A Functional Model for Quantum Mechanics: Unbounded Operators

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Abstract

We extend the recently developed Riesz-Clifford monogenic functional calculus (based on Clifford analysis) for a set of *unbounded non-commuting operators*. Connections with quantum mechanics are discussed.

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1 Introduction

The present paper continues the development of the functional calculus for non-commuting operators started in [14], [17], [18], [19]. Our investigation, as well as a huge part of functional analysis, is motivated by a search for suitable models for quantum mechanics [29]. A very abstract formulation of the quantization problem is exactly the construction of a functional calculus for several non-commuting operators.

There exist already at least two models of functional calculus for several operators. They are the holomorphic calculus of TAYLOR [31], [32] based on several complex variable theory and the Weyl calculus generalized by ANDERSON [2] for arbitrary finite set of self-adjoint operators. Only the last one is able to handle with non-commuting operators. In spite of its indisputable advantages the Weyl calculus has an important shortcoming: it was not connected with any algebraic property like an algebra homomorphism. This lack greatly reduced the applicabilty of the generalized Weyl calculus in quantum mechanics. The Riesz-Clifford functional calculus [19] could handle an n-tuple of bounded self-adjoint non-commuting operators and has a property of $\times$-algebra homomorphism. Connection between the Weyl and the Riesz-Clifford calculi was found in [17], where functional calculi were labeled with associated group representations.

However the technique of [19] could not yet support a wide range of applications in quantum mechanics due to the boundedness restriction on operators. Consideration of quantum field theory suggests that the following particular model is important. Let the quantum mechanical system at hand be described by means of the primary observables $X_1, X_2, \ldots, X_{n-1}, H$

(see Section 4 for details). Here the observables X_i behave like coordinates, i.e., have unbounded spectrum, and H is the energy operator, which can be semibounded (has a positive spectrum). Then quantization is the mapping

$$f(x_1, \ldots, x_{n-1}, x_n) \to f(X_1, \ldots, X_{n-1}, H)$$

from functions defined on the upper half space to operators. We will see that exactly this situation can be naturally described via Riesz-Clifford functional calculus based on Clifford analysis.

The key ingredient of this approach is the use of Clifford analysis with a related structure as a model for functional calculus. The explicit use of Clifford analysis in such a role can be traced back at least to the paper [24] of McIntosh and Pryde, where the case of mutually commuting operators was considered. But even the earlier definition of the Taylor spectrum [31], as it was pointed in [17], Lemma 3.2, already contains the Clifford algebras in hidden form. Thus the use of Clifford analysis in operator theory already has a long history and seems to be a reasonable approach (see also [28]). Nevertheless, the results for non-commuting operators[1] obtained in [17], [18], [19] are mainly based on original ideas, namely the notion of $\times$-algebras [19] and the covariance definition [17], [18] of the functional calculus.

The main objective of the present paper is to extend the Riesz-Clifford functional calculus for unbounded operators. To this end we will follow to the classical path, which leads from Section VII.3 to Section VII.9 of [10]. The underlying idea is to use the already constructed functional calculus for bounded operators and reduce the unbounded case to the previous one by a suitable transformation. For reasons which will be explained later on, here *a*

[1]We are also free from the limitation of [24] that number of operators should be even.

suitable transformation means the Cayley transform of upper half space to the unit ball in $\mathbb{R}^n$ [17], Remark 4.2. Our task is greatly simplified by because much work connected with Moebius transformations was very recently done in the context of Clifford analysis (see, for example, papers of CNOPS [4] and RYAN [26], [27]). But the specific features of the unbounded *and* non-commutative case requires careful study (see discussion at the beginning of Section 3).

In Section 2 of this work we give the main results about Clifford analysis, Moebius transformation (especially the Cayley transform) and Riesz-Clifford functional calculus [19] in a refined form. We have tried to find a reasonable balance between self-consistency and modest length of preliminaries. Due to the large amount of material involved we were often enforced to give only references to original papers. In Section 3 we construct the Riesz-Clifford functional calculus for unbounded operators from the bounded case with the help of the Cayley transform. Some technical difficulties enforce us to modify the approach of [10], § VII.9. We conclude the paper by Section 4 where we discuss our results in connections with quantum mechanics. We make in this Section also some remarks about connections between the Riesz-Clifford [19] and monogenic [17], [18] functional calculi.

2 Definitions, Notations and Preliminary Results

In this Section we give a short overview of notions and results, which will be used later on.

