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# COEXISTENCE OF QUBIT EFFECTS

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Abstract. Two quantum events, represented by positive operators (effects), are *coexistent* if they can occur as possible outcomes in a single measurement scheme. Equivalently, the corresponding effects are coexistent if and only if they are contained in the ranges of a single (joint) observable. Here we give several equivalent characterizations of coexistent pairs of qubit effects. We also establish the equivalence between our results and those obtained independently by other authors. Our approach makes explicit use of the Minkowski space geometry inherent in the four-dimensional real vector space of selfadjoint operators in a two-dimensional complex Hilbert space.

## 1. Introduction

It is a fundamental result of the quantum theory of measurement that pairs of observables represented by noncommuting selfadjoint operators cannot be measured together. The joint measurability of two observables $A, B$ entails that for every state (density operator) $\rho$ there is a joint probability distribution of the observables such that the probability of obtaining a value of $A$ in a (Borel) subset $X$ of $\mathbb{R}$ and a value of $B$ in a subset $Y$ of $\mathbb{R}$ is given by $\mathrm{tr}[\rho G(X \times Y)]$, where $G(X \times Y)$ is a positive operator. The probabilities for $A$ and $B$ alone are included as the *marginal* distributions $X \mapsto \mathrm{tr}[\rho G(X \times \mathbb{R})]$, $Y \mapsto \mathrm{tr}[\rho G(\mathbb{R} \times Y)]$, respectively. The operators $G(X \times \mathbb{R})$ and $G(\mathbb{R} \times Y)$ coincide with the spectral projections $E^A(X)$ and $E^B(Y)$ of $A$ and $B$, respectively. From this it follows that $A$ and $B$ commute and that the operators $G(X \times Y)$ are the projection operators $E^A(X)E^B(Y)$.

For observables $E, F$ represented as positive operator measures (POMs) (say with values in $\mathbb{R}$), the existence of a joint observable does not in general require the commutativity of $E$ and $F$. Observables $E, F$ are said to be jointly measurable if there exists a joint observable $G$ (with values in $\mathbb{R}^2$) of which they are marginals. The positive operators (effects) $E(X)$, $F(Y)$ in the ranges of $E$ and $F$ are then contained in the range of a single observable ($G$). A collection of effects are called *coexistent* if they are contained in the range of a single POM.

The fact that not all pairs of observables are jointly measurable marks a fundamental distinction between quantum mechanics and classical mechanics. The extension of the notion of observables to include general POMs gives room for many families of observables to be jointly measurable, and it becomes important to determine what price is to be paid for reconciling this classical feature with the underlying quantum structure. Since noncommuting *sharp* observables (i.e. projection valued measures) are never jointly measurable it is clear that the joint measurability of noncommuting observables requires these observables to be *unsharp* (i.e. POMs that are not projection valued).

The impossibility of joint measurements of noncommuting sharp observables can be presented as a consequence of the *no-cloning theorem* [1]. In fact, if unknown states could be cloned, this could be utilized to send identical copies into measuring devices for two or more noncommuting observables, thus rendering simultaneously the distributions of values for the original system. The relationship between *approximate* quantum cloning and joint measurements has been a subject of subsequent investigations (see, e.g., [2, 3, 4, 5]). It will be interesting to explore further connections between joint measurability and quantum information tasks.

It is an open problem to give general, operationally significant conditions for the joint measurability of two observables. The relationship between joint measurability and uncertainty relations has been studied in some depth in the past two decades and is reviewed in [6] for the position-momentum case and in [7] for qubit experiments in the specific manifestation of Mach-Zehnder interferometry. In these studies it has been shown that the relation of joint measurability is an important structural feature of the set of quantum observables that is intimately linked with other features, notably the degree of unsharpness of an observable and appropriate metric structures.

The present paper is a contribution to the emerging programme of investigating the structure of the set of observables, which should complement current studies of the dual structure of the set of quantum states. We will address the special case of two *simple* observables (having just two possible values) for a qubit system (represented by a two-dimensional Hilbert space). In this simplest possible case the joint measurability of two simple observables is equivalent to the coexistence of a pair of effects.

The special case of two qubit effects of trace equal to unity had been solved by one of the authors a number of years ago [8]. In this case a simple operational interpretation of the coexistence condition has been given [9]: it can be cast in the form of a trade-off relation for the degrees of unsharpness of the two coexistent observables, required by their noncommutativity. The problem was revisited in [10] in the context of an outline theory of *approximate* joint measurements of noncommuting sharp qubit observables. A coexistence condition for a wider (though not fully general) class of pairs of qubit effects was found subsequently in [11].

Here we deduce necessary and sufficient conditions for the coexistence of two arbitrary effects of a qubit system. In fact we give various alternative, equivalent forms of such conditions which arise from different choices of bases in the space of selfadjoint operators. Since an earlier version of the present paper was made available as arXiv:0802.4167v1, two other papers presented independently different formulations of criteria for the coexistence of qubit effects, using different approaches [12, 13]. The first of these appeared on the same day as our result (note that the coauthor T. Heinosaari of that paper is the same person as the coauthor T. Heinonen of [6]). The authors of [13] proved equivalence between their result and that of [12], and provided partly numerical evidence suggesting equivalence with our results. Here we have obtained the coexistence condition in a form that will explicitly be shown to be equivalent with the condition of [13]. We believe that our approach, which is based on the order and convex structures of the set of effects, lends itself best to generalizations to higher dimensions.

The notions of effects and their coexistence were introduced by G. Ludwig in the 1960s in his fundamental work on the axiomatic foundation of quantum mechanics [14]. We dedicate this work to the memory of Günther Ludwig (1918–2007).

## 2. Coexistent pairs of effects

Let $\mathcal{K}$ be a complex Hilbert space with inner product $(\ |\ )$, and let $\mathcal{L} \equiv [\mathbb{O}, \mathbb{1}]$ denote the set of effects, that is, all operators $a$ such that $\mathbb{O} \prec a \prec \mathbb{1}$. Here $\mathbb{O}$ and $\mathbb{1}$ represent the null and identity operators, respectively, and $\prec$ denotes the usual ordering of selfadjoint operators: $a \prec b$ (equivalently, $b \succ a$) if $(\varphi \,|\, a\varphi) \leq (\varphi \,|\, b\varphi)$ for all $\varphi \in \mathcal{K}$.

Any effect $e$ together with its complement effect $e' = \mathbb{1} - e$ forms a simple observable. In general, an observable with finitely many values is determined essentially by a set of effects $\{a_1, a_2, \ldots, a_n\}$, where the indices label the values, $a_k$ is the effect that determines the probabilities for the outcome labeled with $k$, and $\sum_k a_k = \mathbb{1}$.

**Lemma 1.** *Two effects $e, f$ are coexistent if and only if there are effects $a, b \in \mathcal{L}$, such that*

$$a \prec e \prec b, \quad a \prec f \prec b, \quad a + b = e + f. \tag{1}$$

*Proof.* In fact, these inequalities are necessary and sufficient for each element of the set of operators

$$\{a, e - a, f - a, \mathbb{1} - e - f + a\}. \tag{2}$$

to be effects. This set thus defines an observable whose range contains the effects $e$ and $f$ as well as $e'$ and $f'$; hence it constitutes a joint observable for the simple observables given by $\{e, e'\}$ and $\{f, f'\}$. □

For later reference we note a few well-known results.

**Lemma 2.** *Effects $e, f \in \mathcal{L}$ are coexistent if (a) or (b) hold:*
*(a) $e \prec f$ or $e \succ f$ or $e \prec f'$ or $e \succ f'$;*
*(b) $[e, f] = \mathbb{O}$.*
*In particular, $e, e'$ are coexistent.*

*Proof.* (a) Let $e \prec f$. Take $a = e, b = f$, then $\mathbb{O} \prec e = a \prec e, f \prec e + f - e = f = b \prec \mathbb{1}$. If $e \prec f'$, take $a = \mathbb{O}, b = e + f$, then $\mathbb{O} \prec e, f \prec e + f - \mathbb{O} = b \prec \mathbb{1}$. The other two cases are treated similarly.
(b) If $e, f$ commute then the operators $ef,\ ef',\ e'f,\ e'f'$ are effects which add up to $\mathbb{1}$ and constitute a joint observable for $e, f$.
Finally, choose $f = e'$, then $e \prec f' = e$, so $e, e'$ are coexistent. □

The cases (a) and (b) will be referred to as the *trivial* cases of coexistence. We also note without proof that if at least one of two effects $e, f$ is a projection, then the effects are coexistent if and only if they commute. In this case the joint observable (2) is uniquely determined by $e, f$ via $a = ef$.

## 3. Geometric preliminaries

3.1. **Minkowski space isomorphism.** In the case $\mathcal{K} = \mathbb{C}^2$, selfadjoint operators are represented as hermitian $2 \times 2$ matrices. These form a 4-dimensional real vector space $M_4$, spanned by the basis

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $x \in M_4$ we have $x \succ \mathbb{O}$ exactly when the eigenvalues of $x$ are non-negative. We note also that $x \succ \mathbb{O}$ is equivalent to $\langle x \,|\, x \rangle \geq 0$ und $x_0 \geq 0$.

We define $x \succ_o \mathbb{O}$ (equivalently $\mathbb{O} \prec_o x$ ) to mean that $x \succ \mathbb{O}$ and at least one eigenvalue of $x$ is equal to zero. Then for $x, y \in M_4$, $x \succ_o y$ (or $y \prec_o x$) is defined to mean $x - y \succ_o \mathbb{O}$.

Next we define the bilinear form

$$\langle\, x \,|\, y \,\rangle := x_0 y_0 - \sum_{i=1}^{3} x_i y_i = x_0 y_0 - \mathbf{x} \cdot \mathbf{y},$$

where $x = \sum_{i=0}^{3} x_i \sigma_i$, $y = \sum_{i=0}^{3} y_i \sigma_i$.

We note without proof the following fact.

**Theorem 1.** $(M_4, \langle\ |\ \rangle, \prec, \prec_o)$ *is isomorphic to the 4-dimensional Minkowski space.*

Accordingly, we will apply freely the terminology of Minkowski geometry and refer to $\langle\ |\ \rangle$ as the (Minkowski) scalar product. We use the same notation for vectors and for points in $M_4$ as an affine space.

The *forward* and *backward light cones* of an element $x \in M_4$ are defined as the sets

$$\mathcal{F}(x) := \{y \in M_4 : x \prec_o y\}, \quad \mathcal{B}(x) := \{y \in M_4 : x \succ_o y\}.$$

A vector $x \in M_4$ is called *lightlike* if $\langle\, x \,|\, x \,\rangle = 0$. If $\langle\, x \,|\, x \,\rangle > 0$ or $< 0$, the vector $x$ is called *timelike* or *spacelike*, respectively. Then $x \prec_o y$ is equivalent to $y - x$ being lightlike and $y_0 - x_0 \geq 0$. Elements $x, y \in M_4$ are called *spacelike related*, $x \,\sigma\, y$, if $\langle\, x - y \,|\, x - y \,\rangle < 0$.

The set of effects can now be written as (conv$(X)$ denotes the convex hull of a set $X$)

$$\mathcal{L} = [\mathbb{O}, \mathbb{1}] = \mathrm{conv}(\mathcal{F}(\mathbb{O})) \cap \mathrm{conv}(\mathcal{B}(\mathbb{1}))$$

$\mathcal{L}$ is convex and compact, that is, it includes its boundary $(\mathcal{F}(\mathbb{O}) \cup \mathcal{B}(\mathbb{1})) \cap \mathcal{L}$.

