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COMPUTATION AND DYNAMICS: CLASSICAL AND QUANTUM

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ABSTRACT. We discuss classical and quantum computations in terms of corresponding Hamiltonian dynamics.

1. INTRODUCTION

It is well known that classical computations are modelled by abstract “machines” first introduced in works of E. Post and A. Turing, see [10, § 1.4.5 and § 2.6] for historical notes and further references.

We are going to demonstrate and exploit an explicit analogy between the process of computation on such abstract machines and a Hamiltonian dynamics of a particle in the phase space. We will use for this purpose the Post machine since its description is simpler. In the following we will call it simply the *machine*.

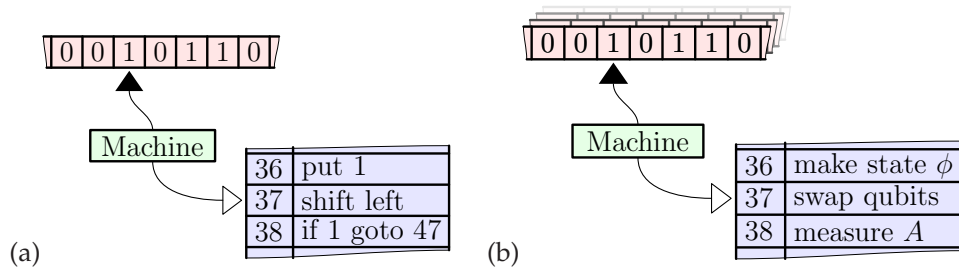


FIGURE 1. (a) A symbolic representation of the classical Post machine (a single tape).

(b) A symbolic representation of computations with a quantum superposition of tapes. The programme contains new types of instructions but it is still classical.

A state of the machine is described by two independent components: the tape (data) and the instruction list (programme), see Fig. 1(a). The tape is assumed to be an infinite sequence of cells with only a finite number of them holding mark 1, all others assumed to be “empty” (holding 0). Another important property of the tape is the *current cell* for observation/modification pointed by a reading head.

The second machine’s component—*programme*—is a finite list of instructions with the second pointer marking the current command. The statements are taken

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from a very limited set and request modifications of the current tape’s cell or respective movements of the reading head and the instruction pointer.

Remark 1.1. The division into “data tape” and “programme” seems to be a fundamental one. This duality is reflected in both—architectures of modern computers and the computer science paradigm of “Algorithms and Data Structures” [16].

A typical quantum computation [3, 14] can be modelled by a *quantisation* of the tape in a machine, see Fig. 1(b). This means that instead of a classical tape holding a sequence of classical bits one considers a quantum tape: a finite number of cells holding *qubits*. Qubits are assumed to be able to store linear combinations (superpositions) of values 0 and 1.

A quantisation of the other half—the programme—is rarely considered: it is still a linear sequence of corresponding instructions, which are unitary operators on qubits in this case. Thus a common quantum computer is strictly speaking semi-classical or quantum-classical computer only. To get fully quantised computer one can additionally request superpositions of computer states and/or programmes. However a realisation of superposition for instructions can be confusing.

In this paper we consider an alternative approach. Firstly we get unification of the tape and the reading head position into a single coordinate. Computer’s programme is linked to the another coordinate. Then we can quantise it in a single move. Computational speed of such a computer cannot be directly compared to a classical one, since it will not only process data in parallel but also perform different computational stages at the same time.

2. PHASE SPACE COMPUTATIONS AND HAMILTONIAN DYNAMICS

To obtain the dynamical description of computations we blend the state of the tape and position of the reading head into a single parameter. We interpret the finite sequence of 1’s and enclosed among them 0’s on the tape as a **dyadic rational** number with the binary point at the immediate right to the current cell. Then the standard actions of the reading head (the first column of Tab. 1) can be translated into operations on the set \mathbb{D} of dyadic numbers (the second column of Tab. 1).

