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# On Classical State Space Realizability of Bilinear Input-Output Differential Equations

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*Abstract*— **This paper studies the realizability property of continuous-time bilinear i/o equations in the classical state space form. Constraints on the parameters of the bilinear i/o model are suggested that lead to realizable models. The paper proves that the 2nd order bilinear i/o differential equation, unlike the discrete-time case, is always realizable in the classical state space form. The complete list of 3rd and 4th order realizable i/o bilinear models is given and two subclasses of realizable i/o bilinear systems are suggested. Our conditions rely basically upon the property that certain combinations of coefficients of the i/o equations are zero or not zero. We provide explicit state equations for all realizable 2nd and 3rd order bilinear i/o equations, and for one realizable subclass of bilinear i/o equations of arbitrary order.**

## I. Introduction

In many practical situations continuous-time input-output (i/o) models of the form

$$y^{(n)} = \varphi(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(s)}) \tag{1}$$

are deduced from i/o data when no information regarding the structure of the observed dynamical system is available *a priori*. Such representations form the basis of much modern identification theory. Identification therefore involves model structure selection prior to parameter estimation. In practice, this involves selecting a form of the multivariate nonlinear function $\varphi(\cdot)$ and the specification of the maximal derivatives for the inputs and outputs that appear in equation (1). Typically, $\varphi$ is assumed to be a low order polynomial, most often a bilinear or quadratic function, and not all possible terms are included since in most cases a more complex model does not necessarily equate to a better model.

The realization problem is defined as follows: given a nonlinear system described by the i/o differential equation of the form (1) with $s \leq n-1$, and $\varphi(\cdot)$ smooth, find, if possible, the state coordinates $x \in \mathbb{R}^n$, $x = \psi(y, \dots, y^{(n-1)}, u, \dots, u^{(s)})$ such that in these coordinates the system takes the classical state space form

$$\dot{x} = f(x, u), \qquad y = h(x), \tag{2}$$

called the realization of (1). It is known that an arbitrarily structured empirical model (1) does not necessarily have a state space realization of the form (2) [1], [2]. Using such a model is highly undesirable for further control design since the state space description is central in modern nonlinear control theory. Thus a basic question is that of deciding when a given i/o equation admits a realization. In this paper our purpose is to find the subsets of bilinear i/o equations

$$y^{(n)} = \sum_{i=1}^{n} a_i y^{(n-i)} + \sum_{i=1}^{n} b_i u^{(n-i)} + \sum_{i,j=1}^{n} c_{ij} y^{(n-i)} u^{(n-j)} \tag{3}$$

that are guaranteed to have a state space representation of order $n$, and as such are good candidate structures to be used in system identification. Since the bilinear model (3) is linear in the parameters, it lends itself easily to the well-established parameter estimation algorithms.

Several (equivalent) necessary and sufficient realizability conditions exist in the literature [1], [2], [3], [4], [5], that allow one to decide if the given i/o equation of the form (1) admits a state space representation or not. These conditions, though transparent and inherently simple, are not helpful if we want to check realizability directly from the knowledge of the bilinear i/o model parameters $a_i$, $b_i$, $c_{ij}$. The objective here is to study further the realizability property of the subclass of i/o bilinear differential models and to suggest constraints on the parameters of the bilinear model that can lead to realizable models. Our results will extend earlier results on realizability of discrete-time bilinear i/o equations [6]. Note that, in the continuous-time case, the applicability of bilinear models is less limited than in the discrete-time case because of their greater generality in approximation, and also because bilinearity often occurs naturally in the continuous-time case [7], [8], [9]. Many relevant physical processes have been satisfactorily modelled by means of bilinear models. Moreover, the results for continuous-time bilinear i/o model realizability are more important than the corresponding results in the discrete-time case, since in the continuous-time case, unlike the discrete-time case [10], no general subclasses of realizable i/o equations have yet been found.

## II. Realizability conditions for i/o bilinear model

Despite the structural simplicity of the bilinear i/o model, the general realizability conditions yield little insight and do not tell us in terms of the parameters $a_i$, $b_i$, $c_{ij}$, which bilinear model is realizable in the classical state space form and which is not. To give a more general view of the nature of parameter restrictions necessary for realizability, it is instructive to consider the special cases where $n = 1, 2, 3, 4$.

