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p-MECHANICS AND FIELD THEORY

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Abstract. The orbit method of Kirillov is used to derive the p-mechanical brackets [25]. They generate the quantum (Moyal) and classical (Poisson) brackets on respective orbits corresponding to representations of the Heisenberg group. The extension of p-mechanics to field theory is made through the De Donder–Weyl Hamiltonian formulation. The principal step is the substitution of the Heisenberg group with Galilean.

Contents

1. Introduction

The paper deals with quantization and bracket structures for mechanical and field-theoretic problems. It extends certain mechanical problems onto the field-theoretic framework, using the finite-dimensional “canonical formalism” based on the concept of *polymomenta* [12, 13, 14, 15, 16, 17, 18, 19, 20]. This finite-dimensional framework is strongly related to the multisymplectic formalism. In this way, the infinite-dimensional symplectic framework and infinite-dimensional canonical formalism traditionally used in field theory *are avoided*. Our approach and results have some relationships with above cited papers by I.V. Kanatchikov.

On mechanical level the main idea is to use the Kirillov orbit method in the case of Heisenberg group [23, 25, 26, 4, 5]. p-Mechanical brackets are constructed using the group algebra (convolution algebra) on the Heisenberg group. This brackets reduce to the quantum (Moyal) or mechanical (Poisson) brackets, depending on the used orbit.

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In this paper generalized brackets are constructed for field theory formulated in finite-dimensional polymomenta terms. The main idea there is to use the Galilei group instead the Heisenberg group. The constructed brackets also may be considered as some field-theoretic Moyal-like brackets (involving several "Planck constants") or classical brackets. This is similar to our construction of p-mechanical brackets. The main difference is that the Heisenberg group is replaced by the Galilei group and some Clifford-algebra-valued quantities are used.

The paper outline is as follows. We start in Section 2.1 from the Heisenberg group and its representations derived through the orbit method of Kirillov. In Section 2.2 we define p-mechanical observables as convolutions on the Heisenberg group $\mathbb{H}^n$ and study their commutators. We modify the commutator of two p-observables by the antiderivative to the central vector field in the Heisenberg Lie algebra in Section 2.3; this produces p-mechanical brackets and corresponding dynamic equation. Then the p-mechanical construction is extended to the De Donder–Weyl Hamiltonian formulation of the field theory [12, 13, 14, 15, 16, 17, 18, 19, 20] in Section 3.1. To this end we replace the Heisenberg group by the Galilean group in Section 3.2. Expanded presentation of Section 2 could be found in [26]. Development of material from Section 3 will follow in subsequent papers.

2. Elements of p-Mechanics

2.1. The Heisenberg Group and Its Representations. Let (s, x, y), where x, $y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [9, 11]. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{1}$$

where the non-commutativity is made by ω—the *symplectic form* [1, § 37] on $\mathbb{R}^{2n}$:

$$\omega(x, y; x', y') = xy' - x'y. \tag{2}$$

The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by left-invariant vector fields

$$S = \partial_s, \qquad X_j = \partial_{x_j} - y_j/2\partial_s, \qquad Y_j = \partial_{y_j} + x_j/2\partial_s \tag{3}$$

on $\mathbb{H}^n$ with the Heisenberg *commutator relations*

$$[X_i, Y_j] = \delta_{i,j} S \tag{4}$$

and all other commutators vanishing. The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ respecting the multiplication (1) and Heisenberg commutators is

$$\exp : sS + \sum_{k=1}^{n} (x_k X_k + y_k Y_k) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n).$$

As any group $\mathbb{H}^n$ acts on itself by the conjugation automorphisms $\mathrm{A}\,(g)h = g^{-1}hg$, which preserve the unit $e \in \mathbb{H}^n$. The differential $\mathrm{Ad}\, : \mathfrak{h}^n \to \mathfrak{h}^n$ of A at e is a linear map which could be differentiated again to the representation ad of the Lie algebra $\mathfrak{h}^n$ by the commutator: $\mathrm{ad}\,(A) : B \mapsto [B, A]$. The dual space $\mathfrak{h}_n^*$ to the Lie algebra $\mathfrak{h}^n$ is realised by the left invariant first order differential forms on $\mathbb{H}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}_n^*$ the map Ad generates the *co-adjoint representation* [22, § 15.1] $\mathrm{Ad}^* : \mathfrak{h}_n^* \to \mathfrak{h}_n^*$:

$$\mathrm{ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \quad \text{where } (s, x, y) \in \mathbb{H}^n \tag{5}$$

and $(h, q, p) \in \mathfrak{h}_n^*$ in bi-orthonormal coordinates to the exponential ones on $\mathfrak{h}^n$. There are two types of orbits in (5) for Ad^*—Euclidean spaces $\mathbb{R}^{2n}$ and single points:

$$\begin{aligned}\mathcal{O}_h &= \{(h,q,p):\ \text{for a fixed } h \neq 0 \text{ and all } (q,p)\in\mathbb{R}^{2n}\}, &(6)\\ \mathcal{O}_{(q,p)} &= \{(0,q,p):\ \text{for a fixed } (q,p)\in\mathbb{R}^{2n}\}. &(7)\end{aligned}$$

