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Late formation of singularities in solutions to the Navier-Stokes equations

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Abstract. We study how late the first singularity can form in solutions of the Navier-Stokes equations and estimate the size of the potentially dangerous time interval, where it can possibly appear. According to Leray (1934), its size is estimated as $O(R^8)$ when normalised by the local existence time, for a general blowup of the enstrophy $Q(t) \geq \dfrac{c\nu^{3/2}}{(t_* - t)^{1/2}}$ at $t = t_*$. Here $R = (E(0)Q(0))^{1/4}/\nu$ is the Reynolds number defined with initial energy $E(0)$ and enstrophy $Q(0)$.

Applying dynamic scaling transformations, we give a general estimate parameterised by the behaviour of the scaled enstrophy. In particular, we show that the size is reduced to $O(R^4)$, for a class of type II blow up of the form $Q(t) \geq \dfrac{c'\nu^{3/2}}{(t_* - t)^{\frac{1}{2}+0}}$.

On the basis on the structure theorem of Leray (1934), we note that the self-similar and asymptotically self-similar blowup are ruled out for any singularities of weak solutions. We also apply the dynamic scaling to weak solutions with more than one singularities to show that the size is estimated as $O(R^4)$ for the type II blowup above.

1. Introduction

Substantial progress has been made recently in the understanding of the regularity properties of the incompressible Navier-Stokes equations. Ruling out self-similar blowup and improving blowup criteria with critical norms are just a few examples of recent results [2, 11, 16, 26, 34, 39]. In this paper we study dynamically scaled version of the Navier-Stokes equations, called non-steady Leray equations in detail. In particular, for a class of blowup (a majority of type II singularities) we study how late a singularity can take place in solutions of the Navier-Stokes equations by combining the energy inequality with the dynamic rescaling.

We consider the Navier-Stokes equations with standard notations in the whole space $\mathbb{R}^3$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \triangle \boldsymbol{u}, \quad (1)$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad (2)$$

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{b}, \tag{3}$$

where $\triangle=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$ and the initial data $\boldsymbol{b}(\boldsymbol{x})$ is smooth and decays sufficiently fast at large distances $|\boldsymbol{b}|\to 0$ as $|\boldsymbol{x}|\to\infty$.

There are a large amount of literature on the mathematical problems of the Navier-Stokes equations, including [1, 5, 6, 7, 8, 12, 14, 19, 27, 31, 36, 37, 38]. For applied mathematical interests, see e.g. [10, 24, 25, 28, 32]. See also [29] for applications of the self-similar ansatz in construction of some exact solutions.

It is well-known that if $\boldsymbol{u}(\boldsymbol{\xi},\tau)$ is a solution to the Navier-Stokes equations in any spatial dimensions, then so is $\lambda\boldsymbol{u}(\lambda\boldsymbol{\xi},\lambda^2\tau)=\boldsymbol{U}(\boldsymbol{\xi},\tau)$, where λ is an arbitrary positive constant. This is because the Navier-Stokes equations are invariant under the following set of scaling transformations:

$$\boldsymbol{x}=\lambda\boldsymbol{\xi}, t=\lambda^2\tau, \tag{4}$$

$$\boldsymbol{u}(\boldsymbol{x},t)=\lambda^{-1}\boldsymbol{U}(\boldsymbol{x}/\lambda,t/\lambda^2), \tag{5}$$

$$p(x,t)=\lambda^{-2}P(\boldsymbol{x}/\lambda,t/\lambda^2), \tag{6}$$

see e.g. [15] for details on self-similarity concepts.

To illustrate the concept of criticality, we recall that the L^q-norm of the velocity in $\mathbb{R}^n$ satisfies

$$\int_{\mathbb{R}^n}|\boldsymbol{u}|^q d\boldsymbol{x}=\lambda^{n-q}\int_{\mathbb{R}^n}|\boldsymbol{U}|^q d\boldsymbol{\xi}.$$

We call it super-critical if $q<n$, critical if $q=n$ and sub-critical if $q>n$. The similar definitions of criticality go for other norms. Historically, the importance of working in a critical space when we study the Navier-Stokes equations dates back to seminal papers of [13, 18].

Study of self-similar blowup solutions in three-dimensions was initiated in [21]. By assuming self-similar evolution of the form

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{\nu}{\sqrt{2a(t_*-t)}}\boldsymbol{U}\left(\boldsymbol{\xi}\right),\ \ \boldsymbol{p}(\boldsymbol{x},t)=\frac{\nu^2}{2a(t_*-t)}\boldsymbol{P}\left(\boldsymbol{\xi}\right),\ \ \boldsymbol{\xi}=\frac{\boldsymbol{x}}{[2a(t_*-t)]^{1/2}}, \tag{7}$$

we obtain the so-called Leray equations

$$\boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\frac{a}{\nu}(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\boldsymbol{U})=-\nabla_{\boldsymbol{\xi}}P+\triangle_{\boldsymbol{\xi}}\boldsymbol{U}, \tag{8}$$

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U}=0. \tag{9}$$

Note that capitalised variables $\boldsymbol{U},P$ are non-dimensional.