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In our proofs below we use a constructive algorithm for finding, if possible, the state variables from the $n$-th order input-output differential equation, where the highest order time derivative $u^{(n-1)}$ of the input $u$ appears linearly, as is the case for bilinear systems. The first step of the algorithm to eliminate $u^{(n-1)}$ is described in [3]. As shown in [11], the generalized state variables, defined in the first step of this algorithm, are the independent invariants of a certain vector field. This vector field is the Lie bracket of a total time derivative operator associated with the i/o differential equation and the partial derivative operator with respect to $u^{(n-1)}$. We write the generalized state equations in terms of the invariants of this vector field, containing now $u^{(n-2)}$ as the highest time derivative of input. We check their linearity with respect to $u^{(n-2)}$ as the necessary and sufficient condition for eliminating $u^{(n-2)}$. If the equations are not linear, we make them linear by suitable selection of the bilinear i/o equation parameters.

Once these generalized state equations are linear, we define a new vector field as the Lie bracket of a total time derivative operator associated with the generalized state equations and the partial derivative operator with respect to $u^{(n-2)}$. The next step is to find the independent invariants of this vector field and use them as the new generalized state variables. By repeating this procedure $n-1$ times, always checking the linearity of the resulting generalized state equations with respect to the highest time derivative of input, and where necessary, making them linear, we finally obtain the classical state equations that do not depend on input derivatives anymore.

At each step of the algorithm, in general, by making the generalized state equations linear with respect to the highest time derivative of the input, we get several alternative restrictions on system parameters which means that there is a branching at each step of the algorithm. Combining the restrictions, obtained at the different steps, we end up with a set of branch-dependent rather complicated realizability conditions.

Now we consider the cases $n = 1, 2, 3, 4$ separately. The first order bilinear input-output model $y^{(1)} = a_1 y + b_1 u + c_{11} y u$ is obviously realizable in the classical state-space form and the choice $x(t) = y(t)$ will yield the state space model. Propositions 1, 2 and 3 below consider the cases $n = 2$, $n = 3$ and $n = 4$, respectively.

**Proposition 1.** *The second order bilinear system described by the i/o equation (3) with $n = 2$ is always realizable in the classical state space form and for the case $c_{11} \neq 0$, the state equations are*

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{c_{11}}\left[b_1 + c_{21}x_1 - c_{11}e^{c_{11}u}x_2\right] \\
\dot{x}_2 &= -\frac{e^{-c_{11}u}}{c_{11}^2}\left[-b_1(a_1c_{11} + c_{21})\right. \\
&+ c_{11}(b_2c_{11} - b_1c_{12})u \\
&+ (a_2c_{11}^2 - a_1c_{11}c_{21} - c_{21}^2)x_1 \\
&+ (c_{11}^2c_{22} - c_{11}c_{12}c_{21})x_1u + c_{11}^2c_{12}e^{c_{11}u}x_2u \\
&+ \left. c_{11}(a_1c_{11} + c_{21})e^{c_{11}u}x_2\right] \qquad (4) \\
y &= x_1.
\end{aligned}$$

*Proof.* According to the theory described above, applied to the second order bilinear i/o equation, the state coordinates can be obtained as the independent invariants of the vector field

$$L_f\frac{\partial}{\partial u^{(1)}} = -\frac{\partial}{\partial u} - \left(b_1 + c_{11}y^{(1)} + c_{21}y\right)\frac{\partial}{\partial y^{(1)}}$$

The latter yields $x_1 = y, \quad x_2 = \frac{e^{-c_{11}u}}{c_{11}}(b_1 + c_{11}\dot{y} + c_{21}y)$. ■

One may investigate what happens to the state equations (4) for the special case $c_{11} = 0$ when the above choice for the state coordinate $x_2$ is impossible. In that case the vector field we are looking for simplifies to

$$L_f\frac{\partial}{\partial u^{(1)}} = -\frac{\partial}{\partial u} - (b_1 + c_{21}y)\frac{\partial}{\partial y^{(1)}}$$

which yields a different choice of the state coordinates $x_1 = y, \quad x_2 = \dot{y} - c_{21}yu - b_1u$. The state equations now become $\dot{x}_1 = x_2 + c_{21}x_1u + b_1u$, $\dot{x}_2 = a_2x_1 + a_1x_2 + (c_{22} + a_1c_{21})x_1u + (c_{12} - c_{21})x_2u + (b_2 + a_1b_1)u + b_1(c_{12} - c_{21})u^2 + (c_{12}c_{21} - c_{21}^2)x_1u^2$, $y = x_1$.

**Proposition 2.** The third order bilinear system described by the i/o equation (3) with $n = 3$, is realizable in the classical state space form if and only if either one of the following conditions is satisfied

(i) $b_1 = c_{11} = c_{21} = c_{31} = 0$,
(ii) $c_{11} = 0, c_{21} = c_{12}$.