The *orbit method* of Kirillov [22, § 15] starts from the observation that the above orbits parametrise irreducible unitary representations of $\mathbb{H}^n$. All representations are *induced* [22, § 13] by a character $\chi_h(s,0,0) = \mathrm{e}^{2\pi \mathrm{i} hs}$ of the centre of $\mathbb{H}^n$ generated by $(h,0,0) \in \mathfrak{h}_n^*$ and shifts (5) from the *left* on orbits. Using [22, § 13.2, Prob. 5] we get a neat formula, which (unlike many other in literature) respects all *physical units* [26]:

$$\rho_h(s,x,y) : f_h(q,p) \mapsto \mathrm{e}^{-2\pi\mathrm{i}(hs+qx+py)} f_h(q - \tfrac{h}{2}y, p + \tfrac{h}{2}x). \tag{8}$$

The derived representation $d\rho_h$ of the Lie algebra $\mathfrak{h}^n$ defined on the vector fields (3) is:

$$d\rho_h(S) = -2\pi\mathrm{i}hI, \quad d\rho_h(X_j) = h\partial_{p_j} + \tfrac{\mathrm{i}}{2} q_j I, \qquad d\rho_h(Y_j) = -h\partial_{q_j} + \tfrac{\mathrm{i}}{2} p_j I \tag{9}$$

Operators D_h^j, $1 \le j \le n$ representing vectors from the complexification of $\mathfrak{h}^n$:

$$D_h^j = d\rho_h(-X_j + \mathrm{i}Y_j) = \tfrac{h}{2}(\partial_{p_j} + \mathrm{i}\partial_{q_j}) + 2\pi(p_j + \mathrm{i}q_j)I = h\partial_{\bar{z}_j} + 2\pi z_j I \tag{10}$$

where $z_j = p_j + \mathrm{i}q_j$ are used to give the following classic result in terms of orbits:

Theorem 2.1 (Stone–von Neumann, cf. [22, § 18.4], [9, Chap. 1, § 5])**.** *All unitary irreducible representations of $\mathbb{H}^n$ are parametrised up to equivalence by two classes of orbits* (6) *and* (7) *of the co-adjoint representation* (5) *in $\mathfrak{h}_n^*$:*

(1) *The infinite dimensional representations by transformation ρ_h* (8) *for $h \neq 0$ in Fock* [9, 11] *space $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ of null solutions to the operators D_h^j* (10)*:*

$$F_2(\mathcal{O}_h) = \{ f_h(p,q) \in L_2(\mathcal{O}_h) \mid D_h^j f_h = 0,\ 1 \le j \le n \}. \tag{11}$$

(2) *The one-dimensional representations as multiplication by a constant on $\mathbb{C} = L_2(\mathcal{O}_{(q,p)})$ which drops out from* (8) *for $h = 0$:*

$$\rho_{(q,p)}(s,x,y) : c \mapsto \mathrm{e}^{-2\pi\mathrm{i}(qx+py)} c. \tag{12}$$

Note that $f_h(p,q)$ is in $F_2(\mathcal{O}_h)$ if and only if the function $f_h(z)\mathrm{e}^{-|z|^2/h}$, $z = p + \mathrm{i}q$ is in the classical Segal–Bargmann space [9, 11], particularly is analytical in z. Furthermore the space $F_2(\mathcal{O}_h)$ is spanned by the Gaussian *vacuum vector* $\mathrm{e}^{-2\pi(q^2+p^2)/h}$ and all *coherent states*, which are "shifts" of the vacuum vector by operators (8).

Commutative representations (12) correspond to the case $h = 0$ in the formula (8). They are always neglected, however their union naturally (see the appearance of Poisson brackets in (22)) acts as the classical *phase space*:

$$\mathcal{O}_0 = \bigcup_{(q,p)\in\mathbb{R}^{2n}} \mathcal{O}_{(q,p)}. \tag{13}$$

Furthermore the structure of orbits of $\mathfrak{h}_n^*$ is implicitly present in equation (23) and its symplectic invariance [26].

2.2. **Convolution and Commutator on $\mathbb{H}^n$.** Using a left invariant measure dg on $\mathbb{H}^n$ the linear space $L_1(\mathbb{H}^n, dg)$ can be upgraded to an algebra with the convolution multiplication:

$$(k_1 * k_2)(g) = \int_{\mathbb{H}^n} k_1(g_1)\, k_2(g_1^{-1}g)\, dg_1 = \int_{\mathbb{H}^n} k_1(gg_1^{-1})\, k_2(g_1)\, dg_1. \tag{14}$$

Inner *derivations* D_k, $k \in L_1(\mathbb{H}^n)$ of $L_1(\mathbb{H}^n)$ are given by the *commutator* for $f \in L_1(\mathbb{H}^n)$:

$$D_k : f \mapsto [k,f] = k * f - f * k = \int_{\mathbb{H}^n} k(g_1)\left(f(g_1^{-1}g) - f(gg_1^{-1})\right)\, dg_1. \qquad (15)$$