The Minkowski scalar product $\langle\, e \,|\, f \,\rangle$ admits a simple physical meaning if $e$ and $f$ are effects: it is equal to the probability of joint occurrence $(\, \Phi \,|\, e \otimes f\, \Phi\,)$ if the effects $e$ and $f$ are measured by, say, Alice and Bob at a two-particle system in the entangled (singlet) state $\Phi = \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+)$.

### 3.2. Properties of spacelike related effects $e, f$ in $M_4$.

**Lemma 3.** *If effects $e, f \in \mathcal{L}$ are spacelike related, $e \sigma f$, then the pairs $e, f$ and $e', f'$ are each linearly independent.*

*Proof.* If $e, f$ are collinear so that (say) $f = \kappa e$ for some $\kappa \geq 0$, then $e - f = (1 - \kappa) e$, and this is a timelike or lightlike vector. Similarly, if (say) $\mathbb{1} - f = \kappa(\mathbb{1} - e)$, then $e - f = -(1 - \kappa)(\mathbb{1} - e)$ is timelike or lightlike. □

**Lemma 4.** *Let $e, f \in \mathcal{L}$ be spacelike related ($e \,\sigma f$), and let $a, b \in \mathcal{L}$ be such that $a \prec e \prec b$, $a \prec f \prec b$ and $a + b = e + f$. Then there exist $\tilde{a}, \tilde{b} \in \mathcal{L}$ such that $\tilde{a} \prec_o e \prec_o \tilde{b}$, $\tilde{a} \prec_o f \prec_o \tilde{b}$ and $\tilde{a} + \tilde{b} = e + f$.*

*Proof.* Let $P$ be the 2-dimensional plane containing $e, f, a$ and hence $b$ (see Fig. 1). In $P$ the forward and backward light cones degenerate to lines. Since $e \,\sigma f$, the forward and backward cones of $e$ and $f$ intersect in exactly one point, respectively. Hence we define $\tilde{a}$ and $\tilde{b}$ by $\mathcal{B}(e) \cap \mathcal{B}(f) \cap P = \{\tilde{a}\}$, $\mathcal{F}(e) \cap \mathcal{F}(f) \cap P = \{\tilde{b}\}$. The lines $\ell(e, \tilde{a})$ and $\ell(f, \tilde{b})$ are parallel, likewise $\ell(e, \tilde{b})$ and $\ell(f, \tilde{a})$. Hence $e, \tilde{b}, f, \tilde{a}$ form the vertices of a parallelogram and $\tilde{a} + \tilde{b} = e + f$. Due to the convexity of $\mathcal{L}$, the element $\tilde{a} \in \mathcal{L}$ since the intersection of $\ell(\tilde{a}, f)$ and the line segment $s(e, a) \subseteq \mathcal{L}$

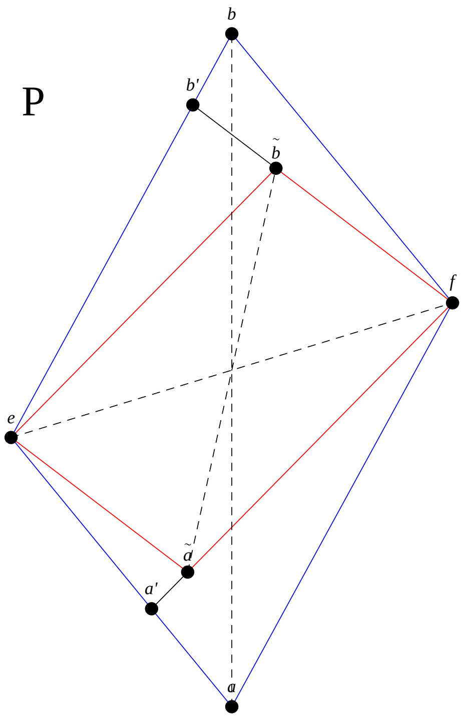


FIGURE 1. Illustration of the proof of Lemma 4.

contains one element $a' \in \mathcal{L}$ and $\tilde{a}$ is in the line segment $s(a', f) \subseteq \mathcal{L}$. Analogously it is shown that $\tilde{b} \in \mathcal{L}$. □

**Lemma 5.** *Effects $e, f$ which are not collinear are mutually commuting if and only if $\mathbb{1}$ lies in the subspace spanned by $e, f$.*

*Proof.* If $e = e_0\mathbb{1} + \mathbf{e} \cdot \boldsymbol{\sigma}$ and $f = f_0\mathbb{1} + \mathbf{f} \cdot \boldsymbol{\sigma}$ commute, then

$$0 = [e, f] = [\mathbf{e} \cdot \boldsymbol{\sigma}, \mathbf{f} \cdot \boldsymbol{\sigma}] = 2i(\mathbf{e} \times \mathbf{f}) \cdot \boldsymbol{\sigma}$$

hence $\mathbf{e}$ and $\mathbf{f}$ are collinear; since $e, f$ are not collinear, it follows that $\mathbb{1}$ is in the span of $e$ and $f$. Conversely, if $\mathbb{1} = xe + yf$ then $x\mathbf{e} + y\mathbf{f} = \mathbb{O}$, so $[e, f] = \mathbb{O}$. □

For two effects $e, f \in M_4$ we define $M(e, f, \mathbb{1})$ as the Minkowski subspace of $M_4$ spanned by $e, f$ and $\mathbb{1}$ and equipped with the orderings $\prec, \prec_o$ inherited from $M_4$. Note that if $e\sigma f$, then $M(e, f, \mathbb{1})$ is 2-dimensional exactly when $e, f$ commute (Lemma 5) and otherwise 3-dimensional.

**Lemma 6.** *Let $T \subset M_4$ be a 3-dimensional timelike subspace, i. e. a subspace containing at least one timelike vector, such that $\mathbb{1} \in T$. Then its $\langle \mid \rangle$-orthogonal complement $T^\perp$ will be a one-dimensional spacelike subspace and $M_4 = T \oplus T^\perp$. The $\langle \mid \rangle$-orthogonal linear projection $\pi : M_4 \to T$ will be monotone, i.e., if $a, b \in M_4$ and $a \prec b$, then $\pi(a) \prec \pi(b)$, and $\langle \mid \rangle$-selfadjoint.*

*Proof.* Each vector $b \in M_4$ can be uniquely written as $b = \pi(b) + b_\perp$, such that $\pi(b) \in T$ and $b_\perp \in T^\perp$. Let $a, b \in M_4$. Then $\langle a \mid \pi(b) \rangle = \langle \pi(a) + a_\perp \mid \pi(b) \rangle =$

$\langle\, \pi(a) \,|\, \pi(b) \,\rangle = \langle\, \pi(a) \,|\, \pi(b) + b_\perp \,\rangle = \langle\, \pi(a) \,|\, b \,\rangle$. This proves $\pi$ being selfadjoint. Concerning monotonicity it suffices to consider the case $\mathbb{O} = a \prec b$, i. e. $\langle\, b \,|\, b \,\rangle \geq 0$ and $b_0 \geq 0$, since $\pi$ is linear. It follows that $\langle\, b_\perp \,|\, b_\perp \,\rangle \leq 0$ since $T^\perp$ is spacelike, and further $0 \leq \langle\, b \,|\, b \,\rangle = \langle\, \pi(b) \,|\, \pi(b) \,\rangle + \langle\, b_\perp \,|\, b_\perp \,\rangle \leq \langle\, \pi(b) \,|\, \pi(b) \,\rangle$. Moreover, $0 \leq b_0 = \langle\, b \,|\, \mathbb{1} \,\rangle = \langle\, b \,|\, \pi(\mathbb{1}) \,\rangle = \langle\, \pi(b) \,|\, \mathbb{1} \,\rangle = \pi(b)_0$, using $\pi$ being selfadjoint. Both inequalities together imply that $\mathbb{O} \prec \pi(b)$, which concludes the proof. □

3.3. **The Minkowski subspace $M_3$.** We will make use of a 3-dimensional Minkowski subspace $M_3(\cong \mathbb{R}^3)$ of $M_4$, defined as the linear span of $\sigma_0, \sigma_1, \sigma_2$, with the orderings $\prec, \prec_0$ carried over from $M_4$. For $x, y \in M_3$, define

$$x \times_o y := \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \times_o \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 y_2 - x_2 y_1 \\ x_0 y_2 - x_2 y_0 \\ x_1 y_0 - x_0 y_1 \end{pmatrix}.$$

Hence $x \times_o y$ is the usual vector product, but with spacelike components inverted. We will use freely the following properties.

**Lemma 7.** *Let $x, y \in M_3$. Then*

$$\begin{aligned} x \times_o y &= -y \times_o x; \\ x \times_o (y \times_o z) &= y\langle\, x \,|\, z \,\rangle - z\langle\, x \,|\, y \,\rangle; \\ \langle\, x \,|\, x \times_o y \,\rangle &= \langle\, y \,|\, x \times_o y \,\rangle = 0; \\ \langle\, x \times_o y \,|\, \tilde{x} \times_o \tilde{y} \,\rangle &= \langle\, x \,|\, \tilde{x} \,\rangle\langle\, y \,|\, \tilde{y} \,\rangle - \langle\, x \,|\, \tilde{y} \,\rangle\langle\, \tilde{x} \,|\, y \,\rangle. \end{aligned}$$

*Furthermore, $x \times_o y = \mathbb{O}$ if and only if $x,\ y$ are collinear.*

We note that the subspace $M(e, f, \mathbb{1})$ can be identified with (a subspace of) $M_3$ since $e, f$ can be unitarily transformed into elements of $M_3$.

We now introduce three basis systems in $M_3$ and give some properties that are useful in what follows.

For $e, f \in \mathcal{L} \cap M_3$, we define two vectors

$$g \equiv e \times_o f, \quad g' \equiv e' \times_o f'. \tag{3}$$

By definition they are in the subspace $\langle\ |\ \rangle$-perpendicular to the vector

$$d \equiv e - f = f' - e' \ , \tag{4}$$

so that the triple $\{g, g', d\}$ forms a basis of $M_3$ if $g, g'$ are linearly independent. Hence,

$$\langle\, g \,|\, d \,\rangle = \langle\, g' \,|\, d \,\rangle = 0. \tag{5}$$

**Lemma 8.** *For $e, f \in \mathcal{L} \cap M_3$, the following statements hold:*

(a) *The vectors $g$ and $g'$ are both nonzero iff neither $e, f$ nor $e', f'$ are collinear. In particular, $g \neq \mathbb{O} \neq g'$ if $e\sigma f$.*
(b) *The vectors $g$ and $g'$ are both spacelike vectors whenever they are nonzero.*
(c) *If $e \neq f$, the vectors $g,\ g'$ are linearly independent iff $[e, f] \neq \mathbb{O}$.*
(d) *$g \times_o g'$ is spacelike iff $e\sigma f$ and $[e, f] \neq \mathbb{O}$.*

*Proof.* (a) If $e\sigma f$, then by Lemma 3 the vector pairs $e, f$ and $e', f'$ are both linearly independent, and thus, by Lemma 7, $g = e \times_o f$ and $g' = e' \times_o f'$ are nonzero.
(b) Two timelike or lightlike vectors are never $\langle\ |\ \rangle$-perpendicular unless they are collinear and lightlike. Thus, since $e$ is timelike or lightlike, then $g$ must be spacelike.

