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ERLANGEN PROGRAMME AT LARGE 3.1  
HYPERCOMPLEX REPRESENTATIONS OF THE HEISENBERG GROUP  
AND MECHANICS

VLADIMIR V. KISIL

*Dedicated to the memory of V.I. Arnold*

ABSTRACT. In the spirit of geometric quantisation we consider representations of the Heisenberg(-Weyl) group induced by hypercomplex characters of its centre. This allows to gather under the same framework, called p-mechanics, the three principal cases: quantum mechanics (elliptic character), hyperbolic mechanics and classical mechanics (parabolic character). In each case we recover the corresponding dynamic equation as well as rules for addition of probabilities. Notably, we are able to obtain whole classical mechanics without any kind of semiclassical limit  $\hbar \rightarrow 0$ .

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On leave from Odessa University.

## 1. INTRODUCTION

Complex valued representations of the Heisenberg group (also known as Weyl or Heisenberg-Weyl group) provide a natural framework for quantum mechanics [12,21]. This is the most fundamental example of the Kirillov orbit method and geometrical quantisation technique [32,33].

Following the pattern we consider representations of the Heisenberg group which are induced by hypercomplex characters of its centre. Besides complex numbers (which correspond to the elliptic case) there are two other types of hypercomplex numbers: dual (parabolic) and double (hyperbolic) [47; 66, App. C].

To describe dynamics of a physical system we use a universal equation based on inner derivations of the convolution algebra [39,41]. The complex valued representations produce the standard framework for quantum mechanics with the Heisenberg dynamical equation [65].

The double number valued representations, with the hyperbolic unit  $j^2 = 1$ , is a natural source of hyperbolic quantum mechanics developed for a while [22,23,25,27,28]. The universal dynamical equation employs hyperbolic commutator in this case. This can be seen as a Moyal bracket based on the hyperbolic sine function. The hyperbolic observables act as operators on a Krein space with an indefinite inner product. Such spaces are employed in study of  $\mathcal{PT}$ -symmetric Hamiltonians and hyperbolic unit  $j^2 = 1$  naturally appear in this setup [17].

The representations with values in dual numbers provide a convenient description of the classical mechanics. For this we do not take any sort of semiclassical limit, rather the nilpotency of the parabolic unit ( $\epsilon^2 = 0$ ) do the task. This removes the vicious necessity to consider the Planck *constant* tending to zero. The dynamical equation takes the Hamiltonian form. We also describe classical non-commutative representations of the Heisenberg group which acts in the first jet space.

*Remark 1.1.* It is commonly accepted that the striking difference between quantum and classical mechanics is non-commutativity of observables in the first case. In particular the Heisenberg commutation relations, see (2.5), imply the uncertainty principle, the Heisenberg equation of motion and other quantum features. However our work shows that quantum mechanics is mainly determined by the properties of complex numbers. Non-commutative representations of the Heisenberg group in dual numbers implies the Poisson dynamical equation and local addition of probabilities in Section 4.2, which are completely classical.

*Remark 1.2.* It is worth to note that our technique is different from contraction technique in the theory of Lie groups [16,55]. Indeed a contraction of the Heisenberg group  $\mathbb{H}^n$  is the commutative Euclidean group  $\mathbb{R}^{2n}$  which does not recreate neither quantum nor classical mechanics.

The approach provides not only three different types of dynamics, it also generates the respective rules for addition of probabilities as well. For example, the quantum interference is the consequence of the same complex-valued structure, which directs the Heisenberg equation. The absence of an interference (a particle behaviour) in the classical mechanics is again the consequence the nilpotency of the parabolic unit. Double numbers creates the hyperbolic law of additions of probabilities which were extensively investigated [25,27]. There are still unresolved issues with positivity of the probabilistic interpretation in the hyperbolic case [22,23].

The work clarifies foundations of quantum and classical mechanics. We recovered from the representation theory the existence of three non-isomorphic model of mechanics already discussed in [22,23] from translation invariant formulation.

It also hinted that hyperbolic counterpart is (at least theoretically) as natural as classical and quantum mechanics are. The approach provides a framework for description of aggregate system which have say both quantum and classical components. This can be used to model quantum computers with classical terminals [49].

Remarkably, simultaneously with the work [22] group-invariant axiomatics of geometry lead R.I. Pimenov [58] to description of  $3^n$  Cayley–Klein constructions. The connection between group-invariant geometry and respective mechanics were explored in many works of N.A. Gromov, see for example [14–16]. Those already highlighted the rôle of three types of hypercomplex units for the realisation of elliptic, parabolic and hyperbolic geometry and kinematic.

There is a further connection between representations of the Heisenberg group and hypercomplex numbers. The symplectomorphism of phase space are also automorphism of the Heisenberg group [12, § 1.2]. Induced representation of the symplectic group naturally lead to hypercomplex numbers [47]. Hamiltonians, which produce those symplectomorphism, are of interest, for example, in quantum optic [61]. An analysis of those Hamiltonians by means of creation/annihilation operators recreate hypercomplex coefficients as well [52, 53].

*Remark 1.3.* This work is performed within the “Erlangen programme at large” framework [45, 51], thus it would be suitable to explain the numbering of various papers. Since the logical order may be different from chronological one the following numbering scheme is used:

Prefix	Branch description
“0” or no prefix	Mainly geometrical works, within the classical field of Erlangen programme by F. Klein
“1”	Papers on analytical functions theories and wavelets
“2”	Papers on operator theory, functional calculi and spectra
“3”	Papers on mathematical physics

For example, this is the first paper in the mathematical physics area.

## 2. HEISENBERG GROUP AND p-MECHANICS

**2.1. The Heisenberg group and induced representations.** Let  $(s, x, y)$ , where  $x, y \in \mathbb{R}^n$  and  $s \in \mathbb{R}$ , be an element of the Heisenberg group  $\mathbb{H}^n$  [12, 21]. The group law on  $\mathbb{H}^n$  is given as follows:

$$(2.1) \quad (s, x, y) \cdot (s', x', y') = (s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y'),$$

where the non-commutativity is due to  $\omega$ —the *symplectic form* on  $\mathbb{R}^{2n}$  [2, § 37]:

$$(2.2) \quad \omega(x, y; x', y') = xy' - x'y.$$

The Heisenberg group is non-commutative Lie group with the centre

$$Z = \{(s, 0, 0) \in \mathbb{H}^n, s \in \mathbb{R}\}.$$

The left shifts

$$(2.3) \quad \Lambda(g) : f(g') \mapsto f(g^{-1}g')$$

act as a representation of  $\mathbb{H}^n$  on a certain linear space of functions. For example, action on  $L_2(\mathbb{H}, dg)$  with respect to the Haar measure  $dg = ds dx dy$  is the *left regular* representation, which is unitary.

The Lie algebra  $\mathfrak{h}^n$  of  $\mathbb{H}^n$  is spanned by left-(right-)invariant vector fields

$$(2.4) \quad S^{1(r)} = \pm\partial_s, \quad X_j^{1(r)} = \pm\partial_{x_j} - \frac{1}{2}y_j\partial_s, \quad Y_j^{1(r)} = \pm\partial_{y_j} + \frac{1}{2}x_j\partial_s$$

on  $\mathbb{H}^n$  with the Heisenberg *commutator relations*

$$(2.5) \quad [X_i^{l(\tau)}, Y_j^{l(\tau)}] = \delta_{ij} S^{l(\tau)}$$

and all other commutators vanishing. We will omit the superscript  $l$  for left-invariant field sometimes.

