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Attributed Graph Transformation via Rule Schemata: Church-Rosser Theorem

Ivaylo Hristakiev* and Detlef Plump

University of York, United Kingdom

Abstract. We present an approach to attributed graph transformation which requires neither infinite graphs containing data algebras nor auxiliary edges that link graph items with their attributes. Instead, we use the double-pushout approach with relabelling and extend it with rule schemata which are instantiated to ordinary rules prior to application. This framework provides the formal basis for the graph programming language GP 2. In this paper, we abstract from the data algebra of GP 2, define parallel independence of rule schema applications, and prove the Church-Rosser Theorem for our approach. The proof relies on the Church-Rosser Theorem for partially labelled graphs and adapts the classical proof by Ehrig and Kreowski, bypassing the technicalities of adhesive categories.

1 Introduction

Traditionally, the theory of graph transformation assumed that labels in graphs do not change in derivations (see, for example, [2]). But in applications of graph transformation it is often necessary to compute with labels. For instance, finding shortest paths in a graph whose edges are labelled with distances requires to determine the shorter of two distances and to add distances.

Graphs in which data elements of some fixed algebra are attached to nodes and edges have been called attributed graphs since [12], the first formal approach to extend graph transformation with computations on labels. In that paper, graphs are encoded as algebras to treat graph structure and algebra data uniformly. With a similar intention, the papers [9,6] go the other way round and encode the data algebra in graphs. Each data element becomes a special data node and auxiliary edges connect ordinary nodes and edges with the data nodes.

The latter approach has become mainstream but has some serious drawbacks (bemoaned as the *akwardness of attributes* in [17]). Firstly, the way attributes are attached to edges leads to the situation of edges having other edges as sources. This requires non-standard graphs and makes the model unusual. Secondly, and more importantly, there is typically an infinite number of data nodes because standard data algebras (such as integers or lists) have infinite domains. This

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means that attributed graphs are usually infinite, leading to a discrepancy between theory and practice as graphs are stored using finite representations. In the approach of [6], even rules are normally infinite because they consist of graphs containing the complete term algebra corresponding to the data algebra.

In this paper, we propose an alternative approach to attributed graph transformation which avoids both infinite graphs and auxiliary attribute edges. Instead of merging graphs with the data algebra, we keep them separate. Host graph items simply get labelled with data elements and rule graph items get labelled with terms. To make this work, rules are instantiated by replacing terms with corresponding data values and then applied as usual. Hence our rules are actually *rule schemata* whose application can be seen as a two-stage process.

In order to modify attributes, it is crucial that interface items in rules can be relabelled. We therefore use the double-pushout approach with partially labelled interface graphs as a formal basis [7]. This approach is also the foundation of the graph programming language GP 2 [15]. The fixed data algebra of GP 2 consists of integers, character strings, and heterogeneous lists of strings and integers. In this paper, we abstract from this particular algebra and consider an arbitrary data algebra (see Subsection 2.2).

In Section 3, we define parallel independence of rule schema applications and prove the so-called Church-Rosser Theorem for our setting. Roughly, this result establishes that independent rule schema applications can be interchanged and result in the same graph. Our proof nicely decomposes into the Church-Rosser Theorem for the double-pushout approach with relabelling plus a simple extension to rule schemata (see Subsection 3.2).

The Church-Rosser Theorem for the relabelling setting was obtained in [8] as a corollary of an abstract result for $\mathcal{M},\mathcal{N}$-adhesive transformation systems. However, we deliberately avoid the categorical machinery of adhesiveness, van Kampen squares, etc. which we believe is difficult to digest for an average reader. Instead, we merely adapt the classical proof of Ehrig and Kreowski [5] to partially labelled graphs, essentially by replacing properties of pushouts and pullbacks in the unlabelled case by properties of natural pushouts in the setting of partially labelled graphs. (A pushout is natural if it is also a pullback.)

The rest of this paper is organized as follows. In Section 2, we describe the general idea of our approach. In Section 3, we present the notions of parallel and sequential independence and formalize the Church-Rosser Theorem at the rule schema level. Section 4 contains the relevant proofs. A conclusion and future work are given in Section 6.

We assume the reader to be familiar with basic notions of the double-pushout approach to graph transformation (see [3]). An extended version of this paper, along with complete proofs, can be found in [10].

2 Attributed Graph Transformation via Rule Schemata

In this section, we present our approach to transforming labelled graphs by rule schemata. We begin by briefly reviewing labelled graphs and the double-pushout approach to graph transformation with relabelling (see [7] for details).

2.1 Double-Pushout Approach with Relabelling

A *partially labelled graph* G over a (possibly infinite) label set $\mathcal{L}$ consists of finite sets V_G and E_G of *nodes* and *edges*, *source* and *target* functions $s_G, t_G \colon E_G \to V_G$, a partial node labelling function $l_{G,V} \colon V_G \to \mathcal{L}$, and a partial edge labelling function $l_{G,E} \colon E_G \to \mathcal{L}$. Given a node or edge x, we write $l_G(x) = \bot$ to express that $l_G(x)$ is undefined[1]. Graph G is *totally labelled* if $l_{G,V}$ and $l_{G,E}$ are total functions. The classes of partially and totally labelled graphs are denoted by $\mathcal{G}_\bot(\mathcal{L})$ and $\mathcal{G}(\mathcal{L})$, respectively.

