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Near-invariance under dynamic scaling for the Navier-Stokes equations in critical spaces: a probabilistic approach to regularity problems

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Abstract. We make a detailed comparison between the Navier-Stokes equations and their dynamically-scaled counterpart, the so-called Leray equations. The Navier-Stokes equations are invariant under *static* scaling transforms, but are not generally invariant under dynamic scaling transforms. We will study how close they can be brought together using the critical dependent variables and discuss the implications on the regularity problems.

Assuming that the Navier-Stokes equations written in the vector potential have a solution that blows up at $t = 1$, we derive the Leray equations by dynamic scaling. We observe: (1) The Leray equations have only one term extra on top of those of the Navier-Stokes equations. (2) We can recast the Navier-Stokes equations as a Wiener path integral and the Leray equations as another Ornstein-Uhlenbeck path integral. By the Maruyama-Girsanov theorem, both equations take the identical form *modulo* the Maruyama-Girsanov density, which is valid up to $t = 2\sqrt{2}$ by the Novikov condition. (3) The global solution of the Leray equations is given by a finite-dimensional projection $\boldsymbol{R}$ of a functional of an Ornstein-Uhlenbeck process and a probability measure. If $\boldsymbol{R}$ remains smooth beyond $t = 1$ under an absolute continuous change of the probability measure, we can rule out finite-time blowup by contradiction. There are two cases: (A) $\boldsymbol{R}$ given by a finite number of Wiener integrals, and (B) otherwise. Ruling out blowup in (A) is straightforward. For (B), a condition based on a limit passage in the Picard iterations is identified for such a contradiction to come out. The whole argument equally holds in $\mathbb{R}^d$ for any $d \geq 2$.

Keywords: Navier-Stokes equations, Leray equations, dynamic scaling, critical spaces, Maruyama-Girsanov theorem, global regularity

1. Introduction

We consider the Navier-Stokes equations with standard notations in the whole space, mainly in $\mathbb{R}^3$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{2} \triangle \boldsymbol{u}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}), \tag{3}$$

where $\triangle = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$. The initial data $\boldsymbol{u}_0(\boldsymbol{x})$ are smooth and well-localised such that $|\boldsymbol{u}_0(\boldsymbol{x})| \to 0$ as $|\boldsymbol{x}| \to \infty$. By choosing spatial and temporal units suitably, we have taken the prefactor as 1/2 in front of the Laplacian, so that applications of probabilistic methods (with a standard Brownian motion) will be simplified. There are lots of publications on the mathematical problems of the Navier-Stokes equations, including [6, 60, 10, 11, 48, 17, 55, 34, 51, 12, 29, 21, 52, 58] on pure analysis side and [53, 13, 39, 36, 43, 57, 14, 47, 3, 45, 46] on applied mathematical aspects.

The Navier-Stokes equations satisfy the following static (i.e. for a fixed time) scale-invariance: if $\boldsymbol{u}(\boldsymbol{x},t)$ is a solution to the Navier-Stokes equations, so is $\lambda\boldsymbol{u}(\lambda\boldsymbol{x},\lambda^2 t)$, where λ (>0) is an arbitrary parameter. Study of self-similar blowup solutions in three-dimensions was initiated in [35]. By assuming self-similar evolution of the form

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{\sqrt{t_*-t}}\boldsymbol{U}\left(\boldsymbol{\xi}\right),\;\; \boldsymbol{p}(\boldsymbol{x},t)=\frac{1}{t_*-t}\boldsymbol{P}\left(\boldsymbol{\xi}\right),\;\; \boldsymbol{\xi}=\frac{\boldsymbol{x}}{\sqrt{t_*-t}} \tag{4}$$

we obtain the steady Leray equations

$$\boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\frac{1}{2}(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\boldsymbol{U})=-\nabla_{\boldsymbol{\xi}}P+\frac{1}{2}\triangle_{\boldsymbol{\xi}}\boldsymbol{U}, \tag{5}$$

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U}=0. \tag{6}$$

Self-similar blowup has been ruled out in [40]; it was proved that if a solution $\boldsymbol{U}$ to the Leray equations satisfies $\boldsymbol{U}\in L^3(\mathbb{R}^3)$ then $\boldsymbol{U}\equiv 0$. It was also proved in [61] that if a solution $\boldsymbol{U}$ to the Leray equations satisfies $\boldsymbol{U}\in L^q(\mathbb{R}^3)$ with $q>3$, then $\boldsymbol{U}\equiv 0$.

In this connection, the following result of a blowup criterion should be mentioned. It was proved in [16] that the L^3-norm must become unbounded if a solution to the three-dimensional Navier-Stokes equations breaks down at a finite time. On this basis, we can confirm the absence of self-similar blowup in a similar manner. Assuming spatial integrability we have by definition

$$\int_{\mathbb{R}^3}|\boldsymbol{u}(\boldsymbol{x},t)|^3 d\boldsymbol{x}=\int_{\mathbb{R}^3}|\boldsymbol{U}(\boldsymbol{\xi})|^3 d\boldsymbol{\xi}. \tag{7}$$

By [16], the left-hand side becomes unbounded at $t=t_*$ if a solution blows up at that time. However, the right-hand side of the above identity is a constant, because it is an integral expressed solely in terms of the similarity variable. This is a contradiction and no self-similar blowup is possible.

In a nutshell, static scale-invariance rules out self-similar blowup with the use of the $\|\boldsymbol{u}\|_{L^3}$ norm. This suggests that there may be a similar contradictory argument, which can rule out blowup in more general cases. One possible idea that we may pursue is that invariance under dynamic scaling, if available and formulated somehow, may constrain more general blowup.

In this paper we propose to make use of the vector potentials $\boldsymbol{\psi}$ (or, their counterparts in higher dimensions) as critical dependent variables, remembering that, unlike norms, they carry complete information of the solutions. We note that the choice of such dependent variables are in line with the so-called Fujita-Kato principle, developed in seminal papers [27, 19].

The rest of this paper is organised as follows. In Section 2, we recall dynamic scaling and the Leray equations. In Section 3, we describe the three-dimensional Navier-Stokes equations using the vector potentials. Applying the Duhamel principle both to the Navier-Stokes and Leray equations, we introduce the notion of *invariance under dynamic scaling.* In Section 4, we discuss how and under what conditions global regularity can be deduced for the Navier-Stokes equations. Section 5 is devoted to summary and discussion. Some technical details are given in Appendices.

2. Dynamic scaling transformations: Leray equations

Assuming that a solution to the Navier-Stokes equations blows up at $t = t_*$, we apply the dynamic scaling transformations

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{\sqrt{t_* - t}} \boldsymbol{U}(\boldsymbol{\xi}, \tau), \tag{8}$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{t_* - t}},\ \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \log \frac{t_*}{t_* - t}, \tag{9}$$

where $\lambda(t) = \sqrt{t_* - t}$, we derive the non-steady version of the Leray equations

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \frac{1}{2}(\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}} P + \frac{1}{2} \triangle_{\boldsymbol{\xi}} \boldsymbol{U}, \tag{10}$$

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{U} = 0. \tag{11}$$

We have taken $t_* = 1$ to make the initial conditions $\boldsymbol{u}$ and $\boldsymbol{U}$ coincide. Note also that $t = 1 - e^{-\tau}$. These equations have been used in many articles, including [7, 25, 38, 8, 9, 24].

As we mentioned above, self-similar blowup has been ruled out in [40, 61]. In [7, 25], asymptotically self-similar blowup has also been ruled out: if the scaled velocity converges in the long time limit

$$\lim_{\tau \to \infty} \|\boldsymbol{U}(\boldsymbol{\xi}, \tau) - \overline{\boldsymbol{U}}(\boldsymbol{\xi})\|_{L^p} = 0,\ \overline{\boldsymbol{U}} \in L^p,\ p \geq 3,$$

then $\overline{\boldsymbol{U}}$ is a steady solution to the Leray equations and hence $\overline{\boldsymbol{U}} \equiv 0$ by [40].