A unitary representation ρ_h of $\mathbb{H}^n$ extends to $L_1(\mathbb{H}^n, dg)$ by the formula:

$$\begin{aligned}[\rho_h(k)f](q,p) &= \int_{\mathbb{H}^n} k(g)\rho_h(g)f(q,p)\, dg \\ &= \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}} k(s,x,y)\mathrm{e}^{-2\pi\mathrm{i}hs}\, ds\right) \mathrm{e}^{-2\pi\mathrm{i}(qx+py)} f(q-hy, p+hx)\, dx\, dy, \qquad (16)\end{aligned}$$

thus $\rho_h(k)$ for a fixed $h \neq 0$ depends only on $\hat{k}_s(h,x,y)$—the partial Fourier transform $s \to h$ of $k(s,x,y)$. Then the representation of the composition of two convolutions depends only from

$$\begin{aligned}(k' * k)\hat{}_s(h,x,y) &= \int_{\mathbb{R}} \int_{\mathbb{H}^n} k'(s',x',y') \\ &\qquad \times k(s-s'-\frac{1}{2}(xy'-yx'), x-x', y-y')\, ds'dx'dy'\mathrm{e}^{2\pi\mathrm{i}hs}\, ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{H}^n} k'(s',x',y')\mathrm{e}^{2\pi\mathrm{i}hs'} \\ &\qquad \times k(s-s'-\frac{1}{2}(xy'-yx'), x-x', y-y')\mathrm{e}^{2\pi\mathrm{i}h(s-s')}\, ds'dx'dy'ds \\ &= \int_{\mathbb{R}^{2n}} \hat{k}'_s(h,x',y')\, \hat{k}_s(h,x-x',y-y')\, \mathrm{e}^{\pi\mathrm{i}h(xy'-yx')}\, dx'dy'. \qquad (17)\end{aligned}$$

The last expression for the full Fourier transforms of k' and k turn out to be the *star product* known in deformation quantisation, cf. [29, (9)–(13)]. Consequently the representation of the commutator (15) depends only from [26]:

$$\begin{aligned}[k',k]\hat{}_s &= \int_{\mathbb{R}^{2n}} \hat{k}'_s(h,x',y')\hat{k}_s(h,x-x',y-y') \\ &\qquad \times \left(\mathrm{e}^{\pi\mathrm{i}h(xy'-yx')} - \mathrm{e}^{-\pi\mathrm{i}h(xy'-yx')}\right)\, dx'dy' \\ &= 2\mathrm{i}\int_{\mathbb{R}^{2n}} \hat{k}'_s(h,x',y')\hat{k}_s(h,x-x',y-y')\, \sin(\pi h(xy'-yx'))\, dx'dy', \qquad (18)\end{aligned}$$

which turn out to be exactly the "Moyal brackets" [29] for the full Fourier transforms of k' and k. Also the expression (18) vanishes for $h = 0$ as can be expected from the commutativity of representations (12).

2.3. **p-Mechanical Brackets on $\mathbb{H}^n$.** An anti-derivation operator $\mathcal{A}$, which is a scalar multiple of a right inverse of the vector field S (3) on $\mathbb{H}^n$, is defined by:

$$S\mathcal{A} = 4\pi^2 I, \qquad \text{where} \quad \mathcal{A}\mathrm{e}^{2\pi\mathrm{i}hs} = \begin{cases} \dfrac{2\pi}{\mathrm{i}h}\mathrm{e}^{2\pi\mathrm{i}hs}, & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0. \end{cases} \qquad (19)$$

It can be extended by the linearity to $L_1(\mathbb{H}^n)$. We introduce [25] a modified convolution operation $\star$ on $L_1(\mathbb{H}^n)$:

$$k_1 \star k_2 = k_1 * (\mathcal{A}k_2),$$

and the associated modified commutator

$$\{[k_1, k_2]\} = k_1 \star k_2 - k_2 \star k_1, \qquad (20)$$

which will be called *p-mechanical brackets* for reasons explained in (23).

One gets from (16) the representation $\rho_h(\mathcal{A}k) = (ih)^{-1}\rho_h(k)$ for any $h \neq 0$. Consequently the modification of (18) for $h \neq 0$ is only slightly different from the original one:

$$\{[k', k]\}_{\hat{s}} = \int_{\mathbb{R}^{2n}} \frac{2\pi}{h} \sin(\pi h(xy' - yx'))\, \hat{k}'_s(h, x', y')\, \hat{k}_s(h, x - x', y - y')\, dx'dy', \quad (21)$$

However the last expression for $h = 0$ is significantly distinct from the vanishing (18). From the natural assignment $\frac{4\pi}{h}\sin(\pi h(xy' - yx')) = 4\pi^2(xy' - yx')$ for $h = 0$ we get the Poisson brackets for the Fourier transforms of k' and k defined on $\mathcal{O}_0$ (13):

$$\rho_{(q,p)} \{[k', k]\} = \frac{\partial \hat{k}'}{\partial q}\frac{\partial \hat{k}}{\partial p} - \frac{\partial \hat{k}'}{\partial p}\frac{\partial \hat{k}}{\partial q}. \quad (22)$$

Furthermore the dynamical equation based on the modified commutator (20) with a suitable Hamilton type function $H(s, x, y)$ for an observable $f(s, x, y)$ on $\mathbb{H}^n$

$$\dot{f} = \{[H, f]\} \text{ is reduced } \begin{cases} \text{by } \rho_h,\ h \neq 0 \text{ on } \mathcal{O}_h \text{ (6) to Moyal's eq. [29, (8)];} \\ \text{by } \rho_{(q,p)} \text{ on } \mathcal{O}_0 \text{ (13) to Poisson's eq. [1, § 39].} \end{cases} \quad (23)$$

The same relationships are true for the solutions of these three equations, see [25] for the harmonic oscillator and [4, 5] for forced oscillator examples.