A *premorphism* $g \colon G \to H$ consists of two functions $g_V \colon V_G \to V_H$ and $g_E \colon E_G \to E_H$ that preserve sources and targets: $g_V(s_G(e)) = s_H(g_E(e)$ and $g_V(t_G(e)) = t_H(g_E(e))$ for all edges e. Premorphism g is a *graph morphism* if it preserves labels, that is, if $l_G(x) = l_H(g(x))$ for all items x such that $l_G(x)$ is defined.

A graph morphism g *preserves undefinedness* if it maps unlabelled items in G to unlabelled items in H. We call g an *inclusion* if $g(x) = x$ for all items x. Note that inclusions need not preserve undefinedness. Finally, g is *injective (surjective)* if g_V and g_E are injective (surjective), and an *isomorphism* if it is injective, surjective and preserves undefinedness.

Partially labelled graphs and graph morphisms constitute a category (which is $\mathcal{M},\mathcal{N}$-adhesive [8] if one picks $\mathcal{M}$ to be the injective morphisms and $\mathcal{N}$ to be the injective morphisms that preserve undefinedness). In this category, pushouts need not exist as can be observed in Figure 2(a).

A *rule* $r = \langle L \leftarrow K \rightarrow R \rangle$ over $\mathcal{L}$ consists of two inclusions $K \to L$ and $K \to R$ such that L and R are graphs in $\mathcal{G}(\mathcal{L})$ and K is a graph in $\mathcal{G}_\bot(\mathcal{L})$.

Definition 1 (Direct derivation). A *direct derivation* between graphs G and H in $\mathcal{G}_\bot(\mathcal{L})$ via a rule $r = \langle L \leftarrow K \rightarrow R \rangle$ consists of two natural pushouts[2] as in Figure 1, where $g \colon L \to G$ is an injective graph morphism.

We denote such a derivation by $G \Rightarrow_{r,g} H$. The requirement that the pushouts in Figure 1 are natural ensures that the pushout complement D in Figure 1 is uniquely determined by rule r, graph G and morphism g [7, Theorem 1]. Figure 2(b) demonstrates that non-natural pushout complements need not be unique. It is worth noting that in the traditional setting of double-pushout graph transformation with totally labelled graphs, the pushouts are automatically natural by the injectivity of $L \leftarrow K$ and $K \to R$.

[1] We do not distinguish between nodes and edges in statements that hold analogously for both sets.

[2] A pushout is *natural* if it is also a pullback.

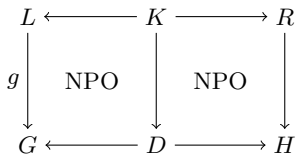


Fig. 1: A direct derivation

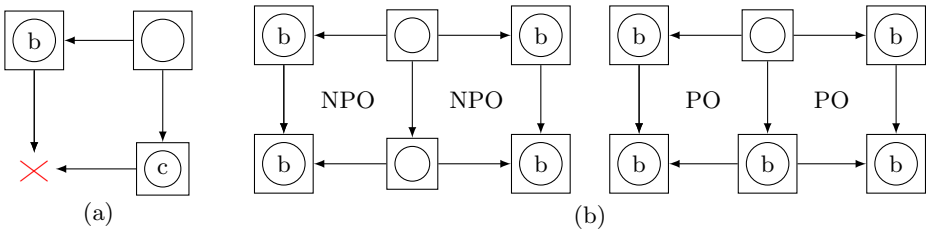


Fig. 2: (a) Pushouts need not exist. (b) A natural and a non-natural double pushout.

Operationally, the application of rule r to graph G proceeds as follows: (1) match L with a subgraph of G by means of an injective graph morphism $g\colon L \to G$ satisfying the *dangling condition*: no node in $g(L)-g(K)$ is incident to an edge in $G-g(L)$; (2) obtain a graph D by removing from G all items in $g(L)-g(K)$ and, for all unlabelled items x in K, making $g(x)$ unlabelled; (3) add disjointly to D all items from $R-K$, keeping their labels, to obtain a graph H; (4) for all unlabelled items x in K, $l_H(g(x))$ becomes $l_R(x)$.

In [7] it is shown that if G is totally labelled, then the resulting graph H is also totally labelled. Moreover, unlabelled items in the interface graph K have unlabelled images in the intermediate graph D by the naturalness condition for pushouts.

2.2 Rule Schemata

Rule schemata for attributed graph transformation were introduced in the context of the graph programming language GP [16]. We first review signatures and algebras (details can be found, for example, in [3, Appendix B]).

Consider a *signature* Σ consisting of a set S of *sorts* and a family of *operation symbols* $OP = (OP_{w,s})_{(w,s)\in S^*\times S}$. A *Σ-algebra* A consists of a family of carrier sets $(A_s)_{s\in S}$ containing data values, and a set of functions implementing the operations of Σ. A *term algebra* $T_\Sigma(X)$ is built up from terms consisting of constants and variables, where X is a family of variables that is disjoint from OP.