2.1 Clifford Algebras and Clifford Analysis

We need some notations from [3], [9]. Let the euclidean vector space $\mathbb{R}^n$ have the orthonormal basis $e_1, e_2, \ldots, e_n$. The Clifford algebra $\mathbf{Cl}(n,0)$ is generated by the elements $e_0 = 1, e_1, \ldots, e_n$ with the usual vector operations and the multiplication defined on the elements[2] of the basis by the following equalities:

$$\begin{aligned} e_j^2 &= -e_0 \qquad (j = 1, \ldots, n), \\ e_j e_k + e_k e_j &= 0 \qquad (j, k = 1, \ldots, n; j \neq k), \end{aligned}$$

and then extended linearly to the whole space. An element of $\mathbf{Cl}(n,0)$ can be written as a linear combination with coefficients $a_\alpha \in \mathbb{R}$ of the monomials $e_\alpha = e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n}$:

$$a = \sum_\alpha a_\alpha e_\alpha = \sum_{j_k = 0 \ or \ 1} a_{j_1 j_2 \cdots j_n} e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n}. \tag{1}$$

The main anti-involution (conjugation) $\bar{a}$ of an element a is defined by the rule:

$$\bar{a} = \sum_\alpha a_\alpha \bar{e}_\alpha = \sum_{j_k = 0 \ or \ 1} a_{j_1 j_2 \cdots j_n} \bar{e}_n^{j_n} \bar{e}_{n-1}^{j_{n-1}} \cdots \bar{e}_1^{j_1}, \tag{2}$$

where $\bar{e}_j = -e_j,\ \bar{e}_0 = e_0,\ 1 \leq j \leq n$. A Clifford algebra valued function f of the variables $(x_1, x_2, \ldots, x_n)$, is called *monogenic* in an open domain $\Omega \subset \mathbb{R}^n$ if it satisfies the Dirac equation

$$Df = \sum_{j=1}^{n} e_j \frac{\partial f}{\partial x_j} = 0. \tag{3}$$

[2] We are considering the only case of a negative definite bilinear form over $\mathbb{R}^n$. The technique for indefinite forms was developed recently and it certainly is very interesting in connection with physical applications. We have also made a shift from the paravectors formalism [19] to the vector one in the present paper. See the discussion in [4].

The main results of Clifford analysis [3], [9] (Cauchy theorem, Cauchy integral formula etc.) have a structure closer that of the complex analysis of one variable than to standard complex analysis of several variables. However, in Clifford analysis *not all polynomials are monogenic functions.* Instead one has to consider the symmetric polynomials of the monomials having the form

$$\vec{x}_j = (e_j x_n + e_n x_j),\ 1 \le j \le n-1. \tag{4}$$

The role of monomials ("regular variables" [8]) is described for quaternionic analysis in [30], for Clifford analysis in [23], for Fueter-Hurwitz analysis in [20], and for solutions of the general Dirac type equation in [14], [15], [22].

We use the following facts and notation. Let $H(\Omega)$ denote the space of all monogenic functions in the domain Ω and by P the space of all monogenic polynomials. The space P has the linear subspaces $P_j,\ 0 \le j < \infty$ of homogeneous polynomials degree j. We will show that P consist of symmetric polynomials constructed from the monomials of the form (4) by symmetric products [23]

$$a_1 \times a_2 \times \cdots \times a_k = \frac{1}{k!} \sum a_{j_1} a_{j_2} \cdots a_{j_k}, \tag{5}$$

where the sum is taken over all of permutations of $(j_1, j_2, \ldots, j_k)$. Clifford valued coefficients are written on the right-hand side.

Lemma 2.1 *The linear subspace P_j has a basis consisting of symmetric polynomials of the form*

$$\begin{aligned} V_\alpha(x) &= (-1)^{|\alpha|}(e_n x_1 + e_1 x_n)^{\alpha_1} \times (e_n x_2 + e_2 x_n)^{\alpha_2} \times \cdots \times (e_n x_{n-1} + e_{n-1} x_n)^{\alpha_{n-1}} \\ &= \vec{x_1}^{\alpha_1} \times \vec{x_2}^{\alpha_2} \times \cdots \times \vec{x_{n-1}}^{\alpha_{n-1}} = \vec{x}^{\alpha}. \end{aligned} \tag{6}$$

Let

$$E(y-x) = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \frac{\overline{y-x}}{|\,y-x\,|^{n+1}} \tag{7}$$

be the Cauchy kernel [9], p. 146 and

$$\begin{aligned} d\sigma &= \sum_{j=1}^{n}(-1)^j e_j dx_1 \wedge \ldots \wedge [dx_j] \wedge \ldots \wedge dx_n. \\ &= \vec{n}ds \end{aligned}$$

be the differential form of the "oriented surface element" ds [9], p. 144, where $\vec{n}$ is the unit normal vector orthogonal to the surface. Then for any $f(x) \in H(\Omega)$ we have the Cauchy integral formula [9], p. 147

$$\int_{\partial\Omega} E(x-y)\, d\sigma_y\, f(y) = \begin{cases} f(x),\ x \in \Omega \\ 0, x \notin \bar{\Omega} \end{cases}. \tag{8}$$

We should point out the universality (with respect to domains) of both the Cauchy kernel $E(y-x)$ and the Cauchy formula in Clifford analysis, in contrast to the case of several complex variables.

If we define as in [3], § 18.6 and [9], Chap. II, Definition 1.5.5

$$W_\alpha^{(a)}(x) = (-1)^{|\alpha|}\partial^\alpha E(x-a), \tag{9}$$

then, for $|\,x\,|<|\,y\,|$ we obtain [9], Chap. II, (1.16)

$$E(y-x) = \sum_{j=0}^{\infty}\left(\sum_{|\alpha|=j} V_\alpha(x) W_\alpha(y)\right). \tag{10}$$

2.2 The Möbius (Conformal) Group

We will give a very short account here. For details the reader should consult the original papers [1], [4], [26], [27].