A similar argument applies to $g'$.
(c) Next we note that

$$\langle\, e \,|\, e' \times_o f' \,\rangle - \langle\, f \,|\, e' \times_o f' \,\rangle = \langle\, d \,|\, e' \times_o f' \,\rangle = 0,$$

and use this to compute:

$$\begin{aligned} g \times_o g' &= (e \times_o f) \times_o (e' \times_o f') = -e\langle\, f \,|\, e' \times_o f' \,\rangle + f\langle\, e \,|\, e' \times_o f' \,\rangle \\ &= -d\langle\, e \,|\, e' \times_o f' \,\rangle = -d\langle\, e \,|\, \mathbb{1} \times_o \mathbb{1} + \mathbb{1} \times_o d + e \times_o f \,\rangle \\ &= -d\langle\, e \,|\, \mathbb{1} \times_o d \,\rangle = -d\langle\, \mathbb{1} \,|\, e \times f \,\rangle, \end{aligned}$$

so

$$g \times_o g' = -d(e_1 f_2 - e_2 f_1) = \pm d\, |\mathbf{e} \times \mathbf{f}|. \tag{6}$$

Thus if $d \neq \mathbb{O}$, then $g \times_o g' \neq \mathbb{O}$ exactly when $e, f$ do not commute.
(d) The last statement follows equally immediately by inspection of (6). □

We compute the inner products:

$$\begin{aligned} C(e,f) &\equiv \langle\, g \,|\, g \,\rangle = \langle\, e \,|\, e \,\rangle\langle\, f \,|\, f \,\rangle - \langle\, e \,|\, f \,\rangle^2\,; && (7)\\ C(e',f') &\equiv \langle\, g' \,|\, g' \,\rangle = \langle\, e' \,|\, e' \,\rangle\langle\, f' \,|\, f' \,\rangle - \langle\, e' \,|\, f' \,\rangle^2\,; && (8)\\ D(e,f) &\equiv D(e',f') \equiv \langle\, g \,|\, g' \,\rangle \\ &= \langle\, e \,|\, e' \,\rangle\langle\, f \,|\, f' \,\rangle - \langle\, e \,|\, f' \,\rangle\langle\, e' \,|\, f \,\rangle\,; && (9)\\ \Delta(e,f) &\equiv \Delta(e',f') \equiv \langle\, g \times_o g' \,|\, g \times_o g' \,\rangle \\ &= \langle\, g \,|\, g \,\rangle\langle\, g' \,|\, g' \,\rangle - \langle\, g \,|\, g' \,\rangle^2 = \langle\, d \,|\, d \,\rangle|\mathbf{e} \times \mathbf{f}|^2\,. && (10) \end{aligned}$$

The first two quantities are non-positive since $e, f, e', f'$ are timelike or lightlike. (One can also directly use the fact that $e \times_o f$ and $e' \times_o f'$ are spacelike whenever they are nonzero.) We will also show that the third term $\langle\, g \,|\, g' \,\rangle > 0$ if $e\sigma f$. Furthermore, as is seen as a direct consequence of eq. (6), the last term is negative if and only if $e\sigma f$ and $[e,f] \neq \mathbb{O}$; given the above explicit expressions, this means that the spacelike vectors $g, g'$ satisfy an inverted Schwarz inequality.

**Lemma 9.** *For all $e, f \in \mathcal{L}$ the following inequalities hold:*

$$\begin{aligned} C(e,f) &= \langle\, e \,|\, e \,\rangle\langle\, f \,|\, f \,\rangle - \langle\, e \,|\, f \,\rangle^2 \le 0\,; && (11)\\ C(e',f') &= \langle\, e' \,|\, e' \,\rangle\langle\, f' \,|\, f' \,\rangle - \langle\, e' \,|\, f' \,\rangle^2 \le 0\,; && (12)\\ D(e,f) &= \langle\, e \,|\, e' \,\rangle\langle\, f \,|\, f' \,\rangle - \langle\, e \,|\, f' \,\rangle\langle\, f \,|\, e' \,\rangle > 0 \quad \text{if } e\sigma f\,; && (13)\\ \Delta(e,f) &= C(e,f)C(e',f') - D(e,f)^2 < 0 \quad \text{iff } e\sigma f \text{ and } [e,f] \neq \mathbb{O}. && (14) \end{aligned}$$

*Proof.* For the purposes of the proof we consider $M(e,f,\mathbb{1})$ as embedded in $M_3$. It remains to verify $D(e,f) > 0$. We show that $D(e,f)$ can be expressed as

$$D(e,f) = \tfrac{1}{2}\langle\, e' + f' \,|\, d \times_o g \,\rangle. \tag{15}$$

In fact, using the rules for $\times_o$ and using $d = f' - e'$ we find: $\langle\, e' + f' \,|\, d \times_o g \,\rangle = \langle\, (e' + f') \times_o d \,|\, g \,\rangle = 2\langle\, g' \,|\, g \,\rangle$. Now, $e\sigma f$, and so the forward-oriented timelike or lightlike vectors $e', f'$ are not collinear, so that $e' + f'$ is timelike. Likewise, $d \times_o g$ is timelike since this vector is $\langle\ |\ \rangle$-perpendicular (in $M_3$) to two spacelike vectors. We show that $d \times_o g$ is forward-oriented. This entails that the inner product of $e' + f'$ and $d \times_o g$ is positive.
Thus we have to show that $\langle\, \mathbb{1} \,|\, d \times_o g \,\rangle > 0$. First note that the vector $\sqrt{f_0}e - \sqrt{e_0}f$

is spacelike. In fact, otherwise one would have (say) $\sqrt{f_0}e \succ \sqrt{e_0}f$, so that $e_0 \geq f_0$ and finally $e \succ \sqrt{e_0/f_0}f \succ f$, which contradicts $e\sigma f$. Now we have:

$$\begin{aligned}0 &\leq -\langle \sqrt{f_0}e - \sqrt{e_0}f \,|\, \sqrt{f_0}e - \sqrt{e_0}f \rangle = 2\sqrt{e_0 f_0}\langle e \,|\, f \rangle - f_0\langle e \,|\, e \rangle - e_0\langle f \,|\, f \rangle \\ &\leq (e_0 + f_0)\langle e \,|\, f \rangle - f_0\langle e \,|\, e \rangle - e_0\langle f \,|\, f \rangle = \langle \mathbb{1} \,|\, e \rangle\langle e - f \,|\, f \rangle - \langle \mathbb{1} \,|\, f \rangle\langle e - f \,|\, e \rangle \\ &= \langle \mathbb{1} \,|\, d \times_o (e \times_o f \rangle = \langle \mathbb{1} \,|\, d \times_o g \rangle . \qquad \square\end{aligned}$$

The vector pair $g$, $g'$ was found to be collinear if $e$ and $f$ commute. To remove this degeneracy, we also consider the basis $\{d,\ g,\ d \times g\}$ of mutually $\langle\ |\ \rangle$-perpendicular vectors in $M_3$. We note the following for later use:

$$\langle e + f \,|\, g \rangle = \langle e' + f' \,|\, g' \rangle = 0\,; \tag{16}$$

$$\begin{aligned}\langle e' + f' \,|\, g \rangle &= \langle e + f \,|\, g' \rangle = 2\langle \mathbb{1} \,|\, g \rangle = 2(e_1 f_2 - e_2 f_1) \\ &= 2\,\mathrm{sign}\,(e_1 f_2 - e_2 f_1)\,|\mathbf{e} \times \mathbf{f}|\,;\end{aligned} \tag{17}$$

$$\langle e + f \,|\, d \times_o g \rangle = -2\langle g \,|\, g \rangle = -2C(e,f)\,; \tag{18}$$

$$\langle e' + f' \,|\, d \times_o g \rangle = 2D(e,f)\,; \tag{19}$$

$$\langle d \times_o g \,|\, d \times_o g \rangle = \langle d \,|\, d \rangle\langle g \,|\, g \rangle - \langle d \,|\, g \rangle^2 = \langle d \,|\, d \rangle\, C(e,f)\,. \tag{20}$$

Using these identities is not hard to verify that

$$g' = \frac{D(e,f)}{C(e,f)}\, g \;-\; \frac{e_1 f_2 - e_2 f_1}{C(e,f)}\, d \times_o g.$$

This confirms that $g$ and $g'$ are collinear exactly when the second term vanishes, that is, when $[e,f] = \mathbb{O}$.

Finally we introduce a basis $\{d, h_+, h_-\}$ of $M_3$ where $h_+, h_-$ are distinct lightlike vectors orthogonal to $d$. Note that this presupposes that $e\sigma f$, so that $\langle d \,|\, d \rangle < 0$. We write $h_\pm = x_\pm g + y_\pm d \times_o g$ and compute:

$$0 = \langle h_\pm \,|\, h_\pm \rangle = x_\pm^2 \langle g \,|\, g \rangle + y_\pm^2 \langle d \times_o g \,|\, d \times_o g \rangle = C(e,f)\left[x_\pm^2 + y_\pm^2 \langle d \,|\, d \rangle\right] .$$

Here we have used the identity (20). Thus we find (using a particular choice of the overall constant factor):

$$h_\pm = \pm\sqrt{|\langle d \,|\, d \rangle|}\; g + d \times_o g\,. \tag{21}$$

We compute:

$$\langle h_+ \,|\, h_- \rangle = 2C(e,f)\langle d \,|\, d \rangle > 0\,; \tag{22}$$

$$\langle e + f \,|\, h_\pm \rangle = -2C(e,f) > 0\,; \tag{23}$$

$$\begin{aligned}\langle e' + f' \,|\, h_\pm \rangle &= 2\left[D(e,f) \pm \sqrt{|\langle d \,|\, d \rangle|}\,(e_1 f_2 - e_2 f_1)\right] \\ &= 2\left[D(e,f) \pm \sqrt{|\langle d \,|\, d \rangle|}|\mathbf{e} \times \mathbf{f}|\mathrm{sign}\,(e_1 f_2 - e_2 f_1)\right] \\ &= 2\left[D(e,f) \pm \sqrt{|\Delta(e,f)|}\,\mathrm{sign}(e_1 f_2 - e_2 f_1)\right] > 0\,.\end{aligned} \tag{24}$$

The last relation follows by application of the identity (17). These quantities are all positive in the present case of $e\sigma f$.

Henceforth we will assume that $\mathrm{sign}\,(e_1 f_2 - e_2 f_1) = +1$ so that we can always replace $e_1 f_2 - e_2 f_1 (> 0)$ with $|\mathbf{e} \times \mathbf{f}|$. This can always be arranged by swapping $e$ and $f$ if necessary. Our main results in the next section will be given in a form that is invariant under this exchange.

## 4. Coexistent pairs of qubit effects

Next we consider the question of the coexistence of qubit effects $e, f$ which are spacelike related and not necessarily mutually commuting. The conditions obtained will ultimately be phrased in such a way that they hold also in the trivial cases of coexistence.

4.1. **Reduction to $M_3$.** We first show that the coexistence of a pair of effects $e, f \in M_4$ can be studied within $M_3$ (taking into account that the relation of coexistence is invariant under unitary transformations). The resulting conditions will be written in a form that is invariant under spatial rotation, using identities such as

$$(e_1 f_2 - e_2 f_1)^2 = |\mathbf{e} \times \mathbf{f}|^2 = -\frac{1}{12}\mathrm{tr}([e,f]^2) = \frac{1}{4}\|[e,f]\|^2 .$$

In this way the result will be generally valid in $M_4$; the reduction to $M_3$ is only made for the sake of simplifying the proofs.