We note in passing that in the critical case the equations for the L^d-norms are degenerate, that is, they coincide completely before and after rescaling ($d \geq 2$). Indeed, we have

$$\frac{1}{d}\frac{d}{dt} \int_{\mathbb{R}^d} |\boldsymbol{u}|^d d\boldsymbol{x} = -\int_{\mathbb{R}^d} |\boldsymbol{u}|^{d-2} \nabla \cdot (\boldsymbol{u} p) d\boldsymbol{x} + \frac{1}{2} \int_{\mathbb{R}^d} |\boldsymbol{u}|^{d-2} \boldsymbol{u} \cdot \triangle \boldsymbol{u} d\boldsymbol{x} \tag{12}$$

and

$$\frac{1}{d}\frac{d}{d\tau} \int_{\mathbb{R}^d} |\boldsymbol{U}|^d d\boldsymbol{\xi} = -\int_{\mathbb{R}^d} |\boldsymbol{U}|^{d-2} \nabla_{\boldsymbol{\xi}} \cdot (\boldsymbol{U} P) d\boldsymbol{\xi} + \frac{1}{2} \int_{\mathbb{R}^d} |\boldsymbol{U}|^{d-2} \boldsymbol{U} \cdot \triangle_{\boldsymbol{\xi}} \boldsymbol{U} d\boldsymbol{\xi}, \tag{13}$$

because of the orthogonality

$$\int_{\mathbb{R}^d} |\boldsymbol{U}|^{d-2}\boldsymbol{U}\cdot(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U}+\boldsymbol{U})d\boldsymbol{\xi}=0, \tag{14}$$

which can be verified by integration by parts. It is clear from this observation that we cannot distinguish $\boldsymbol{u}$ and $\boldsymbol{U}$ in their behaviour by analysing these equations for the critical norm.

3. 3D Navier-Stokes equations written in a vector potential

3.1. Governing equations

With vector potentials $\boldsymbol{\psi}$ such that $\boldsymbol{u} = \nabla\times\boldsymbol{\psi}$ and $\nabla\cdot\boldsymbol{\psi} = 0$, the Navier-Stokes equations read [44]

$$\frac{\partial\boldsymbol{\psi}}{\partial t} - \frac{1}{2}\triangle\boldsymbol{\psi} = \frac{3}{4\pi}\,⨍_{\mathbb{R}^3}\frac{\boldsymbol{r}\times(\nabla\times\boldsymbol{\psi}(\boldsymbol{y}))\,\boldsymbol{r}\cdot(\nabla\times\boldsymbol{\psi}(\boldsymbol{y}))}{|\boldsymbol{r}|^5}\,\mathrm{d}\boldsymbol{y}, \tag{15}$$

where $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}$ and $⨍$ denotes a principal-value integral. We assume that $|\boldsymbol{\psi}(\boldsymbol{x},t)| \to 0$ as $|\boldsymbol{x}| \to \infty$ for $^\forall t \geq 0$. Because a solution to the Navier-Stokes equations breaks down at $t = t_*$ by assumption, $\boldsymbol{\psi}(\boldsymbol{x},t)$ is no longer smooth by that time and we have $\|\nabla\times\boldsymbol{\psi}\|_{L^3} \to \infty$ as $t \to t_*$ by [16].

The dynamic scaling transformation for $\boldsymbol{\psi}(\boldsymbol{x},t)$ is given by

$$\boldsymbol{\psi}(\boldsymbol{x},t) = \widetilde{\boldsymbol{\psi}}(\boldsymbol{\xi},\tau), \tag{16}$$

and $\widetilde{\boldsymbol{\psi}}(\boldsymbol{\xi},\tau)$ satisfies

$$\frac{\partial\widetilde{\boldsymbol{\psi}}}{\partial\tau} - \frac{1}{2}\triangle_{\boldsymbol{\xi}}\widetilde{\boldsymbol{\psi}} + \frac{1}{2}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\widetilde{\boldsymbol{\psi}} = \frac{3}{4\pi}\,⨍_{\mathbb{R}^3}\frac{\boldsymbol{\rho}\times(\nabla\times\widetilde{\boldsymbol{\psi}}(\boldsymbol{\xi}'))\,\boldsymbol{\rho}\cdot(\nabla\times\widetilde{\boldsymbol{\psi}}(\boldsymbol{\xi}'))}{|\boldsymbol{\rho}|^5}\,\mathrm{d}\boldsymbol{\xi}', \tag{17}$$

where $\boldsymbol{\rho} = \boldsymbol{\xi}-\boldsymbol{\xi}'$. Note that $\widetilde{\boldsymbol{\psi}}$ itself has no temporal scaling prefactor in (16) because of its criticality. For this reason, there is only one extra term in (17) in comparison with (15). By construction, $\widetilde{\boldsymbol{\psi}}(\boldsymbol{\xi},\tau)$ is smooth all the time with an asymptotic behaviour $\|\nabla_{\boldsymbol{\xi}}\times\widetilde{\boldsymbol{\psi}}\|_{L^3} \to \infty$ as $\tau \to \infty$. We observe that the extra term $\frac{1}{2}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\widetilde{\boldsymbol{\psi}}$ vanishes at local maxima and minima, so that the equations (15) and (17) are degenerate (that is, they coincide) at those extremal points.

3.2. Duhamel principle

Now we apply Duhamel principle to both of the equations. For simplicity, let us denote the nonlinear term by

$$\boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{x},t) \equiv \frac{3}{4\pi}\,⨍_{\mathbb{R}^3}\frac{\boldsymbol{r}\times(\nabla\times\boldsymbol{\psi}(\boldsymbol{y}))\,\boldsymbol{r}\cdot(\nabla\times\boldsymbol{\psi}(\boldsymbol{y}))}{|\boldsymbol{r}|^5}\,\mathrm{d}\boldsymbol{y}.$$

By regarding the nonlinear term as an external forcing, we can recast the governing equations

$$\left(\frac{\partial}{\partial t} - A\right)\boldsymbol{\psi} = \boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{x},t)$$

formally as

$$e^{tA}\frac{\partial}{\partial t}\left(e^{-tA}\boldsymbol{\psi}\right) = \boldsymbol{T}[\nabla\boldsymbol{\psi}],$$

or

$$\boldsymbol{\psi}(t) = e^{tA}\boldsymbol{\psi}_0 + \int_0^t e^{(t-s)A}\boldsymbol{T}[\nabla\boldsymbol{\psi}](\cdot,s)ds, \tag{18}$$

where we have put $A = \frac{1}{2}\triangle$ and $\boldsymbol{\psi}_0(\boldsymbol{x}) = \boldsymbol{\psi}(\boldsymbol{x},0)$. More explicitly, we can write

$$\begin{aligned}\boldsymbol{\psi}(\boldsymbol{x},t) =& \frac{1}{(2\pi t)^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2t}\right)\boldsymbol{\psi}_0(\boldsymbol{y})d\boldsymbol{y}\\ &+\int_0^t ds\frac{1}{(2\pi(t-s))^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2(t-s)}\right)\boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{y},s)d\boldsymbol{y},\end{aligned} \tag{19}$$

(see **Appendix A**). In the spirit of the Feynman-Kac formula, we can also put (19) in a path integral form as‡

$$\boldsymbol{\psi}(\boldsymbol{x},t) = E[\boldsymbol{\psi}_0(\boldsymbol{W}_t)] + \int_0^t E[\boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{W}_s,t-s)]ds, \text{ for } 0 \le t < t_*, \tag{20}$$

where we have dropped $'$ after changing the time variable to $s' = t-s$, $\boldsymbol{W}_t$ denotes a Wiener process and E an average $E_{\boldsymbol{P}}[F(\boldsymbol{W})] = \int F(\boldsymbol{W})\mu(d\boldsymbol{W})$ with respect the the standard Gaussian probability measure $\boldsymbol{P}$. § An average E without subscript denotes the one with respect to the standard Gaussian probability measure $\boldsymbol{P}$; $E[\cdot] = E_{\boldsymbol{P}}[\cdot]$. When it is necessary to distinguish the measure, e.g. $\boldsymbol{Q}$ used in taking the average, we write $E_{\boldsymbol{Q}}[\cdot]$.

Similarly we can also put the Leray equations in a path integral form. We write $(\boldsymbol{x},t)$ for $(\boldsymbol{\xi},\tau)$ to place the Leray equations on the equal footing as the Navier-Stokes equations, forgetting about how the Leray equations have been derived. We hence write

$$\frac{\partial\widetilde{\boldsymbol{\psi}}}{\partial t} - \frac{1}{2}\triangle\widetilde{\boldsymbol{\psi}} + \frac{1}{2}\boldsymbol{x}\cdot\nabla\widetilde{\boldsymbol{\psi}} = \boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{x}), \tag{21}$$

or

$$\left(\frac{\partial}{\partial t} - \widetilde{A}\right)\widetilde{\boldsymbol{\psi}} = \boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{x},t),$$

‡ The expression should be interpreted as $E[\boldsymbol{f}(\boldsymbol{W}_s,t-s)] = \frac{1}{(2\pi s)^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2s}\right)\boldsymbol{f}(\boldsymbol{y},t-s)d\boldsymbol{y}$, where $\boldsymbol{f}(\boldsymbol{x},t) = \boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{x},t)$. In taking an average, both t and s in $t-s$ should be fixed and we evaluate the Wiener path integral up to s for the thus given function $\boldsymbol{f}(\cdot,t-s)$.