We have two anti-involutions $-$ and $*$ in $\mathbf{Cl}(n,0)$ defined on vectors by $\bar{x} = -x$ and $x^* = x$. It is easy to see that $xy = yx = 1$ for any $x \in \mathbb{R}^n$ and $y = \bar{x}\,\|\,x\,\|^{-2}$, which is the *Kelvin inverse* of x. Finite products of vectors are invertible in $\mathbf{Cl}(n,0)$ and form the *Clifford group* Γ_n. Elements $a \in \Gamma_n$ such that $a\bar{a} = \pm 1$ form the $\mathrm{Pin}(n)$ group—the double cover of the group of orthogonal rotations $\mathrm{O}(n)$.

Let (a,b,c,d) be a quadruple of elements of $\Gamma_n \cup \{0\}$ with the properties:

1. $(ad^* - bc^*) \in \mathbb{R} \setminus 0$;

2. ab^*, cd^*, c^*a, d^*b are vectors.

Then *Vahlen* [1], [4], [27] 2×2-matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form a semisimple [12], § 6.2 group $V(n)$ under the usual matrix multiplication. It has a representation $\pi_{\mathbb{R}^n}$ by transformations of $\mathbb{R}^n \cup \{\infty\}$ given by:

$$\pi_{\mathbb{R}^n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x \mapsto (ax+b)(cx+d)^{-1}, \qquad (11)$$

which form the *Möbius* (or *conformal* [33], Chap. 10) group of $\mathbb{R}^n$. The analogy with fractional-linear transformations of the complex line $\mathbb{C}$ is useful as well as representations of shifts $x \mapsto x+y$, orthogonal rotations $x \mapsto k(a)x$, dilatations $x \mapsto \lambda x$, and the Kelvin inverse $x \mapsto x^{-1}$ by the matrixes $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$, $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ correspondingly.

2.3 Riesz-Clifford Calculus

We will start from the definition of the $\times$-algebra, which we give in an easier form than in [19].

Definition 2.2 Let $\mathfrak{A}$ be a (topological) algebra with the operations of addition $+_{\mathfrak{A}}$ and multiplication $\cdot_{\mathfrak{A}}$, generated by a finite set of elements $a_1, \ldots, a_k$. Define a new operation $\times_{\mathfrak{A}}$ of symmetric multiplication associated with them as follows. Let

$$\begin{aligned} A &= a_1^{\alpha_1} \times a_2^{\alpha_2} \times \cdots \times a_k^{\alpha_k}, \\ B &= a_1^{\beta_1} \times a_2^{\beta_2} \times \cdots \times a_k^{\beta_k}. \end{aligned} \tag{12}$$

Then

$$A \times B = a_1^{\alpha_1+\beta_1} \times a_2^{\alpha_2+\beta_2} \times \cdots \times a_k^{\alpha_k+\beta_k}. \tag{13}$$

Let $\mathfrak{A}_\times$ be the closure in $\mathfrak{A}$ of elements of the form (12); then this operation is continuously extended to $\mathfrak{A}_\times$. The resulting set will be called an $\times$-*algebra* (corresponding to the algebra $\mathfrak{A}$). Let $\mathfrak{A}_\times$ and $\mathfrak{B}_\times$ be two $\times$-algebras. We say that $\phi : \mathfrak{A}_\times \to \mathfrak{B}_\times$ is an $\times$-*homomorphism* of two $\times$-algebras if the following holds

1. $\phi(a_j) = b_j,\ 1 \le j \le k$, where $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ are generators of $\mathfrak{A}$ and $\mathfrak{B}$ correspondingly.

2. $\phi(\lambda a_1 + a_2) = \lambda\phi(a_1) + \phi(a_2)$ for any $a_1, a_2 \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$.

3. $\phi(a_1 \times \cdots \times a_n) = \phi(a_1) \times \cdots \times \phi(a_n)$ for any set $a_1, \ldots, a_n$ of (not necessarily distinct) generators of $\mathfrak{A}$.

Note that a $\times$-algebra with the binary operation $\times$ becomes an associative algebra, which is commutative or non-commutative depending of the nature of the *scalars* in it. The definition of $\times$-algebra could be seen on the first glance as a cumbersome one, nevertheless it should be noted that such different mathematical constructions as the Weyl quantization [2] and Cauchy-Kovalevskaya extension [3], [9] are both examples of $\times$-algebras. For a justification of the product (13) see [16] or Section 4.

By the definition of the symmetric product one can obviously deduce

Lemma 2.3 *Any homomorphism of two algebras is a $\times$-homomorphism of the corresponding $\times$-algebras.*

The converse, of course, is not true (see Example 3.5 of the Weyl quantization in [19]), so the $\times$-homomorphism is *weaker* property than algebra homomorphism.