3. De Donder–Weyl Field Theory

We extend p-mechanics to the De Donder–Weyl field theory, see [12, 13, 14, 15, 16, 17, 18, 19, 20] for detailed exposition and further references. We will be limited here to the preliminary discussion which extends the comment 5.2.(1) from the earlier paper [26]. Our notations will slightly different from the used in the papers [12, 13, 14, 15, 16, 17, 18, 19, 20] to make it consistent with the used above and avoid clashes.

3.1. **Hamiltonian Form of Field Equation.** Let the underlying space-time has the dimension $n + 1$ and be parametrised by coordinates u^μ, $\mu = 0, 1, \ldots, n$ (with u^0 parameter traditionally associated with a time-like direction). Let us consider a field described by some m-component tensor q^a, $a = 1, \ldots, m$. For a system defined by a Lagrangian density $L(q^a, \partial_\mu q^a, u^\mu)$ De Donder–Weyl theory suggests new set of *polymomenta* p_a^μ and *DW Hamiltonian function* $H(q^a, p_a^\mu, u^\mu)$ defined as follows:

$$p_a^\mu := \frac{\partial L(q^a, \partial_\mu q^a, u^\mu)}{\partial(\partial_\mu y^a)} \quad \text{and} \quad H(q^a, p_a^\mu, u^\mu) = p_a^\mu\, \partial_\mu q^a - L(q^a, \partial_\mu q^a, u^\mu). \quad (24)$$

A multidimensional variational problem for the Lagrangian $L(q^a, \partial_\mu q^a, u^\mu)$ leads to the Euler–Lagrange field equations:

$$\frac{d}{du^\mu}\left(\frac{\partial L}{\partial(\partial_\mu q^a)}\right) - \frac{\partial L}{\partial q^a} = 0.$$

Just as in particle mechanics polymomenta (24) help us to transform the above Euler–Lagrange field equations to the Hamilton form:

$$\frac{\partial q^a}{\partial u^\mu} = \frac{\partial H}{\partial p_a^\mu}, \qquad \frac{\partial p_a^\mu}{\partial u^\mu} = -\frac{\partial H}{\partial q^a}, \quad (25)$$

with the standard summation (over repeating *Greek* indexes) convention. The main distinction from a particle mechanics is the existence of $n+1$ different polymomenta p_a^μ associated to each field variable q^a. Therefore, the particle mechanics could be considered as a particular case when $n + 1$ dimensional space-time degenerates for $n = 0$ to "time only".

The next two natural steps [12, 13, 14, 15, 16, 17, 18, 19, 20] inspired by particle mechanics are:

(1) Introduce an appropriate Poisson structure, such that the Hamilton equations (25) will represent the Poisson brackets.
(2) Quantise the above Poisson structure by some means, e.g. Dirac-Heisenberg-Shrödinger-Weyl technique or geometric quantisation.

We use here another path: first to construct a p-mechanical model for equations (25) and then deduce its quantum and classical derivatives as was done for the particle mechanics above. To simplify presentation we will start from the scalar field, i.e. $m = 1$. Thus we drop index a in q^a and p_a^μ and simply write q and p^μ instead.

We also assume that the underlying space-time is flat with a constant metric tensor $\eta^{\mu\nu}$. This metric defines a related *Clifford algebra* [3, 6, 8] with generators e_μ satisfying the relations

$$e_\mu e_\nu + e_\nu e_\mu = \eta^{\mu\nu}. \tag{26}$$

Remark 3.1. For the Minkowski space-time (i.e. in the context of special relativity) a preferable choice may be *quaternions* [28] with generators i, j, k instead the general Clifford algebra.

Remark 3.2. To avoid the possibility of confusion with imaginary units e_ν we will use the another font for the base of natural logarithms e.

Since q and p^μ look like conjugated variables p-mechanics suggests that they should generate a Lie algebra with relations similar to (4). The first natural assumption is the $n+3(=1+(n+1)+1)$-dimensional Lie algebra spanned by X, Y_μ, and S with the only non-trivial commutators $[X, Y_\mu] = S$. However as follows from the Kirillov theory [21] any its unitary irreducible representation is limited to a representation of $\mathbb{H}^1$ listed by the Stone–von Neumann Theorem 2.1. Consequently there is a little chance that we could obtain the field equations (25) in this way.