§ This measure is defined by

$$\boldsymbol{P}\{\boldsymbol{W}_1\in\boldsymbol{E}_1,\ldots,\boldsymbol{W}_N\in\boldsymbol{E}_N\} = \frac{1}{\left[(2\pi)^N\Pi_{k=1}^N(t_k-t_{k-1})\right]^{3/2}}\int_{\boldsymbol{E}_1}\cdots\int_{\boldsymbol{E}_N}\exp\left(-\sum_{k=1}^N\frac{|\boldsymbol{x}_k-\boldsymbol{x}_{k-1}|^2}{2(t_k-t_{k-1})}\right)d\boldsymbol{x}_1\ldots d\boldsymbol{x}_N,$$

where $\boldsymbol{W}_j = \boldsymbol{W}_{t_j},\ \boldsymbol{E}_j\in\mathbb{R}^3$ for $j=1,\ldots,N,\ 0\le t_1<t_2<\ldots<t_N$ with $t_0=\boldsymbol{x}_0=0$.

where we have introduced

$$\widetilde{A} = \frac{1}{2}\left(\triangle - \boldsymbol{x}\cdot\nabla\right).$$

Because the nonlinear term in (21) is smooth all the time, we can write

$$e^{t\widetilde{A}}\frac{\partial}{\partial t}\left(e^{-t\widetilde{A}}\widetilde{\boldsymbol{\psi}}\right) = \boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}],$$

or

$$\widetilde{\boldsymbol{\psi}}(\boldsymbol{x},t) = e^{t\widetilde{A}}\boldsymbol{\psi}_0(\boldsymbol{x},t) + \int_0^t e^{(t-s)\widetilde{A}}\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\cdot,s)ds,$$

where

$$e^{t\widetilde{A}}\boldsymbol{\psi}_0 = \frac{1}{(2\pi(1-e^{-t}))^{3/2}}\int_{\mathbb{R}^3}\boldsymbol{\psi}_0(\boldsymbol{y})\exp\left(-\frac{|e^{-\frac{t}{2}}\boldsymbol{x}-\boldsymbol{y}|^2}{2(1-e^{-t})}\right)\boldsymbol{dy}$$

denotes the Ornstein-Uhlenbeck semi-group (see **Appendix A**). We can put the above form of the equations into a path integral

$$\widetilde{\boldsymbol{\psi}}(\boldsymbol{x},t) = E[\boldsymbol{\psi}_0(\boldsymbol{X}_t)] + \int_0^t E[\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{X}_s,t-s)]ds, \quad \text{for} \;\; t \geq 0, \tag{22}$$

where $\widetilde{\boldsymbol{\psi}}(\boldsymbol{x},0) = \boldsymbol{\psi}(\boldsymbol{x},0) = \boldsymbol{\psi}_0(\boldsymbol{x})$ is the common initial condition. Here $\boldsymbol{X}_t$ denotes the Ornstein-Uhlenbeck process which satisfies

$$d\boldsymbol{X}_t = -\frac{1}{2}\boldsymbol{X}_t dt + d\boldsymbol{W}_t.$$

It should be noted that the equations (20) and (22) take the identical form, except for a difference in the underlying stochastic processes used to evaluate the expectation values. We note that this can be achieved only when we use critical dependent variables. Using vector potentials and path integral representations, the Navier-Stokes and Leray equations have exactly the same form, *modulo* stochastic processes. We will refer to this fact *near-invariance under dynamic scaling* and discuss its implications that it entails below.

We also note that the behaviours of the solutions to those equations are markedly different; a finite-time blowup for one and global regularity for the other. However, generally speaking, the more similar two equations look, the harder it is for their solutions to behave in completely different manners.

4. Probabilistic approach

4.1. Maruyama-Girsanov and related theorems

Consider the Navier-Stokes equations in path integral representation. Assuming that a solution to the Navier-Stokes equations in $\mathbb{R}^3$ breaks down in finite time, by probabilistic methods we aim to study how and under what conditions we can rule out blowup by contradiction.

The main tool we will utilise is the Maruyama-Girsanov and related theorems (**Appendix B**), see e.g. [23, 18, 26, 20, 4, 49, 42, 2, 1, 28, 30] for details. For their other applications to fluid mechanics, see e.g. [5, 15]. See also [37, 54, 63, 62, 50, 41, 31, 32, 33, 56] for stochastic analysis in general. These theorems are often applied in the following forms (e.g. see [1, 62]),
the Cameron-Martin theorem:

$$\int F(\boldsymbol{W}+\boldsymbol{h})\mu(d\boldsymbol{W}) = \int F(\boldsymbol{W}) \exp\left(\int_0^t \dot{\boldsymbol{h}}(s)\cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\dot{\boldsymbol{h}}(s)|^2 ds\right)\mu(d\boldsymbol{W}), \tag{23}$$

the Maruyama-Girsanov theorem:

$$\int F(\boldsymbol{W})\mu(d\boldsymbol{W}) = \int F(\boldsymbol{W}+\boldsymbol{h}) \exp\left(-\int_0^t \dot{\boldsymbol{h}}(s)\cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\dot{\boldsymbol{h}}(s)|^2 ds\right)\mu(d\boldsymbol{W}), \tag{24}$$

where $\mu(d\boldsymbol{W})$ denotes a Wiener measure, F an arbitrary function and $\boldsymbol{h}$ drift.

The simplest, but nevertheless instructive example of their applications is the following. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u,$$

whose solution can be represented by a Wiener integral (i.e. an average of Wiener functional) as

$$u(\boldsymbol{x},t) = E\left[u_0(\boldsymbol{W}_t)\right], \tag{25}$$

which is globally smooth. If we consider the other equation with a drift term

$$\frac{\partial u}{\partial t} = \boldsymbol{b}(\boldsymbol{x})\cdot\nabla u + \frac{1}{2}\triangle u, \tag{26}$$

it can be solved as

$$u(\boldsymbol{x},t) = E\left[u_0(\boldsymbol{X}_t)\right], \tag{27}$$

$$= E\left[u_0(\boldsymbol{W}_t)G_t\right] \quad \text{for } \ 0 \le t < {}^{\exists}T, \tag{28}$$

where $\boldsymbol{X}_t = \boldsymbol{W}_t + \boldsymbol{h}(t),\ \boldsymbol{h}(t) = \int_0^t \boldsymbol{b}(\boldsymbol{W}(s))ds$, and $G_t = \exp\left(\int_0^t \boldsymbol{b}(\boldsymbol{W}_s)\cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right)$ is the Maruyama-Girsanov density. The time T is chosen to satisfy the so-called Novikov condition (see below).

It should be noted that we can work backward; starting from the smooth expression (25) and we can assert after inserting G_t that $E\left[u_0(\boldsymbol{W}_t)G_t\right]$ remains smooth at least for $0 \le t < T$. Then, tracing the transformations (28) $\to$ (27) backward, we see that it solves the second equation (27) with the drift.

To be used in what follows, we make a note of the following

Remark 4.1. *Given a Wiener integral which is smooth for all time, under an absolute continuous change in the probability measure, it remains smooth on a time interval subject to the Novikov condition.*

4.2. Comparison of equations

Using those tools, we compare the equations before and after dynamic scaling. We begin by checking the Novikov condition. Taking $\boldsymbol{b}(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{x}$, we compute the Maruyama-Girsanov density as

$$G_t = \exp\left(-\frac{1}{2}\int_0^t \boldsymbol{W}_s \cdot d\boldsymbol{W}_s - \frac{1}{8}\int_0^t |\boldsymbol{W}_s|^2 ds\right)$$
$$= \exp\left(-\frac{1}{4}(|\boldsymbol{W}_t|^2 - t) - \frac{1}{8}\int_0^t |\boldsymbol{W}_s|^2 ds\right),$$

where the first term in the exponent is a result of Itô calculus. It is important to estimate how long T can be. The Novikov condition

$$E\left[\exp\left(\frac{1}{2}\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right)\right] < \infty,$$

which assures G_t to serve as a martingale, becomes

$$E\left[\exp\left(\frac{1}{8}\int_0^t |\boldsymbol{W}_s|^2 ds\right)\right] = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} d\boldsymbol{\eta} \exp\left(-\frac{|\boldsymbol{\eta}|^2}{2} + \frac{1}{8}\int_0^t |\boldsymbol{x} - \sqrt{s}\boldsymbol{\eta}|^2 ds\right) < \infty.$$

The dangerous contribution comes from quadratic term in the exponent and is given by

$$\exp\left\{\left(-1 + \frac{t^2}{8}\right)\frac{|\boldsymbol{\eta}|^2}{2}\right\},$$

from which we conclude that the Maruyama-Girsanov transform is valid at least up to

$$T = 2\sqrt{2}.$$

Clearly, the same estimate is equally valid for the counterpart density

$$\widehat{G}_t = \exp\left(\frac{1}{2}\int_0^t \boldsymbol{W}_s \cdot d\boldsymbol{W}_s - \frac{1}{8}\int_0^t |\boldsymbol{W}_s|^2 ds\right).$$

On this basis, we state

Proposition 4.1. *The difference between the Navier-Stokes and Leray equations in their path integral representations lies only in the absence or presence of the Maruyama-Girsanov densities.*

Proof. We apply the Maruyama-Girsanov theorem to the two different terms of (22). Taking components of $\boldsymbol{\psi}_0$ or $\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}]$ as F in $E[F(\boldsymbol{X}_t)] = E[F(\boldsymbol{W}_t)G_t]$, we have

$$E[\boldsymbol{\psi}_0(\boldsymbol{X}_t)] = E[\boldsymbol{\psi}_0(\boldsymbol{W}_t)G_t] \text{ for } 0 \le t < T,$$

$$E[\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{X}_s, t-s)] = E[\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{W}_s, t-s)G_s] \text{ for } 0 \le s \le t, \text{ for fixed } t(< T).$$

In the first line we make use of the fact that the initial condition $\boldsymbol{\psi}_0$ is smooth. In the second, we make use of the fact that the nonlinear term in the Leray equations is smooth all the time, as a result of the assumed blow up followed by dynamic scaling.