We can construct linear representations by induction [31, § 13] from a character  $\chi$  of the centre  $Z$ . There are several models for induced representations, here we prefer the following one, which is presented stripping off all generalities, cf. [31, § 13; 60, Ch. 5]. Let  $F_2^X(\mathbb{H}^n)$  be the space of functions on  $\mathbb{H}^n$  having the properties:

$$(2.6) \quad f(gh) = \chi(h)f(g), \quad \text{for all } g \in \mathbb{H}^n, h \in Z$$

and

$$(2.7) \quad \int_{\mathbb{R}^{2n}} |f(0, x, y)|^2 dx dy < \infty.$$

Then  $F_2^X(\mathbb{H}^n)$  is invariant under the left shifts and those shifts restricted to  $F_2^X(\mathbb{H}^n)$  make a representation  $\rho_\chi$  of  $\mathbb{H}^n$  induced by  $\chi$ .

If the character  $\chi$  is unitary, then the induced representation is unitary as well. However the representation  $\rho_\chi$  is not necessarily irreducible. Indeed, left shifts are commuting with the right action of the group. Thus any subspace of null-solutions of a linear combination  $aS + \sum_{j=1}^n (b_j X_j + c_j Y_j)$  of left-invariant vector fields is left-invariant and we can restrict  $\rho_\chi$  to this subspace. The left-invariant differential operators define analytic condition for functions, cf. [65].

**Example 2.1.** The function  $f_0(s, x, y) = e^{ihs - h(x^2 + y^2)/4}$ , where  $h = 2\pi\hbar$ , belongs to  $F_2^X(\mathbb{H}^n)$  for the character  $\chi(s) = e^{ihs}$ . It is also a null solution for all the operators  $X_j - iY_j$ . The closed linear span of functions  $f_g = \Lambda(g)f_0$  is invariant under left shifts and provide a model for Segal–Bargmann type representation of the Heisenberg group, which will be considered below.

*Remark 2.2.* An alternative construction of induced representations is as follow [31, § 13.2]. Consider a subgroup  $H$  of a group  $G$ . Let a smooth section  $s : G/H \rightarrow G$  be a left inverse of the natural projection  $\mathbf{p} : G \rightarrow G/H$ . Thus any element  $g \in G$  can be uniquely decomposed as  $g = s(\mathbf{p}(g)) * \mathbf{r}(g)$  where the map  $\mathbf{r} : G \rightarrow H$  is defined by the previous identity. For a character  $\chi$  of  $H$  we can define a *lifting*  $\mathcal{L}_\chi : L_2(G/H) \rightarrow L_2^X(G)$  as follows:

$$(2.8) \quad [\mathcal{L}_\chi f](g) = \chi(\mathbf{r}(g))f(\mathbf{p}(g)) \quad \text{where } f(x) \in L_2(G/H).$$

The image space of the lifting  $\mathcal{L}_\chi$  is invariant under left shifts. We also define the *pulling*  $\mathcal{P} : L_2^X(G) \rightarrow L_2(G/H)$ , which is a left inverse of the lifting and explicitly can be given, for example, by  $[\mathcal{P}F](x) = F(s(x))$ . Then the induced representation on  $L_2(G/H)$  is generated by the formula  $\rho_\chi(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}$ .

**2.2. Convolutions (observables) on  $\mathbb{H}^n$  and commutator.** Using a left invariant measure  $dg = ds dx dy$  on  $\mathbb{H}^n$  we can define the convolution of two functions:

$$(2.9) \quad (k_1 * k_2)(g) = \int_{\mathbb{H}^n} k_1(g_1) k_2(g_1^{-1}g) dg_1.$$

This is a non-commutative operation, which is meaningful for functions from various spaces including  $L_1(\mathbb{H}^n, dg)$ , the Schwartz space  $S$  and many classes of distributions, which form algebras under convolutions. Convolutions on  $\mathbb{H}^n$  are used as *observables* in p-mechanic [36, 41].

A unitary representation  $\rho$  of  $\mathbb{H}^n$  extends to  $L_1(\mathbb{H}^n, dg)$  by the formula:

$$(2.10) \quad \rho(k) = \int_{\mathbb{H}^n} k(g)\rho(g) dg.$$

This is also an algebra homomorphism of convolutions to linear operators.

For a dynamics of observables we need inner *derivations*  $D_k$  of the convolution algebra  $L_1(\mathbb{H}^n)$ , which are given by the *commutator*:

$$(2.11) D_k : f \mapsto [k, f] = k * f - f * k \\ = \int_{\mathbb{H}^n} k(g_1) (f(g_1^{-1}g) - f(gg_1^{-1})) dg_1, \quad f, k \in L_1(\mathbb{H}^n).$$

To describe dynamics of a time-dependent observable  $f(t, g)$  we use the universal equation, cf. [35, 36]:

$$(2.12) \quad S\dot{f} = [H, f],$$

where  $S$  is the left-invariant vector field (2.4) generated by the centre of  $\mathbb{H}^n$ . The presence of operator  $S$  fixes the dimensionality of both sides of the equation (2.12) if the observable  $H$  (Hamiltonian) has the dimensionality of energy [41, Rem 4.1]. If we apply a right inverse  $\mathcal{A}$  of  $S$  to both sides of the equation (2.12) we obtain the equivalent equation

$$(2.13) \quad \dot{f} = \llbracket H, f \rrbracket,$$

based on the universal bracket  $\llbracket k_1, k_2 \rrbracket = k_1 * \mathcal{A}k_2 - k_2 * \mathcal{A}k_1$  [41].

**Example 2.3** (Harmonic oscillator). Let  $H = \frac{1}{2}(m\omega^2q^2 + \frac{1}{m}p^2)$  be the Hamiltonian of a one-dimensional harmonic oscillator, where  $\omega$  is a constant frequency and  $m$  is a constant mass. Its *p-mechanisation* will be the second order differential operator on  $\mathbb{H}^n$  [5, § 5.1]:

$$H = \frac{1}{2}(m\omega^2X^2 + \frac{1}{m}Y^2),$$

where we dropped sub-indexes of vector fields (2.4) in one dimensional setting. We can express the commutator as a difference between the left and the right action of the vector fields:

$$\llbracket H, f \rrbracket = \frac{1}{2}(m\omega^2((X^r)^2 - (X^l)^2) + \frac{1}{m}((Y^r)^2 - (Y^l)^2))f.$$

Thus the equation (2.12) becomes [5, (5.2)]:

$$(2.14) \quad \frac{\partial}{\partial s}\dot{f} = \frac{\partial}{\partial s} \left( m\omega^2y \frac{\partial}{\partial x} - \frac{1}{m}x \frac{\partial}{\partial y} \right) f.$$

Of course, the derivative  $\frac{\partial}{\partial s}$  can be dropped from both sides of the equation and the general solution is found to be:

$$(2.15) \quad f(t; s, x, y) = f_0 \left( s, x \cos(\omega t) + m\omega y \sin(\omega t), -\frac{x}{m\omega} \sin(\omega t) + y \cos(\omega t) \right),$$

where  $f_0(s, x, y)$  is the initial value of an observable on  $\mathbb{H}^n$ .