3.2. **Convolutions and Commutator on $\mathbb{G}^{n+1}$.** The next natural candidate is the Galilean group $\mathbb{G}^{n+1}$, i.e. a nilpotent step 2 Lie group with the $2n + 3(= 1 + (n+1) + (n+1))$-dimensional Lie algebra. It has a basis X, Y_μ, and S_μ with $n+1$-dimensional centre spanned by S_μ. The only non-trivial commutators are

$$[X, Y_\mu] = S_\mu, \qquad \text{where } \mu = 0, 1, \ldots, n. \tag{27}$$

Again the Kirillov theory [21] assures that any its *complex-valued* irreducible representation is a representation of $\mathbb{H}^1$. However the multidimensionality of the centre of $\mathbb{G}^{n+1}$ offers an option [7, 24] to consider *Clifford valued* representations of $\mathbb{G}^{n+1}$.

Remark 3.3. The appearance of Clifford algebra in connection with field theory and space-time geometry is natural. For example, the conformal invariance of space-time has profound consequences in astrophysics [27] and, in their turn, conformal (Möbius) transformations are most naturally represented by linear-fractional transformations in Clifford algebras [6]. Some other links between nilpotent Lie groups and Clifford algebras are listed in [24].

The Lie group $\mathbb{G}^{n+1}$ is a manifold homeomorphic to $\mathbb{R}^{2n+3}$ with coordinates (s, x, y), where $x \in \mathbb{R}$ and $s, y \in \mathbb{R}^{n+1}$. The group multiplication in these coordinates is defined by, cf. (1):

$$\begin{aligned}&(s, x, y) * (s', x', y') \\ =\ &(s_0 + s'_0 + \tfrac{1}{2}\omega(x, y_0; x', y'_0), \ldots, s_n + s'_n + \tfrac{1}{2}\omega(x, y_n; x', y'_n), x + x', y + y').\end{aligned} \tag{28}$$

The Lie algebra $\mathfrak{g}^{n+1}$ is realised by the left (right) invariant vector fields on $\mathbb{G}^{n+1}$ which satisfy to (27):

$$S^j_{l(r)} = \pm\frac{\partial}{\partial s_j}, \quad X_{l(r)} = \pm\frac{\partial}{\partial x} - \sum_{j=0}^{n}\frac{y_j}{2}\frac{\partial}{\partial s_j}, \quad Y^j_{l(r)} = \pm\frac{\partial}{\partial y_j} + \frac{x}{2}\frac{\partial}{\partial s_j}. \tag{29}$$

The dual $\mathfrak{g}^*_{n+1}$ of the Lie algebra $\mathfrak{g}^{n+1}$ coincides with $\mathbb{R}^{2n+3}$ with coordinates $(h_0,\ldots,h_n,q,p_0,\ldots,p_n)$ in the basis biorthogonal to the basis S^j, X, Y^j, $j=0,1,\ldots,n$ of $\mathfrak{g}^{n+1}$. Similarly to the realisation (11) of Fock type spaces on $\mathfrak{h}^n$ we define the vacuum vector $v_{h,0}$ parametrised by the $(n+1)$-tuple of the Planck constants $h=(h_0,h_1,\ldots,h_n)$ as follows (cf. [26, (2.25)]):

$$v_{h,0}(s,x,y) = \sum_{j=0}^{n}\exp 2\pi h_k\left(-e_j s_j - \tfrac{1}{4}(x^2+y_0^2+y_1^2+\cdots+y_n^2)\right). \tag{30}$$

Here one can use the Euler formula $\mathrm{e}^{a+e_jb}=\mathrm{e}^a(\cos b+e_j\sin b)$ for a, $b\in\mathbb{R}$ as a definition. The coherent states $v_{h,g}$ are obtained as left shifts of $v_{h,0}$ by $g=(s,x,y)$ on $\mathbb{G}^{n+1}$:

$$\begin{aligned} v_{h,g}(g') &= \rho_l(s,x,y)v_{h,0}(s',x',y') \qquad (31)\\ &= \sum_{j=0}^{n}\exp\, 2\pi h_j\left(-e_j\left(s'_j - s_j - \frac{1}{2}\omega(x,y_j;x',y'_j)\right)\right.\\ &\qquad\left. -\frac{1}{4}\left((x'-x)^2+\sum_{l=0}^{n}(y'_l-y_l)^2\right)\right). \end{aligned}$$

Note that the vacuum vector (30) and coherent states (31) are different from ones used in [7].

Definition 3.4. For a fixed parameter $h=(h_0,h_1,\ldots,h_n)\in\mathbb{R}^{n+1}$ we define Segal–Bargmann space $F_2^h(\mathbb{G}^{n+1})$ of functions on $\mathbb{G}^{n+1}$ as the closure of the linear span of vectors $v_{h,g}$ for all $g\in\mathbb{G}^{n+1}$ in the norm defined by the inner product:

$$\begin{aligned} \langle f_1,f_2\rangle_h &= \int_{\mathbb{R}^{n+1}}\int_{\mathbb{R}}\sum_{j=0}^{n}h_j\left(\int_{\mathbb{R}^{n+1}}\bar{f}_1(s,x,y)\,\mathrm{e}^{2\pi e_j h_j s_j}\,ds\right) \qquad (32)\\ &\qquad\times\left(\int_{\mathbb{R}^{n+1}}\mathrm{e}^{-2\pi e_j h_j s_j}\,f_2(s,x,y)\,ds\right)dx\,dy. \end{aligned}$$

The linear combinations of coherent states (31) with Clifford coefficients multiplied from the *right*.