Thus, the Leray equations can be written (see (23)),

$$\begin{aligned}
&\widetilde{\boldsymbol{\psi}}(\boldsymbol{x},t)\\
&= E_{\boldsymbol{P}}[\boldsymbol{\psi}_0(\boldsymbol{X}_t)] + \int_0^t E_{\boldsymbol{P}}[\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{X}_s, t-s)]ds \ \text{ for } \ t \geq 0, \quad (29)\\
&= E_{\boldsymbol{P}}[\boldsymbol{\psi}_0(\boldsymbol{W}_t)G_t] + \int_0^t E_{\boldsymbol{P}}[\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{W}_s, t-s)G_s]ds \text{ for } 0 \leq t < T, \quad (30)
\end{aligned}$$

where $T = 2\sqrt{2}$. We recall that the nonlinear term $\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}](\boldsymbol{x},t)$ is a smooth function of $\boldsymbol{x}$ and t by the assumed blowup and rescaling. Here we make explicit the kind of probability measure used in the evaluation of expectation values. (See Subsection 3.1 for the definition $E_{\boldsymbol{P}}$.)

For the Navier-Stokes equations, we find accordingly (see (24)),

$$\begin{aligned}
&\boldsymbol{\psi}(\boldsymbol{x},t)\\
&= E_{\boldsymbol{P}}[\boldsymbol{\psi}_0(\boldsymbol{W}_t)] + \int_0^t E_{\boldsymbol{P}}[\boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{W}_s, t-s)]ds \text{ for } 0 \leq t < 1, \quad (31)\\
&= E_{\boldsymbol{P}}[\boldsymbol{\psi}_0(\boldsymbol{X}_t)\widehat{G}_t] + \int_0^t E_{\boldsymbol{P}}[\boldsymbol{T}[\nabla\boldsymbol{\psi}](\boldsymbol{X}_s, t-s)\widehat{G}_s]ds \ \text{ for } \ 0 \leq t < 1. \quad (32)
\end{aligned}$$

See **Appendix B** for notations and a sketch of derivations.

Comparing (30) and (31), we confirm that the difference between the Navier-stokes and Leray equations lies in G_t only. Alternatively, by comparing (29) and (32), we confirm the same with $\widehat{G}_t$ only. □

Because $T = 2\sqrt{2} > 1 = t_*$, we note the drastic roles that G_t and $\widehat{G}_t$ play regarding the property of equations. By definition we note that if $\boldsymbol{X}_t$ is replaced by $\boldsymbol{W}_t$ in (29), the solution to the equation (31) becomes non-smooth at $t = 1$. Moreover, we observe that‖

(i) if $\boldsymbol{P}$ is replaced by $\int G_t d\boldsymbol{P}$ in (31), ψ recovers smoothness; cf. (30) and (31),

(ii) if $\boldsymbol{P}$ is replaced by $\int \widehat{G}_t d\boldsymbol{P}$ in (29), $\widetilde{\psi}$ becomes non-smooth; cf. (29) and (32).

There are two path integral equations defined with probability measures that are mutually absolutely continuous. One has a short-lived solution by assumption and the other a globally smooth solution as a result of scaling. At the level of equations, the Leray equations, which can be written similar to the Navier-Stokes equations with G_t, have a smooth solution for $0 \leq t < 2\sqrt{2}$, whereas the corresponding Navier-Stokes equations cease to have a smooth solution already at $t = 1$.¶ This looks a bit strange as it is not expected that the Maruyama-Girsanov density makes such a drastic change,

‖ Here $Q = \int G_t d\boldsymbol{P}$ means $Q(A) = E_{\boldsymbol{P}}[\mathbf{1}_A G_t]$, where $\mathbf{1}_A$ is an indicator function for a Borel set A.

¶ If we start from the Navier-Stokes equations written in dimensional variables, the same conclusion comes out, as it should (see **Appendices C**.)

see e.g. section 4.1 of [49]. However, no immediate conclusion can be drawn from this observation. We will investigate the properties of their solutions in the next subsection.

4.3. Comparison of solutions

We now take a look at what is happening at the level of solutions. In the path integral formulation, the Leray equations are determined when we specify the Ornstein-Uhlenbeck process $\boldsymbol{X}_t$, the probability measure $\boldsymbol{P}$ and the initial data $\boldsymbol{\psi}_0$. Hence its globally smooth solution must also be determined by a combination of $\boldsymbol{X}_t$, $\boldsymbol{P}$ and $\boldsymbol{\psi}_0$. We can thus write in principle

$$\widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t) = \boldsymbol{R}(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0),$$

where

$$\lim_{t\to 0} \boldsymbol{R}(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0) = \boldsymbol{\psi}_0.$$

Now let us consider what $\boldsymbol{R}$ means. Because $\boldsymbol{X}_t$ is an infinite-dimensional quantity (essentially, Brownian motion) while $\widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t)$ is a finite-dimensional quantity (just a vector field in (3+1)-dimensions), we can in principle represent $\widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t)$ as a finite-dimensional projection $\boldsymbol{R}$ of some functional, which depends on $\boldsymbol{X}_t$, $\boldsymbol{P}$ and $\boldsymbol{\psi}_0$. The projection $\boldsymbol{R}$ also depends on $\boldsymbol{T}[\nabla\widetilde{\boldsymbol{\psi}}]$, but that dependence can be subsumed into the arbitrariness of $\boldsymbol{R}$. We will show how $\boldsymbol{R}$ allows a construction by a successive approximation in **Proposition** 4.3. Typically, the projection can be achieved e.g. by a composite function of Wiener integrals.

It is important to distinguish the following two cases:

- (A) $\boldsymbol{R}$ given by a finite number of Wiener integrals,
- (B) otherwise.

Because the Navier-Stokes and Leray equations are nonlinear, successive approximations (i.e. Picard iterations) contain in principle infinitely many Wiener integrals (see **Proposition** 4.3). This is the reason why we need to consider (B) separately. For the case of the Burgers equations, we know its explicit form of the exact solution by a finite number of Wiener integrals, because of the Cole-Hopf transform (see **Appendix D**).

By changing stochastic processes and probability measures, the Cameron-Martin theorem states

$$\begin{aligned}\widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t) &= \boldsymbol{R}(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0) \text{ for } t \geq 0 &(33)\\ &= \boldsymbol{R}\left(\boldsymbol{W}_t, \int G_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \text{ for } 0 \leq t < 2\sqrt{2}, &(34)\end{aligned}$$

Here, it should be noted that $\boldsymbol{R}$ is *not* necessarily given by a composite function of a finite number of Wiener integrals.

Using the same $\boldsymbol{R}$, we can also write $\boldsymbol{\psi}(\boldsymbol{x}, t)$ as a finite-dimensional projection of some functional of $\boldsymbol{W}_t$, which depends on $\boldsymbol{P}$ and $\boldsymbol{\psi}_0$. By the Maruyama-Girsanov theorem

$$\boldsymbol{\psi}(\boldsymbol{x},t) = \boldsymbol{R}(\boldsymbol{W}_t, \boldsymbol{P}, \boldsymbol{\psi}_0) \text{ for } 0 \leq t < 1, \tag{35}$$

$$= \boldsymbol{R}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \text{ for } 0 \leq t < 1. \tag{36}$$

We know some properties that the projection $\boldsymbol{R}$ must satisfy. By definition, if $\boldsymbol{X}_t$ in (33) is replaced by $\boldsymbol{W}_t$, $\boldsymbol{R}$ becomes non-smooth at $t = 1$. We also know that:

(i) if $\boldsymbol{P}$ is replaced by $\int G_t d\boldsymbol{P}$ in (35), $\boldsymbol{\psi}$ recovers smoothness; cf. (34) and (35),

(ii) if $\boldsymbol{P}$ is replaced by $\int \widehat{G}_t d\boldsymbol{P}$ in (33), $\widetilde{\boldsymbol{\psi}}$ becomes non-smooth; cf. (33) and (36).