*Proof.* One starts by looking for the conditions that allow one to eliminate the second order input time derivatives as the necessary conditions for realizability. Note that this can always be done for the 3rd order bilinear input-output equation, since it is linear with respect to $u^{(2)}$. Using the total time derivative operator $f$, associated with the 3rd order bilinear i/o equation, we define a vector field

$$L_f\frac{\partial}{\partial u^{(2)}} = -\frac{\partial}{\partial u^{(1)}} - \left(b_1 + \sum_{i=1}^{3}c_{i1}y^{(3-i)}\right)\frac{\partial}{\partial y^{(2)}}$$

and use its independent invariants $x_i^{[1]}$, $i = 1, ..., 3$ as the generalized state variables:

$$\begin{aligned}
x_1^{[1]} &= y, \quad x_2^{[1]} = y^{(1)} \\
x_3^{[1]} &= -\frac{1}{c_{11}}\exp(-c_{11}u^{(1)})\left[b_1 + \sum_{i=1}^{3}c_{i1}y^{(3-i)}\right].
\end{aligned} \qquad (5)$$

Using the generalized state variables $x_i^{[1]}$ defined by (5), the corresponding first two generalized state equations are

$$\begin{aligned}
\dot{x}_1^{[1]} &= x_2^{[1]} \\
\dot{x}_2^{[1]} &= x_3^{[1]}\exp(c_{11}u^{(1)}) - \frac{b_1}{c_{11}} - \frac{1}{c_{11}}\sum_{i=2}^{3}c_{i1}x_{4-i}^{[1]}.
\end{aligned} \qquad (6)$$

Next, we look for the conditions that allow us to eliminate $u^{(1)}$ from the generalized state equations (6) by defining the new generalized state variables. For that to be possible, according to the theory, the equations (6) must be linear with respect to $u^{(1)}$. The second equation of (6) is linear only if the coefficient $c_{11}$ equals zero. This condition has no alternative, because every generalized state equation must be linear with respect to $u^{(1)}$. Consequently, the first necessary realizability condition for the 3rd order bilinear input-output equation reads

$$c_{11} = 0. \tag{7}$$

With the above condition the generalized state variables have the following form

$$\begin{aligned} x_1^{[1]} &= y, \quad x_2^{[1]} = y^{(1)} \\ x_3^{[1]} &= y^{(2)} - \left[ b_1 + \sum_{i=1}^{3} c_{i1} y^{(3-i)} \right] u^{(1)}. \end{aligned} \tag{8}$$

and the generalized state equations in variables (8) become

$$\begin{aligned} x_1^{[1]} &= x_2^{[1]} \\ x_2^{[1]} &= x_3^{[1]} + \left( b_3 + \sum_{i=2}^{3} c_{i1} x_{(4-i)}^{[1]} \right) u^{(1)} \\ x_3^{[1]} &= \sum_{i=2}^{3} a_i x_{4-i}^{[1]} + \sum_{i=2}^{3} b_\alpha u^{(3-\alpha)} \\ &+ \sum_{i=2}^{3} \sum_{\alpha=2}^{3} c_{i\alpha} x_{(4-i)}^{[1]} u^{(3-\alpha)} - c_{31} x_2^{[1]} u^{(1)} \\ &+ \left[ a_1 + c_{13} u (c_{12} - c_{21}) u^{(1)} \right] \cdot \\ &\cdot \left[ x_3^{[1]} + \left( b_1 + \sum_{i=2}^{3} c_{i1} x_{(4-i)}^{[1]} \right) u^{(1)} \right]. \end{aligned} \tag{9}$$

Still, the third generalized state equation in (9) is nonlinear with respect to the highest time derivative $u^{(1)}$ of the control variable. To make it linear, one has to put the restrictions on the system parameters. Together with condition (7) we obtain two sets of conditions, being necessary and sufficient for realization of equation (5) and given in the formulation of the proposition. ■

**Remark.** For identification purposes, the condition $c_{21} = c_{12}$, unless both parameters are equal to zero, is unnatural since there is no reason to assume that the terms $\dot{y}\ddot{u}$ and $\ddot{y}\dot{u}$ should have equal coefficients. For that reason we suggest the following 3rd order realizable i/o equations to be used in modelling

(i) $b_1 = c_{11} = c_{21} = c_{31} = 0$,
(ii) $c_{11} = c_{21} = c_{12} = 0$.

The state equations, corresponding to (i) are given as a special case of equations (25) for $n = 3$ (see below).