We review fundamental results regarding the absence of self-similar blowup. In [26] it was proved that if a solution $\boldsymbol{U}$ to the Leray equations satisfies $\boldsymbol{U}\in L^3(\mathbb{R}^3)$ then $\boldsymbol{U}\equiv 0$. Also, in [39] it was proved that if a solution $\boldsymbol{U}$ to the Leray equations satisfies $\boldsymbol{U}\in L^q(\mathbb{R}^3)$ with $q>3$, then $\boldsymbol{U}\equiv 0$.

The Prodi-Serrin criterion for regularity of the three-dimensional Navier-Stokes solutions states that as long as

$$\|\boldsymbol{u}\|_{L^s(0,T;L^r)}=\left(\int_0^T\|\boldsymbol{u}(t)\|_r^s dt\right)^{1/s}<\infty,$$

where $\frac{2}{s}+\frac{3}{r}=1$ and $3<r\le\infty$, no singularities can form in solutions of the Navier-Stokes equations up to time T. Whether the L^3-norm (i.e. the case of $r=3$ and $s=\infty$) serves as a blowup criterion for the three-dimensional Navier-Stokes equations or not, has been a long-standing challenging problem. It was finally proved in [11] that this is indeed the case. Later this result has been generalised to n-dimensions in [9]. See also [34] for a related result using another critical norm of $\dot{H}^{1/2}(\mathbb{R}^3)$.

2. Late formation of singularities

We define the energy $E(t)$ and the enstrophy $Q(t)$ as follows

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^3}|\boldsymbol{u}(\boldsymbol{x},t)|^2d\boldsymbol{x},$$

$$Q(t)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\boldsymbol{u}(\boldsymbol{x},t)|^2d\boldsymbol{x}$$

and notations with a tilde denote corresponding integrals after scaling. The energy inequality on a time interval $[0,t]$ implies

$$E(t)+2\nu\int_0^t Q(t')dt'\le E(0).$$

Leray's version of weak solutions (called a "turbulent solution") satisfies the energy inequality in a stronger form

$$E(t)+2\nu\int_s^t Q(t')dt'\le E(s),$$

which holds for almost all $s\ge 0$ including $s=0$ and all t such that $t\ge s$. They exist globally in time, but they can go singular many times. A natural question to ask is whether or not the subsequent blow-up can take place in a self-similar, or in an asymptotically self-similar manner. We discuss these questions on the basis of Leray's structure theorem (see below). The rest of this section places some known results in a proper perspective to be used in the subsequent sections.

2.1. First singularity

We recall the well-known bound on the enstrophy due to [21]

$$\frac{dQ}{dt}\le C\frac{Q(t)^3}{\nu^3}. \tag{10}$$

(Hereafter C, c etc. denote positive constants.) We note that the instantaneous growth rate (10) has been proven to be actually sharp and is associated with vortex rings in physical space [22].

We consider a problem of determining how late a singularity shows up in solutions of the Navier-Stokes equations. By solving the differential inequality (10) under the assumption of blowup $Q(t_*) = \infty$ at $t = t_*$, we obtain a lower-bound

$$Q(t) \geq c \frac{\nu^{3/2}}{\sqrt{t_* - t}}. \tag{11}$$

We recall general classifications of possible singularities here; a singularity which satisfies

$$Q(t) \leq c' \frac{\nu^{3/2}}{\sqrt{t_* - t}}$$

is called Type I and any blowup other than Type I is called Type II, see e.g. [35].

By substituting the lower-bound into the energy inequality

$$E(t) \leq E(0) - 2\nu \int_0^t Q(t')dt',$$

we find

$$E(t) \leq E(0) - 2c\nu^{5/2}(\sqrt{t_*} - \sqrt{t_* - t}).$$

Taking the limit $t \to t_*$, we have

$$E(t_*) \leq E(0) - 4c\nu^{5/2}\sqrt{t_*}.$$

This gives a contradiction, if t_* is larger than

$$t_* \simeq \frac{E(0)^2}{\nu^5}, \tag{12}$$

where $\simeq$ implies that both sides are on the same order, that is, $c_1 \frac{E(0)^2}{\nu^5} < t_* < c_2 \frac{E(0)^2}{\nu^5}$, for some $c_1, c_2\,(> 0)$. Hence we conclude that no singularity can form beyond this time scale.

2.2. Structure theorem

After the first singularity, no uniqueness is guaranteed for weak solutions. For the three-dimensional (and in fact, four-dimensional [17]) Navier-Stokes equations, we have Leray's structure theorem, which characterises the nature of time singularities as follows.

There are disjoint, countable time intervals $J_k, \quad k = 0, 1, 2, \ldots$ such that the following properties hold

$$|[0, \infty) - \cup_{k=0}^{\infty} J_k| = 0,$$

and

$$\sum_{k=1}^{\infty} |J_k|^{1/2} < \infty,$$

where $J_0 = [T, \infty)$ for some $T(> 0)$, is a semi-infinite time interval after the final singularity.

It was proved in [33] that the Hausdorff dimension of singularities on the time axis does not exceed 1/2. (See also [30].) This was proved by first fixing a sequence of singular times by the structure theorem and then applying usual blowup criterion to each singularity. It should be noted that while singular times are not uniquely determined, the above properties hold for each realisation.