**Example 2.4** (Unharmonic oscillator). We consider unharmonic oscillator with cubic potential, see [6] and references therein:

$$(2.16) \quad H = \frac{m\omega^2}{2}q^2 + \frac{\lambda}{6}q^3 + \frac{1}{2m}p^2.$$

Due to absence of non-commutative products *p-mechanisation* is straightforward:

$$H = \frac{m\omega^2}{2}X^2 + \frac{\lambda}{6}X^3 + \frac{1}{m}Y^2.$$

Similarly to the harmonic case the dynamic equation, after cancellation of  $\frac{\partial}{\partial s}$  on both sides, becomes:

$$(2.17) \quad \dot{f} = \left( m\omega^2y \frac{\partial}{\partial x} + \frac{\lambda}{6} \left( 3y \frac{\partial^2}{\partial x^2} + \frac{1}{4}y^3 \frac{\partial^2}{\partial s^2} \right) - \frac{1}{m}x \frac{\partial}{\partial y} \right) f.$$

Unfortunately, it cannot be solved analytically as easy as the harmonic case.

**2.3. States and Probability.** Let an observable  $\rho(k)$  (2.10) is defined by a kernel  $k(g)$  on the Heisenberg group and its representation  $\rho$  at a Hilbert space  $\mathcal{H}$ . A state on the convolution algebra is given by a vector  $v \in \mathcal{H}$ . A simple calculation:

$$\begin{aligned} \langle \rho(k)v, v \rangle_{\mathcal{H}} &= \left\langle \int_{\mathbb{H}^n} k(g) \rho(g)v \, dg, v \right\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{H}^n} k(g) \langle \rho(g)v, v \rangle_{\mathcal{H}} \, dg \\ &= \int_{\mathbb{H}^n} k(g) \overline{\langle v, \rho(g)v \rangle_{\mathcal{H}}} \, dg \end{aligned}$$

can be restated as:

$$\langle \rho(k)v, v \rangle_{\mathcal{H}} = \langle k, l \rangle, \quad \text{where } l(g) = \langle v, \rho(g)v \rangle_{\mathcal{H}}.$$

Here the left-hand side contains the inner product on  $\mathcal{H}$ , while the right-hand side uses a skew-linear pairing between functions on  $\mathbb{H}^n$  based on the Haar measure integration. In other words we obtain, cf. [5, Thm. 3.11]:

**Proposition 2.5.** *A state defined by a vector  $v \in \mathcal{H}$  coincides with the linear functional given by the wavelet transform*

$$(2.18) \quad l(g) = \langle v, \rho(g)v \rangle_{\mathcal{H}}$$

of  $v$  used as the mother wavelet as well.

The addition of vectors in  $\mathcal{H}$  implies the following operation on states:

$$(2.19) \quad \begin{aligned} \langle v_1 + v_2, \rho(g)(v_1 + v_2) \rangle_{\mathcal{H}} &= \langle v_1, \rho(g)v_1 \rangle_{\mathcal{H}} + \langle v_2, \rho(g)v_2 \rangle_{\mathcal{H}} \\ &\quad + \langle v_1, \rho(g)v_2 \rangle_{\mathcal{H}} + \overline{\langle v_1, \rho(g^{-1})v_2 \rangle_{\mathcal{H}}} \end{aligned}$$

The last expression can be conveniently rewritten for kernels of the functional as

$$(2.20) \quad l_{12} = l_1 + l_2 + 2A\sqrt{l_1 l_2}$$

for some real number  $A$ . This formula is behind the contextual law of addition of conditional probabilities [26] and will be illustrated below. Its physical interpretation is an interference, say, from two slits. The mechanism of such interference can be both causal and local, see [30, 40].

### 3. ELLIPTIC CHARACTERS AND QUANTUM DYNAMICS

In this section we consider the representation  $\rho_{\hbar}$  of  $\mathbb{H}^n$  induced by the elliptic character  $\chi_{\hbar}(s) = e^{i\hbar s}$  in complex numbers parametrised by  $\hbar \in \mathbb{R}$ . We also use the convenient agreement  $\hbar = 2\pi\hbar$ .

**3.1. Segal–Bargmann and Schrödinger Representations.** The realisation of  $\rho_{\hbar}$  by the left shifts (2.3) on  $L_2^1(\mathbb{H}^n)$  is rarely used in quantum mechanics. Instead two unitary equivalent forms are more common: the Schrödinger and Segal–Bargmann representations.

The Segal–Bargmann representation can be obtained from the orbit method of Kirillov [32]. It allows spatially separate irreducible components of the left regular representation, each of them is located on the orbit of the co-adjoint representation, see [32; 41, § 2.1] for details, we only present a brief summary here.

We identify  $\mathbb{H}^n$  and its Lie algebra  $\mathfrak{h}_n$  through the exponential map [31, § 6.4]. The dual  $\mathfrak{h}_n^*$  of  $\mathfrak{h}_n$  is presented by the Euclidean space  $\mathbb{R}^{2n+1}$  with coordinates  $(\hbar, q, p)$ . The pairing  $\mathfrak{h}_n^*$  and  $\mathfrak{h}_n$  given by

$$\langle (s, x, y), (\hbar, q, p) \rangle = \hbar s + q \cdot x + p \cdot y.$$

This pairing defines the Fourier transform  $\hat{\cdot} : L_2(\mathbb{H}^n) \rightarrow L_2(\mathfrak{h}_n^*)$  given by [33, § 2.3]:

$$(3.1) \quad \hat{\phi}(F) = \int_{\mathfrak{h}^n} \phi(\exp X) e^{-2\pi i \langle X, F \rangle} dX \quad \text{where } X \in \mathfrak{h}^n, F \in \mathfrak{h}_n^*.$$

For a fixed  $\hbar$  the left regular representation (2.3) is mapped by the Fourier transform to the Segal–Bargmann type representation [11, (1); 41, (2.9)]:

$$(3.2) \quad \rho_{\hbar}(s, x, y) : f(q, p) \mapsto e^{-2\pi i (\hbar s + qx + py)} f\left(q - \frac{\hbar}{2}y, p + \frac{\hbar}{2}x\right).$$

The collection of points  $(\hbar, q, p) \in \mathfrak{h}_n^*$  for a fixed  $\hbar$  is naturally identified with the phase space of the system.

*Remark 3.1.* It is possible to identify the case of  $\hbar = 0$  with classical mechanics [41]. Indeed, a substitution of the zero value of  $\hbar$  into (3.2) produces the commutative representation:

$$(3.3) \quad \rho_0(s, x, y) : f(q, p) \mapsto e^{-2\pi i (qx + py)} f(q, p).$$

It can be decomposed into the direct integral of one-dimensional representations parametrised by the points  $(q, p)$  of the phase space. The classical mechanics, including the Hamilton equation, can be recovered from those representations [41]. However the condition  $\hbar = 0$  (as well as  $\hbar \rightarrow 0$ ) is not completely physical. Commutativity (and subsequent relative triviality) of those representation is the main reason why they are often neglected. The commutativity can be outweighed by special arrangements, e.g. an antiderivative [41, (4.1)], but the procedure is not straightforward, see discussion in [1, 44, 48]. A direct approach using dual numbers will be discussed below, cf. Rem. 4.5.

