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A NOTE ON RATIONAL POINTS NEAR PLANAR CURVES

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ABSTRACT. Under fairly natural assumptions, Huang counted the number of rational points lying close to an arc of a planar curve. He obtained upper and lower bounds of the correct order of magnitude, and conjectured an asymptotic formula. In this note, we establish the conjectured asymptotic formula.

1. INTRODUCTION

Let f be a real-valued function defined on a compact interval $I = [\rho, \xi] \subseteq \mathbb{R}$. For positive real numbers $\delta \leqslant 1/2$ and $Q \geqslant 1$, define

$$\tilde{N}_f(Q, \delta) = \# \left\{ \begin{array}{ll} (a, b, q) \in \mathbb{Z}^3 : & 1 \leqslant q \leqslant Q, a/q \in I, \gcd(a, b, q) = 1, \\ & |f(a/q) - b/q| < \delta/Q \end{array} \right\}.$$

Roughly speaking, this counts the number of rational points with denominator at most Q that lie within δQ^{-1} of the curve $\mathcal{C}_f = \{(x, f(x)) : x \in I\}$. Huang [3, Theorem 2] estimated this quantity. As discussed in [3], such estimates are readily applied to the Lebesgue theory of metric diophantine approximation.

Theorem 1.1 (Huang). *Let $0 < c_1 \leqslant c_2$. Assume that $f : I \to \mathbb{R}$ is a C^2 function satisfying*

$$c_1 \leqslant |f''(x)| \leqslant c_2 \qquad (x \in I),$$

with Lipschitz second derivative. Assume further that

$$1/2 \geqslant \delta > Q^{\varepsilon - 1}, \tag{1.1}$$

for some $\varepsilon \in (0, 1)$. Then

$$\frac{2\sqrt{3}}{9\zeta(3)} + O(Q^{-\varepsilon/2}) \leqslant \frac{\tilde{N}_f(Q, \delta)}{|I|\delta Q^2} \leqslant \frac{1}{\zeta(3)} + O(Q^{-\varepsilon/2}). \tag{1.2}$$

The implied constant depends on I, c_1, c_2, ε and the Lipschitz constant; it is independent of f, δ and Q.

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Theorem 1.1 sharpened the upper bounds obtained by Huxley [4] and Vaughan–Velani [5], as well as the lower bounds obtained by Beresnevich–Dickinson–Velani [1] and Beresnevich–Zorin [2].

The purpose of this note is to squeeze together the constants in (1.2), so as to confirm Huang's conjectured asymptotic formula

$$\tilde{N}_f(Q,\delta) \sim \frac{2}{3\zeta(3)}|I|\delta Q^2 \qquad (Q \to \infty), \tag{1.3}$$

within the range (1.1). The asymptotic formula (1.3) follows straightforwardly from our theorem, which we state below and establish in the next section.

Theorem 1.2. *Assume the hypotheses of Theorem 1.1. Let* $\eta > 0$ *and*

$$0 < \tau < \varepsilon/2.$$

Then

$$\frac{2}{3\zeta(3)} - \eta + O(Q^{-\tau}) \leqslant \frac{\tilde{N}_f(Q,\delta)}{|I|\delta Q^2} \leqslant \frac{2}{3\zeta(3)} + \eta + O(Q^{-\tau}).$$

The implied constant depends on $I, c_1, c_2, \varepsilon, \eta$ *and the Lipschitz constant.*

We use Landau and Vinogradov notation: for functions f and positive-valued functions g, we write $f \ll g$ or $f = O(g)$ if there exists a constant C such that $|f(x)| \leqslant Cg(x)$ for all x. If S is a set, we denote the cardinality of S by $\#S$.

2. The count

In this section, we prove Theorem 1.2. For positive real numbers $\delta \leqslant 1/2$ and $Q \geqslant 1$, define the auxiliary counting function

$$\hat{N}_f(Q,\delta) = \# \left\{ \begin{array}{ll} (a,b,q) \in \mathbb{Z}^3 : & 1 \leqslant q \leqslant Q, a/q \in I, \\ & \gcd(a,b,q) = 1, |f(a/q) - b/q| < \delta/q \end{array} \right\}.$$

With the same assumptions as in Theorem 1.1, Huang [3, Corollary 1] showed that

$$\hat{N}_f(Q,\delta) = (\zeta(3)^{-1} + O(Q^{-\varepsilon/2})) \cdot |I|\delta Q^2. \tag{2.1}$$

Let $t \in \mathbb{N}$, $1/2 < \alpha < 1$ and

$$\alpha_i = \alpha^i \qquad (0 \leqslant i \leqslant t).$$

We will have $t \ll_\eta 1$, so the hypothesis (1.1) is satisfied with 2τ in place of ε and $(\alpha_i Q, \alpha_j \delta)$ in place of (Q,δ), whenever Q is large and $0 \leqslant i,j \leqslant t$. In particular (2.1) holds with these adjustments, so

$$\hat{N}_f(\alpha_i Q, \alpha_j \delta) = \Big(\frac{\alpha_i^2 \alpha_j}{\zeta(3)} + O(Q^{-\tau}) \Big) \cdot |I|\delta Q^2 \qquad (0 \leqslant i,j \leqslant t). \tag{2.2}$$