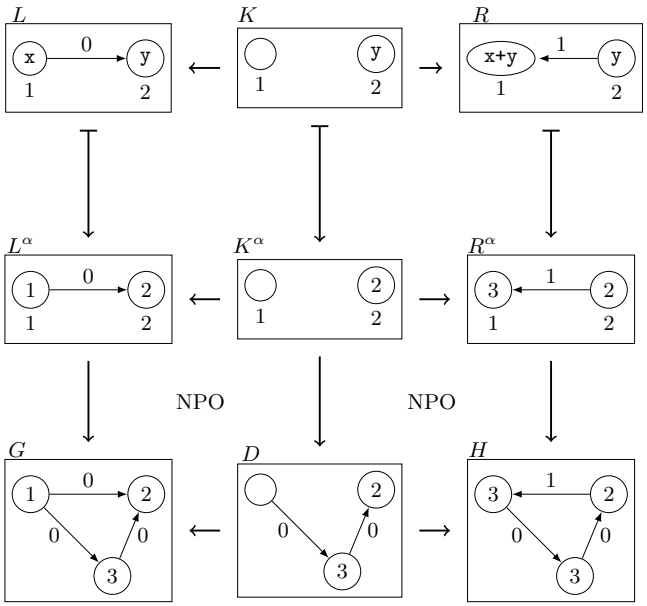


Fig. 3: Example of a rule schema direct derivation

An *assignment* $\alpha\colon X \to A$ is a family of mappings $(\alpha_s\colon X_s \to A_s)_{s\in S}$, giving a value to each variable in X. Its unique extension $\alpha^*\colon T_\Sigma(X) \to A$ evaluates terms according to α.

We assume a fixed Σ-algebra A whose elements are used as host graph labels, and a corresponding term algebra $T_\Sigma(X)$ whose terms are used as labels in rule schemata. To avoid an inflation of symbols, we sometimes equate A or $T_\Sigma(X)$ with the union of its carrier sets.

Definition 2 (Rule schema). A *rule schema* $r = \langle L \leftarrow K \rightarrow R\rangle$ consists of two inclusions $K \to L$ and $K \to R$ such that L and R are graphs in $\mathcal{G}(T_\Sigma(X))$ and K is a graph in $\mathcal{G}_\perp(T_\Sigma(X))$.

To apply a rule schema r to a graph, the schema is first instantiated by evaluating its labels according to some assignment $\alpha\colon X \to A$.

Definition 3 (Rule schema instance and direct derivation). Consider a graph G in $\mathcal{G}_\perp(T_\Sigma(X))$ and an assignment $\alpha\colon X \to A$. The *instance* G^α is the graph in $\mathcal{G}_\perp(A)$ obtained from G by replacing each label l with $\alpha^*(l)$. The instance of a rule schema $r = \langle L \leftarrow K \rightarrow R\rangle$ is the rule $r^\alpha = \langle L^\alpha \leftarrow K^\alpha \rightarrow R^\alpha\rangle$.

A *rule schema direct derivation* via r between graphs G and H in $\mathcal{G}_\perp(A)$ is a direct derivation $G \Rightarrow_{r^\alpha,g} H$ via the instance r^α according to Definition 1.

We write $G \Rightarrow_{r,g,\alpha} H$ if there exists a direct derivation from G to H with rule schema r, graph morpshism g and assignment α. Note that we use $\Rightarrow$ for the application of both rule schemata and rules.

Figure 3 shows an example of a rule schema direct derivation, where we assume that algebra A contains the integers with addition ($+$). The variables `x` and `y` are of sort `int` and are mapped by assignment α to 1 and 2, respectively. This allows for the relabelling of node 1 to 3. Note that this rule schema gives rise to infinitely many instances because the carrier set of integers is infinite.

Given an injective premorphism $g\colon L \to G$ and an assignment $\alpha\colon X \to A$, a graph morphism $g'\colon L^\alpha \to G$ is *induced* by g and α if $g'_V = g_V$ and $g'_E = g_E$. In other words, the application of α to L must turn g into a label-preserving graph morphism. The following proposition gives a necessary and sufficient condition for a rule schema with left-hand side L to be applicable with a morphism induced by g and α. The proof relies on a result in [7] about the existence and uniqueness of direct derivations in the double-pushout approach with relabelling.

Proposition 1 (Existence and uniqueness of direct derivations). *Consider a rule schema $r = \langle L \leftarrow K \rightarrow R \rangle$, an injective premorphism $g\colon L \to G$ with G in $\mathcal{G}_\perp(A)$, and an assignment $\alpha\colon X \to A$. Then there exists a direct derivation $G \Rightarrow_{r,g',\alpha} H$ such that g' is induced by g and α, if and only if g satisfies the dangling condition and each item x in L satisfies*

$$l_G(g(x)) = \alpha^*(l_L(x)).$$

Moreover, in this case H is determined uniquely up to isomorphism.