To recover the Schrödinger representation we use Rem. 2.2, see [37, Ex. 4.1] for details. The subgroup  $H = \{(s, 0, y) \mid s \in \mathbb{R}, y \in \mathbb{R}^n\} \subset \mathbb{H}^n$  defines the homogeneous space  $X = G/H$ , which coincides with  $\mathbb{R}^n$  as a manifold. The natural projection  $\mathbf{p} : G \rightarrow X$  is  $\mathbf{p}(s, x, y) = x$  and its left inverse  $s : X \rightarrow G$  can be as simple as  $s(x) = (0, x, 0)$ . For the map  $\mathbf{r} : G \rightarrow H$ ,  $\mathbf{r}(s, x, y) = (s - xy/2, 0, y)$  we have the decomposition

$$(s, x, y) = s(\mathbf{p}(s, x, y)) * \mathbf{r}(s, x, y) = (0, x, 0) * (s - \frac{1}{2}xy, 0, y).$$

For a character  $\chi_{\hbar}(s, 0, y) = e^{i\hbar s}$  of  $H$  the lifting  $\mathcal{L}_{\chi} : L_2(G/H) \rightarrow L_2^{\chi}(G)$  is as follows:

$$[\mathcal{L}_{\chi} f](s, x, y) = \chi_{\hbar}(\mathbf{r}(s, x, y)) f(\mathbf{p}(s, x, y)) = e^{i\hbar(s - xy/2)} f(x).$$

Thus the representation  $\rho_{\chi}(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}$  becomes:

$$(3.4) \quad [\rho_{\chi}(s', x', y') f](x) = e^{-2\pi i \hbar (s' + xy' - x'y'/2)} f(x - x').$$

After the Fourier transform  $x \mapsto q$  we get the Schrödinger representation on the configuration space:

$$(3.5) \quad [\rho_{\chi}(s', x', y') \hat{f}](q) = e^{-2\pi i \hbar (s' + x'y'/2) - 2\pi i x'q} \hat{f}(q + \hbar y').$$

Note that this again turns into a commutative representation (multiplication by an unimodular function) if  $\hbar = 0$ . To get the full set of commutative representations in this way we need to use the character  $\chi_{(\hbar, p)}(s, 0, y) = e^{2\pi i (\hbar s + py)}$  in the above consideration.

**3.2. Commutator and the Heisenberg Equation.** The property (2.6) of  $F_2^X(\mathbb{H}^n)$  implies that the restrictions of two operators  $\rho_\chi(k_1)$  and  $\rho_\chi(k_2)$  to this space are equal if

$$\int_{\mathbb{R}} k_1(s, x, y) \chi(s) ds = \int_{\mathbb{R}} k_2(s, x, y) \chi(s) ds.$$

In other words, for a character  $\chi(s) = e^{2\pi i \hbar s}$  the operator  $\rho_\chi(k)$  depends only on

$$\hat{k}_s(\hbar, x, y) = \int_{\mathbb{R}} k(s, x, y) e^{-2\pi i \hbar s} ds,$$

which is the partial Fourier transform  $s \mapsto \hbar$  of  $k(s, x, y)$ . The restriction to  $F_2^X(\mathbb{H}^n)$  of the composition formula for convolutions is [41, (3.5)]:

$$(3.6) \quad (k' * k)_{\hat{s}} = \int_{\mathbb{R}^{2n}} e^{i\hbar(xy' - yx')/2} \hat{k}'_s(\hbar, x', y') \hat{k}_s(\hbar, x - x', y - y') dx' dy'.$$

Under the Schrödinger representation (3.5) the convolution (3.6) defines a rule for composition of two pseudo-differential operators (PDO) in the Weyl calculus [12, § 2.3; 21].

Consequently the representation (2.10) of commutator (2.11) depends only on its partial Fourier transform [41, (3.6)]:

$$(3.7) \quad [k', k]_{\hat{s}} = 2i \int_{\mathbb{R}^{2n}} \sin\left(\frac{\hbar}{2}(xy' - yx')\right) \times \hat{k}'_s(\hbar, x', y') \hat{k}_s(\hbar, x - x', y - y') dx' dy'.$$

Under the Fourier transform (3.1) this commutator is exactly the Moyal bracket [67] for  $\hat{k}'$  and  $\hat{k}$ .

For observables in the space  $F_2^X(\mathbb{H}^n)$  the action of  $S$  is reduced to multiplication, e.g. for  $\chi(s) = e^{i\hbar s}$  the action of  $S$  is multiplication by  $i\hbar$ . Thus the equation (2.12) reduced to the space  $F_2^X(\mathbb{H}^n)$  becomes the Heisenberg type equation [41, (4.4)]:

$$(3.8) \quad \dot{f} = \frac{1}{i\hbar} [H, f]_{\hat{s}},$$

based on the above bracket (3.7). The Schrödinger representation (3.5) transforms this equation to the original Heisenberg equation.

**Example 3.2.** (i) Under the Fourier transform  $(x, y) \mapsto (q, p)$  the  $p$ -dynamic equation (2.14) of the harmonic oscillator becomes:

$$\dot{f} = \left( m\omega^2 q \frac{\partial}{\partial p} - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

The same transform creates its solution out of (2.15).

(ii) Since  $\frac{\partial}{\partial s}$  acts on  $F_2^X(\mathbb{H}^n)$  as multiplication by  $i\hbar$ , the quantum representation of unharmonic dynamics equation (2.17) is:

$$(3.9) \quad \dot{f} = \left( m\omega^2 q \frac{\partial}{\partial p} + \frac{\lambda}{6} \left( 3q^2 \frac{\partial}{\partial p} - \frac{\hbar^2}{4} \frac{\partial^3}{\partial p^3} \right) - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

This is exactly the equation for the Wigner function obtained in [6, (30)].

**3.3. Quantum Probabilities.** For the elliptic character  $\chi_h(s) = e^{i\hbar s}$  we can use the Cauchy–Schwartz inequality to demonstrate that the real number  $A$  in the identity (2.20) is between  $-1$  and  $1$ . Thus we can put  $A = \cos \alpha$  for some angle (phase)  $\alpha$  to get the formula for counting quantum probabilities, cf. [27, (2)]:

$$(3.10) \quad l_{12} = l_1 + l_2 + 2 \cos \alpha \sqrt{l_1 l_2}$$

*Remark 3.3.* It is interesting to note that the both trigonometric functions are employed in quantum mechanics: sine is in the heart of the Moyal bracket (3.7) and cosine is responsible for the addition of probabilities (3.10). In the essence the commutator and probabilities took respectively the odd and even parts of the elliptic character  $e^{i\hbar s}$ .