Employing (2.2), we have

$$\begin{aligned}
&\tilde{N}_f(Q,\delta) \\
&\geqslant \sum_{i=1}^{t} \# \left\{ \begin{array}{ll} (a,b,q)\in\mathbb{Z}^3: & \alpha_i Q < q \leqslant \alpha_{i-1}Q, a/q \in I, \\ & \gcd(a,b,q)=1, |f(a/q)-b/q| < \alpha_i\delta/q \end{array} \right\} \\
&= \sum_{i=1}^{t} (\hat{N}_f(\alpha_{i-1}Q,\alpha_i\delta) - \hat{N}_f(\alpha_i Q,\alpha_i\delta)) \\
&= \sum_{i=1}^{t} \Big(\frac{\alpha_{i-1}^2\alpha_i - \alpha_i^3}{\zeta(3)} + O(Q^{-\tau}) \Big) \cdot |I|\delta Q^2.
\end{aligned}$$

Now

$$\tilde{N}_f(Q,\delta) \geqslant \Big(\frac{X(\boldsymbol{\alpha})}{\zeta(3)} + O(tQ^{-\tau}) \Big) \cdot |I|\delta Q^2, \tag{2.3}$$

where

$$X(\boldsymbol{\alpha}) = \sum_{i\leqslant t} (\alpha_{i-1}^2\alpha_i - \alpha_i^3).$$

We compute that

$$\begin{aligned}
X(\boldsymbol{\alpha}) &= (\alpha-\alpha^3)\sum_{j=0}^{t-1}(\alpha^3)^j = \frac{(\alpha-\alpha^3)(1-\alpha^{3t})}{1-\alpha^3} \\
&= (1-\alpha^{3t})(1-(1+\alpha+\alpha^2)^{-1}).
\end{aligned}$$

Choosing α close to 1, and then choosing $t \ll_\eta 1$ large, gives

$$X(\boldsymbol{\alpha}) \geqslant 2/3 - \zeta(3)\eta.$$

Substituting this into (2.3) yields the desired lower bound.

We attack the upper bound in a similar fashion, but there is an extra term to consider. By (2.2), we have

$$\begin{aligned}
&\tilde{N}_f(Q,\delta) - \tilde{N}_f(\alpha_t Q,\alpha_t\delta) \\
&\leqslant \sum_{i=1}^{t} \# \left\{ \begin{array}{ll} (a,b,q)\in\mathbb{Z}^3: & \alpha_i Q < q \leqslant \alpha_{i-1}Q, a/q \in I, \gcd(a,b,q)=1, \\ & |f(a/q)-b/q| < \alpha_{i-1}\delta/q \end{array} \right\} \\
&= \sum_{i=1}^{t} (\hat{N}_f(\alpha_{i-1}Q,\alpha_{i-1}\delta) - \hat{N}_f(\alpha_i Q,\alpha_{i-1}\delta)) \\
&= \sum_{i=1}^{t} \Big(\frac{\alpha_{i-1}^3 - \alpha_{i-1}\alpha_i^2}{\zeta(3)} + O(Q^{-\tau}) \Big) \cdot |I|\delta Q^2.
\end{aligned}$$

Now

$$\tilde{N}_f(Q,\delta) - \tilde{N}_f(\alpha_t Q,\alpha_t\delta) \leqslant \Big(\frac{Y(\boldsymbol{\alpha})}{\zeta(3)} + O(tQ^{-\tau}) \Big) \cdot |I|\delta Q^2,$$

where

$$Y(\boldsymbol{\alpha}) = \sum_{i \leqslant t} (\alpha_{i-1}^3 - \alpha_{i-1}\alpha_i^2).$$

Here

$$Y(\boldsymbol{\alpha}) = \alpha^{-1} X(\boldsymbol{\alpha}) \leqslant \frac{1-\alpha^2}{1-\alpha^3} = \frac{1+\alpha}{1+\alpha+\alpha^2}.$$

Choosing α close to 1 gives $Y(\boldsymbol{\alpha}) \leqslant 2/3 + \zeta(3)\eta/2$, and so

$$\tilde{N}_f(Q,\delta) \leqslant \tilde{N}_f(\alpha_t Q, \alpha_t \delta) + \Big(\frac{2}{3\zeta(3)} + \frac{\eta}{2} + O(tQ^{-\tau}) \Big) \cdot |I| \delta Q^2. \tag{2.4}$$

For the first term on the right hand side of (2.4), we bootstrap Huang's upper bound (1.2). This gives

$$\tilde{N}_f(\alpha_t Q, \alpha_t \delta) \leqslant \Big(\frac{\alpha_t^3}{\zeta(3)} + O(Q^{-\tau}) \Big) \cdot |I| \delta Q^2.$$

Choosing $t \ll_\eta 1$ large, so that $\alpha_t^3 \leqslant \zeta(3)\eta/2$, we now have

$$\tilde{N}_f(\alpha_t Q, \alpha_t \delta) \leqslant \Big(\frac{\eta}{2} + O(Q^{-\tau}) \Big) \cdot |I| \delta Q^2.$$

Substituting this into (2.4) provides the sought upper bound, completing the proof of the theorem.

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