**Theorem 2.** *If $e, f \in \mathcal{L}$ are coexistent, then they are also coexistent in $M(e, f, \mathbb{1})$, that is, there is an effect $a \in \mathcal{L} \cap M(e, f, \mathbb{1})$ such that*

$$\mathbb{O} \prec a \prec e, f \prec e + f - a \prec \mathbb{1}.$$

*Proof.* If $e \prec f$ or $f \prec e$ or $[e, f] = \mathbb{O}$ the claim follows directly. Hence we may consider the case where $e\sigma f$ and $M(e, f, \mathbb{1})$ is a 3-dimensional timelike subspace of $M_4$ and $\pi : M_4 \to M(e, f, \mathbb{1})$ the corresponding linear projection which, by Lemma 6, is monotone. Let $a \in M_4$, $a \prec e, f$ and $e + f - a \prec \mathbb{1}$. It follows that

$$\begin{aligned} \pi(a) &\prec \pi(e) = e, \\ \pi(a) &\prec \pi(f) = f, \\ \pi(e) + \pi(f) - \pi(a) = \pi(e + f - a) &\prec \pi(\mathbb{1}) = \mathbb{1}. \end{aligned}$$

Hence $e, f$ are coexistent in $M(e, f, \mathbb{1})$. □

An obvious corollary is that if effects $e, f \in M_4$ are coexistent and $M(e, f, \mathbb{1}) \subseteq M_3$, then $e, f$ are also coexistent as elements of $M_3$, and vice versa.

4.2. **Characterization of coexistence in $M_3$.** We will use the same notation as in $M_4$ for the forward and backward cone of an element $x$ in $M_3$, namely, $\mathcal{F}(x)$, $\mathcal{B}(x)$. The coexistence criterion of Lemma 4 then states that effects $e, f$ in $M_3$ are coexistent if and only if there is an effect $a \in \mathcal{B}(e) \cap \mathcal{B}(f)$ such that $b = e + f - a$ is also an effect. This is trivially satisfied if $e \prec f$ or $e \succ f$, for then one can choose $a = e, b = f$ in the first case and $a = f, b = e$ in the second. In these trivial cases one of the backward cones encloses the other, and they are disjoint unless $e, f$ are lightlike related. The case $e\sigma f$ is less trivial. We recall the following familiar fact.

**Lemma 10.** *Let $e, f \in \mathcal{L} \cap M_3$ be spacelike related effects, $e\sigma f$. Let $H$ be the plane passing through $\frac{1}{2}(e + f)$ which is $\langle\ |\ \rangle$-perpendicular to $d$. Then the intersections $\mathcal{H}_a \equiv \mathcal{B}(e) \cap \mathcal{B}(f)$ and $\mathcal{H}_b \equiv \mathcal{F}(e) \cap \mathcal{F}(f)$ are the two branches of a hyperbola $\mathcal{H}$ lying in $H$.*

*Proof.* Each of the conditions $a \in \mathcal{B}(e) \cap \mathcal{B}(f)$ and $b = e + f - a \in \mathcal{F}(e) \cap \mathcal{F}(f)$ are equivalent to $\langle\, e-a \,|\, e-a \,\rangle = 0 = \langle\, f-a \,|\, f-a \,\rangle$ and this gives $\langle\, \frac{1}{2}(e+f)-a \,|\, e-f \,\rangle = 0$. Hence this intersection of the two cones lies actually in the plane $H$ and thus is a conic section. □

Let $e, f \in M_3$, with $e\,\sigma f$. Writing $a = \frac{1}{2}(e+f) - v$, the coexistence condition is now spelled out as follows:
(i) $a \in H$ is equivalent to

$$\langle\, v \,|\, d \,\rangle = 0. \tag{25}$$

(ii) $a \in \mathcal{H}_a = \mathcal{B}(e) \cap \mathcal{B}(f)$ and $b = e + f - a \in \mathcal{H}_b = \mathcal{F}(e) \cap \mathcal{F}(f)$ are both equivalent to

$$\langle\, v \pm \tfrac{1}{2}d \,|\, v \pm \tfrac{1}{2}d \,\rangle \;=\; \langle\, v \,|\, v \,\rangle + \tfrac{1}{4}\langle\, d \,|\, d \,\rangle = 0, \tag{26}$$

$$v_0 \;\geq\; \tfrac{1}{2}|d_0|. \tag{27}$$

(Note that here we have utilized (i). We also remark that (27) can be replaced by the weaker $v_0 > 0$. The sharper bound arises from the fact that $a \prec e, f$ implies $a_0 = \frac{1}{2}(e_0 + f_0) - v_0 \leq e_0,\ f_0$.)
(iii) The conditions $a \succ \mathbb{O}$ and $b = e + f - a \prec \mathbb{1}$ specify two bounded segments

$$\mathcal{S}_a \equiv \mathcal{B}(e) \cap \mathcal{B}(f) \cap \mathrm{conv}(\mathcal{F}(\mathbb{O})), \quad \mathcal{S}_b \equiv \mathcal{B}(e) \cap \mathcal{B}(f) \cap \mathrm{conv}(\mathcal{F}(e - f')) \tag{28}$$

of admissible elements $a$ on the hyperbola branch $\mathcal{H}_a = \mathcal{B}(e) \cap \mathcal{B}(f)$.

Note that $\mathcal{S}_a \neq \emptyset$ since $\mathbb{O} \prec e,\ f$, so that $\mathcal{B}(e) \cap \mathcal{B}(f)$ cannot fall entirely outside $\mathcal{L}$. Similarly, $\mathcal{S}_b \neq \emptyset$ since $e,\ f \prec \mathbb{1}$, so that $\mathcal{F}(e) \cap \mathcal{F}(f)$ cannot fall entirely outside of $\mathcal{L}$. But it may happen that $\mathcal{S}_a$ as well as $\mathcal{S}_b$ degenerate into a single point. The coexistence conditions can thus be characterized geometrically.

**Lemma 11.** *Let $e, f \in \mathcal{L} \cap M_3$, $e\sigma f$. Then*

$$e, f \text{ coexistent} \Longleftrightarrow \mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset\,.$$

Since $e, f$ are coexistent if $[e, f] = \mathbb{O}$, it follows that $\mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset$ in the commutative case. The following confirms further trivial cases of coexistence:

$$e \prec f' \iff \mathrm{conv}(\mathcal{F}(\mathbb{O})) \subseteq \mathrm{conv}(\mathcal{F}(e - f')) \Longrightarrow \mathcal{S}_a \subseteq \mathcal{S}_b\,, \tag{29}$$

$$e \succ f' \iff \mathrm{conv}(\mathcal{F}(e - f')) \subseteq \mathrm{conv}(\mathcal{F}(\mathbb{O})) \Longrightarrow \mathcal{S}_b \subseteq \mathcal{S}_a\,. \tag{30}$$

This means conversely that

$$\mathcal{S}_a \not\subseteq \mathcal{S}_b \text{ and } \mathcal{S}_b \not\subseteq \mathcal{S}_a \implies e\sigma f'\,. \tag{31}$$

The remaining trivial cases of coexistence, $e \prec f$ or $e \succ f$, cannot be characterized in terms of $\mathcal{S}_a$, $\mathcal{S}_b$ since the hyperbola and hence these sets are only defined if $e\sigma f$.

We proceed to find necessary and sufficient conditions for $\mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset$ to be true. The end points of the segments $\mathcal{S}_a$, $\mathcal{S}_b$ are determined by $a \succ_o \mathbb{O}$, that is,

$$\begin{aligned} 0 \;&=\; \langle\, a \,|\, a \,\rangle = \tfrac{1}{4}\langle\, e + f \,|\, e + f \,\rangle + \langle\, v \,|\, v \,\rangle - \langle\, e + f \,|\, v \,\rangle \\ &=\; \langle\, e \,|\, f \,\rangle - \langle\, e + f \,|\, v \,\rangle \quad \text{for } \mathcal{S}_a, \quad \text{[using (26)]} \qquad (32) \\ 0 \;&\leq\; a_0 = \tfrac{1}{2}(e_0 + f_0) - v_0, \qquad (33) \end{aligned}$$

and $e + f - a \prec_o \mathbb{1}$, that is,

$$\begin{aligned} 0 \;&=\; \langle\, \mathbb{1} + a - e - f \,|\, \mathbb{1} + a - e - f \,\rangle \\ &=\; 1 + \tfrac{1}{4}\langle\, e + f \,|\, e + f \,\rangle + \langle\, v \,|\, v \,\rangle - \langle\, \mathbb{1} \,|\, e + f \,\rangle \\ &\qquad -2\langle\, \mathbb{1} \,|\, v \,\rangle + \langle\, e + f \,|\, v \,\rangle \\ &=\; \langle\, e' \,|\, f' \,\rangle - \langle\, e' + f' \,|\, v \,\rangle \quad \text{for } \mathcal{S}_b, \quad \text{[using (26)]} \qquad (34) \\ 0 \;&\leq\; 1 - e_0 - f_0 + a_0 = \tfrac{1}{2}(e_o' + f_0') - v_0. \qquad (35) \end{aligned}$$

Note that in (32) and (34) we have used (26).

Inequalities (33) and (35) already follow from the remaining conditions and hence can be neglected as far as equivalence to the coexistence of $e$ and $f$ is concerned. In fact, geometrically, (33) holds since the hyperbola $\mathcal{B}(e) \cap \mathcal{B}(f)$ intersects $\mathcal{F}(\mathbb{O})$ in exactly two points, or touches $\mathcal{F}(\mathbb{O})$ at a single point in special cases. On the other hand, $\mathcal{B}(e) \cap \mathcal{B}(f) \cap \mathcal{B}(\mathbb{O}) = \emptyset$ or, in special cases, $\mathcal{B}(e) \cap \mathcal{B}(f) \cap \mathcal{B}(\mathbb{O}) = \{\mathbb{O}\}$. Analogous arguments apply to (35).

4.3. **Main result.** We now deduce a set of coexistence conditions using the lightlike vectors $h_\pm$ (eq. (21)) for the parametrization of the plane $H$ containing $\mathcal{H}$. The vectors $h_\pm$ will be found to determine the directions of the asymptotes of the hyperbola $\mathcal{H}$, which intersect in the point $\frac{1}{2}(e+f)$.

We start with the hyperbola condition (26) and the linear equations (32) and (34) which specify the segments $\mathcal{S}_a$ and $\mathcal{S}_b$, respectively. The end points of $\mathcal{S}_a, \mathcal{S}_b$ will be determined using the parametrization $a = \frac{1}{2}(e+f) - \lambda h_+ - \mu h_-$.[1]

The equation (26) of the hyperbola now becomes (using (22)):

$$\lambda\mu = -\frac{1}{8}\frac{\langle\, d \,|\, d \,\rangle}{\langle\, h_+ \,|\, h_- \,\rangle} = \frac{1}{16|C(e,f)|}\,. \tag{36}$$

Note that now the asymptotes of the hyperbola in the $\lambda$-$\mu$-plane are perpendicular, and $\mu(\lambda)$ is a monotonic function. If $\lambda \to \infty$, then $\mu \to 0$, and the vector $v = \lambda h_+ + \mu h_-$ pointing from $\frac{1}{2}(e+f)$ to a point on the hyperbola (in $M_3$) approaches $\lambda h_+$, which is thus seen to be in the direction of an asymptote. Similarly, the direction of the other asymptote is given by $h_-$.