**Example 3.4.** Take a vector  $v_{(a,b)} \in L_2^{\hbar}(\mathbb{H}^n)$  defined by a Gaussian with mean value  $(a, b)$  in the phase space for a harmonic oscillator of the mass  $m$  and the frequency  $\omega$ :

$$(3.11) \quad v_{(a,b)}(q, p) = \exp\left(-\frac{2\pi\omega m}{\hbar}(q-a)^2 - \frac{2\pi}{\hbar\omega m}(p-b)^2\right).$$

A direct calculation shows:

$$\begin{aligned} \langle v_{(a,b)}, \rho_{\hbar}(s, x, y)v_{(a',b')} \rangle &= \frac{4}{\hbar} \exp\left(\pi i(2s\hbar + x(a+a') + y(b+b'))\right) \\ &\quad - \frac{\pi}{2\hbar\omega m}((\hbar x + b - b')^2 + (b - b')^2) - \frac{\pi\omega m}{2\hbar}((\hbar y + a' - a)^2 + (a' - a)^2) \\ &= \frac{4}{\hbar} \exp\left(\pi i(2s\hbar + x(a+a') + y(b+b'))\right) \\ &\quad - \frac{\pi}{\hbar\omega m}\left((b-b' + \frac{\hbar x}{2})^2 + (\frac{\hbar x}{2})^2\right) - \frac{\pi\omega m}{\hbar}\left((a-a' - \frac{\hbar y}{2})^2 + (\frac{\hbar y}{2})^2\right) \end{aligned}$$

Thus the kernel  $l_{(a,b)} = \langle v_{(a,b)}, \rho_{\hbar}(s, x, y)v_{(a,b)} \rangle$  (2.18) for a state  $v_{(a,b)}$  is:

$$(3.12) \quad l_{(a,b)} = \frac{4}{\hbar} \exp\left(2\pi i(s\hbar + xa + yb) - \frac{\pi\hbar}{2\omega m}x^2 - \frac{\pi\omega m\hbar}{2\hbar}y^2\right)$$

An observable registering a particle at a point  $q = c$  of the configuration space is  $\delta(q - c)$ . On the Heisenberg group this observable is given by the kernel:

$$(3.13) \quad X_c(s, x, y) = e^{2\pi i(s\hbar + xc)}\delta(y).$$

The measurement of  $X_c$  on the state (3.11) (through the kernel (3.12)) predictably is:

$$\langle X_c, l_{(a,b)} \rangle = \sqrt{\frac{2\omega m}{\hbar}} \exp\left(-\frac{2\pi\omega m}{\hbar}(c-a)^2\right).$$

**Example 3.5.** Now take two states  $v_{(0,b)}$  and  $v_{(0,-b)}$ , where for the simplicity we assume the mean values of coordinates vanish in the both cases. Then the corresponding kernel (2.19) has the interference terms:

$$\begin{aligned} l_i &= \langle v_{(0,b)}, \rho_{\hbar}(s, x, y)v_{(0,-b)} \rangle \\ &= \frac{4}{\hbar} \exp\left(2\pi i s\hbar - \frac{\pi}{2\hbar\omega m}((\hbar x + 2b)^2 + 4b^2) - \frac{\pi\hbar\omega m}{2}y^2\right). \end{aligned}$$

The measurement of  $X_c$  (3.13) on this term contains the oscillating part:

$$\langle X_c, l_i \rangle = \sqrt{\frac{2\omega m}{\hbar}} \exp\left(-\frac{2\pi\omega m}{\hbar}c^2 - \frac{2\pi}{\omega m\hbar}b^2 + \frac{4\pi i}{\hbar}cb\right)$$

Therefore on the kernel  $l$  corresponding to the state  $v_{(0,b)} + v_{(0,-b)}$  the measurement is

$$\langle X_c, l \rangle = 2\sqrt{\frac{2\omega m}{\hbar}} \exp\left(-\frac{2\pi\omega m}{\hbar}c^2\right) \left(1 + \exp\left(-\frac{2\pi}{\omega m\hbar}b^2\right) \cos\left(\frac{4\pi}{\hbar}cb\right)\right).$$

The presence of the cosine term in the last expression can generate an interference picture. In practise it does not happen for the minimal uncertainty state (3.11) which we are using here: it rapidly vanishes outside of the neighbourhood of zero, where oscillations of the cosine occurs, see Fig. 1(a).

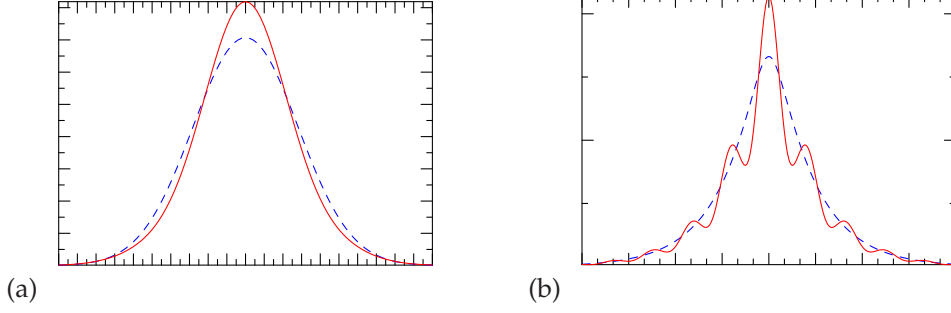


FIGURE 1. Quantum probabilities: the blue (dashed) graph shows the addition of probabilities without interaction, the red (solid) graph present the quantum interference. Left picture shows the Gaussian state (3.11), the right—the rational state (3.14)

**Example 3.6.** To see a traditional interference pattern one can use a state which is far from the minimal uncertainty. For example, we can consider the state:

$$(3.14) \quad u_{(a,b)}(q, p) = \frac{\hbar^2}{((q-a)^2 + \hbar/\omega m)((p-b)^2 + \hbar\omega m)}.$$

To evaluate the observable  $X_c$  (3.13) on the state  $l(g) = \langle u_1, \rho_{\hbar}(g)u_2 \rangle$  (2.18) we use the following formula:

$$\langle X_c, l \rangle = \frac{2}{\hbar} \int_{\mathbb{R}^n} \hat{u}_1(q, 2(q-c)/\hbar) \overline{\hat{u}_2(q, 2(q-c)/\hbar)} dq,$$

where  $\hat{u}_i(q, x)$  denotes the partial Fourier transform  $p \mapsto x$  of  $u_i(q, p)$ . The formula is obtained by swapping order of integrations. The numerical evaluation of the state obtained by the addition  $u_{(0,b)} + u_{(0,-b)}$  is plotted on Fig. 1(b), the red curve shows the canonical interference pattern.

#### 4. HYPERCOMPLEX REPRESENTATIONS OF THE HEISENBERG GROUP

The group of symmetries of classical mechanics—the group preserving the symplectic form (2.2)—generates automorphisms of the Heisenberg group in a natural way [12, § 1.2]. Those common symmetries of quantum and classical mechanics are behind many important connections, e.g. between classical “symplectic camel” and the Heisenberg uncertainty relations [10].

The symplectic group of  $\mathbb{R}^2$  is isomorphic to the celebrated group  $SL_2(\mathbb{R})$  [54]. Both groups  $\mathbb{H}^n$  and  $SL_2(\mathbb{R})$  contributes to the symmetries of the paraxial wave equation [61]. There are many other physical links between the Heisenberg group and  $SL_2(\mathbb{R})$ , e.g. metaplectic representation [12, Ch. 4].

It was demonstrated in [50] that dual and double numbers appears very naturally within the induced representations of the group  $SL_2(\mathbb{R})$ . Special relativity [62] and global space-time model [19, 46] also link the representation theory to hypercomplex numbers. Physical significance of hypercomplex numbers and representation theory of Clifford algebras was recently highlighted as well [4, 59, 63, 64]. There is an explicit similarity between the commutators in the Heisenberg-Weyl Lie algebra and anticommutators defining Clifford algebra [34, 38], which can be unified as a superspace [3, 9]. Thus it would be an omission to restrict linear representations of  $\mathbb{H}^n$  to complex numbers only.