Proof. "If": By assumption, each item x in L satisfies $l_G(g(x)) = \alpha^*(l_L(x)) = l_{L^\alpha}(x)$ and hence $g'\colon L^\alpha \to G$ with $g'_V = g_V$ and $g'_E = g_E$ is a graph morphism.

Moreover, it is clear that g' satisfies the dangling condition with respect to r^α because g satisfies the dangling condition with respect to r. Thus, by [7, Theorem 1], there is a direct derivation $G \Rightarrow_{r^\alpha,g'} H$ where H is determined uniquely up to isomorphism by r^α and g'. Since r^α and g' are uniquely determined by r, α and g, it follows that H is uniquely determined by r, α and g, too.

"Only if": Suppose that $G \Rightarrow_{r,g',\alpha} H$ where g' is induced by g and α. Then, by definition, $G \Rightarrow_{r^\alpha,g'} H$. Hence, by [7, Theorem 1], g' satisfies the dangling condition. Since $g'_V = g_V$ and $g'_E = g_E$, it is clear that g satisfies the dangling condition with respect to r. Moreover, since g' is label-preserving, each item x in L satisfies $l_G(g(x)) = l_G(g'(x)) = l_{L^\alpha}(x) = \alpha^*(l_L(x))$. □

As indicated above, a rule schema $r = \langle L \leftarrow K \rightarrow R \rangle$ may have infinitely many instances. Even if one restricts to instances that are compatible with a given premorphism $g\colon L \to G$, there may be infinitely many instances to choose from. For example, consider a premorphism that maps a node in L labelled with $x + y$ to a node in G labelled with the integer 3 (assuming the conventions of Figure 3). There are infinitely many assignments meeting the labelling condition of Proposition 1 because the equation $x + y = 3$ has infinitely many solutions over the integers.

Example 1 (GP 2 rule schemata). Labels in the graph programming language GP 2 [15,1] are integers, character strings or heterogeneous lists of integers and

character strings. Lists are constructed by concatenation: given lists `x` and `y`, their concatenation is written `x:y`.

Expressions in the left-hand side L of a GP 2 rule schema are syntactically restricted to ensure that at most one instance of the schema is compatible with a given premorphism $g\colon L \to G$. To this end, left-hand expressions must neither contain arithmetic operators (except unary minus) nor repeated list variables, and all variables occurring on the right-hand side of a rule schema must also occur on the left-hand side.

Figure 4 shows the declaration of a GP 2 rule schema `inc`. Its left-hand labels contain typed variables which are instantiated with concrete values during graph matching. By convention, the interface of the rule schema consists of two unlabelled nodes. The effect of `inc` is to increment the rightmost element in the list of node 2.

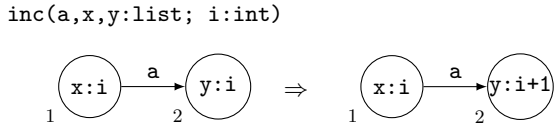


Fig. 4: Declaration of a GP 2 rule schema

In this paper we are not concerned with implementation issues and do not impose any restrictions on rule schemata. Abstracting from GP 2's label algebra, other possible data types for labels include (multi)sets, stacks, queues and records.

3 Church-Rosser Theorem

In this section, we present the notion of parallel independence for direct derivations with relabelling and then extend it to applications of rule schemata.

3.1 Independence of Direct Derivations with Relabelling

Let each of the diagrams in Figure 5 represent two direct derivations according to Definition 1.

Definition 4 (Parallel and sequential independence). Two direct derivations $H_1 \Leftarrow_{r_1,m_1} G \Rightarrow_{r_2,m_2} H_2$ as in Figure 5 (top) are *parallel independent* if there exist morphisms $i : L_1 \to D_2$ and $j : L_2 \to D_1$ such that $f_2 \circ i = m_1$ and $f_1 \circ j = m_2$.

Two direct derivations $G \Rightarrow_{r_1,m_1} H_1 \Rightarrow_{r_2,m_2} H_2$ as in Figure 5 (bottom) are *sequentially independent* if there exist morphisms $i : R_1 \to D_2$ and $j : L_2 \to D_1$ such that $f_2 \circ i = m'_1$ and $f_1 \circ j = m_2$.

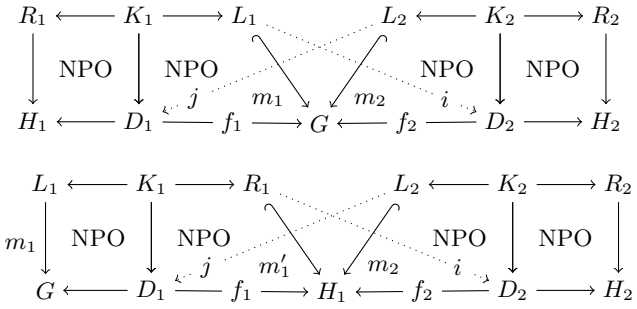


Fig. 5: Parallel and sequential independence (top and bottom, respectively)

It will turn out that parallel and sequential independence are related: two direct derivations $H_1 \Leftarrow_{r_1,m_1} G \Rightarrow_{r_2,m_2} H_2$ are parallel independent if and only if the direct derivations $H_1 \Rightarrow_{r_1^{-1},m_1'} G \Rightarrow_{r_2,m_2} H_2$ are sequentially independent, where r_1^{-1} denotes the inverse rule of r_1 and m_1' is the comatch of m_1.