**Proposition 2A.** The state equations, corresponding to the case (ii) are

$$\begin{aligned} \dot{x}_1 &= x_2 + b_1 u + c_{31} x_1 u \\ \dot{x}_2 &= x_3 + a_1 b_1 b_2 u + (a_1 b_2 c_{31} + c_{32}) x_1 u \\ &+ (c_{22} - 2c_{31}) x_2 u + (0.5 b_1 c_{13} \\ &+ 0.5 b_1 c_{22} - 1.5 b_1 c_{31}) u^2 \\ &+ (0.5 c_{13} c_{31} + 0.5 c_{22} c_{31} - 1.5 c_{31}^2) x_1 u^2 \\ \dot{x}_3 &= a_3 x_1 + a_2 x_2 + a_1 x_3 + (a_2 b_1 + a_1 b_2 + b_3 \\ &+ a_1^2 b_1) u + (a_2 c_{31} + a_1 c_{32} + c_{33} + a_1^2 c_{31}) u x_1 \\ &+ (a_1 c_{22} + c_{23} - 2a_1 c_{31}) u x_2 + (b_2 c_{13} - b_2 c_{22} \\ &+ b_1 c_{23} + 0.5 a_1 b_1 (3c_{13} - c_{22} - c_{31}) + b_2 c_{31} \\ &- b_1 c_{32}) u^2 + (1.5 a_1 c_{13} c_{31} - 0.5 a_1 c_{22} c_{31} + c_{23} c_{31} \\ &- 0.5 a_1 c_{31}^2 + c_{13} c_{32} - c_{22} c_{32}) x_1 u^2 \\ &+ (c_{13} c_{22} - c_{22}^2 - 1.5 c_{13} c_{31} + 2.5 c_{22} c_{31} x_2 \\ &- 1.5 c_{31}^2 - c_{32}) x_2 u^2 + (c_{13} - c_{22} + c_{31}) x_3 u^2 \\ &+ \tfrac{1}{2} (b_1 c_{13}^2 - b_1 c_{22}^2 - b_1 c_{13} c_{31} + 3 b_1 c_{22} c_{31} \\ &- 2 b_1 c_{31}^2) u^3 + 0.5 (c_{13}^2 c_{31} - c_{22}^2 c_{31} - c_{13} c_{31}^2 \\ &+ 3 c_{22} c_{31}^2 - 2 c_{31}^3) x_1 u^3 \\ y &= x_1. \end{aligned}$$

**Proposition 3.** The fourth order bilinear system described by the i/o equation (3) with $n = 4$, is realizable in the classical state space form if and only if one of the following conditions is satisfied

(i) $b_1 = b_2 = 0$,
$c_{11} = c_{12} = c_{21} = c_{22} = c_{31} = c_{32} = c_{41} = c_{42} = 0$
(ii) $b_1 = c_{11} = c_{12} = c_{21} = c_{31} = c_{41} = 0$, $c_{13} = c_{22}$
(iii) $c_{11} = c_{12} = c_{13} = c_{21} = c_{22} = c_{31} = 0$, $c_{14} - c_{23} + c_{32} - c_{41} = 0$

*Proof.* According to theory, one has to look for the conditions that allow one to eliminate the third order input time derivatives as the necessary conditions for realizability. Note that this can always be done for 4th order bilinear input-output equations, since it is linear with respect to $u^{(3)}$. Using the total time derivative operator $f$ associated with the i/o equation, we define a vector field

$$\begin{aligned} L_f \frac{\partial}{\partial u^{(3)}} &= -\frac{\partial}{\partial u^{(2)}} \\ &- \left( b_1 + c_{11} y^{(3)} + c_{21} y^{(2)} + c_{31} y^{(1)} + c_{41} y \right) \frac{\partial}{\partial y^{(3)}} \end{aligned}$$

and use its independent invariants $x_i^{[1]}$, $i = 1, ..., n$ as the generalized state variables:

$$\begin{aligned} &x_1^{[1]} = y, \quad x_2^{[1]} = y^{(1)}, \quad x_3^{[1]} = y^{(2)}, \\ &x_4^{[1]} = \frac{1}{c_{11}} \exp(-c_{11} u^{(2)}) \cdot \\ &\left[ b_1 + c_{11} y^{(3)} + c_{21} y^{(2)} + c_{31} y^{(1)} + c_{41} y \right]. \end{aligned} \tag{10}$$

Using the generalized state variables $x_i^{[1]}$ defined by (10), the corresponding first three generalized state equations are

$$\begin{aligned} \dot{x}_1^{[1]} &= x_2^{[1]}, \quad \dot{x}_2^{[1]} = x_3^{[1]}, \\ \dot{x}_3^{[1]} &= x_4^{[1]} \exp\left( c_{11} u^{(2)} \right) \\ &- \frac{1}{c_{11}} \left( b_1 + c_{21} y^{(2)} + c_{31} y^{(1)} + c_{41} y \right). \end{aligned} \tag{11}$$