2.3. Case of multiple singularities

If a weak solution has many singularities, we first fix the sequence of time singularities by the structure theorem: $t_1, t_2, \ldots, t_n, \ldots$, ordered as $t_1 < t_2 < t_3 < \ldots$, which is in general *not* unique except for t_1. Applying the blowup criterion repeatedly to the energy inequality, we obtain

$$2c\nu^{5/2}\left(\int_0^{t_1}\frac{dt}{\sqrt{t_1-t}}+\int_{t_1}^{t_2}\frac{dt}{\sqrt{t_2-t}}+\ldots+\int_{t_{n-1}}^{t_n}\frac{dt}{\sqrt{t_n-t}}+\ldots\right)\le E(0),$$

where a common constant $c(>0)$ has been redefined as the minimum of constants for all possible blowup. Thus we have

$$4c\nu^{5/2}\left(\sqrt{t_1}+\sqrt{t_2-t_1}+\ldots+\sqrt{t_n-t_{n-1}}+\ldots\right)\le E(0).$$

When t_1 is small, $\sqrt{t_2-t_1}$ is the next candidate to be dominant. If $t_2 \gg t_1$ we have $4c'\nu^{5/2}\sqrt{t_2} \le E(0)$, for some $c'(>0)$. This will give the t_2 the same bound as (12). If t_2 is also small, then we consider $\sqrt{t_3-t_2}$ and so on. Therefore we conclude that none of the $t_1, t_2, \ldots$ can exceed (12).

The same conclusion can obtained without referring to intermediate singularities step by step [21]. We have shown this awkward proof, because the similar kind of argument will be used in this paper.

3. Dynamic scaling transformations

Assuming that a solution to the Navier-Stokes equations blows up at $t = t_*$, we apply the dynamical rescaling transformations

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{\nu}{\sqrt{2a(t_*-t)}}\boldsymbol{U}(\boldsymbol{\xi},\tau),$$

$$\boldsymbol{\xi}=\frac{\boldsymbol{x}}{\sqrt{2a(t_*-t)}},\ \tau=\nu\int_0^t\frac{ds}{\lambda(s)^2}=\frac{\nu}{2a}\log\frac{t_*}{t_*-t},$$

where $\lambda(t)=\sqrt{2a(t_*-t)}$. We then obtain the non-steady version of the Leray equations

$$\frac{\partial\boldsymbol{U}}{\partial\tau}+\boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\frac{a}{\nu}(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\boldsymbol{U})=-\nabla_{\boldsymbol{\xi}}P+\triangle_{\boldsymbol{\xi}}\boldsymbol{U}, \tag{13}$$

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U}=0. \tag{14}$$

Here the zooming-in parameter $a(>0)$ has the same physical dimension as kinematic viscosity ν, and therefore we can write $2at_* = 1/k^2$, using a constant k which has the dimension of wavenumber. We pick $k = \sqrt{\frac{Q(0)}{E(0)}}$ for comparison with the known results. Note also that $t = \dfrac{1 - e^{-2a\tau/\nu}}{2ak^2}$. We note that these equations have been used e.g. in [2, 16, 23].

We make a simple observation regarding the blowup criteria using the enstrophy. It follows form (10) that

$$\frac{d}{dt} \log Q(t) \leq \frac{C}{\nu^3} Q(t)^2,$$

or

$$\log \frac{Q(t)}{Q(0)} \leq \frac{C}{\nu^3} \int_0^t Q(t')^2 dt'.$$

This time integral itself is critical (i.e. scale-invariant) under rescaling transforms and it is necessary for a blow up that

$$\int_0^{t_*} Q(t)^2 dt = \nu^3 \int_0^{\infty} \tilde{Q}(\tau)^2 d\tau = \infty. \tag{15}$$

We observe that the borderline case $Q(t) \simeq \frac{1}{\sqrt{t_* - t}}$ can take place in (10), but the other borderline $\tilde{Q}(\tau) \simeq \frac{1}{\sqrt{\tau}}$ cannot, because the former implies a $\log t$-behaviour in the first criterion, that is, a linear in τ behaviour for the second criterion. In fact, $\int_0^\infty \tilde{Q}(\tau)^2 d\tau = \infty$ is not sufficient, but only necessary. This can be verified by the scaled version of (10)

$$\frac{d\tilde{Q}}{d\tau} + \frac{a}{\nu}\tilde{Q} \leq C\tilde{Q}(\tau)^3, \tag{16}$$

from which it follows

$$\tilde{Q}(\tau) \leq \tilde{Q}(0) \exp\left(C \int_0^\tau \tilde{Q}(\tau')^2 d\tau' - \frac{a\tau}{\nu} \right).$$

This shows that the exponential should not decay to zero as $\tau \to \infty$, because if it does, we would have $\frac{\nu^2}{\sqrt{2a}}\tilde{Q}(\tau) = \sqrt{t_* - t}\, Q(t) \to 0$ as $t \to t_*$, which contradicts the Leray bound (11). More precisely, by solving (16) we have

$$\tilde{Q}(\tau) \geq \sqrt{\frac{a}{C\nu}},$$

hence the exponent cannot be negative.