The above properties again show how remarkable a role that the Maruyama-Girsanov densities play. Addition or removal of these martingales affects the property of the solutions drastically, that is, in the presence of G_t, a solution to the Leray equations remains smooth beyond $t = 1$, at which the corresponding solution to the Navier-Stokes equations breaks down. Alternatively, in the presence of $\widehat{G}_t$, a solution to the Navier-Stokes equations becomes non-smooth at $t = 1$, beyond which the corresponding solution to the Leray equations remains smooth. For a better understanding of those observations, we recall the **Remark** 4.1 at the end of subsection 4.1. Assuming (37) below, we trace the footsteps we have taken as (36) $\to$ (35), which are valid on a longer time interval, to conclude that $\boldsymbol{\psi}(\boldsymbol{x},t)$ is smooth for $0 \leq t < 2\sqrt{2}$. This contradicts with the assumed blowup at $t = 1$. Hence, no blowup is possible under the condition (37).

Remark 4.2. *In (33) we know that*

$$\boldsymbol{R}(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0) \;\; \textit{is smooth for} \;\; {}^{\forall}t \geq 0.$$

If it remains smooth under the absolute continuous change of the measure, that is, if

$$\boldsymbol{R}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \quad \textit{is smooth for} \;\; 0 \leq t < 2\sqrt{2}, \tag{37}$$

then no solutions of the Navier-Stokes equations blow up.

Note that the number $2\sqrt{2}$ above can be replaced by any number (> 1). For example, a weaker condition

$$E\left[\exp\left((1+\epsilon)\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right)\right] < \infty \text{ with } \epsilon > 0,$$

discussed e.g. in [23], gives rise to $0 \leq t < 2$. This is good enough to obtain the same conclusion as $2 > 1$.

The next task is to check whether and how (37) is satisfied in the two different cases (A) and (B) above. We first rule out blowup in the case (A), which is straightforward.

Proposition 4.2. *If $\boldsymbol{R}$ is given by a composite function of a finite number of Wiener integrals, the condition (37) is satisfied.*

Proof. For simplicity, assume that a globally smooth $\boldsymbol{R}$ is given by a single Wiener integral of the form

$$\boldsymbol{R}\,(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0) = E_{\boldsymbol{P}}[\boldsymbol{F}(\boldsymbol{X}_t)].$$

We then consider whether

$$\boldsymbol{R}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) = E_{\boldsymbol{P}}[\boldsymbol{F}(\boldsymbol{X}_t)\widehat{G}_t]$$

is smooth with respect to spatial and time variables. We note that $E_{\boldsymbol{P}}[\boldsymbol{F}(\boldsymbol{X}_t)]$ solves (26) with $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$. With the same choice of $\boldsymbol{F}$, $E_{\boldsymbol{P}}[\boldsymbol{F}(\boldsymbol{W}_t)]$ solves (25) and equals $E_{\boldsymbol{P}}[\boldsymbol{F}(\boldsymbol{X}_t)\widehat{G}_t]$ for $0 \le t < T$ because of the Maruyama-Girsanov theorem. We deduce that $E_{\boldsymbol{P}}[\boldsymbol{F}(\boldsymbol{X}_t)\widehat{G}_t]$ is smooth as long as $\widehat{G}_t$ serves as a martingale. The same conclusion holds for more general cases of composite functions of a finite number of Wiener integrals, by making changes to the measures one by one. □

This proves absence of blowup in the case (A). If the Navier-Stokes equations (hence also the Leray equations) are linearisable to e.g. the heat equations, then $\boldsymbol{R}$ consists of a finite number of Wiener integrals. However, because it is not known whether the converse holds true or not, the case (A) is nontrivial.

In order to handle the case (B), we begin by noting

Proposition 4.3. *For all $n \ge 0$, the successive approximants $\widetilde{\boldsymbol{\psi}}^{(n)}(\boldsymbol{x}, t)$ are given by a composite function made up of a finite number of Wiener integrals.*

Proof. We prove this by mathematical induction.
We define the approximations $\widetilde{\boldsymbol{\psi}}^{(n)}(\boldsymbol{x}, t)$ by the following Picard iteration scheme for the Leray equations

$$\widetilde{\boldsymbol{\psi}}^{(n+1)}(\boldsymbol{x}, t) = E_{\boldsymbol{P}}[\boldsymbol{\psi}_0(\boldsymbol{X}_t)] + \int_0^t E_{\boldsymbol{P}}[\boldsymbol{T}[\nabla \widetilde{\boldsymbol{\psi}}^{(n)}](\boldsymbol{X}_s, t-s)]ds, \quad \text{for } \; n = 0, 1, 2, \dots \tag{38}$$

together with the initial step

$$\widetilde{\boldsymbol{\psi}}^{(0)}(\boldsymbol{x}, t) = E_{\boldsymbol{P}}[\boldsymbol{\psi}_0(\boldsymbol{X}_t)]. \tag{39}$$

For $n = 0$, the statement is true by (39). Assume it is true for $n = k \ge 0$, then by (38), $\widetilde{\boldsymbol{\psi}}^{(k+1)}(\boldsymbol{x}, t)$ is also made up of a finite number of Wiener integrals. Hence the statement is true for all $n \ge 0$. □

This iteration scheme is convergent $\lim_{n\to\infty} \widetilde{\boldsymbol{\psi}}^{(n)}(\boldsymbol{x}, t) = \widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t)$ for all $t > 0$, because $\widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t) = \boldsymbol{R}(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0)$ is smooth for all time and satisfies (29) and hence the limit exists. This argument is, however, vague in that little is known about the nature of the convergence; we do not know precisely in what sense it converges, but we do know the limit is smooth. (In view of $\|\nabla\widetilde{\psi}\|_{L^3} \to \infty$ as $t \to \infty$, the convergence is not expected to be uniform in t.)

Now we introduce the approximants $\boldsymbol{R}^{(n)}\,(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0)$ by defining $\boldsymbol{R}^{(n)}\,(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0) \equiv \widetilde{\boldsymbol{\psi}}^{(n)}(\boldsymbol{x}, t)$, to prove

Proposition 4.4. *For* $^\forall n \geq 0$,

$$\boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0\right) \quad \textit{is smooth for } {}^\forall t \geq 0$$
$$\Longrightarrow \boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \quad \textit{is smooth for } 0 \leq t < 2\sqrt{2}.$$

Proof. It follows by combining **Propositions** 4.2 and 4.3. □

It should be noted that the upper-end $2\sqrt{2}$ of the time interval does *not* depend on n. We also note that for finite n, the smoothness of $\widetilde{\boldsymbol{\psi}}^{(n)}(\boldsymbol{x}, t)$ itself follows from the linearity of the equations in the Picard iterations.

We consider a condition under which a contradiction can be obtained for the case (B). One possibility is to accept the following passing to the limit procedure. Because **Proposition** 4.4 holds for all n and the time intervals are independent of n, if we formally pass to the limit of $n \to \infty$, we find

$$\lim_{n\to\infty} \boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0\right) \text{ is convergent and smooth for } {}^\forall t \geq 0$$
$$\Longrightarrow \lim_{n\to\infty} \boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \text{ is convergent and smooth for } 0 \leq t < 2\sqrt{2}.$$

While the details of its convergence are not known, the former limit does exist for all time by the global smoothness of the solution to the Leray equations. Therefore, under the condition the latter also holds. The function $\boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right)$ defines the approximants for the Navier-Stokes equations. The details of its convergence are not known either, but we know that it converges to a smooth function at least for $0 \leq t < 1$.

By **Remark** 4.2 we note the following

Remark 4.3. *If the passage to the limit* $n \to \infty$ *is accepted in* **Proposition** *4.4, we can rule out blowup of solutions to the Navier-Stokes equations in* $\mathbb{R}^3$.