Next, we look for the conditions that allow us to eliminate $u^{(2)}$ from the generalized state equations (11) via defining the new generalized state variables. For this to be possible, the equations (11) must be linear with respect to $u^{(2)}$. The third equation of (11) is linear only if the coefficient $c_{11}$ equals zero. This condition has no alternative, because every generalized state equation must be linear with respect to $u^{(2)}$. Consequently, the first necessary realizability condition for the 4th order bilinear input-output equation reads

$$c_{11} \quad = \quad 0. \tag{12}$$

With the above condition the generalized state variables have the following form

$$\begin{aligned} &x_1^{[1]} = y, \quad x_2^{[1]} = y^{(1)}, \quad x_3^{[1]} = y^{(2)}, \\ &x_4^{[1]} = y^{(3)} - \left( b_1 + c_{21} y^{(2)} + c_{31} y^{(1)} + c_{41} y \right) u^{(2)}. \end{aligned}$$

The generalized state equations now become

$$\begin{aligned} \dot{x}_1^{[1]} &= x_2^{[1]}, \quad \dot{x}_2^{[1]} = x_3^{[1]}, \\ \dot{x}_3^{[1]} &= x_4^{[1]} + \left( b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right) u^{(2)} \\ \dot{x}_4^{[1]} &= \left[ a_1 + (c_{12} - c_{21}) u^{(2)} + c_{13} u^{(1)} + c_{14} u \right] \cdot \\ &\quad \cdot \left[ x_4^{[1]} + \left( b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right) u^{(2)} \right] \\ &+ a_2 x_3^{[1]} + a_3 x_2^{[1]} + a_4 x_1^{[1]} + b_2 u^{[2]} + b_3 u^{[1]} + b_4 u \\ &+ (c_{22} - c_{31}) x_3^{[1]} u^{(2)} + c_{23} x_3^{[1]} u^{(1)} + c_{24} x_3^{[1]} \\ &+ (c_{32} - c_{41}) x_2^{[1]} u^{(2)} + c_{33} x_2^{[1]} u^{(1)} + c_{34} x_2^{[1]} u \\ &+ c_{42} x_1^{[1]} u^{(2)} + + c_{43} x_1^{[1]} u^{(1)} + c_{44} x_1^{[1]} u. \end{aligned} \tag{13}$$

Note that the fourth equation of (13) is still nonlinear with respect to $u^{(2)}$ and will be linear only if $c_{12} = c_{21}$ or $b_1 = c_{21} = c_{31} = c_{41} = 0$, which together with the condition (12), gives two sets of conditions necessary to hold for elimination of $u^{(2)}$

$$c_{11} = 0, \quad c_{12} = c_{21}, \tag{14}$$

or alternatively,

$$b_1 = c_{11} = c_{21} = c_{31} = c_{41} = 0. \tag{15}$$

So, we have a first branching of conditions at this point. We continue first with the set of conditions (14) and return later to conditions (15). With (14), equations (13) read

$$\begin{aligned} \dot{x}_1^{[1]} &= x_2^{[1]}, \quad \dot{x}_2^{[1]} = x_3^{[1]}, \\ \dot{x}_3^{[1]} &= x_4^{[1]} + \left( b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right) u^{(2)} \\ \dot{x}_4^{[1]} &= a_2 x_3^{[1]} + a_3 x_2^{[1]} + a_4 x_1^{[1]} + b_3 u^{(1)} + b_4 u \\ &+ c_{23} x_3^{[1]} u^{(1)} + c_{24} x_3^{[1]} u + c_{33} x_2^{[1]} u^{(1)} + c_{34} x_2^{[1]} u \\ &+ c_{43} x_1^{[1]} u^{(1)} + c_{44} x_1^{[1]} u \\ &+ \left( a_1 + c_{13} u^{(1)} + c_{14} u \right) x_4^{[1]} \\ &+ \left[ \left( b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right) \cdot \right. \\ &\quad \cdot \left( a_1 + c_{13} u^{(1)} + c_{14} u \right) + b_2 \\ &+ \left. (c_{22} - c_{31}) x_3^{[1]} + (c_{32} - c_{41}) + c_{42} x_1^{[1]} \right] u^{(2)}. \end{aligned} \tag{16}$$