4. Main result

The purpose of this section is to show that for a certain class of blowup, the first singularity cannot take place as late as the known estimate $E(0)^2/\nu^5$ obtained in [21].

Proposition 4.1. *A parametrisation of the scaled enstrophy*

$$\tilde{Q}(\tau) \geq c\tau^{\alpha}, \tag{17}$$

is equivalent to

$$Q(t) \geq c\left(\frac{\nu}{2a}\right)^{\alpha+\frac{1}{2}} \frac{\nu^{3/2}}{\sqrt{t_* - t}} \left(\log \frac{t_*}{t-t_*}\right)^{\alpha}, \tag{18}$$

where $\alpha \geq -\frac{1}{2}$ *and the extra prefactor satisfies*

$$\left(\frac{\nu}{2a}\right)^{\alpha+\frac{1}{2}} \leq \left(\frac{R^2}{2^{\alpha+2}c\Gamma(\alpha+1)}\right)^{\frac{2\alpha+1}{2(\alpha+1)}}.$$

Using the same α*, the size of the time interval which can contain the first singularity is estimated as*

$$\frac{t_*}{T} \simeq \left(\frac{R^{2(2\alpha+3)}}{2^{\alpha+2}c\Gamma(\alpha+1)}\right)^{\frac{1}{\alpha+1}}.$$

Proof. Assuming that a Navier-Stokes solution blows up at $t = t_*$, we consider the non-steady Leray equations (13). Before and after rescaling, the energy and enstrophy are related by

$$E(t) = \frac{\nu^2}{k}e^{-a\tau/\nu}\tilde{E}(\tau), \quad Q(t) = \nu^2 k e^{a\tau/\nu}\tilde{Q}(\tau).$$

By the condition of finite energy, we have

$$\int_0^{t_*} Q(t)dt = \frac{\nu}{k}\int_0^{\infty} e^{-a\tau/\nu}\tilde{Q}(\tau)d\tau < \infty,$$

whereas the blowup criterion requires

$$\int_0^{t_*} Q(t)^2 dt = \nu^3 \int_0^{\infty} \tilde{Q}(\tau)^2 d\tau = \infty. \tag{19}$$

Note that the left integral in (19), with a physical dimensions of $[\nu^3]$, is critical.

Integrating the energy equation for the scaled variables

$$\frac{d}{d\tau}\left(e^{-a\tau/\nu}\tilde{E}\right) = -2e^{-a\tau/\nu}\tilde{Q},$$

we obtain the scaled version of the energy inequality

$$0 \leq \tilde{E}(\tau) \leq e^{a\tau/\nu}\left(\tilde{E}(0) - \underbrace{2\int_0^{\tau} e^{-a\tau'/\nu}\tilde{Q}(\tau')d\tau'}_{=I}\right).$$

By (17), we are led to consider the following integral

$$I = 2c\int_0^{\infty} e^{-a\tau/\nu}\tau^{\alpha}d\tau$$

$$= 2c \left(\frac{\nu}{a}\right)^{\alpha+1} \Gamma(\alpha+1),$$

where Γ denotes the gamma function. Balancing two terms in the energy inequality, the borderline is determined by

$$\frac{kE(0)}{\nu^2} \geq 2c \left(\frac{\nu}{a}\right)^{\alpha+1} \Gamma(\alpha+1),$$

or

$$\left(\frac{\nu}{2a}\right)^{\alpha+1} \leq \frac{R^2}{2^{\alpha+2} c\Gamma(\alpha+1)}, \tag{20}$$

where $R = (E(0)Q(0))^{1/4}/\nu$ denotes the Reynolds number. It follows for $\alpha \geq -1/2$ that

$$\left(\frac{\nu}{2a}\right)^{\alpha+\frac{1}{2}} \leq \left(\frac{R^2}{2^{\alpha+2} c\Gamma(\alpha+1)}\right)^{\frac{2\alpha+1}{2(\alpha+1)}}. \tag{21}$$

On the other hand, we have

$$Q(t) \geq c\nu^2 k e^{a\tau/\nu} \tau^\alpha \tag{22}$$

$$= \frac{c\nu^2}{\sqrt{2a(t_*-t)}} \left(\frac{\nu}{2a} \log \frac{t_*}{t_*-t}\right)^\alpha \tag{23}$$

$$= \left(\frac{\nu}{2a}\right)^{\alpha+\frac{1}{2}} \frac{c\nu^{3/2}}{\sqrt{t_*-t}} \left(\log \frac{t_*}{t_*-t}\right)^\alpha. \tag{24}$$

It should be noted that the extra prefactor $\left(\frac{\nu}{2a}\right)^{\alpha+\frac{1}{2}}$ is bounded from above by (21). Furthermore, we have

$$t_* = \frac{1}{2ak^2} = \frac{E(0)}{2\nu Q(0)} \left(\frac{R^2}{2c\Gamma(\alpha+1)}\right)^{\frac{1}{\alpha+1}}.$$

Normalising by $T \simeq \nu^3/Q(0)^2$, we estimate the size of the dangerous time interval as

$$\frac{t_*}{T} \simeq \frac{1}{2(2c\Gamma(\alpha+1))^{\frac{1}{\alpha+1}}} R^{\frac{2(2\alpha+3)}{\alpha+1}}.$$

□

We consider special cases where the prefactor in (24) is $O(1)$ and discuss general cases. If the Reynolds number is of $O(1)$, i.e. for small initial data, singularities are absent and t_*/T is of $O(1)$ irrespective of α. If not, the extra prefactor in the above enstrophy bound (24) becomes large.