The equations defining the line segments $\mathcal{S}_a, \mathcal{S}_b$ are:

$$\lambda\langle\, e+f \,|\, h_+ \,\rangle + \mu\langle\, e+f \,|\, h_- \,\rangle = \langle\, e \,|\, f \,\rangle \quad \text{for } \mathcal{S}_a\,, \tag{37}$$

$$\lambda\langle\, e'+f' \,|\, h_+ \,\rangle + \mu\langle\, e'+f' \,|\, h_- \,\rangle = \langle\, e' \,|\, f' \,\rangle \quad \text{for } \mathcal{S}_b\,. \tag{38}$$

These represent straight lines with negative gradients and intersecting the hyperbola in the first quadrant, see Fig. 2. Using the expressions (23), (24) for the coefficients, these linear equations can be rewritten as:

$$\mu = \frac{\langle\, e \,|\, f \,\rangle}{2|C(e,f)|} - \lambda \quad \text{for } \mathcal{S}_a\,, \tag{39}$$

$$\mu = \frac{\langle\, e' \,|\, f' \,\rangle}{2(D(e,f) - \sqrt{|\Delta(e,f)|})} - \lambda\frac{D(e,f) + \sqrt{|\Delta(e,f)|}}{D(e,f) - \sqrt{|\Delta(e,f)|}} \quad \text{for } \mathcal{S}_b\,. \tag{40}$$

The first line has a fixed negative slope $-1$ and the second is always steeper downward. We denote the intersection points of these lines with the hyperbola $(\lambda_a^\pm,\, \mu_a^\pm)$

[1] We will use the same notation $\mathcal{S}_a, \mathcal{S}_b$ for the representations of the segments in the $\lambda$−$\mu$−plane.

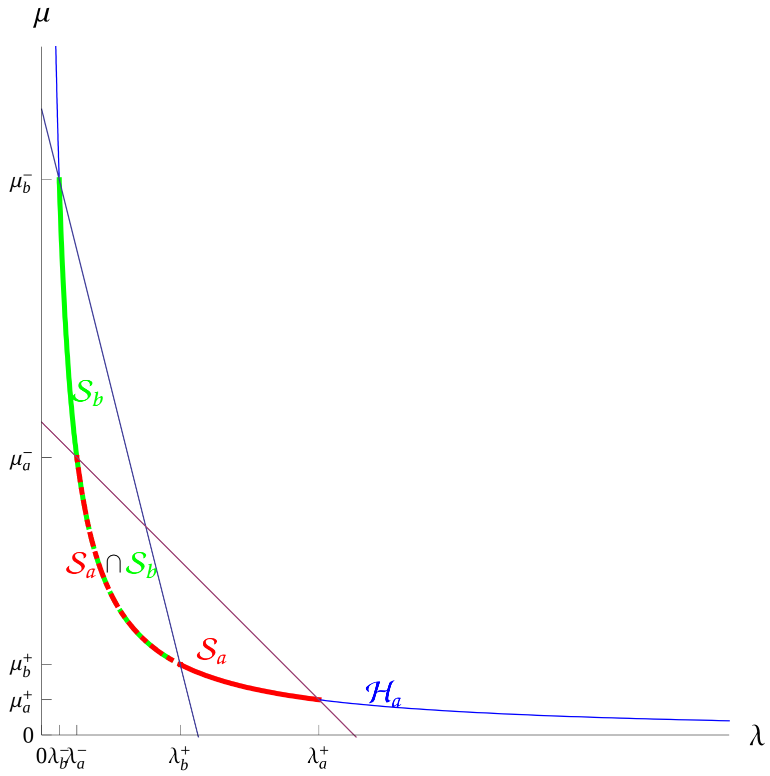


FIGURE 2. Representation of the hyperbola branch $\mathcal{B}(e) \cap \mathcal{B}(f)$ in the $\lambda - \mu-$plane, with an indication of the coordinates of the end points of the segments $\mathcal{S}_a, \mathcal{S}_b$.

and $(\lambda_b^\pm, \mu_b^\pm)$. Using the expressions (23), (24) for the coefficients, we find:

$$\begin{aligned}
\lambda_a^\pm &= \frac{1}{2\langle\, e+f \,|\, h_+ \,\rangle} \left\{ \langle\, e \,|\, f \,\rangle \pm \sqrt{\langle\, e \,|\, f \,\rangle^2 - \frac{\langle\, e+f \,|\, h_+ \,\rangle \langle\, e+f \,|\, h_- \,\rangle}{4|C(e,f)|}} \right\} \\
&= \frac{1}{4|C(e,f)|} \left\{ \langle\, e \,|\, f \,\rangle \pm \sqrt{\langle\, e \,|\, e \,\rangle \langle\, f \,|\, f \,\rangle} \right\} > 0\,; \\
\lambda_b^\pm &= \frac{1}{2\langle\, e'+f' \,|\, h_+ \,\rangle} \left\{ \langle\, e' \,|\, f' \,\rangle \pm \sqrt{\langle\, e' \,|\, f' \,\rangle^2 - \frac{\langle\, e'+f' \,|\, h_+ \,\rangle \langle\, e'+f' \,|\, h_- \,\rangle}{4|C(e,f)|}} \right\} \\
&= \frac{1}{4(D(e,f) + \sqrt{|\Delta(e,f)|})} \left\{ \langle\, e' \,|\, f' \,\rangle \pm \sqrt{\langle\, e' \,|\, e' \,\rangle \langle\, f' \,|\, f' \,\rangle} \right\} > 0\,.
\end{aligned}$$

Similar expressions are found for $\mu_a^\pm$ and $\mu_b^\pm$. We will write this briefly as

$$\lambda_a^\pm = \frac{\Gamma_\pm(e,f)}{4|C(e,f)|}\,, \quad \mu_a^\pm = \frac{\Gamma_\mp(e,f)}{4|C(e,f)|}\,, \tag{41}$$

$$\lambda_b^\pm = \frac{\Gamma_\pm(e',f')}{4(D(e,f) + \sqrt{|\Delta(e,f)|})}\,, \quad \mu_b^\pm = \frac{\Gamma_\mp(e',f')}{4(D(e,f) - \sqrt{|\Delta(e,f)|})}\,, \tag{42}$$

$$\Gamma_\pm(e,f) \equiv \langle e\,|\,f\,\rangle \pm \sqrt{\langle e\,|\,e\,\rangle\langle f\,|\,f\,\rangle} > 0\,, \tag{43}$$

$$\Gamma_\pm(e',f') \equiv \langle e'\,|\,f'\,\rangle \pm \sqrt{\langle e'\,|\,e'\,\rangle\langle f'\,|\,f'\,\rangle} > 0\,. \tag{44}$$

It is now straightforward to observe given the slopes of the two straight lines intersecting the hyperbola that the two segments $\mathcal{S}_a, \mathcal{S}_b$ are nonintersecting exactly when $\lambda_a^- \le \lambda_b^+$ and $\lambda_b^- \le \lambda_a^+$. But since the slope of the second line is always greater in magnitude than that of the first line, the second of these inequalities is always satisfied (since the lines always intersect the hyperbola). Thus we have:

**Lemma 12.** *Let $e, f \in \mathcal{L}$, with $e\sigma f$, $[e,f] \neq \mathbb{O}$. Then*

$$\mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset \iff \lambda_a^- \le \lambda_b^+\,. \tag{45}$$

This inequality is evaluated as follows:

$$\Big(D(e,f) + \sqrt{|\Delta(e,f)|}\Big)\,\Gamma_-(e,f) \le |C(e,f)|\,\Gamma_+(e',f')\,. \tag{46}$$

We note the following relations:

$$\Gamma_+(e,f)\Gamma_-(e,f) = |C(e,f)|\,, \tag{47}$$

$$\Gamma_+(e',f')\Gamma_-(e',f') = |C(e',f')|\,, \tag{48}$$

$$\Big(D(e,f) + \sqrt{|\Delta(e,f)|}\Big)\Big(D(e,f) - \sqrt{|\Delta(e,f)|}\Big) = |C(e,f)C(e',f')|\,, \tag{49}$$

$$\Gamma_+(e,f)\,\Gamma_+(e',f')\Gamma_-(e,f)\,\Gamma_-(e',f') = |C(e,f)C(e',f')|\,. \tag{50}$$

Using these identities, (46) is found to be equivalent to

$$D(e,f) + \sqrt{|\Delta(e,f)|} \le \Gamma_+(e,f)\,\Gamma_+(e',f')\,. \tag{51}$$

We proceed to transform this inequality further. First rearrange terms so that only $\sqrt{|\Delta(e,f)|}$ remains on the left hand side, then square the expressions on both sides to obtain the inequality:

$$-\Delta(e,f) \le (\Gamma_+(e,f)\,\Gamma_+(e',f') - D(e,f))^2\,. \tag{52}$$

Using the form of $\Delta(e,f)$ given in (14), then eq. (52) becomes after some rearrangement:

$$2D(e,f)\Gamma_+(e,f)\,\Gamma_+(e',f') \le (\Gamma_+(e,f)\,\Gamma_+(e',f'))^2 + |C(e,f)C(e',f')|\,. \tag{53}$$

Using (50), the last inequality becomes after cancellation of $\Gamma_+(e,f)\,\Gamma_+(e',f')$:

$$2D(e,f) \le \Gamma_+(e,f)\,\Gamma_+(e',f') + \Gamma_-(e,f)\,\Gamma_-(e',f')\,. \tag{54}$$

Now observe that this last inequality entails that $D(e,f) \le \Gamma_+(e,f)\Gamma_+(e',f')$ since otherwise $D(e,f) > \Gamma_+(e,f)\Gamma_+(e',f') \ge \Gamma_-(e,f)\Gamma_-(e',f')$, in contradiction to (54). Thus we can transform back to the equivalent (52), and using that $D(e,f) \le \Gamma_+(e,f)\Gamma_+(e',f')$, we finally obtain (51).

We note that although this characterization of the coexistence of $e, f$ was deduced under the assumption $e\sigma f$, $[e,f] \neq \mathbb{O}$, it is trivially fulfilled if these assumptions are violated, since then the left-hand side of (52) is zero or negative. Using the definitions of $D(e,f)$ and $\Gamma_\pm(e,f)$, $\Gamma_\pm(e',f')$, inequality (54) can be given in explicit form, leading to the following result.