The following could be directly verified:

Lemma 3.5. (1) *The left regular action of* $\mathbb{G}^{n+1}$ *is unitary with respect to the inner product* (32).
(2) *The vacuum vector* (30) *and consequently all coherent states* (31) *have the unit norm in* $F_2^h(\mathbb{G}^n)$.

Let P_j be the operator $F_2^h(\mathbb{G}^n)\to F_2^h(\mathbb{G}^n)$ defined through the partial Fourier transform $s_j\to h_j$ as follows:

$$[P_jk](s,x,y) = e^{2\pi e_j s_j h_j}\int_{\mathbb{R}^{n+1}}k(s,x,y)\,\mathrm{e}^{-2\pi e_j s_j h_j}\,ds.$$

The following properties can be directly verified.

Lemma 3.6. *If all components* h_j *of* $h=(h_0,h_1,\ldots,h_n)\in\mathbb{R}^{n+1}$ *are non-zero then*

(1) *Each operator* P_j *is an orthogonal projection on* $F_2^h(\mathbb{G}^n)$.
(2) *The sum of all* P_j *is the identity operator on* $F_2^h(\mathbb{G}^n)$: $\sum_0^n P_j = I$.
(3) *All operators* P_j *commute with the left and right action of* $\mathbb{G}^{n+1}$ *on* $F_2^h(\mathbb{G}^n)$.

Definition 3.7. Observables are defined as convolution K operators on $L_2(\mathbb{G}^{n+1})$ with a kernel $k(s,x,y)$, which has a property:

$$\int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds = \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{2\pi e_j h_j s_j}\,ds. \tag{33}$$

Due to the resolution $\sum_0^n P_j = I$ on $F_2^h(\mathbb{G}^n)$ from 2 we see that the restriction of a convolution operator $K = K\sum_0^n P_j = \sum_0^n KP_j$ to $F_2^h(\mathbb{G}^n)$ is completely defined by the set of operators KP_j. Kernels of operators KP_j are given by partial Fourier transforms $(s_0,\ldots,s_j,\ldots,s_n)\to(0,\ldots,h_j,\ldots,0)$:

$$\hat{k}_j(0,\ldots,h_j,\ldots,0,x,y) = \int_{\mathbb{R}^{n+1}} k(s_0,\ldots,s_n,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds_0\ldots ds_j\ldots ds_n.$$

The following Lemma simplify further calculations:

Lemma 3.8. *Let a kernel* $k(s,x,y)$ *satisfy to the identity* (33)*, then for any constant* $c\in\mathcal{C}\ell(n+1)$:

$$\int_{\mathbb{R}^{n+1}} k(s,x,y)\,c\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds = \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds\,c.$$

Proof. In a trivial way we can present $c\,\mathrm{e}^{-2\pi e_j h_j s_j} = \mathrm{e}^{-2\pi e_j h_j s_j}c_1 + \mathrm{e}^{2\pi e_j h_j s_j}c_2$, where $c_1 = \frac{1}{2}(c - e_j c e_j)$ and $c_2 = \frac{1}{2}(c + e_j c e_j)$, thus $c = c_1 + c_2$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} k(s,x,y)\,c\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds\\
&\quad= \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds\,c_1 + \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{2\pi e_j h_j s_j}\,ds\,c_2\\
&\quad= \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds\,c_1 + \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds\,c_2 \qquad (34)\\
&\quad= \int_{\mathbb{R}^{n+1}} k(s,x,y)\,\mathrm{e}^{-2\pi e_j h_j s_j}\,ds\,c,
\end{aligned}$$

where transformation (34) follows from the property (33). □

Using this Lemma we found that the restriction of the composition $k' * k$ of two convolutions to $F_2^h(\mathbb{G}^n)$ depends only (cf. (17)) on:

$$\begin{aligned}
(k' * k)\hat{}_j &= \int_{\mathbb{R}^{n+2}} \hat{k}'_j(0,\ldots,h_j,\ldots,0,x',y')\,\hat{k}_j\left(0,\ldots,h_j,\ldots,0,x-x',y-y'\right)\\
&\qquad\times \mathrm{e}^{\pi e_j h_j (xy'_j - y_j x')}\,dx'\,dy'.
\end{aligned}$$