It is yet to be checked whether the Navier-Stokes equations actually satisfy this condition or not. (See Section 6 for discussion.)

$$\begin{array}{ccc}
\text{Leray equations} & \longrightarrow & \text{Navier-Stokes equations} \\
\widetilde{\boldsymbol{\psi}}^{(n)}(\boldsymbol{x}, t) = \boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0\right) & \xrightarrow{(c)} & \boldsymbol{\psi}^{(n)}(\boldsymbol{x}, t) = \boldsymbol{R}^{(n)}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \\
\text{smooth for } {}^\forall t \geq 0 & & \text{smooth for } 0 \leq t < 2\sqrt{2} \\
\Big\downarrow {\scriptstyle (a)\ n\to\infty} & & \Big\downarrow {\scriptstyle (d)\ n\to\infty} \\
\widetilde{\boldsymbol{\psi}}(\boldsymbol{x}, t) = \boldsymbol{R}\left(\boldsymbol{X}_t, \boldsymbol{P}, \boldsymbol{\psi}_0\right) & \xrightarrow{(b)} & \boldsymbol{\psi}(\boldsymbol{x}, t) = \boldsymbol{R}\left(\boldsymbol{X}_t, \int \widehat{G}_t d\boldsymbol{P}, \boldsymbol{\psi}_0\right) \\
\text{smooth for } {}^\forall t \geq 0 & & \text{smooth for } 0 \leq t < 2\sqrt{2} \text{ (target)}
\end{array}$$

Figure 1: Exchanging the orders of the iteration procedure ($n \to \infty$) and the transformation of the measure. To consider (a) followed by (b), we first assert (c) and then take the limit of (a) to let (b) follow. This should be distinguished from considering (c), followed by taking the limit (d) on its own.

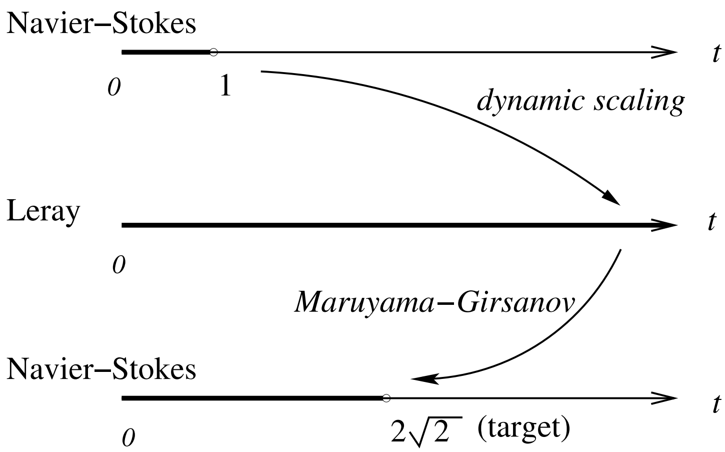

Figure 2. The Maruyama-Girsanov theorem retrieves the short-lived solution of the Navier-Stokes equations as a pull-back from the long-lived solution of the Leray equations. If the pull-back outlives the original local solution under the limit passage, it gives rise to a contradiction.

Let us summarise our approach here schematically. There are two operations; the absolute continuous change of the measure and the passage to the limit of large n. **Figure** 1 shows how we may retrieve the solution of the Navier-Stokes equations from that of the Leray equations, by exchanging the iteration procedure and the transformation of the measure. It should be noted that the limiting procedure in **Remark** 4.3 differs from a combination of (c) followed by the limit (d) on its own. Here we assert (c) first and then aim to claim (b) via the limit (a). Also, in **Figure** 2 we show how a possible contradiction is brought about by the reconstructed solution remaining smooth beyond $t = 1$, at which the original local solution blows up.

5. Navier-Stokes equations in d-dimensions

The above argument equally works in any other spatial dimensions i.e. $d = 2$ and $d \geq 4$. To see this, it is sufficient to observe the following two facts.

1) The equation for the stream function ψ ($d = 2$) and those for the tensor potentials ($d \geq 4$) take the same structure. Indeed, we have for $d = 2$

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\triangle\right)\psi = \frac{1}{\pi}\fint_{\mathbb{R}^2} \frac{(\boldsymbol{r} \times \nabla\psi(\boldsymbol{y}))\boldsymbol{r} \cdot \nabla\psi(\boldsymbol{y})}{|\boldsymbol{r}|^4}\,\mathrm{d}\boldsymbol{y},$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, [43]. More generally, the equations for the tensorial potentials $\boldsymbol{\psi} = (\psi_{ij}),\ i, j = 1, 2, \dots, d$ can be written (see **Proposition** 5.1 for derivations)

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = \boldsymbol{T}[\nabla\boldsymbol{\psi}] + \frac{1}{2}\triangle\boldsymbol{\psi},$$

where

$$T_{ij}[\nabla\boldsymbol{\psi}] \equiv -\frac{1}{\sigma_d}\fint_{\mathbb{R}^d}\left(\frac{\delta_{ki}}{r^d}-\frac{r_k\,r_i}{r^{d+2}}\right)\frac{\partial\psi_{kl}}{\partial y_l}\,\frac{\partial\psi_{jm}}{\partial y_m}d\boldsymbol{y} + \frac{1}{\sigma_d}\fint_{\mathbb{R}^d}\left(\frac{\delta_{kj}}{r^d}-\frac{r_k\,r_j}{r^{d+2}}\right)\frac{\partial\psi_{kl}}{\partial y_l}\,\frac{\partial\psi_{im}}{\partial y_m}d\boldsymbol{y},$$

with $\sigma_d = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$.

2) The heat kernel in d-dimensions

$$\frac{1}{(2\pi t)^{d/2}}\exp\left(-\frac{|\boldsymbol{x}|^2}{2t}\right) \tag{40}$$

depends on the spatial dimension d in its prefactor, but *not* in its exponent. Hence the Novikov condition yields the same estimate $T = 2\sqrt{2}$ in any spatial dimensions.

We conclude that

Collorary 5.1. Proposition *4.4 holds equally in $\mathbb{R}^d$. If the passage to the limit $n\to\infty$ in* **Remark** *4.3 is accepted, we can rule out blowup of solutions to the Navier-Stokes equations in $\mathbb{R}^d$ $(d\geq 2)$.*

Proposition 5.1. *For the Navier-Stokes equations in $\mathbb{R}^d$*

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{1}{2}\triangle u_i,\ \ (i=1,2,\dots,d) \tag{41}$$

The equations for the vorticity tensor $\omega_{ij} = \dfrac{\partial u_j}{\partial x_i} - \dfrac{\partial u_i}{\partial x_j}$ and the tensor potential ψ_{ij} are given $(i,j=1,2,\dots,d)$, respectively, by

$$\frac{\partial \omega_{ij}}{\partial t} + u_k\frac{\partial \omega_{ij}}{\partial x_k} = \omega_{jk}\frac{\partial u_k}{\partial x_i} - \omega_{ik}\frac{\partial u_k}{\partial x_j} + \frac{1}{2}\triangle\omega_{ij} \tag{42}$$

and

$$\frac{\partial\psi_{ij}}{\partial t} = -\frac{1}{\sigma_d}\fint_{\mathbb{R}^d}\left(\frac{\delta_{ki}}{r^d}-\frac{(x_k-y_k)(x_i-y_i)}{r^{d+2}}\right)\frac{\partial\psi_{kl}}{\partial y_l}\,\frac{\partial\psi_{jm}}{\partial y_m}d\boldsymbol{y} + \frac{1}{\sigma_d}\fint_{\mathbb{R}^d}\left(\frac{\delta_{kj}}{r^d}-\frac{(x_k-y_k)(x_j-y_j)}{r^{d+2}}\right)\frac{\partial\psi_{kl}}{\partial y_l}\,\frac{\partial\psi_{im}}{\partial y_m}d\boldsymbol{y} + \frac{1}{2}\triangle\psi_{ij}, \tag{43}$$

where $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{y}$, $u_k = \frac{\partial\psi_{kl}}{\partial x_l}$ and $\omega_{ij} = -\triangle\psi_{ij}$.

Proof. By taking cross-derivatives, we find

$$\frac{\partial \omega_{ij}}{\partial t} + u_k\frac{\partial \omega_{ij}}{\partial x_k} + \underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k}}_{=I} = \frac{1}{2}\triangle\omega_{ij}. \tag{44}$$

Noting

$$I = \frac{\partial u_k}{\partial x_i}\left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\right) - \frac{\partial u_k}{\partial x_j}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)$$

we obtain

$$\frac{\partial \omega_{ij}}{\partial t} + u_k \frac{\partial \omega_{ij}}{\partial x_k} = \omega_{jk} \frac{\partial u_k}{\partial x_i} - \omega_{ik} \frac{\partial u_k}{\partial x_j} + \frac{1}{2} \triangle \omega_{ij}. \tag{45}$$

On the other hand, by writing

$$I = \frac{\partial}{\partial x_k} \left(\frac{\partial u_k}{\partial x_i} u_j \right) - \frac{\partial}{\partial x_k} \left(\frac{\partial u_k}{\partial x_j} u_i \right) = \frac{\partial^2 (u_k u_j)}{\partial x_k \partial x_i} - \frac{\partial^2 (u_k u_i)}{\partial x_k \partial x_j} - \frac{\partial (u_k \omega_{ij})}{\partial x_k},$$

we find

$$\frac{\partial \omega_{ij}}{\partial t} + \frac{\partial^2 (u_k u_j)}{\partial x_k \partial x_i} - \frac{\partial^2 (u_k u_i)}{\partial x_k \partial x_j} = \frac{1}{2} \triangle \omega_{ij}.$$