**Theorem 3.** *Let $e, f \in \mathcal{L}$. Then $e, f$ are coexistent if and only if*

$$\begin{aligned} &\langle e\,|\,e'\,\rangle\langle f\,|\,f'\,\rangle - \langle e\,|\,f'\,\rangle\langle e'\,|\,f\,\rangle - \langle e\,|\,f\,\rangle\langle e'\,|\,f'\,\rangle \\ &\qquad = D(e,f) - \langle e\,|\,f\,\rangle\langle e'\,|\,f'\,\rangle \le \sqrt{\langle e\,|\,e\,\rangle\langle f\,|\,f\,\rangle\langle e'\,|\,e'\,\rangle\langle f'\,|\,f'\,\rangle}\,. \end{aligned} \tag{55}$$

We give yet another reformulation of the inequality (55) which highlights the significance of the noncommutativity of $e, f$. To this end we recall that $D(e,f) = [|\Delta(e,f)| + |C(e,f)C(e',f')|]^{1/2}$, isolate this term in (55) on the left-hand side, and square both sides of the inequality. This gives:

$$\begin{aligned}|\Delta(e,f)| &\le \Big(\langle\, e\,|\,f\,\rangle\langle\, e'\,|\,f'\,\rangle + \sqrt{NN'}\Big)^2 - |C(e,f)C(e',f')| \\ &= \Big(\langle\, e\,|\,f\,\rangle\langle\, e'\,|\,f'\,\rangle + \sqrt{NN'}\Big)^2 - \big(\langle\, e\,|\,f\,\rangle^2 - N\big)\big(\langle\, e'\,|\,f'\,\rangle^2 - N'\big) \\ &= \Big(\langle\, e\,|\,f\,\rangle\sqrt{N'} + \langle\, e'\,|\,f'\,\rangle\sqrt{N}\Big)^2 .\end{aligned}$$

Here we have introduced the abbreviations

$$N \equiv \langle\, e\,|\,e\,\rangle\langle\, f\,|\,f\,\rangle, \quad N' \equiv \langle\, e'\,|\,e'\,\rangle\langle\, f'\,|\,f'\,\rangle . \tag{56}$$

Thus we have established the following, recalling that $|\Delta(e,f)| = |\langle\, d\,|\,d\,\rangle|\mathbf{e}\times\mathbf{f}|^2 = |\langle\, d\,|\,d\,\rangle|\frac{1}{4}\|[e,f]\|^2$.

**Corollary 1.** *Effects $e, f \in \mathcal{L} \subset M_4$ are coexistent if and only if the following inequality holds:*

$$-\tfrac{1}{4}\langle\, d\,|\,d\,\rangle\,\|[e,f]\|^2 \le \Big(\langle\, e\,|\,f\,\rangle\sqrt{\langle\, e'\,|\,e'\,\rangle\langle\, f'\,|\,f'\,\rangle} + \langle\, e'\,|\,f'\,\rangle\sqrt{\langle\, e\,|\,e\,\rangle\langle\, f\,|\,f\,\rangle}\Big)^2 . \tag{57}$$

The left-hand side has been written in a way such that the inequality becomes automatically true in the trivial cases of coexistence where $e, f$ are not spacelike related. It is also manifest that the inequality holds if $e, f$ commute.

## 5. Special cases

5.1. **Commutative case.** It is instructive to consider the case where $e\sigma f$ and $[e,f] = \mathbb{O}$. In view of (10) this entails that $g$ and $g'$ are collinear, $g' = \kappa g$, so that $\Delta(e,f) = 0$. It follows from eqs. (39), (40) that the lines defining the end points of the segments $\mathcal{S}_a, \mathcal{S}_b$ are parallel. Thus, $\mathcal{S}_a \subseteq \mathcal{S}_b$ or $\mathcal{S}_a \supseteq \mathcal{S}_b$.

This result can also be obtained directly from the defining equations (25), (32) and (34) for the end points of the segments in $M_3$. The vector $v$ being $\langle\ |\ \rangle$-orthogonal to $d$ means that it lies in the subspace spanned by $g$ and $d \times_o g$. As $e\sigma f$, the commutativity $[e,f] = \mathbb{O}$ entails that this subspace is also spanned by $g'$ and $d \times_o g'$ since then $g' = \kappa g$. Writing $v = \alpha g + \beta g d \times_o g$ it then follows that $\langle\, e' + f'\,|\,v\,\rangle = \frac{\beta}{\kappa}\langle\, e' + f'\,|\,d \times_o g'\,\rangle = -2\frac{\beta}{\kappa}\langle\, g'\,|\,g'\,\rangle = -2\beta\kappa\langle\, g\,|\,g\,\rangle = -\kappa\langle\, e+f\,|\,v\,\rangle$. This shows that the two equations (32) and (34) define two parallel lines, so the conclusion is once more that $\mathcal{S}_a \subseteq \mathcal{S}_b$ or $\mathcal{S}_a \supseteq \mathcal{S}_b$.

5.2. **Pairs of unbiased effects.** We consider a special case of interest where the two effects $e, f$ (as well as their complements have zero-components equal to $\frac{1}{2}$. This case was treated in [8]. Effects of that form are sometimes called *unbiased*.

**Corollary 2.**

$$\begin{aligned} & e = \tfrac{1}{2}(\mathbb{1} + \tilde{\mathbf{e}}\cdot\sigma), \quad f = \tfrac{1}{2}(\mathbb{1} + \tilde{\mathbf{f}}\cdot\sigma) \quad \textit{are coexistent} \\ & \iff |\tilde{\mathbf{e}}|^2 + |\tilde{\mathbf{f}}|^2 \le 1 + (\tilde{\mathbf{e}}\cdot\tilde{\mathbf{f}})^2 && (58) \\ & \iff |\tilde{\mathbf{e}}\times\tilde{\mathbf{f}}|^2 \le (1-|\tilde{\mathbf{e}}|^2)(1-|\tilde{\mathbf{f}}|^2) && (59) \\ & \iff |\tilde{\mathbf{e}}+\tilde{\mathbf{f}}| + |\tilde{\mathbf{e}}-\tilde{\mathbf{f}}| \le 2 . && (60) \end{aligned}$$

*Proof.* For the above form of effects it is straightforward to verify that inequality (55) assumes the explicit form (58) after appropriate rearrangement of terms. □

The coexistence condition in the form (59) has a simple operational meaning as explained in [10]: the quantities $1-|\tilde{\mathbf{e}}|^2$ and $1-|\tilde{\mathbf{f}}|^2$ are measures of the *unsharpness* of $e, f$, so that according to this inequality the degrees of unsharpness of a coexistent pair of effects $e, f$ cannot simultaneously be made small if $e, f$ do not commute.

## 6. Comparison with other approaches

In [13] the coexistence of $e, f$ is expressed in the form of a single inequality which reads in our notation:

$$(1 - F(e)^2 - F(f)^2)\left(1 - \frac{x^2}{F(e)^2} - \frac{y^2}{F(f)^2}\right) \leq (xy - 4\mathbf{e}\cdot\mathbf{f})^2 \tag{61}$$

Here

$$\begin{aligned} x &= e_0 - (1-e_0) = 2e_0 - 1 = \langle\, e \,|\, e \,\rangle - \langle\, e' \,|\, e' \,\rangle\,, \\ y &= f_0 - (1-f_0) = 2f_0 - 1 = \langle\, f \,|\, f \,\rangle - \langle\, f' \,|\, f' \,\rangle \end{aligned}$$

are measures of *bias* (for example, $x = 0$ iff $e_0 = 1 - e_0 = \frac{1}{2}$), and

$$F(e) := \sqrt{\langle\, e \,|\, e \,\rangle} + \sqrt{\langle\, e' \,|\, e' \,\rangle}\,, \quad F(f) := \sqrt{\langle\, f \,|\, f \,\rangle} + \sqrt{\langle\, f' \,|\, f' \,\rangle}\,. \tag{62}$$

For later use we introduce the abbreviations

$$B(e) := \sqrt{\langle\, e \,|\, e \,\rangle} - \sqrt{\langle\, e' \,|\, e' \,\rangle}\,, \quad B(f) := \sqrt{\langle\, f \,|\, f \,\rangle} - \sqrt{\langle\, f' \,|\, f' \,\rangle} \tag{63}$$

and note the following:

$$x = B(e)F(e)\,, \quad y = B(f)F(f)\,.$$

In [13] it is shown that the condition (61) is equivalent to the condition found in [12]. We now proceed to establish the equivalence of (55) and (61).

**Theorem 4.** *Inequalities (55) and (61) are equivalent.*

*Proof.* A lengthy calculation gives the following reformulation of the left-hand side (abbreviated $LHS$) of (61):

$$\begin{aligned} LHS &= 1 + x^2 + y^2 + 2(\langle\, e \,|\, e \,\rangle + \langle\, e' \,|\, e' \,\rangle)(\langle\, f \,|\, f \,\rangle + \langle\, f' \,|\, f' \,\rangle) - 8\sqrt{NN'} \\ &= x^2 + y^2 - 1 + 8\langle\, e \,|\, e' \,\rangle\langle\, f \,|\, f' \,\rangle - 8\sqrt{NN'} \\ &= 8\langle\, e \,|\, e' \,\rangle\langle\, f \,|\, f' \,\rangle - 8\sqrt{NN'} + 1 + 4(e_0^2 + f_0^2) - 4(e_0 + f_0). \end{aligned}$$

Next we note:

$$xy - 4\mathbf{e}\cdot\mathbf{f} = \langle\, \mathbb{1} - 2e \,|\, \mathbb{1} - 2f \,\rangle = \langle\, e - e' \,|\, f - f' \,\rangle,$$

so that the right-hand side ($RHS$) becomes:

$$\begin{aligned} RHS &= (\langle\, e \,|\, f \,\rangle + \langle\, e' \,|\, f' \,\rangle - \langle\, e \,|\, f' \,\rangle - \langle\, e' \,|\, f \,\rangle)^2 \\ &= 8\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle + 8\langle\, e \,|\, f' \,\rangle\langle\, e' \,|\, f \,\rangle + 1 + 4(e_0^2 + f_0^2) - 4(e_0 + f_0). \end{aligned}$$

Now it is immediately seen that $LHS \leq RHS$ is equivalent to (55). □

## 7. Interpretation

We prove that $F(e)$ defined in (62) is a measure of the *unsharpness*, or *fuzziness*, of the effect $e$.

**Lemma 13.** *Let $e \in \mathcal{L}$. Then*

$$0 \leq F(e) \leq 1\,. \tag{64}$$

*Furthermore,*

(a) *$F(e) = 0$ iff $e_0 = |\mathbf{e}| = \frac{1}{2}$, that is, $e$ is a rank-1 projection (a nontrivial sharp effect);*
(b) *$F(e) = 1$ iff $e = e_0\mathbb{1}$, that is, $e$ is a trivial effect.*

*Proof.* Write

$$\begin{aligned} F(e)^2 &= \langle\, e \,|\, e\,\rangle + \langle\, e' \,|\, e'\,\rangle + 2\sqrt{\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle} \\ &= 1 + 2\sqrt{\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle} - [1 - \langle\, e \,|\, e\,\rangle - \langle\, e' \,|\, e'\,\rangle] \\ &= 1 + 2\sqrt{\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle} - 2\langle\, e \,|\, e'\,\rangle \\ &= 1 + 2\left[\sqrt{\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle} - \langle\, e \,|\, e'\,\rangle\right]. \end{aligned} \tag{65}$$

Now we consider $e, e'$ temporarily as elements of $M_3$ so that we may write:

$$\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle - \langle\, e \,|\, e'\,\rangle^2 = \langle\, e \times_o e' \,|\, e \times_o e'\,\rangle \leq 0$$

Note that $e, e'$ are timelike or lightlike and so this expression is zero if $e, e'$ are collinear; if $e, e'$ are not collinear, then $e \times_o e'$ must be spacelike. (Recall that $\langle\, e \,|\, k\,\rangle = 0$ is only possible for timelike or lightlike $e, k$ if both are lightlike and $k = \alpha e$ with $\alpha \geq 0$.)
Hence we have $\sqrt{\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle} \leq \langle\, e \,|\, e'\,\rangle$, and therefore $F(e) \leq 1$.

(a) $F(e) = 0$ is equivalent to $e_0^2 - |\mathbf{e}|^2 = 0 = (1 - e_0)^2 - |\mathbf{e}|^2$, which holds iff $|\mathbf{e}| = e_0 = 1 - e_0 = \frac{1}{2}$.