Fix an $(n-1)$-tuple of bounded self-adjoint operators $T = (T_1, \ldots, T_{n-1})$ on the Hilbert space H. The following is a definition of the Riesz-Clifford monogenic calculus.

Definition 2.4 We say that an $(n-1)$-tuple of operators $T \in \mathfrak{A}$ *has a monogenic functional calculus* $(\mathcal{A}, \Phi)$ *based on* $\mathbb{R}^n$ whenever the following conditions hold: $\mathcal{A}$ is a topological vector space of monogenic functions from $\Omega \subset \mathbb{R}^n$ to $\mathbf{Cl}(n, 0)$, with addition defined pointwise and (symmetric) $\times$-multiplication, and $\Phi : \mathcal{A} \to \mathfrak{A}$ is a continuous $\times$-homomorphism such that

$$\Phi : \vec{x}_j (= e_j x_1 + e_1 x_j) \mapsto T_j, \ 1 \le j \le n-1 \tag{14}$$

There, to extend calculus from commuting operators to non-commuting we ones we relax the requirement from homomorphism to $\times$-homomorphism.

Theorem 2.5 (Uniqueness) *For a given simply-connected domain Ω and an $(n-1)$-tuple of operators T, there exists no more than one monogenic functional calculus.*

Theorem 2.6 *For any $(n-1)$-tuple T of bounded self-adjoint operators there exist a monogenic calculus on $\mathbb{R}^n$.*

To construct an integral formula for the monogenic calculus we should define the Cauchy kernel of the operators T_j. It may be done as follows

Definition 2.7 Let (cf. (10))

$$E(y,T) = \sum_{j=0}^{\infty} \left(\sum_{|\alpha|=j} V_\alpha(T) W_\alpha(y) \right) \tag{15}$$

where

$$V_\alpha(T) = T_1^{\alpha_1} \times \cdots \times T_{n-1}^{\alpha_{n-1}}; \tag{16}$$

i.e., we have formally substituted in (6) for $(\vec{x}_1, \ldots, \vec{x}_{n-1})$ the $(n-1)$-tuple of operators $(T_1, \ldots, T_{n-1})$.

We have [17]

Lemma 2.8 *Let $| T |= \lim_{j\to\infty} \sup_\sigma \| T_{\sigma(1)} \cdots T_{\sigma(j)} \|^{1/j}$, $1 \le \sigma(i) \le n-1$ be the Rota joint spectral radius [25]. Then for fixed $| y |>| T |$, equation* (15) *defines a bounded operator in $\mathfrak{A}$.*

Definition 2.9 The maximal open subset $R_C(T)$ of $\mathbb{R}^n$ such that for any $y \in R_C(T)$ the series in (15) converges in the uniform operator topology to a bounded operator on $\widetilde{H}$ will be called the *(Cauchy) resolvent set* of T. The complement of $R_C(T)$ in $\mathbb{R}^n$ will be called the *(Cauchy) spectral set* of T and denoted by $\sigma_C(T)$.

From Lemma 2.8 it follows that $R_C(T)$ is always non-empty and $\sigma_C(T)$ is bounded; from Definition 2.9 one can see that $\sigma_C(T)$ is closed. Moreover, it is easy to see that from Liouville's theorem ([24], Theorem 5.5) it follows that $\sigma_C(T)$ is non-empty, thus

Lemma 2.10 *The Cauchy spectral set $\sigma_C(T)$ is compact.*

Recall formula [9], Chap. II, Lemma 1.5.7(i):

$$\int_{\partial\mathbb{B}(0,r)} W_\beta(y)\, d\sigma\, V_\alpha(y) = \delta_{\alpha\beta}. \tag{17}$$

It is easily follows from it that

Lemma 2.11 *Let $r >|T|$ and let Ω be the ball $\mathbb{B}(0,r) \in \mathbb{R}^n$. Then for any symmetric polynomial $P(\vec{x})$ we have*

$$P(T) = \int_{\partial\Omega} E(y,T)\, d\sigma_y\, P(y) \tag{18}$$

where $P(T)$ is the symmetric polynomial of the $(n-1)$-tuple T.

Lemma 2.12 *For any domain Ω, which does not contain $\sigma_C(T)$, and any $f \in H(\Omega)$, we have*

$$\int_{\partial\Omega} E(y,T)\, d\sigma_y\, f(y) = 0 \tag{19}$$

Due to this Lemma we can replace the domain $\mathbb{B}(0,r)$ at Lemma 2.11 with an arbitrary domain Ω containing the spectral set $\sigma_C(T)$. An application of Lemma 2.11 gives the main

Theorem 2.13 *Let $T = (T_1, \ldots, T_{n-1})$ be an $(n-1)$-tuple of bounded self-adjoint operators. Let the domain Ω with piecewise smooth boundary have a connected complement and suppose the spectral set $\sigma_C(T)$ lies inside a domain Ω. Then the mapping*

$$\Phi : f(x) \mapsto f(T) = \int_{\partial\Omega} E(y,T)\, d\sigma_y\, f(y) \tag{20}$$

defines a monogenic calculus for T.