Lemma 1 (Characterization of parallel independence). *Two direct derivations $H_1 \Leftarrow_{r_1,m_1} G \Rightarrow_{r_2,m_2} H_2$ are parallel independent if and only if for all items $x_1 \in L_1$ and $x_2 \in L_2$ such that $m_1(x_1) = m_2(x_2)$,*

- *$x_1 \in K_1$ and $x_2 \in K_2$, and*
- *$l_{K_1}(x_1) \neq \bot$ and $l_{K_2}(x_2) \neq \bot$.*

The first condition states that every common item is an interface item. The second condition states that no common item is relabelled by either derivation.

Example 2 (Counterexample to parallel independence). Figure 6 shows two direct derivations $H_1 \Leftarrow G \Rightarrow H_2$ that use instances of the rule schema of Figure 3. The derivations are *not* parallel independent: there are no morphisms $L_1 \to D_2$ and $L_2 \to D_1$ with the desired properties. The problem is that node 1 gets relabelled, breaking the second independence condition.

Our main result (Theorem 2) will show that rule schema direct derivations that are parallel independent can be interchanged to obtain a common result graph. First, we state the Church-Rosser theorem for plain rules in the sense of Definition 1. This has been proved in [8] as a corollary of the Church-Rosser theorem for $\mathcal{M},\mathcal{N}$-adhesive transformation systems. However, we obtain the result directly without using the notions of adhesiveness and van Kampen square.

The proof follows the original Church-Rosser proof of [5]. At specific points it will be necessary to show that the results for NPO decomposition apply to the given setting. See Section 4 for the complete proof.

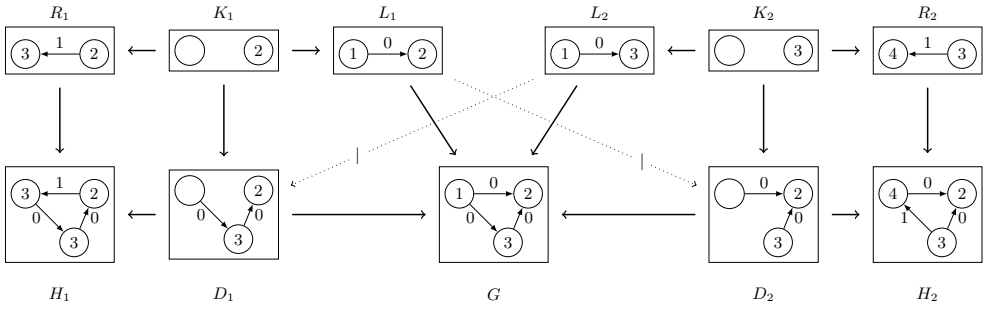


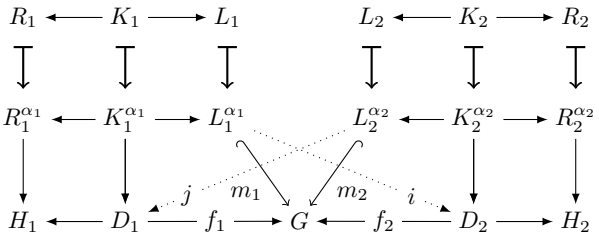
Fig. 6: Counterexample to parallel independence

Theorem 1 (Church-Rosser theorem for plain rules). *Given two parallel independent direct derivations* $G \Rightarrow_{r_1,m_1} H_1$ *and* $G \Rightarrow_{r_2,m_2} H_2$*, there are a graph* $\tilde{H}$ *and direct derivations* $H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ *and* $H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$*. Moreover,* $G \Rightarrow_{r_1,m_1} H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ *as well as* $G \Rightarrow_{r_2,m_2} H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$ *are sequentially independent.*

3.2 Church-Rosser Theorem for Rule Schema Derivations

This subsection lifts the previous independence result to rule schema applications. The main idea is to simply add instantiation on top of plain direct derivations.

Definition 5 (Parallel independence of rule schema derivations). Two rule schema direct derivations $G \Rightarrow_{r_1,m_1,\alpha_1} H_1$ and $G \Rightarrow_{r_2,m_2,\alpha_2} H_2$ are *parallel independent* if the plain derivations with relabelling $G \Rightarrow_{r_1^{\alpha_1},m_1} H_1$ and $G \Rightarrow_{r_2^{\alpha_2},m_2} H_2$ are parallel independent according to Definition 4.