To eliminate the variables $u^{(2)}$ from equations (16), we calculate the Lie bracket of a vector field

$$f^{[1]} \quad = \quad \sum_{i=1}^{4} \dot{x}_i^{[1]} \frac{\partial}{\partial x_i^{[1]}} + \sum_{\alpha=1}^{2} u^{(\alpha)} \frac{\partial}{\partial u^{(\alpha-1)}}$$

with vector field $\dfrac{\partial}{\partial u^{(2)}}$:

$$\begin{aligned} &L_{f^{[1]}} \frac{\partial}{\partial u^{(2)}} = \\ &-\frac{\partial}{\partial u^{(1)}} - \left( b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right) \frac{\partial}{\partial x_3^{[1]}} \\ &- \left[ \left( b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right) \cdot \right. \\ &\cdot \left( a_1 + c_{13} u^{(1)} + c_{14} u \right) \\ &+ \left. b_2 + (c_{22} - c_{31}) x_3^{[1]} + (c_{32} - c_{41}) x_2^{[1]} + c_{42} x_1^{[1]} \right] \frac{\partial}{\partial x_4^{[1]}} \end{aligned} \tag{17}$$

and define the new generalized state variables $x_i^{[2]}$ as the independent invariants of vector field (17). First three of them are

$$x_1^{[2]} = x_1^{[1]}, \quad x_2^{[2]} = x_2^{[1]},$$

$$x_1^{[2]} = \frac{1}{c_{11}} \exp(-c_{21} u^{(1)}) (b_1 + c_{21} x_3^{[1]} + c_{31} x_2^{[1]} + c_{41} x_1^{[1]}),$$

yielding the first two generalized state equations

$$\begin{aligned} \dot{x}_1^{[2]} &= x_2^{[2]}, \\ \dot{x}_2^{[2]} &= x_3^{[2]} \exp\left( c_{21} u^{(1)} \right) \\ &- \frac{1}{c_{11}} \left[ b_1 + c_{31} x_2^{[1]} + c_{41} x_1^{[1]} \right]. \end{aligned} \tag{18}$$

Since the second equation of (18) is nonlinear with respect to $u^{(1)}$, one cannot eliminate it unless $c_{21} = 0$. The latter yields, together with conditions (14), the following conditions necessary for elimination of $u^{(1)}$

$$c_{11} = c_{12} = c_{21} = 0. \tag{19}$$

If conditions (19) hold, the equations (16) read

$$
\begin{aligned}
\dot{x}_1^{[1]} &= x_2^{[1]}, \quad \dot{x}_2^{[1]} = x_3^{[1]}, \\
\dot{x}_3^{[1]} &= x_4^{[1]} + \left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right) u^{(2)}, \\
\dot{x}_4^{[1]} &= \left(a_1 + c_{13}u^{(1)} + c_{14}u\right) x_4^{[1]} \\
&+ a_2 x_3^{[1]} + a_3 x_2^{[1]} + a_4 x_1^{[1]} + b_3 u^{(1)} \\
&+ b_4 u + c_{23}x_3^{[1]}u^{(1)} + c_{24}x_3^{[1]}u \quad + c_{33}x_2^{[1]}u^{(1)} \\
&+ c_{34}x_2^{[1]}u \quad + c_{43}x_1^{[1]}u^{(1)} + c_{44}x_1^{[1]}u \\
&+ \left[\left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right)\left(a_1 + c_{13}u^{(1)} + c_{14}u\right)\right. \\
&+ b_2 + (c_{22} - c_{31})\, x_1^{[1]} + (c_{32} - c_{41})\, x_2^{[1]} \\
&+ \left. c_{42}x_1^{[1]}\right] u^{(2)}.
\end{aligned}
$$

and the vector field (17) has the following form

$$
\begin{aligned}
&L_{f^{[1]}} \frac{\partial}{\partial u^{(2)}} = \\
&-\frac{\partial}{\partial u^{(1)}} - \left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right) \frac{\partial}{\partial x_3^{[1]}} \\
&- \Big[(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]})\,(a_1 + c_{14}u) \\
&+ b_2 + (c_{32} - c_{41})x_2^{[1]} + c_{42}x_1^{[1]} + (c_{22} - c_{31})x_3^{[1]} \\
&+ \left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right) c_{13}u^{(1)}\Big] \frac{\partial}{\partial x_4^{[1]}}
\end{aligned}
\tag{20}
$$

The new generalized state variables $x_i^{[2]}$, as the independent invariants of vector field (20), are

$$
\begin{aligned}
x_1^{[2]} &= x_1^{[1]}, \quad x_2^{[2]} = x_2^{[1]}, \\
x_3^{[2]} &= x_3^{[1]} - \left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right) u^{(1)}, \\
x_4^{[2]} &= x_4^{[1]} - \Big[\left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right)(a_1 + c_{14}u) + b_2 \\
&+ (c_{32} - c_{41})x_2^{[1]} + c_{42}x_1^{[1]} + (c_{22} - c_{31})x_3^{[1]}\Big] u^{(1)} \\
&+ \frac{1}{2}\left(b_1 + c_{31}x_2^{[1]} + c_{41}x_1^{[1]}\right) \cdot \\
&\cdot (c_{22} - c_{31} - c_{13})\,(u^{(1)})^2.
\end{aligned}
$$