3 Functional Calculus Semibounded Operators

We recall some considerations of [10], § VII.9. One can observe that for an operator T with a non-empty resolvent set $R(T)$ and arbitrary $a \in R(T)$, the operator $A = (T - aI)^{-1}$ is bounded (by the very definition of a resolvent set). Having a functional calculus for the bounded operator A as a given, we can define the functional calculus for operator T by the formula

$$f(T) = \phi(A), \text{ where } \phi(z) = f(z^{-1} + a).$$

This scheme was successfully applied in [27] to extend results of [24] for a commuting n-tuple of operators T_j to unbounded case. It was possible mainly because for commuting operators T_j the "Möbius transformation"

$$(aT + b)(cT + d)^{-1} = P_1 e_1 + \cdots + P_n e_n, \text{ where } T = T_1 e_1 + \cdots + T_n e_n$$

again produces an n-tuple of commuting operators P_j (let us say, it maps vector-operators again to vector-operators) and thus the classical scheme could be produced without modifications.

This is no longer true for an arbitrary n-tuple of non-commuting operators as it could be easily seen. Thus one should look for a way to avoid this obstacle. For example, one can try to map by means of the Cayley transform functions over the sphere to unbounded domains instead of operators to bounded case. Here we again meet some difficulties generated by the multi-dimensional and non-commutative nature of Clifford analysis, however they could be overcome by an adequate modification of our definitions. So we will proceed in this way.

To illustrate our method we again firstly consider the case of classic functional calculus of an operator. Working in the commutative setting of one or several complex variables one enjoys the following freedom: for a given function $f(z)$ in domain Ω and an arbitrary holomorphic mapping $\phi : \Omega' \to \Omega$ and an arbitrary holomorphic function $g(z)$ on Ω', the function $f'(z) = g(z)f(\phi(z))$ is again holomorphic in Ω'. However such types of mappings are mappings of holomorphic functions as linear spaces, not as functional algebras unless $g(z) \equiv 1$. Thus even in the one-variable case if we forced to set $g(z)$ not identically equal to 1 (to preserve some additional structure) we will lose the algebraic structure.

Example 3.1 (see [21], Chap. IX) We want to construct a holomorphic mapping, which will be an intertwining operator between two discreet series representations with a lowest weight $m \geq 2$ in the unit disc $\mathbb{D}$ and upper half plane $\mathbb{H}^+$. Having the Cayley transform

$$w = \frac{z-i}{z+i}, \qquad z = -i\frac{w+1}{w-1}, \quad z \in \mathbb{H}, w \in \mathbb{D}$$

one can define desired transformation on functions $f(w)$ on unit disc by the formula:

$$f(w) \mapsto f\left(\frac{z-i}{z+i}\right)(z-i)^{-m}.$$

In particular,

$$w^k \mapsto \left(\frac{z-i}{z+i}\right)^k (z-i)^{-m}.$$

From the Cauchy kernel decomposition on the unit disc,

$$f(w) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} f(t)\frac{dt}{w-t} = \frac{1}{2\pi i}\int_{\mathbb{S}^1} f(t)\sum_{k=0}^{\infty}\frac{w^k}{t^{k+1}}dt.$$

By means of the Cayley transform we can obtain a decomposition

$$f'(z) = f\left(\frac{z-i}{z+i}\right)(z-i)^{-m} = \frac{1}{2\pi i}\frac{1}{(z-i)^m}\int_\Gamma \sum_{k=0}^{\infty}\left(\frac{z-i}{z+i}\right)^k \frac{1}{t^{k+1}} f(t)dt. \tag{21}$$

Here we could already observe that the described transformation destroys the simple algebraic relations $g_k(w)g_l(w) = g_{k+l}(w)$ between function $g_k(w) = w^k$.

The situation in Clifford analysis is very similar (we repeatedly use here various results of [4], [27] without specific references). Having a Möbius transformation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the Euclidian space $\mathbb{R}^n$ we have almost no choice for constructing transformations of the space of monogenic functions unless

$$[\widehat{g}f](x) = \frac{(cx+d)^*}{\| cx+d \|^n} f\left(\frac{ax+b}{cx+d}\right). \tag{22}$$

We will fix now[3] g to be the Cayley transform[4] $g = \begin{pmatrix} e_n & 1 \\ 1 & e_n \end{pmatrix}$ of the upper half space $\mathbb{H}^+$ to the unit ball $\mathbb{B}$. Its inverse is $g = \frac{1}{2}\begin{pmatrix} -e_n & 1 \\ 1 & -e_n \end{pmatrix}$. Analogously to (21) we could see (compare with integral form for the adjoint function in [3], § 13.3)

$$[\widehat{g}f](x) = \int_{\mathbb{S}^{n-1}} \sum_{j=0}^{\infty}\left(\sum_{|\alpha|=j} \frac{x+e_n}{|x+e_n|^n} V_\alpha\left(\frac{e_n x+1}{x+e_n}\right) W_\alpha(y)\right) d\sigma_y f(y), \tag{23}$$

[3]Another interesting unbounded case, the exterior of a ball, could be treated analogously to upper half space.