Theorem 2 (Church-Rosser theorem for rule schemata). *Given two parallel independent rule schema direct derivations* $G \Rightarrow_{r_1,m_1} H_1$ *and* $G \Rightarrow_{r_2,m_2} H_2$*, there is a graph* $\tilde{H}$ *and rule schema direct derivations* $H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ *and* $H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$*. Moreover* $G \Rightarrow_{r_1,m_1} H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ *as well as* $G \Rightarrow_{r_2,m_2} H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$ *are sequentially independent.*

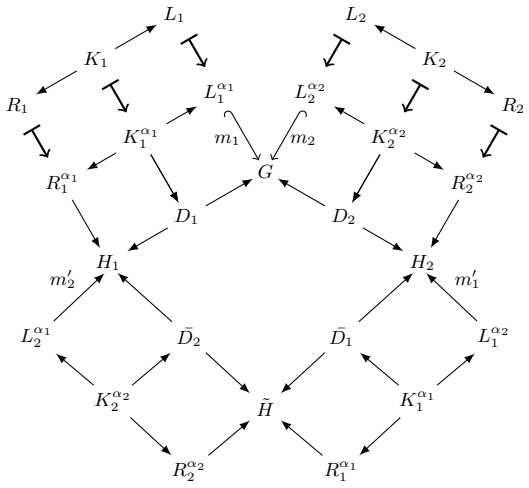


Fig. 7: Church-Rosser theorem for rule schemata

Proof. From Theorem 1, we know that independence of the plain derivations with relabelling $G \Rightarrow_{r_1^{\alpha_1},m_1} H_1$ and $G \Rightarrow_{r_2^{\alpha_2},m_2} H_2$ implies the existence of a graph $\tilde{H}$ and direct derivations $H_1 \Rightarrow_{r_2^{\alpha_2},m_2} \tilde{H}$ and $H_2 \Rightarrow_{r_1^{\alpha_1},m_1} \tilde{H}$. This is illustrated in Figure 7.

The direct derivations $G \Rightarrow_{r_1^{\alpha_1},m_1} H_1$ and $G \Rightarrow_{r_2^{\alpha_2},m_2} H_2$ use instances of the rule schemata r_1 and r_2, and therefore there are rule schema direct derivations $H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ and $H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$. With Theorem 1 follows that both $G \Rightarrow_{r_1^{\alpha_1},m_1} H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ and $G \Rightarrow_{r_2^{\alpha_2},m_2} H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$ are sequentially independent. □

4 Proof of Theorem 1

The proof follows the original Church-Rosser proof of [5]. At specific points it will be necessary to show that the results for NPO decomposition apply to the given setting. This is because for partially labelled graphs, pushouts need not always exist, and not all pushouts along injective morphisms are natural. These facts have been observed in Figure 2.

Using the definition of parallel independence (Definition 4), we start by decomposing the derivations as shown in Figure 8.

The graph D_0 is obtained as a pullback of $(D_1 \to G \leftarrow D_2)$. The universal property of pullbacks gives us that $K_1 \to D_1$ and $K_2 \to D_2$ decompose into $K_1 \to D_0 \to D_1$ and $K_2 \to D_0 \to D_2$ respectively. We also have that $(1+2)$ and $(1+3)$ are NPOs because they are left-hand sides of derivations. Furthermore, $D_1 \to G$ and $D_2 \to G$ are injective and jointly surjective which makes (1) an NPO ([10, Lemma 4]).

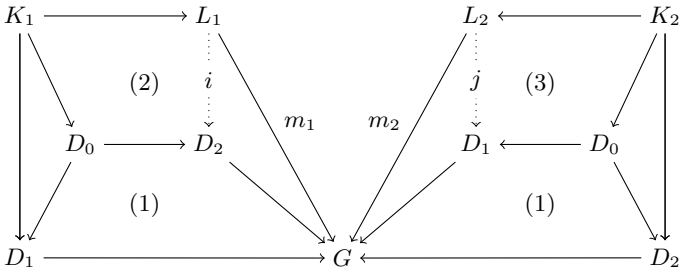


Fig. 8: First decomposition diagram

$D_1 \to G$ and $D_2 \to G$ injective imply that $D_0 \to D_2$ and $D_0 \to D_1$ are also injective. The subsequent parts of the proof contain four claims which are proven afterwards.

Claim 1. The squares (2) and (3) are NPOs.

Next, the pushouts $\overline{D_1}$ of $(D_0 \leftarrow K_1 \to R_1)$ (5) and $\overline{D_2}$ of $(D_0 \leftarrow K_2 \to R_2)$ (6) are constructed. These exist by the following claim:

Claim 2. In Figure 9, the pushouts $\overline{D_1}$ of $(D_0 \leftarrow K_1 \to R_1)$ (5) and $\overline{D_2}$ of $(D_0 \leftarrow K_2 \to R_2)$ (6) exist.

Again using uniqueness, the morphisms $R_1 \to H_1$ and $R_2 \to H_2$ decompose into $R_1 \to \overline{D_2} \to H_1$ and $R_2 \to \overline{D_1} \to H_2$. We also have that $(5+7)$ and $(6+8)$ are NPOs because they are right-hand sides of derivations.