**4.1. Hyperbolic Representations and Addition of Probabilities.** Now we turn to double numbers also known as hyperbolic, split-complex, etc. numbers [29; 62; 66, App. C]. They form a two dimensional algebra  $\mathbb{O}$  spanned by 1 and  $j$  with the property  $j^2 = 1$ . There are zero divisors:

$$j_{\pm} = \frac{1}{\sqrt{2}}(1 \pm j), \quad \text{such that} \quad j_+j_- = 0 \quad \text{and} \quad j_{\pm}^2 = j_{\pm}.$$

Thus double numbers algebraically isomorphic to two copies of  $\mathbb{R}$  spanned by  $j_{\pm}$ . Being algebraically dull double numbers are nevertheless interesting as a homogeneous space [47, 51] and they are relevant in physics [25, 62, 63]. The combination of p-mechanical approach with hyperbolic quantum mechanics was already discussed in [5, § 6].

For the hyperbolic character  $\chi_{jh}(s) = e^{jhs} = \cosh hs + j \sinh hs$  of  $\mathbb{R}$  one can define the hyperbolic Fourier-type transform:

$$\hat{k}(q) = \int_{\mathbb{R}} k(x) e^{-jqx} dx.$$

It can be understood in the sense of distributions on the space dual to the set of analytic functions [28, § 3]. Hyperbolic Fourier transform intertwines the derivative  $\frac{d}{dx}$  and multiplication by  $jq$  [28, Prop. 1].

**Example 4.1.** For the Gaussian the hyperbolic Fourier transform is the ordinary function (note the sign difference!):

$$\int_{\mathbb{R}} e^{-x^2/2} e^{-jqx} dx = \sqrt{2\pi} e^{q^2/2}.$$

However the opposite identity:

$$\int_{\mathbb{R}} e^{x^2/2} e^{-jqx} dx = \sqrt{2\pi} e^{-q^2/2}$$

is true only in a suitable distributional sense. To this end we may note that  $e^{x^2/2}$  and  $e^{-q^2/2}$  are null solutions to the differential operators  $\frac{d}{dx} - x$  and  $\frac{d}{dq} + q$  respectively, which are intertwined (up to the factor  $j$ ) by the hyperbolic Fourier transform. The above differential operators  $\frac{d}{dx} - x$  and  $\frac{d}{dq} + q$  are images of the *ladder operators* in the Lie algebra of the Heisenberg group. They are intertwining by the Fourier transform, since this is an automorphism of the Heisenberg group [20]. A careful study of ladder operators reveals connections with hypercomplex numbers [52, 53].

An elegant theory of hyperbolic Fourier transform may be achieved by a suitable adaptation of [20], which uses representation theory of the Heisenberg group.

**4.1.1. Hyperbolic Representations of the Heisenberg Group.** Consider the space  $\mathbb{F}_h^j(\mathbb{H}^n)$  of  $\mathbb{O}$ -valued functions on  $\mathbb{H}^n$  with the property:

$$(4.1) \quad f(s + s', h, y) = e^{jhs'} f(s, x, y), \quad \text{for all } (s, x, y) \in \mathbb{H}^n, s' \in \mathbb{R},$$

and the square integrability condition (2.7). Then the hyperbolic representation is obtained by the restriction of the left shifts to  $\mathbb{F}_h^j(\mathbb{H}^n)$ . To obtain an equivalent representation on the phase space we take  $\mathbb{O}$ -valued functional of the Lie algebra  $\mathfrak{h}_n$ :

$$(4.2) \quad \chi_{(h,q,p)}^j(s, x, y) = e^{j(hs + qx + py)} = \cosh(hs + qx + py) + j \sinh(hs + qx + py).$$

The hyperbolic Segal—Bargmann type representation is intertwined with the left group action by means of the Fourier transform (3.1) with the hyperbolic functional (4.2). Explicitly this representation is:

$$(4.3) \quad \rho_{\hbar}(s, x, y) : f(q, p) \mapsto e^{-j\hbar(s+qx+py)} f\left(q - \frac{\hbar}{2}y, p + \frac{\hbar}{2}x\right).$$

For a hyperbolic Schrödinger type representation we again use the scheme described in Rem. 2.2. Similarly to the elliptic case one obtains the formula, resembling (3.4):

$$(4.4) \quad [\rho_{\chi}^j(s', x', y')f](x) = e^{-j\hbar(s'+xy'-x'y'/2)} f(x - x').$$

Application of the hyperbolic Fourier transform produces a Schrödinger type representation on the configuration space, cf. (3.5):

$$(4.5) \quad [\rho_{\chi}^j(s', x', y')\hat{f}](q) = e^{-j\hbar(s'+x'y'/2)-jx'q} \hat{f}(q + \hbar y').$$

The extension of this representation to kernels according to (2.10) generates hyperbolic pseudodifferential operators introduced in [28, (3.4)].

4.1.2. *Hyperbolic Dynamics.* Similarly to the elliptic (quantum) case we consider a convolution of two kernels on  $\mathbb{H}^n$  restricted to  $F_{\hbar}^j(\mathbb{H}^n)$ . The composition law becomes, cf. (3.6):

$$(4.6) \quad (k' * k)_{\hat{s}} = \int_{\mathbb{R}^{2n}} e^{j\hbar(xy' - yx')} \hat{k}'_s(\hbar, x', y') \hat{k}_s(\hbar, x - x', y - y') dx' dy'.$$

This is close to the calculus of hyperbolic PDO obtained in [28, Thm. 2]. Respectively for the commutator of two convolutions we get, cf. (3.7):

$$(4.7) \quad [k', k]_{\hat{s}} = \int_{\mathbb{R}^{2n}} \sinh(\hbar(xy' - yx')) \hat{k}'_s(\hbar, x', y') \hat{k}_s(\hbar, x - x', y - y') dx' dy'.$$

This the hyperbolic version of the Moyal bracket, cf. [28, p. 849], which generates the corresponding image of the dynamic equation (2.12).

**Example 4.2.** (i) For a quadratic Hamiltonian, e.g. harmonic oscillator from Example 2.3, the hyperbolic equation and respective dynamics is identical to quantum considered before.

(ii) Since  $\frac{\partial}{\partial s}$  acts on  $F_2^j(\mathbb{H}^n)$  as multiplication by  $j\hbar$  and  $j^2 = 1$ , the hyperbolic image of the unharmonic equation (2.17) becomes:

$$\dot{f} = \left( m\omega^2 q \frac{\partial}{\partial p} + \frac{\lambda}{6} \left( 3q^2 \frac{\partial}{\partial p} + \frac{\hbar^2}{4} \frac{\partial^3}{\partial p^3} \right) - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

The difference with quantum mechanical equation (3.9) is in the sign of the cubic derivative.

4.1.3. *Hyperbolic Probabilities.* To calculate probability distribution generated by a hyperbolic state we are using the general procedure from Section 2.3. The main differences with the quantum case are as follows:

(i) The real number  $A$  in the expression (2.20) for the addition of probabilities is bigger than 1 in absolute value by. Thus it can be associated with the hyperbolic cosine  $\cosh \alpha$ , cf. Rem. 3.3, for certain phase  $\alpha \in \mathbb{R}$  [28].