[4]We use the Cnops convention [4] under which $\frac{a}{b}$ for $a, b \in \mathbf{Cl}(n,0)$ is always understood as ab^{-1}.

where $x \in \mathbb{H}^+$ and $y \in \mathbb{B}$. From here one could see that functions $V'_\alpha(x) = \frac{x+e_n}{|x+e_n|^n} V_\alpha\left(\frac{e_n x+1}{x+e_n}\right)$ are an appropriate substitution in the upper half space for functions $V_\alpha(y)$ in the unit ball just as $\left(\frac{z-i}{z+i}\right)^k (z-i)^{-m}$ is for w^k in the one dimensional case.

Now we arrive at the above mentioned difficulty: the described transformation destroys the structure of the $\times$-algebra. The solution could be carried out by the following simple result, which is closely connected to many *symbolical calculi* as was pointed by Howe [11].

Lemma 3.2 *Let $\mathfrak{A}$ be an associative algebra with binary operation $\cdot$ and $t \in \mathfrak{A}$. Then $\mathfrak{A}$ with the binary operation $a \circ_t b = a \cdot t \cdot b$ is again an associative algebra. Moreover the mapping $l_t : a \to at$ and $r_t : a \to ta$ are both algebras homomorphisms $(\mathfrak{A}, \circ_t) \to (\mathfrak{A}, \cdot)$:*

$$l_t(a \circ_t b) = l_t(a) \cdot l_t(b), \quad r_t(a \circ_t b) = r_t(a) \cdot r_t(b).$$

We easily conclude

Corollary 3.3 *Let $\mathfrak{A}$ be an associative algebra with binary operation $\cdot$ and $t \in \mathfrak{A}$. Then there is an associated structure of $\times$-algebra with $\times = \times_t$ product defined as symmetrical product in $\mathfrak{A}$ with respect to the modified multiplication $\circ_t$.*

Proof. Lemma 3.2 could be checked by the streightforward substitution. Then the corrolary immediatly foolows from Lemma 2.3. $\square$

In our case one could select $t = (x - e_n)\,|x - e_n|^{n-2}$ and elements

$$a_j(x) = \widehat{g}(e_n y_j + e_j y_n) = \frac{x+e_n}{|x+e_n|^n} \frac{2e_n x_j + e_j(x^2+1)}{(x+e_n)^2} \tag{24}$$

as generators of a modified $\times$-algebra. It easy to see, particularly that

$$V'_\alpha \times_t V'_\beta = V'_{\alpha+\beta}.$$

Now we could give a definition:

Definition 3.4 For a set of (unbounded) operators T *the functional calculus* $(\mathcal{A}, \Phi)$ *on* $\mathbb{H}^+$ should satisfy to the following conditions: $\mathcal{A}$ is a $\times$-algebra generated by $a_j(x)$ of monogenic functions from $\mathbb{H}^+$ to $\mathbf{Cl}(n,0)$, with addition defined pointwise and (symmetric) $\times_t$-multiplication, and $\Phi : \mathcal{A} \to \mathfrak{A}$ is a continuous $\times$-homomorphism such that

$$\Phi : a_j \mapsto T_j,\ 1 \leq j \leq n-1 \tag{25}$$

Because such a definition again does not lead to any uncertainty, we have

Theorem 3.5 (Uniqueness) *For an $n-1$-tuple of operators T, there exists no more than one monogenic functional calculus on $\mathbb{H}^+$.*

An integral formula for the monogenic calculus on $\mathbb{H}^+$ is obtained by means of the Cayley transform, so is the Cauchy kernel of the operators T_j.

Definition 3.6 Let (cf. (21))

$$E'(y,T) = \sum_{j=0}^{\infty} \left(\sum_{|\alpha|=j} V_\alpha(T) W_\alpha \left(\frac{e_n y + 1}{y + e_n} \right) \frac{-y - e_n}{|y + e_n|^n} \right) \tag{26}$$

where

$$V_\alpha(T) = T_1^{\alpha_1} \times \cdots \times T_{n-1}^{\alpha_{n-1}}; \tag{27}$$

and $y \in \mathbb{R}^n \setminus \mathbb{H}^+$.

Definition 3.7 We will say that an $(n-1)$-tuple T of operators are semibounded if equation (26) defines a bounded operator in $\mathfrak{A}$ for all y such that $y_n \le 0$.

We now present the main theorem about unbounded functional calculus.

Theorem 3.8 *Let $T = (T_1, \ldots, T_{n-1})$ be an $(n-1)$-tuple of semibounded self-adjoint operators. Then the mapping*

$$\Phi : f(x) \mapsto f(T) = \int_{\mathbb{R}^{n-1}} E'(y, T)\, d\sigma_y\, f(y) \tag{28}$$

defines a monogenic calculus for T in $\mathbb{H}^+$.