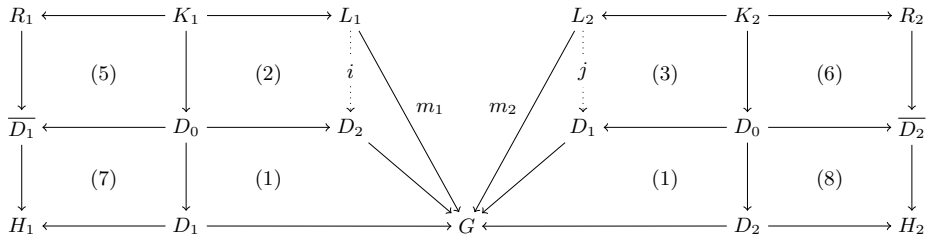


Fig. 9: Second decomposition diagram

Also, (7) and (8) become NPOs by the NPO Decomposition Lemma [10, Lemma 5.3].

Claim 3. In Figure 9, the squares (7) and (8) are NPOs.

The graph $\tilde{H}$ is constructed as a pushout of $(\overline{D_1} \leftarrow D_0 \to \overline{D_2})$ (4). (See square (4) in Figure 10.)

Claim 4. The pushout of $(\overline{D_1} \leftarrow D_0 \to \overline{D_2})$ exists.

This pushout becomes NPO by [10, Lemma 3] and the arguments in the proof of Claim 4. Furthermore, the graph $\tilde{H}$ is totally labelled due to the way D_0, $\overline{D_1}$ and $\overline{D_2}$ are constructed - $\overline{D_1}$ can contain unlabelled items only from $D_0 - K_1$ which are labelled in $\overline{D_2}$, and vice versa.

The pushouts can be rearranged as in Figure 10 to show that $G \Rightarrow_{r_1,m_1} H_1 \Rightarrow_{r_2,m_2'} \tilde{H}$ as well as $G \Rightarrow_{r_2,m_2} H_2 \Rightarrow_{r_1,m_1'} \tilde{H}$ are sequentially independent. Note that the graph $\tilde{H}$ is totally labelled.

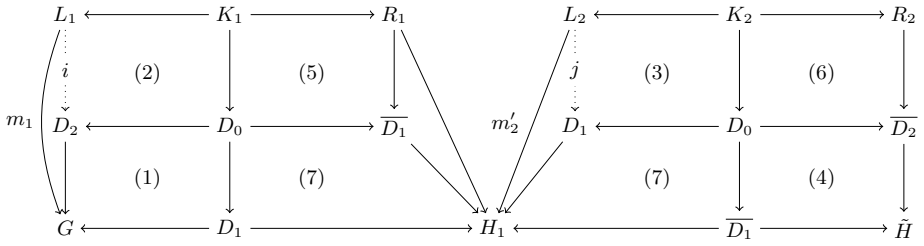
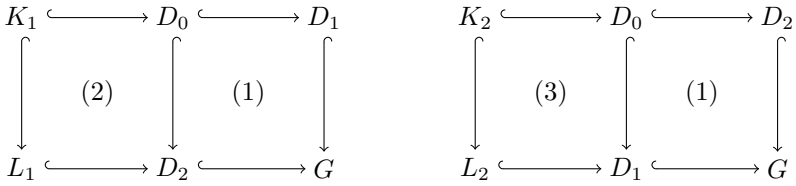


Fig. 10: Rearranged pushouts

This concludes the proof of the Church-Rosser theorem. □

Next, we present the proofs of the above claims.

Proof of Claim 1 We need to show that the conditions of the NPO Decomposition Lemma [10, Lemma 5.2] hold for the following diagrams.



$D_0 \to D_1$ is injective because $D_2 \to G$ is injective by definition and (1) is PB. (1) has already been proven to be NPO (at the start of this section). Pushout exists over $(L_1 \leftarrow K_1 \to D_0)$ as L_1 is totally labelled, both morphisms are injective and $K_1 \to D_0$ preserves undefinedness, all by the definition of direct derivation with relabelling. The square $K_1 L_1 D_0 D_2$ commutes because $(1 + 2)$ and (1) are NPOs. Therefore, all conditions of the NPO Decomposition Lemma [10, Lemma 5.2] hold.

The proof for the second diagram is analogous.

This concludes the proof that the squares (2) and (3) are NPOs. □

Proof of Claim 2 As in the previous proof, R_1 and R_2 are totally labelled, all morphisms are injective and both $K_1 \to D_0$ and $K_2 \to D_0$ preserve undefinedness, all by the definition of direct derivation with relabelling. Therefore, the pushouts (5) and (6) exist by [10, Lemma 2.2]. □

Proof of Claim 3 In the context of Figure 11, we need to show that the conditions of the NPO Decomposition Lemma [10, Lemma 5.3] hold.

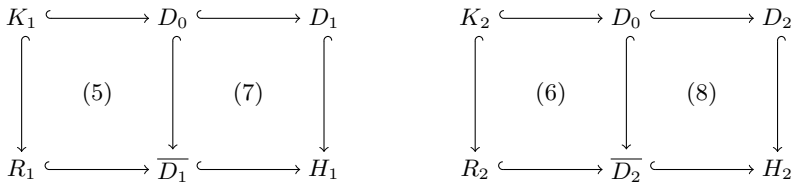
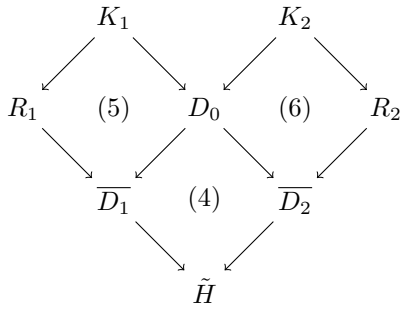


Fig. 11: Pushouts (5), (6), (7) and (8).