For a special value of $\alpha = -1/2$, we have

$$Q(t) \geq \frac{c\nu^{3/2}}{\sqrt{t_*-t}} \left(\log \frac{t_*}{t_*-t}\right)^{-1/2} \tag{25}$$

together with

$$\frac{t_*}{T} \simeq \frac{R^8}{8\pi c^2}.$$

This behavior $O(R^8)$ is the same as the result obtained in [21]. It should be noted that the same result has come out from a weaker condition (25). The value $\alpha = -1/2$ corresponds to the borderline case, where the necessary condition for a blow up in (15) is marginally satisfied.

Up to here the Reynolds number R and the exponent α are independent parameters. We can make use of this freedom to keep the prefactor at $O(1)$. For example, if we are interested in flows with Reynolds number up to $R = 1000$, we can choose $\alpha = 7$ because $2^9 \times 7! \simeq 2.5 \times 10^6$. If we want to handle flows with higher Reynolds numbers, we must increase α accordingly. If we do not limit Reynolds numbers, we must allow unbounded α. More precisely, we find the following.

Proposition 4.2. *For a class of slightly faster blow up*

$$Q(t) \geq \frac{c' \nu^{3/2}}{(t_* - t)^{\frac{1}{2}+0}}, \tag{26}$$

the size of the potentially dangerous time interval is given by $O(R^4)$.

Proof. Taking $R^2 = 2^{\alpha+1} 2c\Gamma(\alpha+1)$, by (21) the extra prefactor of the enstrophy bound in (18) remains at $O(1)$;

$$Q(t) \geq \frac{c\nu^{3/2}}{(t_* - t)^{\frac{1}{2}}} \left(\log \frac{t_*}{t_* - t} \right)^{\alpha},$$

for any $\alpha(> 0)$. The condition (26) is sufficient for this. We then have

$$\frac{t_*}{T} \simeq R^{-\frac{2}{\alpha+1}} R^{\frac{2(2\alpha+3)}{\alpha+1}} = R^4.$$

□

We note that the above estimate corresponds to

$$t_* \simeq \frac{E(0)}{\nu Q(0)}.$$

We also note that this argument does not apply to type I singularity, but covers a majority of type II singularity.

5. Beyond the first singularity

After the first singularity, i.e. the breakdown of a classical solution, we consider a 'turbulent solution,' which is a weak solution with a strong form of the energy inequality.

Dynamic scaling transformations have been used to discuss how late the first singularity t_* can happen under the condition (26). In principle weak solutions can blow up many times and it is of interest to study possible singularities beyond the first one.

As we mentioned above, self-similar blowup [26, 39] and asymptotically self-similar blowup [2, 16] have been excluded for the first singularity. See also [3, 4] for related works. If the scaled velocity converges in the long time limit

$$\lim_{\tau\to\infty} \|\boldsymbol{U}(\boldsymbol{\xi},\tau) - \overline{\boldsymbol{U}}(\boldsymbol{\xi})\|_{L^p} = 0,\ \overline{\boldsymbol{U}} \in L^p,\ p \geq 3,$$

then $\overline{\boldsymbol{U}}$ is a steady solution to the Leray equations and hence $\overline{\boldsymbol{U}} \equiv 0$ by [26]. This is the argument developed in [2, 16]. Note that weak solutions to the *steady* Leray equations are known to be actually smooth [26].

In this section, we address the following two questions on the basis of Leray's structure theorem. 1) We will rule out self-similar and asymptotically self-similar blowup for the singularities beyond the first singularity. 2) We will study how late singularities can form in the weak solutions under the assumption of faster blowup (26).

5.1. Exclusion of self-similar blowup

Assuming that a weak solution blow up many times, by the structure theorem we choose an ordered sequence of times $t_1, t_2, \ldots, t_n$. (We have denoted $t_* = t_1$ in previous sections.) The first question is straightforward: because in three-dimensions weak solutions are known to be piece-wise smooth on each interval, we can apply the results [2, 16].

It was proved in [20] that a weak solution $\boldsymbol{u} \in L^\infty(0, t_*; L^3)$ is right continuous on $[0, t_*)$. Because the structure theorem states that a weak solution (turbulent solution) is piece-wise smooth in time and has the same property on each interval (t_n, t_{n+1}) as on $(0, t_1)$, applying the result of [20] to each t_n, we find that there exists a function $\boldsymbol{a}_n(\boldsymbol{x})$ such that

$$\lim_{t\to t_n+0} \|\boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{a}_n(\boldsymbol{x})\|_{L^3} = 0,$$

for $n \geq 1$. On the first interval $[0, t_*) \equiv [t_0, t_1)$ prior to the first singularity, the corresponding function $\boldsymbol{a}_0(\boldsymbol{x})$ coincides with the initial data $\boldsymbol{a}_0(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}, 0)$. Even though $\boldsymbol{u}(\boldsymbol{x}, t)$ is undefined at $t = t_n$, we can make use of $\boldsymbol{a}_n(\boldsymbol{\xi})$ to define initial data for the Leray equations by

$$\boldsymbol{U}(\boldsymbol{\xi}, 0) = \boldsymbol{a}_n(\boldsymbol{\xi}),$$

thereby enabling us to consider non-steady Leray equations on $[t_n, t_{n+1})$. We then conclude that self-similar and asymptotically self-similar blowup is excluded at $t = t_{n+1}$ by [2, 16].