(b) From the above expression for $F(e)^2$ it is clear that $F(e) = 1$ happens if and only if the term in square brackets is zero, that is, if and only if $e, e'$ are collinear, so that $e = e_0\mathbb{1}$. □

Note that $F(e)^2 = 1 - 4|\mathbf{e}|^2 = 1 - |\tilde{\mathbf{e}}|^2$ in the case of an unbiased effect $e$, which confirms the interpretation sketched in the preceding subsection.

In [12] a measure $S(e)$ of the *sharpness* of an effect $e$ is introduced that is crucial for the formulation of the coexistence condition. It is defined as follows (in our notation):

$$S(e) = 2\left[\langle\, e \,|\, e'\,\rangle - \sqrt{\langle\, e \,|\, e\,\rangle\langle\, e' \,|\, e'\,\rangle}\right]. \tag{66}$$

In light of the calculation (65) it is immediately seen that the sharpness $S(e)$ is closely related to $F(e)$:

$$S(e) = 1 - F(e)^2\,. \tag{67}$$

The properties desired of a measure $S(e)$ of sharpness of an effect were proposed in a brief paper of one of the present authors [15] and are satisfied in the present case as shown in [12] and evident from Lemma 13:

(68) $0 \le S(e) \le 1\,;$

(69) $S(e) = 0 \iff e$ is a trivial effect;

(70) $S(e) = 1 \iff e$ is a nontrivial projection (nontrivial sharp effect);

(71) $S(e) = S(e')$.

The quantities $B(e), B(f)$ are the measures of bias $x, y$, scaled with the fuzziness parameters. It is not hard to show that $F(e) \in [-1, 1]$ and $B(e) \in [-1, 1]$. Introducing the abbreviations

$$F := F(e)^2 + F(f)^2\,, \quad B := B(e)^2 + B(f)^2\,, \tag{72}$$

inequality (61) can be recast in the form

$$1-(1-F)(1-B) = F+B-FB = \tfrac{1}{2}\left[F(2-B) + B(2-F)\right] \ge 1-(xy-4\mathbf{e}\cdot\mathbf{f})^2\,. \tag{73}$$

It can be shown that the expressions on both sides are non-negative. For the left-hand side this follows from the fact that $0 \le B, F \le 2$. In the case of the right-hand side, one may note that

$$|(\sqrt{|x|}, \sqrt{2|\mathbf{e}|})|^2 = |x| + 2|\mathbf{e}| \le 1\,, \quad |(\sqrt{|y|}, \sqrt{2|\mathbf{f}|})|^2 = |y| + 2|\mathbf{f}| \le 1\,,$$

and then use the Cauchy-Schwarz inequality for these two-dimensional vectors to obtain:

$$|xy| + 4|\mathbf{e}||\mathbf{f}| \le 1.$$

We rewrite the right-hand side of (73) as follows:

$$\begin{aligned} 1-(xy-4\mathbf{e}\cdot\mathbf{f})^2 &= 1 - x^2y^2 - 16(\mathbf{e}\cdot\mathbf{f})^2 + 8xy\mathbf{e}\cdot\mathbf{f} \\ &= 1 - x^2y^2 + 16\left[|\mathbf{e}|^2|\mathbf{f}|^2 - (\mathbf{e}\cdot\mathbf{f})^2\right] - 16|\mathbf{e}|^2|\mathbf{f}|^2 + 8xy\mathbf{e}\cdot\mathbf{f} \\ &= \left[1 - (|xy| + 4|\mathbf{e}||\mathbf{f}|)^2\right] + 8\left[|xy||\mathbf{e}||\mathbf{f}| + xy\mathbf{e}\cdot\mathbf{f}\right] + 16[\mathbf{e}\times\mathbf{f}]^2\,. \end{aligned} \tag{74}$$

This shows that the lower bound of inequality (73) is composed of three non-negative contributions, one of which is directly related to the commutator of $e$ and $f$. What emerges from the form (73) of the coexistence condition is that *biasedness* figures as a third factor in the interplay between *noncommutativity* and *unsharpness* deciding over the coexistence of $e$ and $f$.

## 8. Discussion

We have deduced an inequality which constitutes a necessary and sufficient condition for the coexistence of a pair of qubit effects. The form of this condition differs from the conditions obtained in [12] and [13], which were shown to be equivalent in the latter publication. The equivalence of the latter conditions with an earlier version of our result (given in arXiv:0802.4167v2 and reproduced here in the Appendix) had only been confirmed using numerical techniques in [13]. We have proven the equivalence analytically by explicitly showing how the condition (55) obtained here directly translates into the condition (61), which was obtained in [13].

We have recast this last condition in a form that highlights the interplay between three significant factors: a measure of noncommutativity and measures of unsharpness and bias, respectively. It would be interesting to find formulations of the latter two measures that are purely in operator terms so as to allow generalization to higher dimensions.

An important difference between our approach and the two other approaches lies in the fact that the latter are based on the standard parametrization of the set of qubit effects, while in the present paper the focus is on the geometric and order structures of the set of effects. This may be of use for the open problems of finding coexistence conditions for more than two effects or two observables and for higher-dimensional Hilbert spaces as well as obtaining generic operational interpretations of such conditions.

## Appendix: Alternative Formulation

An alternative formulation of the main result that uses the basis vectors $g, g'$ was developed in an early version of the present work (arXiv:0802.4167). As the authors of [12] and [13] refer to this, it is reproduced here for comparison. We recall that in this formulation we are working under the assumption $e, f \in \mathcal{L}$ with $e\sigma f$ and $[e,f] \neq \mathbb{O}$.

We use $g, g'$ to parametrize $a = \frac{1}{2}(e+f) - v = \frac{1}{2}(e+f) - \lambda g - \mu g' \in H$. Then the conditions (i)-(iii) of subsection 4.2 read as follows. Eq. (25) is automatically fulfilled since $g,\ g'$ are $\langle\ |\ \rangle$-perpendicular to $d$ (eq. (5)). The equation (26) for the hyperbola $\mathcal{H}$ becomes

$$H(\lambda,\mu) \equiv \lambda^2 \langle\, g \,|\, g\,\rangle + \mu^2 \langle\, g' \,|\, g'\,\rangle + 2\lambda\mu \langle\, g \,|\, g'\,\rangle + \tfrac{1}{4}\langle\, d \,|\, d\,\rangle = 0. \tag{75}$$

Note that inequality (14), that is, $\Delta(e,f) < 0$, is the determinant condition ensuring that (75) describes a hyperbola in the $\lambda - \mu-$plane.

The conditions (32) and (34) for the end points of $\mathcal{S}_a, \mathcal{S}_b$ translate into

$$\mu \langle\, e + f \,|\, g'\,\rangle = \langle\, e \,|\, f\,\rangle \quad \text{for } \mathcal{S}_a \tag{76}$$

and

$$\lambda \langle\, e' + f' \,|\, g\,\rangle = \langle\, e' \,|\, f'\,\rangle \quad \text{for } \mathcal{S}_b. \tag{77}$$

Recalling eq. (17), these linear equations now can be written in the form

$$\mu \;=\; \frac{\langle\, e \,|\, f\,\rangle}{2(e_1 f_2 - e_2 f_1)} = \frac{\langle\, e \,|\, f\,\rangle}{2|\mathbf{e}\times\mathbf{f}|} = \frac{1}{2}\langle\, e \,|\, f\,\rangle \sqrt{\frac{|\langle\, d \,|\, d\,\rangle|}{|\Delta(e,f)|}} \equiv \mu_0 \quad \text{for } \mathcal{S}_a\,, \tag{78}$$

$$\lambda \;=\; \frac{\langle\, e' \,|\, f'\,\rangle}{2(e_1 f_2 - e_2 f_1)} = \frac{\langle\, e' \,|\, f'\,\rangle}{2|\mathbf{e}\times\mathbf{f}|} = \frac{1}{2}\langle\, e' \,|\, f'\,\rangle \sqrt{\frac{|\langle\, d \,|\, d\,\rangle|}{|\Delta(e,f)|}} \equiv \lambda_0 \quad \text{for } \mathcal{S}_b\,. \tag{79}$$

These two equations describe a horizontal and a vertical line in the $\lambda - \mu-$plane each of which intersects the hyperbola (75), thus cutting out the two bounded segments $\mathcal{S}_a$ and $\mathcal{S}_b$, see Fig. 3.[2] By assuming $0 < e_1 f_2 - e_2 f_1 = |\mathbf{e}\times\mathbf{f}|$, as we did in (17), we have ensured that $\mathcal{H}_a$ lies in the first quadrant ($\lambda \geq 0, \mu \geq 0$). We will eventually write the resulting inequalities in a symmetric fashion with respect to $e$ and $f$; hence the case where $\mathcal{H}_a$ lies in the third quadrant need not be considered separately.

The coexistence condition $\mathcal{S}_a \cap \mathcal{S}_b \neq \mathbb{O}$ can be explored by considering the location of the intersection point $(\lambda_0, \mu_0)$ of the two lines described by $\lambda = \lambda_0$ and $\mu = \mu_0$ relative to the hyperbola branch $\mathcal{H}_a$. Let $P$ and $Q$ be the points

[2] We use the same notation $\mathcal{H}$, $\mathcal{H}_a$, $\mathcal{S}_a$ and $\mathcal{S}_b$ for the representations in the $\lambda - \mu-$plane of the hyperbola, its branch $\mathcal{B}(e) \cap \mathcal{B}(f)$, and its segments $\mathcal{S}_a$, $\mathcal{S}_b$.

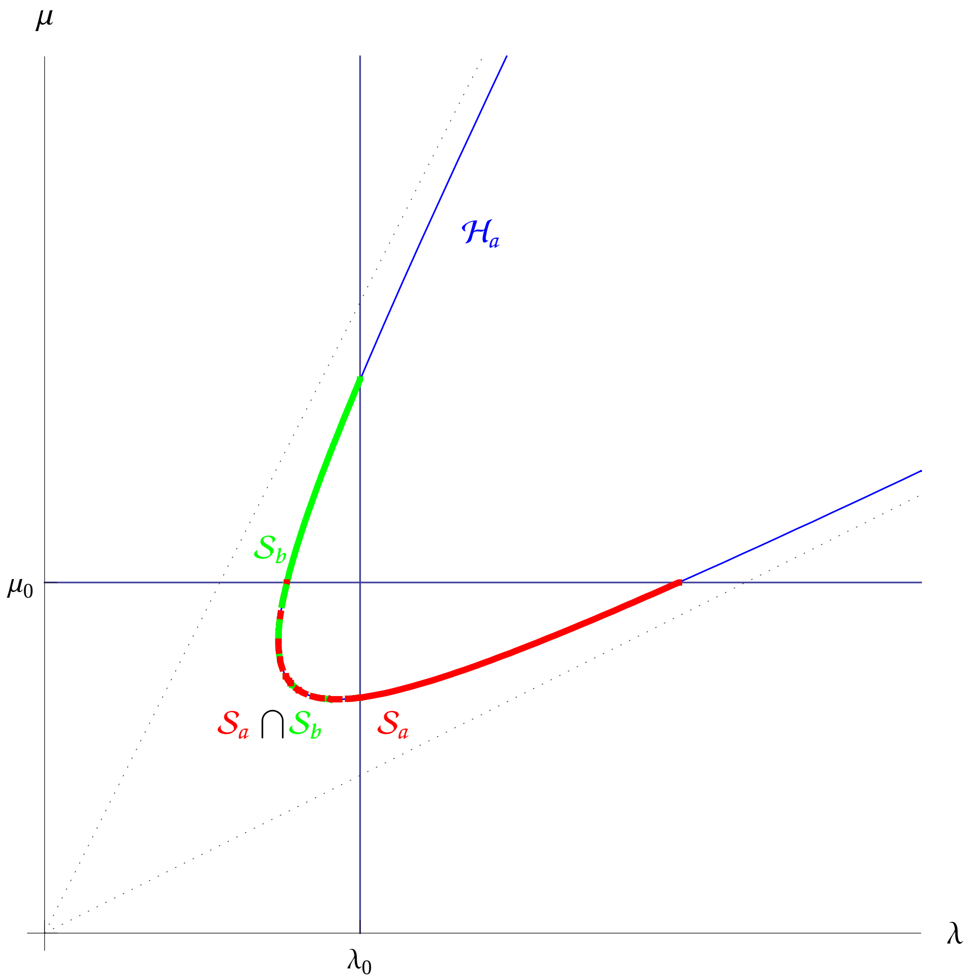


FIGURE 3. One branch $\mathcal{H}_a$ of the hyperbola described by Eq.(75) together with the segments $\mathcal{S}_a$ and $\mathcal{S}_b$ defined by $\mu \leq \mu_0$ and $\lambda \leq \lambda_0$, respectively. To every point in $\mathcal{S}_a \cap \mathcal{S}_b$ there exist effects $a$ and $b$ satisfying Eq.(1) and hence $e$ and $f$ are coexistent.

