbb{T}) \rightarrow H_2(\mathbb{D}) : f(z) \mapsto \mathcal{W}f(g) = \langle f, v_g \rangle$ is the *Cauchy integral*:

$$(2.4) \quad \mathcal{W}f(u) = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \frac{1}{u - z} dz.$$

Other classical objects of complex analysis (the Cauchy-Riemann equation, the Taylor series, the Bergman space, etc.) can be also obtained [5, 8] from representation ρ_1 but are not used and considered here.

3. Representations of $SL_2(\mathbb{R})$ in Banach Algebras

A simple but important observation is that the Möbius transformations (2.2) can be easily extended to any Banach algebra. Let \mathfrak{A} be a Banach algebra with the unit e , an element $a \in \mathfrak{A}$ with $\|a\| < 1$ be fixed, then

$$(3.1) \quad g : a \mapsto g \cdot a = (\bar{\alpha}a - \bar{\beta}e)(\alpha e - \beta a)^{-1}, \quad g \in SL_2(\mathbb{R})$$

is a well defined $SL_2(\mathbb{R})$ action on a subset $\mathbb{A} = \{g \cdot a \mid g \in SL_2(\mathbb{R})\} \subset \mathfrak{A}$, i.e. \mathbb{A} is a $SL_2(\mathbb{R})$ -homogeneous space. Let us define the *resolvent* function $R(g, a) : \mathbb{A} \rightarrow \mathfrak{A}$:

$$R(g, a) = (\alpha e - \beta a)^{-1}$$

then

$$(3.2) \quad R_1(g_1, \mathfrak{a})R_1(g_2, g_1^{-1}\mathfrak{a}) = R_1(g_1g_2, \mathfrak{a}).$$

The last identity is well known in representation theory [3, § 13.2(10)] and is a key ingredient of *induced representations*. Thus we can again linearise (3.1) (cf. (2.3)) in the space of continuous functions $C(\mathbb{A}, M)$ with values in a left \mathfrak{A} -module M , e.g. $M = \mathfrak{A}$:

$$(3.3) \quad \begin{aligned} \rho_a(g_1) : f(g^{-1} \cdot a) &\mapsto R(g_1^{-1}g^{-1}, a)f(g_1^{-1}g^{-1} \cdot a) \\ &= (\alpha'e - \beta'a)^{-1} f\left(\frac{\bar{\alpha}' \cdot a - \bar{\beta}'e}{\alpha'e - \beta'a}\right). \end{aligned}$$

For any $m \in M$ we can again define a K -invariant *vacuum vector* as $v_m(g^{-1} \cdot a) = m \otimes v_0(g^{-1} \cdot a) \in C(\mathbb{A}, M)$. It generates the associated with v_m family of *coherent states* $v_m(u, a) = (ue - a)^{-1}m$, where $u \in \mathbb{D}$.

The *wavelet transform* defined by the same common formula based on coherent states (cf. (2.4)):

$$\mathcal{W}_m f(g) = \langle f, \rho_a(g)v_m \rangle,$$

is a version of Cauchy integral, which maps $L_2(\mathbb{A})$ to $C(SL_2(\mathbb{R}), M)$. It is closely related (but not identical!) to the Riesz-Dunford functional calculus: the traditional functional calculus is given by the case:

$$\Phi : f \mapsto \mathcal{W}_m f(0) \quad \text{for } M = \mathfrak{A} \text{ and } m = e.$$

The both conditions—the intertwining property and initial value—required by Definition 1.2 easily follows from our construction.

4. Jet Bundles and Prolongations of ρ_1

Spectrum was defined in 1.3 as the *support* of our functional calculus. To elaborate its meaning we need the notion of a *prolongation* of representations introduced by S. Lie, see [11, 12] for a detailed exposition.

DEFINITION 4.1. [12, Chap. 4] Two holomorphic functions have *n*th order contact in a point if their value and their first n derivatives agree at that point, in other words their Taylor expansions are the same in first $n + 1$ terms.