5.2. Late time singularities

For a general blow up, the late time scale beyond which no singularity can form has been confirmed in Subsection 2.3. We consider a class of faster blow up (26) here. To discuss the second question of the (possible) subsequent singularities, we need to handle dynamically scaled Leray equations in weak sense. By the structure theorem, again we choose an ordered sequence of time singularities $t_1, t_2, \ldots, t_n$ then apply the dynamic scaling transforms. (These singularities are not unique, except for t_1.)

In order to apply dynamic scaling transforms to the weak form of the Navier-Stokes equations, we establish a weak formulation of the Leray equations. There are two different ways of the weak formulation; one is deriving a weak form of the non-steady Leray equations from the Navier-Stokes equations in weak form and the other is generalising the Leray equations in classical sense to weak sense. By considering the first singularity, they must match for consistency. We confirm that this is the case in the **Appendix** below.

Taking the origin of time at $t = 0$, not at $t_n \ \ (n \geq 1)$, we apply dynamic scaling transformations to singular points $t_* = t_1, t_2, t_3 \dots$ on the basis of the Leray equations in weak form, that is, on a time interval $[0, t_n)$. The above argument for the first singularity can be carried over to these cases. We conclude that

$$\frac{t_n}{T} \simeq R^4.$$

for

$$Q(t) \geq \frac{c' \nu^{3/2}}{(t_n - t)^{\frac{1}{2}+0}},$$

where $n \geq 1$ and $t_{n-1} < t < t_n$.

6. Summary and discussion

In this paper, we have studied how late singularities can form in solutions to the Navier-Stokes equations.

Firstly, we considered the first singularity on the basis of dynamic scaling transformations. For a majority of type II singularities, we have proved that no blowup is possible beyond $t_* \simeq E(0)/\nu Q(0)$, which is smaller than the known time scale $t_* \simeq E(0)^2/\nu^5$ for a general blowup. Equivalently, for this class of blowup, the size of dangerous time interval is reduced by 50% on a logarithmic scale from $\frac{t_*}{T} \simeq R^8$ to $\frac{t_*}{T} \simeq R^4$, where $R = (E(0)Q(0))^{1/4}/\nu$ is the Reynolds number.

Secondly, we apply the same argument to subsequent singularities that can take place in weak solutions to the Navier-Stokes equations. To this end, we formulate the non-steady Leray equations in weak sense and show that no singularity in that class cannot happen later than $t_* \simeq E(0)/\nu Q(0)$ on the basis of the structure theorem.

While this argument cannot be applied to Type I singularity, it does place limitations on the appearance of singularities because Type II singularities are not ruled out at the moment, even for the axisymmetric case. Our argument works in the whole space $\mathbb{R}^3$. For other boundary conditions, say on a bounded domain or under periodic boundaries, it is not known if the corresponding result is available or not because of lack of Leray equations for those cases.

Appendix A. Leray equations in weak form

The Navier-Stokes equations in weak form take the following form

$$\int_0^{T'} \left[-\left(\boldsymbol{u}, \frac{\partial \boldsymbol{\phi}}{\partial t} \right) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\phi}) + \nu (\nabla \boldsymbol{u}, \nabla \boldsymbol{\phi}) \right] dt = (\boldsymbol{b}, \boldsymbol{\phi}), \tag{A.1}$$

where $\boldsymbol{\phi}(\boldsymbol{x}, t)$ denotes a smooth test function compactly supported in space-time, $(,)$ an inner-product in $\boldsymbol{x}$-space and n the spatial dimension to clarify where it appears ($n = 3$). Applying dynamic scaling, the Leray equations in weak form take the form

$$\int_0^{T''} e^{-(n-1)a\tau/\nu} \left[-\left(\boldsymbol{U}, \frac{\partial \boldsymbol{\Phi}}{\partial \tau} + \frac{a}{\nu} \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right)' + (\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' + (\nabla \boldsymbol{U}, \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi})' \right] d\tau \tag{A.2}$$

$=(\boldsymbol{B}, \boldsymbol{\Phi}(\cdot, 0))'$,where $\boldsymbol{\Phi}(\boldsymbol{\xi}, \tau) \equiv \boldsymbol{\phi}\left(\frac{e^{-a\tau/\nu}}{k} \boldsymbol{\xi}, \frac{1 - e^{-2a\tau/\nu}}{2ak^2} \right)$, $\boldsymbol{B}(\boldsymbol{\xi}) = \frac{\sqrt{2at_*}}{\nu} \boldsymbol{b}(\boldsymbol{x})$ and $(,)'$ an inner-product in $\boldsymbol{\xi}$-space.