(ii) The nature of hyperbolic interference on two slits is affected by the fact that  $e^{j\hbar s}$  is not periodic and the hyperbolic exponent  $e^{jt}$  and cosine  $\cosh t$  do not oscillate. It is worth to notice that for Gaussian states the hyperbolic interference is exactly the same as quantum one, cf. Figs. 1(a) and 2(a). This is similar to coincidence of quantum and hyperbolic dynamics of harmonic oscillator.

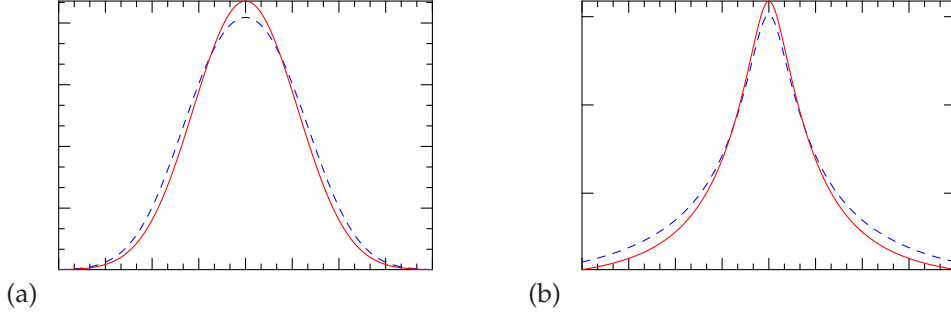


FIGURE 2. Hyperbolic probabilities: the blue (dashed) graph shows the addition of probabilities without interaction, the red (solid) graph present the quantum interference. Left picture shows the Gaussian state (3.11), with the same distribution as in quantum mechanics, cf. Fig. 1(a). The right picture shows the rational state (3.14), note the absence of interference oscillations in comparison with the quantum state on Fig. 1(b).

The contrast between two types of interference is prominent for the rational state (3.14), which is far from the minimal uncertainty, see the different patterns on Figs. 1(b) and 2(b).

**4.2. Parabolic (Classical) representations on the phase space.** After the previous two cases it is natural to link classical mechanics with dual numbers generated by the parabolic unit  $\varepsilon^2 = 0$ . Connection of the parabolic unit  $\varepsilon$  with the Galilean group of symmetries of classical mechanics is around for a while [66, App. C].

However the nilpotency of the parabolic unit  $\varepsilon$  make it difficult if we will work with dual number valued functions only. To overcome this issue we consider a commutative real algebra  $\mathfrak{C}$  spanned by  $1, i, \varepsilon$  and  $i\varepsilon$  with identities  $i^2 = -1$  and  $\varepsilon^2 = 0$ . A seminorm on  $\mathfrak{C}$  is defined as follows:

$$|a + bi + c\varepsilon + di\varepsilon|^2 = a^2 + b^2.$$

**4.2.1. Classical Non-Commutative Representations.** We wish to build a representation of the Heisenberg group which will be a classical analog of the Segal–Bargmann representation (3.2). To this end we introduce the space  $F_h^\varepsilon(\mathbb{H}^n)$  of  $\mathfrak{C}$ -valued functions on  $\mathbb{H}^n$  with the property:

$$(4.8) \quad f(s + s', h, y) = e^{\varepsilon h s'} f(s, x, y), \quad \text{for all } (s, x, y) \in \mathbb{H}^n, s' \in \mathbb{R},$$

and the square integrability condition (2.7). It is invariant under the left shifts and we restrict the left group action to  $F_h^\varepsilon(\mathbb{H}^n)$ .

There is an unimodular  $\mathfrak{C}$ -valued function on the Heisenberg group parametrised by a point  $(h, q, p) \in \mathbb{R}^{2n+1}$ :

$$E_{(h,q,p)}(s, x, y) = e^{2\pi(\varepsilon s h + i x q + i y p)} = e^{2\pi i(x q + y p)}(1 + \varepsilon s h).$$

This function, if used instead of the ordinary exponent, produces a modification  $\mathcal{F}_\varepsilon$  of the Fourier transform (3.1). The transform intertwines the left regular representation with the following action on  $\mathfrak{C}$ -valued functions on the phase space:

$$(4.9) \quad \rho_h^\varepsilon(s, x, y) : f(q, p) \mapsto e^{-2\pi i(x q + y p)} \left( f(q, p) + \varepsilon h (s f(q, p) + \frac{y}{2\pi i} f'_q(q, p) - \frac{x}{2\pi i} f'_p(q, p)) \right).$$

*Remark 4.3.* Comparing the traditional infinite-dimensional (3.2) and one-dimensional (3.3) representations of  $\mathbb{H}^n$  we can note that the properties of the representation (4.9) are a non-trivial mixture of the former:

- (i) The action (4.9) is non-commutative, similarly to the quantum representation (3.2) and unlike the classical one (3.3). This non-commutativity will produce the Hamilton equations below in a way very similar to Heisenberg equation, see Rem. 4.5.
- (ii) The representation (4.9) does not change the support of a function  $f$  on the phase space, similarly to the classical representation (3.3) and unlike the quantum one (3.2). Such a localised action will be responsible later for an absence of an interference in classical probabilities.
- (iii) The parabolic representation (4.9) can not be derived from either the elliptic (3.2) or hyperbolic (4.3) by the plain substitution  $\hbar = 0$ .

We may also write a classical Schrödinger type representation. According to Rem. 2.2 we get a representation formally very similar to the elliptic (3.4) and hyperbolic versions (4.4):

$$(4.10) \quad \begin{aligned} [\rho_{\chi}^{\varepsilon}(s', x', y')f](x) &= e^{-\varepsilon\hbar(s'+xy'-x'y'/2)}f(x-x') \\ &= (1-\varepsilon\hbar(s'+xy'-\frac{1}{2}x'y'))f(x-x'). \end{aligned}$$

However due to nilpotency of  $\varepsilon$  the (complex) Fourier transform  $x \mapsto q$  produces a different formula for parabolic Schrödinger type representation in the configuration space, cf. (3.5) and (4.5):

$$(4.11) \quad [\rho_{\chi}^{\varepsilon}(s', x', y')\hat{f}](q) = e^{2\pi i x' q} \left( (1-\varepsilon\hbar(s'-\frac{1}{2}x'y'))\hat{f}(q) + \frac{\varepsilon\hbar y'}{2\pi i}\hat{f}'(q) \right).$$

This representation shares all properties mentioned in Rem. 4.3 as well.