A point $(z, u^{(n)}) = (z, u, u_1, \dots, u_n)$ of the *jet space* $\mathbb{J}^n \sim \mathbb{D} \times \mathbb{C}^n$ is the equivalence class of holomorphic functions having n th contact at the point z with the polynomial:

$$(4.1) \quad p_n(w) = u_n \frac{(w-z)^n}{n!} + \dots + u_1 \frac{(w-z)}{1!} + u.$$

For a fixed n each holomorphic function $f : \mathbb{D} \rightarrow \mathbb{C}$ has n th *prolongation* (or *n-jet*) $j_n f : \mathbb{D} \rightarrow \mathbb{C}^{n+1}$:

$$(4.2) \quad j_n f(z) = (f(z), f'(z), \dots, f^{(n)}(z)).$$

The graph $\Gamma_f^{(n)}$ of $j_n f$ is a submanifold of \mathbb{J}^n which is section of the *jet bundle* over \mathbb{D} with a fibre \mathbb{C}^{n+1} . We also introduce a notation J_n for the map $J_n : f \mapsto \Gamma_f^{(n)}$ of a holomorphic f to the graph $\Gamma_f^{(n)}$ of its n -jet $j_n f(z)$ (4.2).

One can prolong any map of functions $\psi : f(z) \mapsto [\psi f](z)$ to a map $\psi^{(n)}$ of n -jets by the formula

$$(4.3) \quad \psi^{(n)}(J_n f) = J_n(\psi f).$$

For example such a prolongation $\rho_1^{(n)}$ of the representation ρ_1 of the group $SL_2(\mathbb{R})$ in $H_2(\mathbb{D})$ (as any other representation of a Lie group [12]) will be again a representation of $SL_2(\mathbb{R})$. Equivalently we can say that J_n *intertwines* ρ_1 and $\rho_1^{(n)}$:

$$J_n \rho_1(g) = \rho_1^{(n)}(g) J_n \quad \text{for all } g \in SL_2(\mathbb{R}).$$

Of course, the representation $\rho_1^{(n)}$ is not irreducible: any jet subspace \mathbb{J}^k , $0 \leq k \leq n$ is $\rho_1^{(n)}$ -invariant subspace of \mathbb{J}^n . However the representations $\rho_1^{(n)}$ are *primary* [3, § 8.3] in the sense that they are not sums of two subrepresentations.

The following statement explains why jet spaces appeared in our study of functional calculus.

PROPOSITION 4.2. *Let matrix a be a Jordan block of a length k with the eigenvalue $\lambda = 0$, and m be its root vector of order k , i.e. $a^{k-1}m \neq a^k m = 0$. Then the restriction of ρ_a on the subspace generated by v_m is equivalent to the representation ρ_1^k .*

5. Spectrum and the Jordan Normal Form of a Matrix

Now we are prepared to describe a spectrum of a matrix. Since the functional calculus is an intertwining operator its support is a decomposition into intertwining operators with prime representations (we could not expect generally that these prime subrepresentations are irreducible).

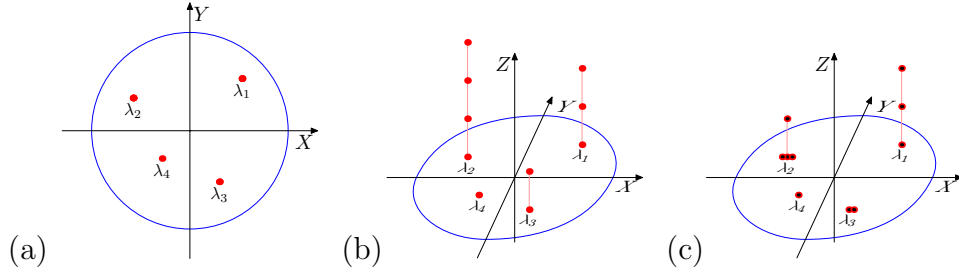


FIGURE 1. Classical spectrum (a) of a matrix vs. the new version (b) with its mapping (c).

Recall the transitive on \mathbb{D} group of inner automorphisms of $SL_2(\mathbb{R})$, which can send any $\lambda \in \mathbb{D}$ to 0 and are actually parametrised by such a λ . This group extends Proposition 4.2 to the complete characterisation of ρ_a for matrices.