Thus the restriction of the commutator $k' * k - k * k'$ of two convolutions to $F_2^h(\mathbb{G}^n)$ depends only (cf. (18)) on:

$$\begin{aligned}
&(k' * k - k * k')\hat{}_j\\
&\quad= \int_{\mathbb{R}^{n+2}} \hat{k}'_j(0,\ldots,h_j,\ldots,0,x',y')\,\hat{k}_j\left(0,\ldots,h_j,\ldots,0,x-x',y-y'\right)\\
&\qquad\qquad\times\left(\mathrm{e}^{\pi e_j h_j (xy'_j - y_j x')} - \mathrm{e}^{-\pi e_j h_j (xy'_j - y_j x')}\right)dx'\,dy'.\\
&\quad= \int_{\mathbb{R}^{n+2}} \hat{k}'_j(0,\ldots,h_j,\ldots,0,x',y')\,\hat{k}_j\left(0,\ldots,h_j,\ldots,0,x-x',y-y'\right)\\
&\qquad\qquad\times 2e_j \sin\left(\pi h_j (xy'_j - y_j x')\right)dx'\,dy'. \qquad (35)
\end{aligned}$$

3.3. **p-Mechanical Brackets on $\mathbb{G}^{n+1}$.** To define an appropriate brackets of two observables k' and k we will again modify the restriction $[k', k\rceil_j$ of their commutator on $F_2^h(\mathbb{G}^{n+1})$. To this end we use (cf. (19)) antiderivative operators $\mathcal{A}_0$, $\mathcal{A}_1$, ..., $\mathcal{A}_n$ which are multiples of right inverse to the vector fields S^0, S^1, ..., S^n (29):

$$S^j\mathcal{A}_j = 4\pi^2 P_j, \qquad \text{where } \mathcal{A}_j e^{2\pi e_k h_k s_k} = \begin{cases} \dfrac{2\pi}{e_j h_j} e^{2\pi e_j h_j s_j}, & \text{if } h_j \neq 0, \text{ and } j = k; \\ 4\pi^2 s_j, & \text{if } h_j = 0, \text{ and } j = k; \\ 0, & \text{if } h_k \neq 0, \text{ and } j \neq k. \end{cases} \tag{36}$$

The definition of the brackets follows the ideas of [7, § 3.3]: to each vector field S_j should be associated a generator e_j of Clifford algebra (26). Thus our brackets are as follows, cf. (20):

$$\{[B_1, B_2]\} = B_1 * B_2\mathcal{A} - B_2 * B_1\mathcal{A}, \qquad \text{where } \mathcal{A} = \sum_{j=0}^{n} e_j\mathcal{A}_j. \tag{37}$$

These brackets will be used in the right-hand side of the p-dynamical equation. Its left-hand side should contain a replacement for the time derivative. As was already mentioned in [12, 13, 14, 15, 16, 17, 18, 19, 20] the space-time play a rôle of multidimensional time in the De Donder–Weyl construction. Thus we replace time derivative by the symmetric pairing $D\circ$ with the *Dirac operator* [3, 6, 8] $D = e_j\partial_j$ as follows:

$$D \circ f = -\frac{1}{2}\left(e_j \frac{\partial f}{\partial u^j} + \frac{\partial f}{\partial u^j} e_j\right), \qquad \text{where } D = e_j\partial_j. \tag{38}$$

Finally the p-mechanical dynamical equation, cf. (23):

$$D \circ f = \{[H, f]\}, \tag{39}$$

is defined through the brackets (37) and the Dirac operator (38).

To "verify" the equation (39) we will find its classical representation and compare it with equations (25). Indeed, combining (35) and (36) for $h_j \neq 0$ we got:

$$\begin{aligned} \{[k', k]\}\,\hat{}_j \;=\; & \int_{\mathbb{R}^{n+2}} \hat{k}'_j(0, \ldots, h_j, \ldots, 0, x', y')\, \hat{k}_j\left(0, \ldots, h_j, \ldots, 0, x - x', y - y'\right) \\ & \times 4\pi^2 e_j \frac{\sin\left(\pi h_j (x y'_j - y_j x')\right)}{\pi h_j} dx'\, dy'. \end{aligned} \tag{40}$$

In the limit $h_j \to 0$ this naturally becomes

$$\begin{aligned} \{[k', k]\}\,\hat{}_j \;=\; & \int_{\mathbb{R}^{n+2}} \hat{k}'_j(0, \ldots, 0, x', y')\, \hat{k}_j\left(0, \ldots, 0, x - x', y - y'\right) \\ & \times 4\pi^2 e_j (x y'_j - y_j x') dx'\, dy'. \end{aligned} \tag{41}$$

Then the representation $\rho_{(q,p)}$ of the $\{[k', k]\}\,\hat{}_j$ is expressed through the complete Fourier transform $(s, x, y) \to (0, q, p)$ as follows:

$$\rho_{(q,p)}\left(\{[k', k]\}\,\hat{}_j\right) = \sum_{j=0}^{n} \left(\frac{\partial \hat{k}'(0, q, p)}{\partial q}\frac{\partial \hat{k}(0, q, p)}{\partial p^j} - \frac{\partial \hat{k}'(0, q, p)}{\partial p^j}\frac{\partial \hat{k}(0, q, p)}{\partial q}\right) e_j.$$