The inverse relations are

$$
\begin{aligned}
x_1^{[1]} &= x_1^{[2]}, \quad x_2^{[1]} = x_2^{[2]}, \\
x_3^{[1]} &= x_3^{[2]} + \left(b_1 + c_{31}x_2^{[2]} + c_{41}x_1^{[2]}\right) u^{(1)}, \\
x_4^{[1]} &= x_4^{[2]} + \Big[\left(b_1 + c_{31}x_2^{[2]} + c_{41}x_1^{[2]}\right)(a_1 + c_{14}u) + b_2 \\
&+ (c_{32} - c_{41})x_2^{[2]} + c_{42}x_1^{[2]} + (c_{22} - c_{31})x_3^{[2]}\Big] u^{(1)} \\
&+ \frac{1}{2}\left(b_1 + c_{31}x_2^{[2]} + c_{41}x_1^{[2]}\right) \cdot \\
&\cdot (c_{22} - c_{31} + c_{13})\,(u^{(1)})^2.
\end{aligned}
$$

The first three generalized state equations have the form

$$
\begin{aligned}
\dot{x}_1^{[2]} &= x_2^{[2]}, \\
\dot{x}_2^{[2]} &= x_3^{[2]} + \left(b_1 + c_{31}x_2^{[2]} + c_{41}x_1^{[2]}\right) u^{(1)}, \\
\dot{x}_3^{[2]} &= x_4^{[2]} + \Big[\left(b_1 + c_{31}x_2^{[2]} + c_{41}x_1^{[2]}\right)(a_1 + c_{14}u) + b_2 \\
&+ \quad (c_{32} - 2c_{41})x_2^{[2]} + c_{42}x_1^{[2]} + (c_{22} - 2c_{31})x_3^{[2]}\Big] u^{(1)} \\
&+ \quad \frac{1}{2}\left(b_1 + c_{31}x_2^{[2]} + c_{41}x_1^{[2]}\right) \cdot \\
&\cdot (c_{22} - 3c_{31} + c_{13})(u^{(1)})^2.
\end{aligned}
\tag{21}
$$

We can eliminate the quantities $u^{(1)}$ from these generalized state equations via a new generalized state transformation, if and only if all these state equations separately are linear with respect to $u^{(1)}$. The third equation of (21) is linear, if either $b_1 = c_{31} = c_{41} = 0$, or

$$
c_{22} - 3c_{31} + c_{13} = 0. \tag{22}
$$

The first condition, together with the earlier conditions (19), yields again conditions (15) and therefore will not cause the new branching. The second new necessary conditions (22), together with (15), have no alternative, because the third equation in system (21) must be linear with respect to $u^{(1)}$ independent on the other equations.

The fourth state equation for $\dot{x}_4^{[2]}$ is extremely complicated and contains quadratic and cubic terms with respect to $u^{(1)}$. The coefficients of the quadratic and cubic terms are zero for two different cases, either

$$
b_1 = c_{31} = c_{41} = 0, \quad c_{22} - c_{31} - c_{13} = 0, \tag{23}
$$

or

$$
c_{13} = c_{31} = c_{22} = 0, \quad -c_{23} + c_{32} - c_{41} + c_{14} = 0. \tag{24}
$$

The conditions (23), together with (19) and (22), yield the realizability conditions (ii). The conditions (24), together with (19) and (22), yield the realizability conditions (iii).

Now we return to the first branching (see (14) and (15)) and continue with conditions (15). Analogous calculations will give us the third set of conditions (i). ∎

The results of Propositions 2 and 3 illustrate the complicated nature of realizability conditions for i/o bilinear models. For arbitrary $n$, we have to go through $n-1$ steps. At each step we obtain several restrictions on system parameters with many equivalent branches. All these conditions can be combined together in very many different ways. They yield peculiar structures and most of them are probably not important for practical applications. We suggest below two realizable subclasses of i/o bilinear models that are diagrammatically shown below:

| | | | | |
|---|---|---|---|---|
| $a_1$ ● . . . | $a_{n-3}$● | $a_{n-2}$● | $a_{n-1}$● | $a_n$● |
| $b_1$ ○ . . . | $b_{n-3}$○ | $b_{n-2}$● | $b_{n-1}$● | $b_n$● |
| $c_{11}$ ○ . . . | $c_{1,n-3}$○ | $c_{1,n-2}$○ | $c_{1,n-1}A$ | $c_{1,n}$● |
| $c_{21}$ ○ . . . | $c_{2,n-3}$○ | $c_{2,n-2}A$ | $c_{2,n-1}$● | $c_{2,n}$● |
| $c_{31}$ ○ . . . | $c_{3,n-3}$○ | $c_{3,n-2}$● | $c_{3,n-1}$● | $c_{3,n}$● |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $c_{n-1,1}$ ○ . . . | $c_{n-1,n-3}$○ | $c_{n-1,n-2}$● | $c_{n-1,n-1}$● | $c_{n-1,n}$● |
| $c_{n,1}$ ○ . . . | $c_{n,n-3}$○ | $c_{n,n-2}$● | $c_{n,n-1}$● | $c_{n,n}$● |