Head action	Arithmetic operation	Value of $\Delta_p H(q_0, p_0)$
Head to the left	Divide the fraction by 2	$-\frac{1}{2}q_0$
Head to the right	Multiply the fraction by 2	q_0
Replace 0 by 1	Add 1 to the fraction	1
Replace 1 by 0	Subtract 1 from the fraction	-1

TABLE 1. The first column lists actions of the reading head of a machine, the second column translates them into dyadic arithmetic. The third column provides values of a Hamiltonian which direct those transformations.

Similarly the set $\mathbb{Z}_n = \{1, 2, \dots, n\}$ can index a programme of n instructions. Thus a full state of a machine is described by a point $(q, p) \in \mathbb{D} \times \mathbb{Z}_n$. Calculation is a dynamic on this set with a discrete time parameter $t \in \mathbb{N}$. An iteration from a current state (q_t, p_t) to the next one (q_{t+1}, p_{t+1}) is given by the pair of finite differences equations:

$$(1) \quad \Delta_t q = \Delta_p H, \quad \Delta_t p = -\Delta_q H.$$

Here $\Delta_q H$ and $\Delta_p H$ is a pair of functions $\mathbb{D} \times \mathbb{Z}_n \rightarrow \mathbb{Z}$ and $\mathbb{D} \times \mathbb{Z}_n \rightarrow \mathbb{D}$ respectively. The function $\Delta_p H$ defines transformations of the tape according to the third column in Tab. 1. The programme flow is directed by $\Delta_q H$ as described in Tab. 2.

Remark 2.1. We intentionally use notations resembling Hamiltonian dynamics in order to exploit the duality between data and algorithms mentioned in Rem. 1.1. However the exact mathematical formalism for this duality is still missing. For example, canonical transformations mixing data and programme can be related to the philosophy behind **Prolog** and **Lisp** programming languages.

Instruction pointer	Value of $\Delta_q H(q_0, p_0)$
Next Instruction	-1
Go to p_1	$p_0 - p_1$
If cell is 1 go to p_1	$\begin{cases} p_0 - p_1, & \text{if } [q_0] = 1 \pmod{2}; \\ 1, & \text{if } [q_0] \neq 1 \pmod{2}. \end{cases}$

TABLE 2. Movements of the programme pointer (the first column) and the corresponding values of a Hamiltonian (the second column).

Theorem 2.2. *Calculations of a Post machine is described by a discrete dynamics in the phase space $\mathbb{D} \times \mathbb{Z}_n$ defined by the equations (1). A programme corresponds to a Hamiltonian governing the dynamic.*

Abstract computing	Hamilton dynamic
Tape state	Coordinate
Inner state	Momentum
Program	Hamiltonian
Execution	Dynamics
Inclusion-Exclusion	Wave superposition

TABLE 3. The correspondence between element of abstract calculations and dynamics in the phase space.

Tab. 3 shows a correspondence between notions of computing and Hamilton dynamics. Developing this approach we can define a *fully* quantum computation through quantisation of the classical discrete dynamics. This gives simultaneous propagation along all possible paths, which means parallel procession of data *and* the programme similarly.

3. EXAMPLE: POLYNOMIAL SEQUENCES OF BINOMIAL TYPE

Classical computations of many combinatorial quantities is based on the **inclusion-exclusion principle** [15, § 2.1]. Its quantum counterpart is the superposition of wave functions: the resulting probability can be anything from the sum (inclusion) to the difference (exclusion) of given probabilities. Thus such combinatorial calculations are very suitable for quantum computations.

For example, let $q_n(x)$ be a *token* [4, 6] from \mathbb{N} to \mathbb{R} , i.e. the sequence of polynomials of $\deg q_n = n$ satisfying to the identity:

$$q_n(x+y) = \sum_{k=0}^n q_k(y)q_{n-k}(x),$$

If $q_n(x)$ is such a token then a polynomial sequence $p_n(x) = n!q_n(x)$ is of *binomial type* [11, § 4.3]. Examples are provided by power monomials, falling (rising) factorials, Abel, Laguerre and many other famous polynomials.