PROPOSITION 5.1. *Representation ρ_a is equivalent to a direct sum of the prolongations $\rho_1^{(k)}$ of ρ_1 in the k th jet space \mathbb{J}^k intertwined with inner automorphisms. Consequently the spectrum of a (defined via the functional calculus $\Phi = \mathcal{W}_m$) labelled exactly by n pairs of numbers (λ_i, k_i) , $\lambda_i \in \mathbb{D}$, $k_i \in \mathbb{Z}_+$, $1 \leq i \leq n$ some of whom could coincide.*

Obviously this spectral theory is a fancy restatement of the *Jordan normal form* of matrices.

EXAMPLE 5.2. Let $J_k(\lambda)$ denote the Jordan block of the length k for the eigenvalue λ . On the Fig. 1 there are two pictures of the spectrum for the matrix

$$a = J_3(\lambda_1) \oplus J_4(\lambda_2) \oplus J_1(\lambda_3) \oplus J_2(\lambda_4),$$

where

$$\lambda_1 = \frac{3}{4}e^{i\pi/4}, \quad \lambda_2 = \frac{2}{3}e^{i5\pi/6}, \quad \lambda_3 = \frac{2}{5}e^{-i3\pi/4}, \quad \lambda_4 = \frac{3}{5}e^{-i\pi/3}.$$

Part (a) represents the conventional two-dimensional image of the spectrum, i.e. eigenvalues of a , and (b) describes spectrum \mathbf{spa} arising from the wavelet construction. The first image did not allow to distinguish a from many other essentially different matrices, e.g. the diagonal matrix

$$\text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

which even have a different dimensionality. At the same time the Fig. 1(b) completely characterise a up to a similarity. Note that each point of \mathbf{spa} on Fig. 1(b) corresponds to a particular root vector, which spans a primary subrepresentation.

6. Spectral Mapping Theorem

As was mentioned in the Introduction a reasonable spectrum should be linked to the corresponding functional calculus by an appropriate spectral mapping theorem. The new version of spectrum is based on prolongation of ρ_1 into jet spaces (see Section 4). Naturally a correct version of spectral mapping theorem should also operate in jet spaces.

Let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic map, let us define its action on functions $[\phi_* f](z) = f(\phi(z))$. According to the general formula (4.3) we can define the prolongation $\phi_*^{(n)}$ onto the jet space \mathbb{J}^n . Its associated action $\rho_1^k \phi_*^{(n)} = \phi_*^{(n)} \rho_1^n$ on the pairs (λ, k) is given by the formula:

$$(6.1) \quad \phi_*^{(n)}(\lambda, k) = \left(\phi(\lambda), \left[\frac{k}{\deg_\lambda \phi} \right] \right),$$

where $\deg_\lambda \phi$ denotes the degree of zero of the function $\phi(z) - \phi(\lambda)$ at the point $z = \lambda$ and $[x]$ denotes the integer part of x .

THEOREM 6.1 (Spectral mapping). *Let ϕ be a holomorphic mapping $\phi : \mathbb{D} \rightarrow \mathbb{D}$ and its prolonged action $\phi_*^{(n)}$ defined by (6.1), then*

$$\mathbf{sp}\phi(a) = \phi_*^{(n)} \mathbf{sp}a$$

The explicit expression of (6.1) for $\phi_*^{(n)}$, which involves derivatives of ϕ upto n th order, is known, see for example [2, Thm. 6.2.25], but was not recognised before as form of spectral mapping.

EXAMPLE 6.2. Let us continue with Example 5.2. Let ϕ map all four eigenvalues $\lambda_1, \dots, \lambda_4$ of the matrix a into themselves. Then Fig. 1(a) will represent the classical spectrum of $\phi(a)$ as well as a .

However Fig. 1(c) shows mapping of the new spectrum for the case ϕ has *orders of zeros* at these points as follows: the order 1 at λ_1 , exactly the order 3 at λ_2 , an order at least 2 at λ_3 , and finally any order at λ_4 .

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