$$
\begin{array}{ccccc}
a_1\bullet & \dots & a_{n-2}\bullet & a_{n-1}\bullet & a_n\bullet \\
\\
b_1\circ & \dots & b_{n-2}\circ & b_{n-1}\bullet & b_n\bullet \\
\\
c_{11}\circ & \dots & c_{1,n-2}\circ & c_{1,n-1}\bullet & c_{1,n}\bullet \\
c_{21}\circ & \dots & c_{2,n-2}\circ & c_{2,n-1}\bullet & c_{2,n}\bullet \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c_{n,1}\circ & \dots & c_{n,n-2}\circ & c_{n,n-1}\bullet & c_{n,n}\bullet
\end{array}
$$

For the second subclass we also give the corresponding state equations, shown below:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \ \dots, \ \dot{x}_{n-2} = x_{n-1} \\
\dot{x}_{n-1} &= -\frac{1}{c_{1,n-1}}\left(b_{n-1} + \sum_{i=1}^{n-1} c_{n-i+1,n-1}x_i\right) \\
&+ e^{c_{1,n-1}u}x_n \\
\dot{x}_n &= \frac{e^{-c_{1,n-1}u}}{c_{1,n-1}^2}\left\{\zeta + \beta u + \sum_{i=1}^{n-1}\alpha_i x_i + \sum_{i=1}^{n-1}\gamma_i x_i u\right\} \\
&+ \frac{e}{c_{1,n-1}^2}\left[c_{1,n-1}\left(c_{2,n-1}+1\right)x_n\right. \\
&+ \left. c_{1,n-1}^2 c_{1,n}x_n u\right]
\end{aligned}
\tag{25}
$$

and $y = x_1$, where $\zeta = -b_{n-1}(c_{2,n-1} - a_1 c_{1,n-1})$, $\beta = b_n c_{1,n-1}^2 - b_{n-1}c_{1,n-1}c_{1,n})$, $\alpha_i = a_{n-i+1}c_{1,n-1}^2 - a_1 c_{1,n-1}c_{n-i+1,n-1} - c_{2,n-1}c_{n-i+1,n-1} + c_{1,n-1}c_{n-i+2,n-1}$ and $\gamma_i = c_{1,n-1}^2 c_{n-i+1,n} - c_{1,n-1}c_{1,n}c_{n-i+1,n-1}$.

Though the state equations, given above were suggested by applying the realization theory in [11], can be checked directly by eliminating $x$ in the state equations.

## III. CONCLUSIONS

In this paper we have studied the class of higher order bilinear i/o differential equations, that may approximate many nonlinear systems, and has been popular in the identification literature. It has been demonstrated that the 2nd order bilinear i/o differential equation, unlike the discrete-time case, is always realizable in the classical state space form. Sufficient and necessary conditions, in terms of restrictions on the bilinear model parameters, have been provided to establish whether it is possible to find a state space representation of the i/o bilinear system, or not, for the cases of 3rd and 4th order models.

When compared to the general realizability conditions (see e.g. [11]), our conditions rely on the property that certain combinations of coefficients of the i/o equations are zero or not. Since, even in low order cases, the necessary and sufficient conditions exhibit quite a peculiar and non-regular structure, it is a very difficult, and probably not a practical task, to find the necessary and sufficient conditions for the general case. Instead, we suggest two subclasses of realizable i/o bilinear systems.

Note also that earlier results do not suggest explicit state equations for i/o models. Though a procedure to find them was given, application in general requires integrating the integrable one-forms which sometimes can be a complicated task. In this paper we provide explicit state equations for all realizable 2nd and 3rd order bilinear i/o equations and for one realizable subclass of bilinear i/o equations for the general case for arbitrary value $n$.

The results indicate that special care should be taken when choosing the model structure in identification if one wants to end up with a realizable i/o model, since the general bilinear i/o model is not realizable in the classical state space form. Future research will be directed towards the development of simple model classes, other than bilinear, which can be put into the state space form, and capture the basic nonlinearities of the plants whilst remaining within limited complexity, like the discrete-time subclass in [10].

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