4.2.2. *Hamilton Equation.* The identity  $e^{\varepsilon t} - e^{-\varepsilon t} = 2\varepsilon t$  can be interpreted as a parabolic version of the sine function, while the parabolic cosine is identically equal to one [18, 50]. From this we obtain the parabolic version of the commutator (3.7):

$$\begin{aligned} [k', k]_{\hat{s}}(\varepsilon\hbar, x, y) &= \varepsilon\hbar \int_{\mathbb{R}^{2n}} (xy' - yx') \\ &\quad \times \hat{k}'_s(\varepsilon\hbar, x', y') \hat{k}_s(\varepsilon\hbar, x - x', y - y') dx' dy', \end{aligned}$$

for the partial parabolic Fourier-type transform  $\hat{k}_s$  of the kernels. Thus the parabolic representation of the dynamical equation (2.12) becomes:

$$(4.12) \quad \varepsilon\hbar \frac{d\hat{f}_s}{dt}(\varepsilon\hbar, x, y; t) = \varepsilon\hbar \int_{\mathbb{R}^{2n}} (xy' - yx') \hat{H}_s(\varepsilon\hbar, x', y') \hat{f}_s(\varepsilon\hbar, x - x', y - y'; t) dx' dy',$$

Although there is no possibility to divide by  $\varepsilon$  (since it is a zero divisor) we can obviously eliminate  $\varepsilon\hbar$  from the both sides if the rest of the expressions are real. Moreover this can be done "in advance" through a kind of the antiderivative operator considered in [41, (4.1)]. This will prevent "imaginary parts" of the remaining expressions (which contain the factor  $\varepsilon$ ) from vanishing.

*Remark 4.4.* It is noteworthy that the Planck constants completely disappeared from the dynamical equation. Thus the only prediction about it following from our construction is  $\hbar \neq 0$ , which was confirmed by experiments, of course.

Using the duality between the Lie algebra of  $\mathbb{H}^n$  and the phase space we can find an adjoint equation for observables on the phase space. To this end we apply the usual Fourier transform  $(x, y) \mapsto (q, p)$ . It turn to be the Hamilton equation [41, (4.7)]. However the transition to phase space is more a custom rather than a necessity and in many cases we can efficiently work on the Heisenberg group itself.

*Remark 4.5.* It is noteworthy, that the non-commutative representation (4.9) allows to obtain the Hamilton equation directly from the commutator  $[\rho_h^\varepsilon(k_1), \rho_h^\varepsilon(k_2)]$ . Indeed its straightforward evaluation will produce exactly the above expression. On the contrast such a commutator for the commutative representation (3.3) is zero and to obtain the Hamilton equation we have to work with an additional tools, e.g. an anti-derivative [41, (4.1)].

**Example 4.6.** (i) For the harmonic oscillator in Example 2.3 the equation (4.12) again reduces to the form (2.14) with the solution given by (2.15). The adjoint equation of the harmonic oscillator on the phase space is not different from the quantum written in Example 3.2(i). This is true for any Hamiltonian of at most quadratic order.  
(ii) For non-quadratic Hamiltonians classical and quantum dynamics are different, of course. For example, the cubic term of  $\partial_s$  in the equation (2.17) will generate the factor  $\varepsilon^3 = 0$  and thus vanish. Thus the equation (4.12) of the unharmonic oscillator on  $\mathbb{H}^n$  becomes:

$$\dot{f} = \left( m\omega^2 y \frac{\partial}{\partial x} + \frac{\lambda y}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{m} x \frac{\partial}{\partial y} \right) f.$$

The adjoint equation on the phase space is:

$$\dot{f} = \left( \left( m\omega^2 q + \frac{\lambda}{2} q^2 \right) \frac{\partial}{\partial p} - \frac{1}{m} p \frac{\partial}{\partial q} \right) f.$$

The last equation is the classical Hamilton equation generated by the cubic potential (2.16). Qualitative analysis of its dynamics can be found in many textbooks [2, § 4.C, Pic. 12; 57, § 4.4].

*Remark 4.7.* We have obtained the Poisson bracket from the commutator of convolutions on  $\mathbb{H}^n$  without any quasiclassical limit  $\hbar \rightarrow 0$ . This has a common source with the deduction of main calculus theorems in [7] based on dual numbers. As explained in [51, Rem. 6.9] this is due to the similarity between the parabolic unit  $\varepsilon$  and the infinitesimal number used in non-standard analysis [8]. In other words, we never need to take care about terms of order  $O(\hbar^2)$  because they will be wiped out by  $\varepsilon^2 = 0$ .

An alternative derivation of classical dynamics from the Heisenberg group is given in the recent paper [56].

4.2.3. *Classical probabilities.* It is worth to notice that dual numbers are not only helpful in reproducing classical Hamiltonian dynamics, they also provide the classic rule for addition of probabilities. We use the same formula (2.18) to calculate kernels of the states. The important difference now that the representation (4.9) does not change the support of functions. Thus if we calculate the correlation term  $\langle v_1, \rho(g)v_2 \rangle$  in (2.19), then it will be zero for every two vectors  $v_1$  and  $v_2$  which have disjoint supports in the phase space. Thus no interference similar to quantum or hyperbolic cases (Subsection 3.3) is possible.

## 5. DISCUSSION

In this paper we derive mathematical models for various physical setup from hypercomplex representations of the Heisenberg group. There are roots for such hypercomplex characters in the structure of ladder operators associated to three non-isomorphic quadratic Hamiltonians [52, 53]. Such hypercomplex representations may be also useful for many other groups as well, see the example of the  $SL_2(\mathbb{R})$  group in [47]. Moreover non-trivial parabolic characters described in [47, 50] are awaiting a further exploration.

There is a connection of our work with the technique of contractions and analytic continuations of groups [14, 15], these papers also highlight the role of hypercomplex numbers of three types. However in our research we do not modify the group (the Heisenberg group more specifically) itself, we rather consider its representations in different functional spaces created by three types of hypercomplex characters. All three cases have a lot of algebraic similarity and can be written in a unified manner with the help of parameter, which takes three values, say  $u = i, \varepsilon, j$ , with  $i^2 = -1, \varepsilon^2 = 0, j^2 = 1$ . For example, representations (3.4), (4.4) and (4.10) can be unified in:

$$(5.1) \quad [\rho_h^u(s', x', y')f](x) = e^{-uh(s'+xy'-x'y'/2)}f(x-x').$$

It is noteworthy that this algebraic similarity exists along with the significant topological and analytic differences between elliptic, parabolic and hyperbolic cases. An illustration is the distinction of the elliptic (3.5) and parabolic (4.11) representations in the configuration space, despite of the fact that both representations are derived from the unified form (5.1).

The parabolic representations (4.10) and (4.11) of the Heisenberg group act in the first order jet spaces. Such spaces have a well established connections with Lagrangian and Hamiltonian formulations of quantum field theory [13, 24, 43], study of aggregate quantum-classical systems [44, 48] and spectral theory of operators [42]. Nevertheless the localised non-commutative representation of  $\mathbb{H}^n$  built in this paper seems to be new and deserve detailed investigation.

We already seen that it may be useful to consider several hypercomplex units in the same time. In the case of classical mechanics we combined  $i$  and  $\varepsilon$ . The algebra generated by  $i$  and  $j$  is known as (commutative) Segre quaternions. Such commutative algebras with hypercomplex units and their physical applications attracted attention of many researchers recently [4, 59, 62, 63].

We may even need to study an algebra which contains all three hypercomplex units simultaneously. The most straightforward way is to take eight dimensional commutative algebra with the basis  $1, i, \varepsilon, j, i\varepsilon, ij, \varepsilon j, i\varepsilon j$ . A reduction of dimensionality from 8 to 6 can be achieved if we replace products  $\varepsilon j$  and  $i\varepsilon j$  through the further identities  $\varepsilon j = \varepsilon$  and  $i\varepsilon j = i\varepsilon$ . This do not affect associativity of the product.

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