of $\mathcal{H}_a$ which have a tangent parallel to the $\lambda$-axis and $\mu$-axis, respectively, with coordinates $(\lambda_P, \mu_P)$ and $(\lambda_Q, \mu_Q)$ (see Fig. 4). After a short calculation we obtain

$$\lambda_P^2 = \frac{1}{4} \frac{|\langle\, d \,|\, d \,\rangle| D(e,f)^2}{|C(e,f)||\Delta(e,f)|}\,, \quad \mu_Q^2 = \frac{1}{4} \frac{|\langle\, d \,|\, d \,\rangle| D(e,f)^2}{|C(e',f')||\Delta(e,f)|}\,.$$

We have always:

$$P \in \mathcal{S}_a \quad \text{and} \quad Q \in \mathcal{S}_b. \tag{80}$$

Hence,

$$\lambda_0 \geq \lambda_P \implies \mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset\,; \tag{81}$$

$$\mu_0 \geq \mu_Q \implies \mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset\,. \tag{82}$$

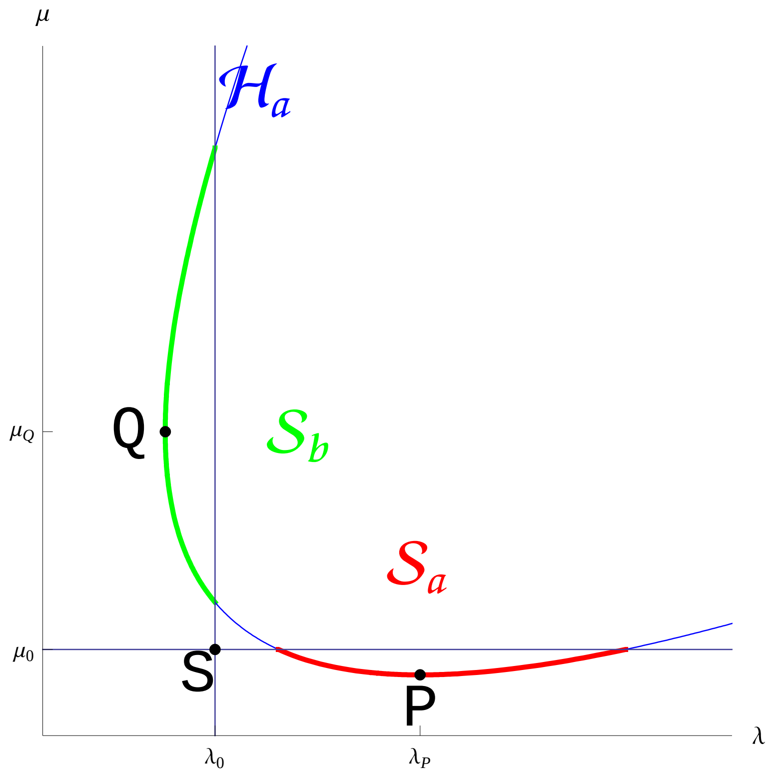


FIGURE 4. The only case where $\mathcal{S}_a \cap \mathcal{S}_b = \emptyset$ occurs if and only if $\lambda_0 < \lambda_P$ and $\mu_0 < \mu_Q$ and $S = (\lambda_0, \mu_0)$ lies outside the convex hull of the hyperbola branch $\mathcal{H}_a$.

In the remaining case of

$$\lambda_0 < \lambda_P \quad \text{and} \quad \mu_0 < \mu_Q \tag{83}$$

we have (see Fig. 4):

$$\mathcal{S}_a \cap \mathcal{S}_b \neq \emptyset \iff (\lambda_0, \mu_0) \in \mathrm{conv}(\mathcal{H}_a) \iff H(\lambda_0, \mu_0) \geq 0. \tag{84}$$

The inequalities $\lambda_0^2 \geq \lambda_P^2$, $\mu_0^2 \geq \mu_Q^2$ and $H(\lambda_0, \mu_0) \geq 0$ are collectively necessary and sufficient (though non-exclusive) conditions for the coexistence of $e, f$. With some rearrangement they assume the form:

$$D(e,f)^2 \;\leq\; -\langle\, e' \,|\, f' \,\rangle^2 C(e,f)\,; \tag{85}$$
$$D(e,f)^2 \;\leq\; -\langle\, e \,|\, f \,\rangle^2 C(e',f')\,; \tag{86}$$
$$-\Delta(e,f) \leq\; 2\langle\, e \,|\, f \,\rangle \langle\, e' \,|\, f' \,\rangle D(e,f) + \langle\, e' \,|\, f' \,\rangle^2 C(e,f) + \langle\, e \,|\, f \,\rangle^2 C(e',f')\,. \tag{87}$$

Noting (11), (12), the first two inequalities can also be written as:

$$-\Delta(e,f) \;\;\leq\;\; -\langle\, e' \,|\, e' \,\rangle \langle\, f' \,|\, f' \,\rangle\, C(e,f)\,, \tag{88}$$
$$-\Delta(e,f) \;\;\leq\;\; -\langle\, e \,|\, e \,\rangle \langle\, f \,|\, f \,\rangle\, C(e',f')\,. \tag{89}$$

These two inequalities are automatically satisfied if $e, f$ are not spacelike related or if they commute since in these cases $-\Delta(e,f) < 0$. Hence we have established the following.

**Theorem 5.** *Let $e, f \in \mathcal{L} \subset M_4$. Then $e$ and $f$ are coexistent if and only if at least one of the following conditions is satisfied:*

(90) $-\Delta(e,f) \;\leq\; \langle\, e' \,|\, e' \,\rangle\langle\, f' \,|\, f' \,\rangle |C(e,f)|\,;$

(91) $-\Delta(e,f) \;\leq\; \langle\, e \,|\, e \,\rangle\langle\, f \,|\, f \,\rangle\, |C(e',f')|\,;$

(92) $-\Delta(e,f) \leq\; 2\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle D(e,f) - \langle\, e' \,|\, f' \,\rangle^2 |C(e,f)| - \langle\, e \,|\, f \,\rangle^2 |C(e',f')|\,.$

Finally we show explicitly that this formulation is equivalent to the main result.

**Theorem 6.** *Inequality (55) is satisfied if and only if at least one of the inequalities (90), (91) or (92) is satisfied.*

*Proof.* We note first that inequalities (90), (91) and (92) of Theorem 5 can be rephrased in the equivalent form

$$D(e,f)^2 \;\leq\; \langle\, e' \,|\, f' \,\rangle^2 |C(e,f)|\,; \tag{93}$$

$$D(e,f)^2 \;\leq\; \langle\, e \,|\, f \,\rangle^2 |C(e',f')|\,; \tag{94}$$

$$\big(D(e,f) - \langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle\big)^2 \;\leq\; NN'\,. \tag{95}$$

Here we use the abbreviations (56) for $N, N'$. We also observe that the inequality (55) of Theorem 3 can be split up into two (non-exclusive) bits: (55) holds if and only if (95) is true or

$$D(e,f) - \langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle \leq 0\,. \tag{96}$$

Assume that at least one of the inequalities of Theorem 5 hold. If the last one, (95), holds, then (55) of Theorem 3 follows.

Next suppose that the last inequality (95) is violated, so that one of (93) or (94) is valid. It follows readily that each of them implies (96). Indeed, one of the following two chains of inequalities applies:

$$D(e,f) \leq \langle\, e' \,|\, f' \,\rangle\sqrt{|C(e,f)|} < \langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle\,;$$
$$D(e,f) \leq \langle\, e \,|\, f \,\rangle\sqrt{|C(e',f')|} < \langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle\,.$$

To prove the converse implication, we note that (55) is also equivalent to the exclusive alternative: either (95), or

$$D(e,f) - \langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle < -\sqrt{NN'}\,. \tag{97}$$

We assume that (97) holds. (In the other case the conclusion follows trivially.) Then (95) is violated and so we have to show that (93) or (94) follows. Suppose these are both violated, i.e., $D(e,f)^2 > \langle\, e' \,|\, f' \,\rangle^2 |C(e,f)|$ and $D(e,f)^2 > \langle\, e \,|\, f \,\rangle^2 |C(e',f')|$. Thus we obtain

$$\langle\, e' \,|\, f' \,\rangle^2 |C(e,f)| < D(e,f)^2 < (\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle - \sqrt{NN'})^2\,,$$
$$\langle\, e \,|\, f \,\rangle^2 |C(e',f')| < D(e,f)^2 < (\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle - \sqrt{NN'})^2\,.$$

After some rearrangements and using

$$|C(e,f)| = \langle\, e \,|\, f \,\rangle^2 - N\,, \quad |C(e',f')| = \langle\, e' \,|\, f' \,\rangle^2 - N'\,,$$

this implies

$$\begin{aligned} 2\sqrt{NN'}\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle &< NN' + N\langle\, e' \,|\, f' \,\rangle^2 \,, \\ 2\sqrt{NN'}\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle &< NN' + N'\langle\, e \,|\, f \,\rangle^2 \,. \end{aligned}$$

Further rearrangement yields:

$$\begin{aligned} \sqrt{NN'}(\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle - \sqrt{NN'}) &< \sqrt{N}\langle\, e' \,|\, f' \,\rangle[\sqrt{N}\langle\, e' \,|\, f' \,\rangle - \sqrt{N'}\langle\, e \,|\, f \,\rangle] \,, \\ \sqrt{NN'}(\langle\, e \,|\, f \,\rangle\langle\, e' \,|\, f' \,\rangle - \sqrt{NN'}) &< \sqrt{N'}\langle\, e \,|\, f \,\rangle[\sqrt{N'}\langle\, e \,|\, f \,\rangle - \sqrt{N}\langle\, e' \,|\, f' \,\rangle] \,. \end{aligned}$$

The expression on the left-hand sides is non-negative. The right-hand sides are not both non-negative, so exactly one of them is non-positive, and the corresponding inequality is thus violated (even in the limiting case of $0 < 0$). This is a contradiction. □

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