By applying $(-\triangle)^{-1}$, we get

$$\frac{\partial \psi_{ij}}{\partial t} - R_k R_i (u_k u_j) + R_k R_j (u_k u_i) = \frac{1}{2} \triangle \psi_{ij},$$

or

$$\frac{\partial \psi_{ij}}{\partial t} = R_k R_i (\partial_l \psi_{kl} \cdot \partial_m \psi_{jm}) - R_k R_j (\partial_l \psi_{kl} \cdot \partial_m \psi_{im}) + \frac{1}{2} \triangle \psi_{ij}.$$

Using a formula for the second-order derivatives

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{d} f(\boldsymbol{x}) + \frac{1}{\sigma_d} \fint_{\mathbb{R}^d} \left(\frac{\delta_{ij}}{r^d} - \frac{(x_i - y_i)(x_j - y_j)}{r^{d+2}} \right) f(\boldsymbol{y}) d\boldsymbol{y}$$

for a potential ϕ satisfying $\triangle \phi = f$, we obtain the required form. □

It should be noted that

$$\partial_i \partial_j \left(R_k R_i (\partial_l \psi_{kl})(\partial_m \psi_{jm}) - R_k R_j (\partial_l \psi_{kl})(\partial_m \psi_{im}) \right) = 0$$

is satisfied consistently by anti-symmetry to ensure the incompressible condition.

6. Summary and discussion

We have introduced the notion of *near-invariance under dynamic scaling* for the Navier-Stokes equations, which is available only when we employ the critical dependent variables and path integral representations. On this basis, using probabilistic methods we have discussed under which conditions global regularity for the Navier-Stokes equations is deduced by contradiction.

What has been done are summarised as follows. Applying dynamic scaling to the Navier-Stokes equations written in the vector potentials, we obtain the Leray equations. Recasting them in path integral forms, we make their differences to a bare minimum. Using the Maruyama-Girsanov technique as a pull-back we remove the effect of the drift term, thereby retrieving a short-lived solution of the Navier-Stokes equations from a long-lived solution of the Leray equations. There are two cases.

(A) If the reconstruction consists of a finite number of Wiener integrals, we get a contradiction, because it outlives the local solution of the Navier-Stokes equations.

(B) In more general cases, we have proved that the Picard approximants for the Leray equations at each order are made up of Wiener integrals. After transformation of the measures, the corresponding Navier-Stokes approximants remain smooth for $t < 2\sqrt{2}$ by the Novikov condition. This suggests that there is room that we may still get a contradiction.

One approach is to accept the limit passage in the Picard iteration scheme. This limit passage, however, seems to be a delicate procedure, which depend on the properties of the Navier-Stokes equations other than scale-invariance. The subtlety can be seen in the fact that blowup has been proven for the modified Navier-Stokes equations which respect scale-invariance, with the energy inequality [59], or without [22].

In order to pursue the current approach, what needs to be studied in connection with **Remark** 4.3 is the following. It is important to discern for which kind of nonlinear terms, if any, out of many possible modified equations, the limit of Picard iterations inherits the property of the approximants.

A few more remarks of general nature may be in order. In analysis, there is a kind of things that we can transform, i.e. "variables." In probability theory, on top of that there are "measures," with which we can play different games. They add additional richness to the study of PDEs, in this case specifically enabling us to close the round-trip in a nontrivial fashion; we use dynamic scaling for the outbound and the Maruyama-Girisanov technique for the inbound in **Figure** 2.

It is of interest to compare the property of the norm $\|\boldsymbol{u}\|_{L^d}$ with that of the dependent variable $\boldsymbol{\psi}$. Under dynamic scaling, the governing equations (12,13) for the L^d-norm are totally indiscernible while those (15,17) for the $\boldsymbol{\psi}$ are marginally discernible, because the unknown $\boldsymbol{\psi}$ carries the full information of the solutions. By sweeping the minimal difference under the rug, we have shown how we may possibly disclose a contradiction out of the blowup assumption.

Not a single bound (i.e. interpolation inequality) has been used in the argument so far. This may explain, at least partially, why the spatial dimensions is irrelevant here. The current argument does not hold under periodic boundaries, or on bounded domains, because dynamically-scaled equations are not available under such circumstances.

Appendix A. Heat kernel and Mehler formula

These matters are trivial, but best stated here in dimensional form for convenience.

The heat equation
$$\frac{\partial u}{\partial t} = \nu \triangle u$$
can be solved by a heat kernel
$$u(\boldsymbol{x}, t) = \frac{1}{(4\pi\nu t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu t}\right) u_0(\boldsymbol{y}) d\boldsymbol{y}$$
$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|\boldsymbol{\eta}|^2}{2}\right) u_0(\boldsymbol{x} - \sqrt{2\nu t}\boldsymbol{\eta}) d\boldsymbol{\eta} = E_{\boldsymbol{P}}[u_0(\boldsymbol{W}_t)],$$

where $\boldsymbol{\eta} = (\boldsymbol{x} - \boldsymbol{y})/\sqrt{2\nu t}$. Suggests a convenient replacement rule $\boldsymbol{W}_t \longrightarrow \boldsymbol{x} - \sqrt{2\nu t}\boldsymbol{\eta}$ in the last expression.

There is a counterpart for the modified heat kernel

$$\frac{\partial u}{\partial t} + a\boldsymbol{x} \cdot \nabla u = \nu \triangle u.$$

By

$$\boldsymbol{X} = e^{-at}\boldsymbol{x}, T = \frac{1 - e^{-2at}}{2a},$$

we reduce it to

$$\frac{\partial \widetilde{u}}{\partial T} = \nu \triangle_{\boldsymbol{X}} \widetilde{u},$$

where $\widetilde{u}(\boldsymbol{X}, T) = u(\boldsymbol{x}, t)$. Hence we have

$$\widetilde{u}(\boldsymbol{X}, T) = \frac{1}{(4\pi\nu T)^{d/2}} \int_{\mathbb{R}^d} u_0(\boldsymbol{y}) \exp\left(-\frac{|\boldsymbol{X} - \boldsymbol{y}|^2}{4\nu T}\right) \boldsymbol{dy},$$

or, in terms of the original variables,

$$u(\boldsymbol{x}, t) = \frac{1}{\left\{\frac{2\pi\nu}{a}(1 - e^{-2at})\right\}^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{a|e^{-at}\boldsymbol{x} - \boldsymbol{y}|^2}{2\nu(1 - e^{-2at})}\right) u_0(\boldsymbol{y})\boldsymbol{dy}.$$

By $\boldsymbol{\eta} = \sqrt{\frac{a}{\nu}} \frac{e^{-at}\boldsymbol{x} - \boldsymbol{y}}{\sqrt{1 - e^{-2at}}}$, we can also write

$$\begin{aligned} u(\boldsymbol{x}, t) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|\boldsymbol{\eta}|^2}{2}\right) u_0\left(e^{-at}\boldsymbol{x} - \boldsymbol{\eta}\sqrt{\frac{\nu}{a}(1 - e^{-2at})}\right) \boldsymbol{d\eta}, \\ &= E_{\boldsymbol{P}}[u_0(\boldsymbol{X}_t)], \end{aligned}$$

which is known as Mehler formula and suggests a replacement rule $\boldsymbol{X}_t \longrightarrow e^{-at}\boldsymbol{x} - \boldsymbol{\eta}\sqrt{\frac{\nu}{a}(1 - e^{-2at})}$. Finally we may equivalently write using stochastic variables

$$\boldsymbol{X}_t = e^{-at}\boldsymbol{W}_{(e^{2at}-1)/2a},$$

or

$$E[u_0(\boldsymbol{X}_t)] = E\left[u_0\left(e^{-at}\boldsymbol{W}_{\frac{e^{2at}-1}{2a}}\right)\right].$$

Appendix B. Cameron-Martin-Maruyama-Girsanov theorems

These can be found in many textbooks, in particular [20, 4, 62]. We recall them here for convenience.