PROOF. The theorem follows from Theorem 2.13 with an application of the Cayley transform. Corollary 3.3 for the particular case of generators (24) garantees that the calculus given by formula (28) will satisfy to the $\times_t$-homomorphism condition of Definition 3.4. □

4 Riesz-Clifford Calculus and Quantum Mechanics

It seems that non-commutativity is not only the distinguished feature of quantum theory but also an important motive of the contemporary mathematics. The search of adequate non-commutative counterparts for classic (commutative) objects is the goal of many important papers and it obviously inspired by physics' demands (see for example [5]). Even more, it is physics which tells us in many important cases, which (from many alternatives) is the "proper" non-commutative twin to a classic notion.

It was noted in [19] that symmetric product (5) has the precise meaning of a quantum simultaneous measurement several non-commuting observables. Moreover, the Jordan symmetrical product (see for example [13], § 1.2) $A \circ B = \frac{1}{2}(AB + BA) = \left(\frac{A+B}{2}\right)^2 - \left(\frac{A-B}{2}\right)^2$ should be considered not as a binary operation subject to generate (non-associative) algebra, but as particular case of the symmetric product (5) with only two multipliers. To be short, *the simultaneous (from macroscopic point of view) measurement of several observables means equal probabilities of measurements in all possible succeeding orders.*

In this vein we would like to give a quantum interpretation[5] of the binary product invented in (13). From a point of view of operator theory this product hardly makes sense because it is not connected with usual operator composition. Thus some justification of it should be useful.

It is often argued in different approaches to quantum field theory, that only a small number of *primary* observables could be measured directly. For example, in axiomatic local quantum field theory they are (for a single particle) four projections Δx_j, $0 \le j \le 3$ of space-time interval Δx in a fixed reference system. All other observables, even if they are measured by a single (but complicated!) device, are composite ones.

For all such approaches the following model will be an adequate one. We have a (small) finite set x_j, $1 \le j \le n$ of directly measured (primary) observables of a system S. Let the device A measures the observables a of the system S. Because linearity in quantum mechanics is commonly accepted, we could assume without lost of generality that a is just a product of several

[5]This is a part of detailed quantum mechanical interpretation from [16].

x_j (without a summation). Under the above assumption about the primary nature of x_j this means that the device A consists (maybe in a non-trivial way) of several subdevices for simultaneous measurements of x_j and forming their product. By the property [19] of "simultaneous" measurements in quantum mechanics this product could be only the symmetric product (5). Thus we arrive at the expression

$$a = x_1^{\alpha_1} \times x_2^{\alpha_2} \times \cdots \times x_n^{\alpha_n}$$

for the observable a and some α_j. Considering another device B for an observable b we could conclude analogously

$$b = x_1^{\beta_1} \times x_2^{\beta_2} \times \cdots \times x_n^{\beta_n}.$$

Let us now put devices A and B together to measure simultaneous by a "product" of observables a and b. Which kind of product it will be? If the devices A and B really constitute an entity for simultaneous measurements, then all their subdevices for measurement of primary observables x_j should again produce for us the symmetric product:

$$b = x_1^{\alpha_1+\beta_1} \times x_2^{\alpha_2+\beta_2} \times \cdots \times x_n^{\alpha_n+\beta_n}.$$

This is exactly the formula (13) from the definition of $\times$-algebras. Thus *$\times$-product* (13) *corresponds to the result of measurement of the product two observables a and b of a system S with fixed set of primary measurable observables x_j.*

Upper half space as a domain for the spectrum naturally arises, for example, in the following problem. Let a physical system be described by some set of non-commuting coordinates (selfadjoint operators) Q_1, P_1, Q_2, P_2, $\ldots$,

Q_n, P_n and a Hamiltonian H. For physical reasons it is naturally assume that the Hamiltonian H is positively defined and thus the joint spectrum of the given $(2n+1)$-tuples of operators is localized in the upper half space. Then reasonable observables of the system in hand should be a function of the given operators. Note also, that a functional calculus based on several complex variables theory could not achieve this goal: the upper half space (as well as the exterior of any ball) is not a domain of holomorphy!

Finally we make some remarks about definition of monogenic functional calculus on the base of its conformal covariance [17], [18]. The connection between functional calculi and group covariance was already known for some of them, but only as a property; see, for example, [6], [7]. To put it as a definition seems to be useful. This approach allows particularly to prove the spectral mapping theorem [17], Theorem 3.19, thus it has close links with the structure of operator algebra, unless a $\times$-homomorphism and $\times$-product. Covariance with respect to conformal group is also important in physical application [16].

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References

[1] L. V. Ahlfors. Möbius transformations in $\mathbb{R}^n$ expressed through 2×2 matrices of Clifford numbers. *Complex Variables Theory Appl.*, 5(2):215–224, 1986.

[2] Robert F. V. Anderson. The Weyl functional calculus. *J. Funct. Anal.*, 4:240–267, 1969.

[3] F. Brackx, R. Delanghe, and F. Sommen. *Clifford Analysis*, volume 76 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, Boston, 1982.

[4] Jan Cnops. *Hurwitz Pairs and Applications of Möbius Transformations.* Habilitation dissertation, Universiteit Gent, Faculteit van de Wetenschappen, 1994.