$D_0 \to D_1$ has already been established as injective. We need that there exists unique NPO complement of $K_1 \to R_1 \to \overline{D_1}$. We have that $K_1 \to R_1$ is injective by definition. $R_1 \to \overline{D_1}$ is injective because $K_1 \to D_0$ and $L_1 \to D_2$ are injective. The pushout (5+7) is a right-hand side of a derivation and $R_1 \to H_1 = R_1 \to \overline{D_1} \to H_1$. Consequently, $R_1 \to \overline{D_1}$ satisfies the dangling condition w.r.t. $K_1 \to R_1$, thus the existence of a unique NPO complement is given by [10, Lemma 2.3]. The proof for the second diagram is analogous.

This concludes the proof that the prerequisites for the NPO Decomposition Lemma [10, Lemma 5.3] hold. Hence, squares (7) and (8) become NPOs. □



Proof of Claim 4 For a pushout to exist, $\overline{D_1}$ and $\overline{D_2}$ have to agree on the labels of the unlabelled nodes of D_0 ([10, Lemma 2.2]).

$\overline{D_1}$ is constructed as the pushout of $(R_1 \leftarrow K_1 \to D_0)$ with R_1 being totally labelled. Its node and edge sets and labelling function is as defined in [10, Lemma 2.2]. Moreover, $R_1 \to \overline{D_1}$ and $D_0 \to \overline{D_1}$ are injective and jointly surjective.

There are 3 main cases for an item x to be labelled in $\overline{D_1}$:

- the item is created by the first derivation $x \in R_1 - K_1$. This means it does not exist in D_1, D_2 or G. Consequently, this item does not have a preimage in D_0 by pullback construction ([10, Lemma 2.1]). Therefore, it cannot be a source of conflict for pushout existence.
- the item is relabelled by the first derivation, meaning its preimage in D_1 (and D_0) is unlabelled $x \in K_1$ and $l_{K_1}(x) = l_{D_1}(x) = l_{D_0}(x) = \perp$. By the definition of parallel independence, no common items are relabelled making the item not have a preimage in R_2. Therefore it is unlabelled in $\overline{D_2}$ (by definition of pushout), making it a non-conflict w.r.t. pushout existence.

– the item is in $\overline{D_1} - R_1$, i.e. a labelled item of $D_0 - K_1$. We have that $D_0 \to \overline{D_1}$ and $D_0 \to \overline{D_2}$ are label preserving, so the label of x in $\overline{D_2}$ is the same as in $\overline{D_1}$.

In all cases, the labels of $\overline{D_1}$ are preserved by the second derivation. The argument for the labelled items of $\overline{D_2}$ is analogous.

This concludes the proof that the pushout (4) over $(\overline{D_1} \leftarrow D_0 \to \overline{D_2})$ exists. □

5 Related Work

We have adapted the classical Church-Rosser proof of Ehrig and Kreowski [5] to partially labelled graphs and extended the result to rule schemata, essentially by replacing properties of pushouts and pullbacks in the unlabelled case by properties of natural pushouts in the setting of partially labelled graphs.

In [6], the theory of attributed graph transformation is developed in the framework of so-called adhesive HLR categories. Among other results, the Church-Rosser Theorem is proved in this setting. The approach is further studied in [4] by adding nested application conditions and proving the previous results for this more expressive approach. Both are a generalized version of the Church-Rosser Theorem of [9].

So-called symbolic graphs are attributed graphs in which all data nodes are variables, combined with a first-order logic formula over these variables. In [13] it is shown that this approach can specify and transform classes of ordinary attributed graphs that satisfy the given formula. The underlying graph structure is the same as in [6], hence the approach shares the issues described in the introduction.

Recently, a generalised Church-Rosser theorem for attributed graph transformation has been proved in [11] by using symbolic graphs. A notion of parallel independence is used that takes into account the semantics of attribute operations, in order to reduce the number of "false positives" in conflict checking.

6 Conclusion

In this paper, we have presented an approach to attributed graph transformation based on partially labelled graphs and rule schemata which are instantiated to ordinary rules prior to application. We have defined parallel independence of rule schema applications and have proved the Church-Rosser theorem for our approach. The proof relies on the Church-Rosser theorem for graph transformation with relabelling and adapts the classical proof by Ehrig and Kreowski, bypassing the technicalities of adhesive categories.

Future work includes establishing other classical graph transformation results in our setting, such as embedding and restriction theorems. Furthermore, we aim at studying critical pairs and confluence both for the particular case of the GP 2 language and for attributed graph transformation over arbitrary label

algebras. In particular, we plan to give a construction of critical pairs (labelled with expressions) that guarantees the set of critical pairs is both finite and complete. Completeness would mean that all possible conflicts of rule schema applications can be represented as embeddings of critical pairs.

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