Proof. By

$$\nabla = \frac{1}{\sqrt{2a(t_* - t)}} \nabla_{\boldsymbol{\xi}}, \quad \frac{\partial}{\partial t} = \frac{\nu}{2a(t_* - t)} \left(\frac{\partial}{\partial \tau} + \frac{a}{\nu} \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \right)$$

we compute term by term of (A.1). For example, the spatial integral of its first term on the left-hand side reads

$$\int \frac{\nu}{\sqrt{2a(t_* - t)}} \boldsymbol{U}(\boldsymbol{\xi}, \tau) \frac{\nu}{2a(t_* - t)} \left(\frac{\partial \boldsymbol{\Phi}}{\partial \tau} + \frac{a}{\nu} \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right) (2a(t_* - t))^{n/2} d\boldsymbol{\xi}$$

$$= \nu^2 (2a(t_* - t))^{(n-3)/2} \left(\boldsymbol{U}, \frac{\partial \boldsymbol{\Phi}}{\partial \tau} + \frac{a}{\nu} \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi} \right)'.$$

To complete the left-hand side, we also note

$$\int_0^{T'} dt = \frac{1}{\nu k^2} \int_0^{T''} d\tau e^{-2a\tau/\nu},$$

together with $(2a(t_* - t))^{1/2} = \frac{1}{k} e^{-a\tau/\nu}$. On the other hand, the right-hand side is

$$\int \frac{\nu}{\sqrt{2at_*}} \boldsymbol{B}(\boldsymbol{\xi}) \cdot \boldsymbol{\Phi}(\boldsymbol{\xi}) (2a(t_* - t))^{n/2} d\boldsymbol{\xi} \bigg|_{t=0} = \frac{\nu}{k^{n-1}} (\boldsymbol{B}, \boldsymbol{\Phi})'.$$

The rest is straightforward. □

We check that the same result comes out when we start from the Leray equations in strong form, i.e. for the first singularity, and move onto a weak form.

Proof. Take inner-product of (13) with $\boldsymbol{\Phi}$ to write

$$(\boldsymbol{U}_\tau, \boldsymbol{\Phi})' + (\boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' + \frac{a}{\nu}(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' + \underline{\frac{a}{\nu}(\boldsymbol{U}, \boldsymbol{\Phi})'}$$

$$= -(\nabla_{\boldsymbol{\xi}} P, \boldsymbol{\Phi})' + (\triangle_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})'.$$

Integrating the above with respect to τ against the weight factor $e^{-(n-1)a\tau/\nu}$, we find that the first term is

$$\int_0^{T'} e^{-(n-1)a\tau/\nu} (\boldsymbol{U}_\tau, \boldsymbol{\Phi})' d\tau$$

$$= (\boldsymbol{B}, \boldsymbol{\Phi}(\cdot, 0))' + \underline{(n-1)\frac{a}{\nu} \int e^{-(n-1)a\tau/\nu} (\boldsymbol{U}, \boldsymbol{\Phi})' d\tau} - \int e^{-(n-1)a\tau/\nu} (\boldsymbol{U}, \boldsymbol{\Phi}_\tau)' d\tau.$$

Noting that the third term is

$$\frac{a}{\nu}(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U}, \boldsymbol{\Phi})' = \underline{-\frac{na}{\nu}(\boldsymbol{U}, \boldsymbol{\Phi})'} - \frac{a}{\nu}(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{\Phi}, \boldsymbol{U})',$$

we see the same equations as (A.1) come out because of cancellation of the three underlined parts. □

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References

[1] Caffarelli, L., Kohn, R., & Nirenberg, L. 1982 Partial regularity of suitable weak solutions of the Navier-Stokes equations *Commun. Pure Appl. Math.* **35**, 771–831.

[2] Chae, D. 2007 Nonexistence of asymptotically self-similar singularities in the Euler and the Navier-Stokes equations *Math. Ann.* **338**, 435–449.

[3] Chae, D. 2010 On the generalized self-similar singularities for the Euler and the Navier-Stokes equations. *J. Func. Anal.* **258**, 2865–2883.

[4] Chae, D. & Shvydkoy, R. 2013 On Formation of a Locally Self-Similar Collapse in the Incompressible Euler Equations *Arch. Rat. Mech. Anal.* **209**, 999–1017.

[5] Constantin, P. 1986 Note on loss of regularity for solutions of the 3D incompressible euler and related equations *Commun. Math. Phys.* **104** 311–326.

[6] Constantin, P. & Foias, C. 1988 *Navier-Stokes Equations* Chicago, University of Chicago Press.

[7] Doering, C.R. 2009 The 3D Navier-Stokes problem *Annu. Rev. Fluid Mech.* **41** 109–128.

[8] Doering, C.R. & Gibbon, J.D. 1995 *Applied Analysis of the Navier-Stokes Equations* Cambridge, Cambridge University Press.

[9] Dong, H. & Du, D. 2009 The Navier-Stokes Equations in the Critical Lebesgue Space *Commun. Math. Phys.* **292**, 811–827.