Finally from the decomposition 2 it follows that $\{[k', k]\} = \sum_{j=0}^{n} \{[k', k]\}\,\hat{}_j$ and we obtain the classical representation of p-mechanical brackets, cf. (22):

$$\rho_{(q,p)}\,\{[k', k]\} = \sum_{j=0}^{n} \left(\frac{\partial \hat{k}'}{\partial q}\frac{\partial \hat{k}}{\partial p^j} - \frac{\partial \hat{k}'}{\partial p^j}\frac{\partial \hat{k}}{\partial q}\right) e_j. \tag{42}$$

Consequently the dynamics from the equation (39) of field observable q with a scalar-valued Hamiltonian H in the classical representation is given by:

$$D \circ q = \sum_{j=0}^{n} \left(\frac{\partial H}{\partial q} \frac{\partial}{\partial p^j} - \frac{\partial H}{\partial p^j} \frac{\partial}{\partial q} \right) e_j q \qquad \Longleftrightarrow \qquad \sum_{j=0}^{n} \frac{\partial q}{\partial u^j} e_j = \sum_{j=0}^{n} \frac{\partial H}{\partial p^j} e_j. \quad (43)$$

After the separation of components in the last equation with different generators e_j we get the first $n+1$ equations from the set (25).

To obtain the last equation for polymomenta (25) we again use the Clifford algebra generators to construct the *combined polymomenta* $p = \sum_0^n e_k p^k$. For them:

$$\begin{aligned} D \circ p &= -\sum_{j=0}^{n} \frac{1}{2} \left(e_j \frac{\partial \sum_0^n e_k p^k}{\partial u^j} + \frac{\partial \sum_0^n e_k p^k}{\partial u^j} e_j \right) = -\sum_{j=0}^{n} \frac{\partial p^j}{\partial u^j} e_j e_j = \sum_{j=0}^{n} \frac{\partial p^j}{\partial u^j}, \\ \{[H, p]\} &= \sum_{j=0}^{n} \left(\frac{\partial H}{\partial q} \frac{\partial \sum_0^n e_k p^k}{\partial p^j} - \frac{\partial H}{\partial p^j} \frac{\sum_0^n \partial e_k p^k}{\partial q} \right) e_j = \sum_{j=0}^{n} \frac{\partial H}{\partial q} e_j e_j = C \frac{\partial H}{\partial q}, \end{aligned}$$

where $C = \sum_0^n e_j e_j$, i.e. $C = -2$ for the Minkowski space-time. Thus the equation (39) for the combined polymomenta $p = \sum_k e_k p^k$ in the classical representation becomes:

$$\sum_{j=0}^{n} \frac{\partial p^j}{\partial u^j} = C \frac{\partial H}{\partial q}, \qquad (44)$$

i.e. coincides with the last equation in (25) up to a constant factor $-C = -\sum_0^n e_j e_j$. If this constant is non-zero the second equation in (25) is equivalent to the equation (44) with the Hamilton function $H_C(q, -\frac{1}{C} p) = -\frac{1}{C} \partial_j q p^j - L(q, \partial_j q, u)$. Thus p-mechanical equation (39) passed the test by its classical representation.

Consequently we may assume that the "quantum" representations $\rho_h = \rho_{(h_0,\ldots,h_n)}$ of $\mathbb{G}^{n+1}$ map the p-mechanical bracket $\{[k', k]\}$ (37) and corresponding dynamic equation (39) to the equation of quantum fields. Such an image $[\cdot,\cdot]_q$ of $\{[\cdot,\cdot]\}$ is easily obtained from (40) and the decomposition 2:

$$\begin{aligned} &[\rho_h(k'), \rho_h(k)]_q = \rho_h \{[k', k]\} \qquad (45) \\ &\quad = \int_{\mathbb{R}^{n+2}} \sum_{j=0}^{n} \hat{k}'_j(0, \ldots, h_j, \ldots, 0, x', y')\, \hat{k}_j \left(0, \ldots, h_j, \ldots, 0, x - x', y - y'\right) \\ &\qquad\qquad \times \frac{4\pi e_j}{h_j} \sin\left(\pi h_j (x y'_j - y_j x')\right) dx'\, dy'. \end{aligned}$$

This brackets are similar to the Moyal brackets (21). Consequently the dynamic equation (39) with a Hamiltonian $\rho_h(H)$ for an observable $\rho_h(k)$ becomes in the quantum representation

$$D \circ \rho_h(k) = [\rho_h(H), \rho_h(k)]_q, \qquad (46)$$

in the notations (45). Therefore the image (46) of the equation (39) could stand for a quantisation of its classical images in (25), (42). A further study of quantum images of the equation (39) as well as extension to vector or spinor fields should follow in subsequent papers. In different spaces of functions these quantisations corresponds to Dirac-Heisenberg-Schrödinger-Weyl, geometric, deformational, etc. quantisations of particle mechanics.

Remark 3.9. To consider vector or spinor fields with components q_a, $a = 1, \ldots, m$ it worths to introduce another Clifford algebra with generators c^a and consider a composite field $q = c^a q_a$. There are different ways to link Clifford and Grassmann algebras, see e.g. [2, 10]. Through such a link the Clifford algebra with generators

e_j corresponds to horizontal differential forms in the sense of [12, 13, 14, 15, 16, 17, 18, 19, 20] and the Clifford algebra generated by c_a—to the vertical.

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