A dynamics in a configurational space Q can be described by the *propagator* $K(q_2, t_2; q_1, t_1)$ —a complex valued function defined on $Q \times \mathbb{R} \times Q \times \mathbb{R}$. It is a probability amplitude for a transition $q_1 \rightarrow q_2$ from a state q_1 at time t_1 to q_2 at time t_2 . The fundamental assumption about the quantum world is the *absence of trajectories* for a system's evolution through the configurational space Q : the system at any time t_i could be found at any point q_i .

R. Feynman developing ideas of A. Einstein, M.V. Smoluhovski and P.A.M. Dirac proposed an expression for the propagator via the “integral over all possible paths”:

$$K(q_2, t_2; q_1, t_1) = \int \frac{\mathcal{D}q \mathcal{D}p}{h} \exp \left(\frac{i}{\hbar} \int_{t_1}^{t_2} dt (p\dot{q} - H(p, q)) \right).$$

Here $H(p, q)$ is the Hamiltonian of the system. The inner integral is over a path in the phase space. The outer integral is taken over “all possible paths between two given points with respect to a measure $\mathcal{D}q \mathcal{D}p$ on paths in the phase space”.

Proposition 3.1. *Any quantum system is a quantum computer for an evaluation of its own propagator K , computation is done simultaneously along all possible paths.*

In this way we obtain the path computation formula for polynomials q_n [5]:

$$q_n(x) = \int \mathcal{D}k \mathcal{D}p \exp \int_0^x (-ipk' + h(p)) dt, \quad \text{where } h(p) = \sum_{k=0}^{\infty} q'_k(0) e^{ipk}.$$

Thus a quantum system with the above Hamiltonian $h(p)$ allows to calculate q_n in a single operation (measurement). This looks unrealistically quick and one can ask: how to compare speeds of quantum and classical computations after all?

4. QUANTUM COMPUTERS WITH CLASSICAL TERMINALS

A discussion of quantum computers is often limited to quantum algorithms. However this is an oversimplification, which does not include the process of qubit preparation (input of data), building sequences of quantum gates (programming) and reading of the final state (data output). Of course, in the classical case these three processes can be done in a negligible time in comparison with the actual computation. However, this is no longer true for quantum computations.

Example 4.1. Let us review two most known quantum algorithms.

- (1) Shor's factorisation algorithm [14] required the quantum circuit to be re-assembled accordingly every time a new random number was chosen for a test. Thus the time of circuit assembling (programming) should be included in the overall computational cost.
- (2) Grover's database search algorithm [3] requires several repeated recalculations, each of which would destroy the database (the projection postulate of quantum measurement [12]). Thus the time for rebuilding a database (data input) and measurement (data output) should be included in the overall computational cost.

For more realistic consideration we have to add *classical interfaces* for input and output to make quantum computations really usable. At present even a simple quantum step like two qbits swapping is done by a millions of classical computational steps. Is it a *present day* technological limitation or *fundamental* exchange rate between cost of a quantum and classical computation? If an *application* of an existent quantum gate is so expensive, how expensive is *to built* a case-specific

quantum circuit for $f(x) = a^x$ [3] or quantum Fourier transform [14]? Such questions are already hinted in [14] but are rarely discussed in depth. Consequently we miss not only clear answers but even the understanding of their importance.

In the first half of this paper we presented classical and quantum computations as dynamics. Then a quantum computer with classical terminals shall be represented by a dynamics of a quantum-classical aggregate system. Is there a consistent theory to describe such a dynamics? This is a debated topics with the majority of physicists believing that this is fundamentally impossible [2,13]. If this is so, shall it be interpreted as our inability (as a macroscopic and thus classical objects) to efficiently interact with quantum computing devices even if they are to be built?

Quantum-classical dynamics is oftenly connected with an existence of special quantum-classic bracket which shall unify (and replace) both quantum commutator and Poisson brackets. A mathematical model for a classical system attached as an input/output terminal to a quantum computer can be attempted from the quantum-classical formalism proposed in [1,7–9]. Such a model would provide an opportunity for effective estimation of the overall cost of quantum computing during the entire cycle: preparation-computation-reading.

To stimulate an attention to this issue we wish to conclude by the following:

Conjecture 4.2 (“Golden rule” of quantum-classic information). A gain in quantum algorithms is outweighed by losses in classical I/O and programing.

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