[10] Dowker, M.D. & Ohkitani, K. 2012 Numerical Study on the Scaling Property of the Intermittency and local Reynolds number in Navier-Stokes turbulence: a cross-over scale in the Caffarelli-Kohn-Nirenberg integral *Phys. Fluids* **24**, 115112-1–19,

[11] Escauriaza, L., Seregin, G. & Sverak, V. 2003 $L_{3,\infty}$-solutions of the Navier-Stokes equations and backward uniqueness *Russ. Math. Surv.* **58**, 211–250.

[12] Foias, C., Manley, O., Rosa, O. & Temam, R. 2001 *Navier-Stokes Equations and Turbulence* Cambridge, Cambridge University Press.

[13] Fujita, H. & Kato, T. 1964 On the Navier-Stokes initial value problem. I *Arch. Rat. Mech. Anal.* **16**, 269–315.

[14] Gibbon, J.D. 2010 Regularity and singularity in solutions of the three-dimensional Navier-Stokes equations *Proc. Roy. Soc. A* **466**, 2587–2604.

[15] Giga, M., Giga, Y. & Saal, J. 2010 *Nonlinear Partial Differential Equations: Asymptotic Behavior of Solutions and Self-Similar Solutions* Birkhäuser.

[16] Hou, T. and Li, R. 2007 Nonexistence of locally self-similar blow-up for the 3D incompressible Navier-Stokes equations *Dis. Cont. Dyn. Sys.* **18**, 637–642.

[17] Kato, T. 1984 Strong L^p-solutions of the Navier-Stokes equation in R^m, with applications to weak solutions *Math. Z.* **187**, 471–480.

[18] Kato, T. & Fujita, H. 1962 On the nonstationary Navier-Stokes system *Rendiconti del Seminario Matematico della Universitá di Padova* **32**, 243–260.

[19] Kozono, H. 2009 Navier-Stokes equations *Sugaku expositions* **22**, 187–213.

[20] Kozono, H. & Sohr, H. 1996 Remark on uniqueness of weak solutions to the Navier-Stokes equations *Analysis* **16**, 255–272.

[21] Leray, J. 1934 Essai sur le mouvement d'un liquide visqueux emplissant l'espace *Acta Math.* **63**, 193–248 (1934).

[22] Lu L and Doering C R 2008 Limits on enstrophy growth for solutions of the three-dimensional Navier-Stokes equations *Indiana Univ. Math. J.* **57** 2693-2728

[23] Mazzucato, A.L. 2005 On the energy spectrum for weak solutions of the Navier Stokes equations *Nonlinearity* **18**, 1–19.

[24] Moffatt, H.K. 2000 The interaction of skewed vortex pairs: a model for blowup of the Navier-Stokes equations *J. Fluid Mech.* **409**, 51–68.

[25] Moffatt, H.K. 2002 Mechanics for a New Millennium in *Local and Global Perspectives in Fluid Dynamics*, 521–540.

[26] Nečas, J., Růžička, M. & Šverák, V. 1996 On Leray's self-similar solutions of the Navier-Stokes equations *Acta Math.* **176**, 283–294.

[27] Ohkitani, K. 2008 A miscellany of basic issues on incompressible fluid equations *Nonlinearity* **21**, 255–271.

[28] Ohkitani, K. & Dowker, M.D. 2012 Numerical Study on Comparison of Navier-Stokes and Burgers Equations *Phys.Fluids* **24**, 055113-1–20.

[29] Okamoto, H. 1997 Exact solutions of the Navier-Stokes equations via Leray's scheme *Japan J. Indust. Appl.Maths.* **14** 169–197.

[30] Robinson, J.C. & Sadowski 2007 Decay of weak solutions and the singular set of the three-dimensional Navier-Stokes equations *Nonlinearity* **20**, 1185–91.

[31] Robinson, J.C., Sadowski, W. & Silva, R.P. 2012 Lower bounds on blow up solutions of the three-dimensional Navier-Stokes equations in homogeneous Sobolev spaces *J. Math. Phys.* **53**, 115618-1–15.

[32] Rosen, G. 1970 Navier-Stokes initial value problem for boundary-free incompressible fluid flow *Phys. Fluids* **12** 2891–2903.

[33] Scheffer, V. 1976 Turbulence and Hausdorff dimension in *Turbulence and Navier Stokes Equations* Lec. Notes Math. **565** 174–183, Springer, Berlin

[34] Seregin, G. 2012 A Certain Necessary Condition of Potential Blow up for Navier-Stokes Equations *C*ommun. Math. Phys. **312**, 833–845.

[35] Seregin, G. and Sverak, V. 2009 On type I singularities of the local axis-symmetric solutions of the Navier-Stokes equations *Commun. Part. Diff. Eqs.* **34** 171–201

[36] Sohr, H. 2001 The Navier-Stokes Equations: An Elementary Functional Analytic Approach

Birkhäuser, Basel.

[37] Tao, T. 2013 Localisation and compactness properties of the Navier-Stokes global regularity problem *Analysis and PDE* **6**, 25–107.

[38] Temam, R. 1984 *Navier-Stokes Equations: Theory and Numerical Analysis* 1984, American Mathematical Society.

[39] Tsai, T.-P. 1998 On Leray's Self-Similar Solutions of the Navier-Stokes Equations Satisfying Local Energy Estimates *Arch. Rat. Mech. Anal.* **143**, 29–51.