Maruyama-Girsanov theorem (to be applied to the Navier-Stokes equations)

Consider a Brownian motion $\boldsymbol{W}_t$ under the probability measure $\boldsymbol{P}$. Assuming that the Novikov condition

$$E\left[\exp\left(\frac{1}{2}\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right)\right] < \infty$$

holds for $0 \leq t < T$, put

$$G_t = \exp\left(\int_0^t \boldsymbol{b}(\boldsymbol{W}_s) \cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right)$$

and

$$d\widehat{\boldsymbol{P}} = G_t d\boldsymbol{P}.$$

Then, $\widehat{\boldsymbol{W}_t} = \boldsymbol{W}_t - \int_0^t \boldsymbol{b}(\boldsymbol{W}_s) ds$ is another Brownian motion under the probability measure $\widehat{\boldsymbol{P}}$. That is, the distribution of $\widehat{\boldsymbol{W}}_t$ with respect to $\widehat{\boldsymbol{P}}$ is the same as that of $\boldsymbol{W}_t$ with respect to $\boldsymbol{P}$. It is noted that these theorems refer to relationship between parings of {stochastic process, probability measure}. We have, in particular, for any Wiener functionals

$$E_{\boldsymbol{P}}[F(\boldsymbol{W})] = E_{\widehat{\boldsymbol{P}}}[F(\widehat{\boldsymbol{W}})] = E_{\boldsymbol{P}}[F(\widehat{\boldsymbol{W}})G_t],$$

where $E_{\boldsymbol{P}}[\cdot]$ and $E_{\widehat{\boldsymbol{P}}}[\cdot]$ denote averages with respect to $\boldsymbol{P}$ and $\widehat{\boldsymbol{P}}$. Or, defining $\boldsymbol{h}(t) = \int_0^t \boldsymbol{b}(\boldsymbol{W}_s) ds$, we may write

$$G_t = \exp\left(\int_0^t \dot{\boldsymbol{h}}(s) \cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\dot{\boldsymbol{h}}(s)|^2 ds\right).$$

By replacing $\boldsymbol{h}$ with $-\boldsymbol{h}$, we have equivalently

$$E_{\boldsymbol{P}}[F(\boldsymbol{W})] = E_{\boldsymbol{Q}}[F(\boldsymbol{W} + \boldsymbol{h})] = E_{\boldsymbol{P}}[F(\boldsymbol{W} + \boldsymbol{h})\widehat{G}_t],$$

where

$$\widehat{G}_t = \exp\left(-\int_0^t \dot{\boldsymbol{h}}(s) \cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t |\dot{\boldsymbol{h}}(s)|^2 ds\right),$$

$$d\boldsymbol{Q} = \widehat{G}_t d\boldsymbol{P},$$

and $\boldsymbol{W} + \boldsymbol{h}$ is yet another Brownian motion under $\boldsymbol{Q}$. This is (24).

THE CAMERON-MARTIN THEOREM (to be applied to the Leray equations)
We have

$$\begin{aligned} E_{\boldsymbol{Q}}[F(\boldsymbol{W})] &= E_{\boldsymbol{P}}[F(\boldsymbol{W} + \boldsymbol{h})] = E_{\widehat{\boldsymbol{P}}}[F(\widehat{\boldsymbol{W}} + \boldsymbol{h})] \\ &= E_{\widehat{\boldsymbol{P}}}[F(\boldsymbol{W})] = E_{\boldsymbol{P}}[F(\boldsymbol{W})G_t], \end{aligned}$$

which is (23). The first equality follows from the definition of $\boldsymbol{Q}$, the second from the fact that $\widehat{\boldsymbol{W}}$ is Brownian motion under $\widehat{\boldsymbol{P}}$ (Maruyama-Girsanov theorem) and the third from the definition $\widehat{\boldsymbol{W}} = \boldsymbol{W} - \boldsymbol{h}$.

Appendix C. Maruyama-Girsanov theorem (more general cases)

We refer [49, 42] on this matter. Consider a stochastic process

$$d\boldsymbol{Y}(t) = \boldsymbol{\beta}(t)dt + \gamma(t)d\boldsymbol{W}(t),$$

where $\boldsymbol{W}$ is a Wiener process with respect to a probability distribution $\boldsymbol{P}$, such that

$$\gamma(t)\boldsymbol{v}(t) = \boldsymbol{\beta}(t) - \boldsymbol{\alpha}(t),$$

$$E\left[\exp\left(\frac{1}{2}\int_0^T |\boldsymbol{v}(s)|^2 ds\right)\right] < \infty.$$

Put

$$G_t = \exp\left(\int_0^t \boldsymbol{v}(s)\cdot d\boldsymbol{W}(s) - \frac{1}{2}\int_0^t |\boldsymbol{v}(s)|^2 ds\right)$$

and define

$$d\boldsymbol{Q} = G_T d\boldsymbol{P}.$$

Then

$$\widetilde{\boldsymbol{W}}(t) = \boldsymbol{W}(t) - \int_0^t \boldsymbol{v}(s)ds \ \text{ for } \ 0 \le t \le T$$

is a Wiener process with respect to $\boldsymbol{Q}$ such that

$$d\boldsymbol{Y}(t) = \boldsymbol{\alpha}(t)dt + \gamma(t)d\widetilde{\boldsymbol{W}}.$$

We apply this to the Navier-Stokes equations written in the following form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\triangle\boldsymbol{u}, \tag{C.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}). \tag{C.2}$$

By dynamic scaling

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{\sqrt{2a(t_* - t)}}\boldsymbol{U}(\boldsymbol{\xi},\tau), \tag{C.3}$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2a(t_* - t)}}, \ \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a}\log\frac{t_*}{t_* - t}, \tag{C.4}$$

where $\lambda(t) = \sqrt{2a(t_* - t)}$, the Leray equations read

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + a(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}}P + \nu\triangle_{\boldsymbol{\xi}}\boldsymbol{U}, \tag{C.5}$$

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U} = 0. \tag{C.6}$$

We take $2at_* = 1$ so that the initial conditions $\boldsymbol{u}$ and $\boldsymbol{U}$ coincide. Note also that $t = \dfrac{1 - e^{-2a\tau}}{2a}$.

By taking $\boldsymbol{\gamma} = \sqrt{2\nu}$, $\boldsymbol{\alpha} = 0$, $\boldsymbol{\beta} = -a\boldsymbol{x}$ above. we have $\boldsymbol{v} = -\dfrac{a\boldsymbol{x}}{\sqrt{2\nu}}$. The Novikov condition gives $T = \dfrac{\sqrt{2}}{a}$, which is greater than $t_* = \dfrac{1}{2a}$ by the same factor of $2\sqrt{2}$, as obtained in **Section** 4.

Appendix D. Burgers equations

We consider the Burgers equations in d-dimensions

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{2} \triangle \boldsymbol{u}. \tag{D.1}$$

It is known that there is no blowup for the Burgers equations, primarily because of the maximum principle on $\|\boldsymbol{u}\|_\infty$. If the velocity has a potential $\boldsymbol{u} = \nabla\phi$, we can rewrite the above equations as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 = \frac{1}{2}\triangle\phi$$

without loss of generality. The current argument is applicable only for such a class of potential flows. (Actually, this is redundant the Hopf-Cole linearisation is explicitly available for this class.) The current argument cannot cover the more general cases, where the velocity does not have a potential. The Cole-Hopf transform yields

$$\exp(-\phi(\boldsymbol{x}, t)) = E[\exp(-\phi_0(\boldsymbol{W}_t))].$$

After applying dynamic scaling to the Burgers equations, we obtain

$$\frac{\partial \widetilde{\phi}}{\partial t} + \frac{1}{2}|\nabla\widetilde{\phi}|^2 = \frac{1}{2}\triangle\widetilde{\phi} - \frac{1}{2}\boldsymbol{x}\cdot\nabla\widetilde{\phi},$$

which can be solved as

$$\begin{aligned} \exp(-\widetilde{\phi}(\boldsymbol{x}, t)) &= E[\exp(-\phi_0(\boldsymbol{X}_t))] && \text{(D.2)} \\ &= E[\exp(-\phi_0(\boldsymbol{W}_t))G_t] \;\text{ for }\; 0 \le t < 2\sqrt{2}. && \text{(D.3)} \end{aligned}$$

More explicitly,

$$\exp(-\widetilde{\phi}(\boldsymbol{x}, t)) = \int_{\mathbb{R}^3} \frac{d\boldsymbol{\eta}}{(2\pi)^{3/2}} e^{-\frac{|\boldsymbol{\eta}|^2}{2}} \exp(-\phi_0(e^{-\frac{1}{2}t}\boldsymbol{x} - \sqrt{1-e^{-t}}\boldsymbol{\eta})) \text{ for } t \ge 0.$$

We may consider more general equations

$$f(\phi(\boldsymbol{x}, t)) = E[f(\phi_0(\boldsymbol{W}_t))],$$

which includes as special cases the Burgers equations with $f(x) = e^{-x}$ and the heat equation with $f(x) = x$. Or, we can think of a functional (i.e. spatial integral transform) $f[\phi]$

$$f[\phi](\boldsymbol{x}, t) = E[f[\phi_0](\boldsymbol{W}_t)]$$

as even more general cases, which are still reducible to the heat equation. However, if we allow explicit time-dependence in the function (or functional) of f,

$$f_t[\widetilde{\phi}](\boldsymbol{x}, t) = E[g_t[\phi_0](\boldsymbol{W}_t)]$$

with $f_0 = g_0$, then it is no longer reducible to the heat equation.

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