[5] Alain Connes. *Non-Commutative Geometry.* Academic Press, New York, 1994.

[6] Raul E. Curto and Florian-Horia Vasilescu. Automorphism invariance of the operator-valued Poisson transform. *Acta Sci. Math. (Szeged)*, 57:65–78, 1993.

[7] Raul E. Curto and Florian-Horia Vasilescu. Standard operator models in several variables. 1994.

[8] Richard Delanghe. On the singularities of functions with values in a Clifford algebra. *Math. Ann.*, 196:293–319, 1972.

[9] Richard Delanghe, Frank Sommen, and Vladimir Souček. *Clifford Algebra and Spinor-Valued Functions*. Kluwer Academic Publishers, Dordrecht, 1992.

[10] Nelson Dunford and Jacob T. Schwartz. *Linears Operators. Part I: General Theory*, volume VII of *Pure and Applied Mathematics*. John Wiley & Sons, Inc., New York, 1957.

[11] Roger Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, 38:188–254, 1980.

[12] Alexander A. Kirillov. *Elements of the Theory of Representations*. Springer-Verlag, New York, 1976.

[13] Alexander A. Kirillov. Geometric quantization. In V. I. Arnold and S. P. Novikov, editors, *Dynamical Systems IV*, volume 4 of *Encyclopedia of Mathematical Sciences*, pages 137–172. Springer-Verlag, Berlin, 1990.

[14] Vladimir V. Kisil. Relative convolutions. I. Properties and applications. Reporte Interno # 162, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 1994. e-print archive `fuct-an/9410001`, to appear in *Advances in Mathamatics*.

[15] Vladimir V. Kisil. Connection between different function theories in clifford analysis. *Adv. in Appl. Clifford Algebras*, 5(1):63–74, 1995. e-print archive `funct-an/9501002`.

[16] Vladimir V. Kisil. Do we need that observables form an algebra? page 15, 1995. Odessa State University, preprint.

[17] Vladimir V. Kisil. Möbius transformations and monogenic functional calculus. *Electr. Research Announcements of AMS*, 2(1), 1996. (To appear).

[18] Vladimir V. Kisil. Spectrum, functional calculi and group representations. *Dokl. Akad. Nauk SSSR*, 1996. (submitted, russian).

[19] Vladimir V. Kisil and Enrique Ramírez de Arellano. The Riesz-Clifford functional calculus for several non-commuting operators and quantum field theory. *Math. Methods Appl. Sci.*, page 16, 1996. e-print archive `funct-an/9502006`.

[20] Wiesław Królikowski and Enrique Ramírez de Arellano. Polynomial solutions of the Fueter-Hurwitz equation. In Alexander Nagel and Edward Lee Stout, editors, *The Madison Symposium on Complex Analysis*, number 137 in Contemprorary Mathematics, pages 297–305. AMS, Providence, Rhode Island, 1992.

[21] Serge Lang. $SL_2(\mathbf{R})$, volume 105 of *Graduate Text in Mathematics*. Springer-Verlag, New York, 1985.

[22] G. Laville. Sur un calcul symbolique de Feynmann. In *Seminar d'Analyse*, volume 1295 of *Lect. Notes in Math.*, pages 132–145. Springer-Verlag, Berlin, 1987.

[23] Helmuth R. Malonek. Hypercomplex differentiability and its applications. In F. Bracks et al., editor, *Clifford Algebras and Applications in Mathematical Physics*, pages 141–150. Kluwer Academic Publishers, Netherlands, 1993.

[24] Alan McIntosh and Alan Pryde. A functional calculus for several commuting operators. *Indiana Univ. Math. J.*, 36:421–439, 1987.

[25] Gian-Carlo Rota and W.G. Strang. A note on the joint spectral radius. *Nederl. Akad. Wetensch. Indag. Math.*, 22:379–381, 1960.

[26] John Ryan. Conformal transformations and Hardy spaces arising in Clifford analysis. 1995. preprint.

[27] John Ryan. Some application of conformal covariance in Clifford analysis. In John Ryan, editor, *Clifford Algebras in Analysis and Related Topics*, pages 128–155. CRC Press, Boca Raton, 1995.

[28] Norberto Salinas. Clifford analysis and joint hyponormality. 1994. preprint.

[29] Irving E. Segal. C^*-algebras and quantization. In Robert Doran, editor, *C^*-Algebras: 1943–1993*, number 167 in Contemporary Mathematics, pages 55–65. AMS, Providence, Rhode Island, 1994.

[30] A. Sudbery. Quaternionic analysis. *Math. Proc. Camb. Phil. Soc.*, 85:197–225, 1979.

[31] Joseph L. Taylor. The analytic-functional calculus for several commuting operators. *Acta Math.*, 125:1–38, 1970.

[32] Joseph L. Taylor. A general framework for a multioperator functional calculus. *Adv. in Math.*, 9:183–252, 1972.

[33] Michael E. Taylor. *Noncommutative Harmonic Analysis*, volume 22 of *Math. Surv. and Monographs*. American Mathematical Society, Providence, Rhode Island, 1986.