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# Mean shear flows generated by nonlinear resonant Alfvén waves

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In the context of resonant absorption, nonlinearity has two different manifestations. The first is the reduction in amplitude of perturbations around the resonant point (wave energy absorption). The second is the generation of mean shear flows outside the dissipative layer surrounding the resonant point. Ruderman *et al.* [Phys. Plasmas **4**, 75 (1997)] studied both these effects at the slow resonance in isotropic plasmas. Clack *et al.* [Astron. Astrophys. **494**, 317 (2009)] investigated nonlinearity at the Alfvén resonance, however, they did not include the generation of mean shear flow. In this present paper, we investigate the mean shear flow, analytically, and study its properties. We find that the flow generated is parallel to the magnetic surfaces and has a characteristic velocity proportional to  $\epsilon^{1/2}$ , where  $\epsilon$  is the dimensionless amplitude of perturbations far away from the resonance. This is, qualitatively, similar to the flow generated at the slow resonance. The jumps in the derivatives of the parallel and perpendicular components of mean shear flow across the dissipative layer are derived. We estimate the generated mean shear flow to be of the order of  $10\text{kms}^{-1}$  in both the solar upper chromosphere and solar corona, however, this value strongly depends on the choice of boundary conditions. It is proposed that the generated mean shear flow can produce a Kelvin–Helmholtz instability at the dissipative layer which can create turbulent motions. This instability would be an additional effect, as a Kelvin–Helmholtz instability may already exist due to the velocity field of the resonant Alfvén waves. This flow can also be superimposed onto existing large scale motions in the solar upper atmosphere.

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## I. INTRODUCTION

It has long been established, from observations, that the solar corona is highly structured and inhomogeneous with temperatures of the order of  $10^6$  K. The solar corona is filled with a large number of discrete magnetic loops (coronal arcades) and there is an abundance of observational evidence showing that magnetohydrodynamic (MHD) waves propagate in, and across, these loops (see, e.g. Refs. [1]–[4]). In order to sustain such high temperatures whilst combatting optically thin radiation and thermal conduction there must exist some mechanism(s) acting as a source of steady heating.

The last few decades saw a multitude of models proposed to tackle the complicated problem of coronal heating, such as heating by waves (e.g. resonant absorption, phase mixing) or heating by magnetic relaxation (e.g. reconnection). There is an increasing consensus that all these processes act simultaneously to different degrees of efficiency throughout the solar corona. The agreement comes as observational data confirms the abundance of MHD wave propagation in the solar corona, along with the complexity of the magnetic configuration enabling magnetic heating to occur (see, e.g. Refs. [5]–[10]).

Resonant absorption of Alfvén waves in coronal loops was first suggested by Ionson (author?) [11] as a non-thermal heating mechanism of the corona and since then resonant absorption has been studied intensively by many authors in the linear regime for a variety of applications (see, e.g. Refs. [12]–[25]). A new understanding of the process of resonant absorption became available after the

paper by Ruderman *et al.* (author?) [26] which was the first analytical study on the nonlinear aspect of resonant absorption. Not only was it shown that resonant absorption of slow waves was an inherently nonlinear phenomenon, they also showed that a mean shear flow is generated outside the dissipative layer. By their calculations, however, they still found the generated flow was much too large compared to the observed velocities. On the other hand, the authors did note that their results should be used with caution in the solar atmosphere as some of their assumptions were not fully realistic for that environment.

From the coronal heating point of view, the slow resonance studied by Ruderman *et al.* (author?) [26] was not expected to contribute significantly as the energy stored in slow waves is much less than the required energy to compensate the losses and it is difficult for slow waves to reach the corona, as they become shocked as they climb due to density stratification. From this point of view, Alfvén waves and Alfvén resonance are of much greater interest as estimations show that the energy carried by Alfvén waves is much higher and they can reach the corona without significant damping. The validity of nonlinear resonant Alfvén waves (under coronal conditions) was studied in great detail by Clack *et al.* (author?) [27], where they showed that in coronal plasmas the nonlinear addition to the result found in linear MHD is so small that the linear approach can be used with great accuracy. Essentially this investigation clarified the upper limit in which linear theory is applicable to resonant Alfvén waves.

Studies have been carried out to investigate the prop-

erties of shear flows generated by velocity field of Alfvén waves, however, nearly all of these have been numerical due to analytical complications when considering non-linearity, turbulence and resonant absorption simultaneously. These studies have found that shear flows could give rise to a Kelvin–Helmholtz instability at the narrow dissipative layer (see, e.g. Ref. [28]). This instability can drive turbulent motions and, in turn, locally enhance transport coefficients which can alter the efficiency of heating (see, e.g. Refs. [28]–[30]). None of these investigations have studied the generation of mean shear flow at the Alfvén resonance by nonlinear interactions. The generation of mean shear flow can supply additional shear enhancing turbulent motions.

Recent advancements in the understanding of nonlinear Alfvén resonance has inspired us to study of the generation of mean shear flows at the Alfvén resonance. In the present paper we will derive the equations describing the generated mean shear flow outside the Alfvén dissipative layer and estimate the magnitude of the shear flow. We already know, from Ofman and Davila (author?) [28], that the plasma velocity at the dissipative layer may be reduced by the turbulent enhancement of the dissipative parameters, implying that for a given heating rate, the wave amplitude is reduced compared to the linear case. This means that any result we produce must be reduced when considered for the solar corona since turbulent motions are likely to be present and this will reduce the mean flow speed.

The paper is organised as follows. In the next section, we introduce the governing equations and discuss the main assumptions. In Sect. III we recall some previous results and derive the solution for the equation governing the mean shear flow outside the dissipative layer. In Sect. IV we derive the governing equations for the mean shear flow inside the dissipative layer. Section V gives the jumps in the derivatives of the mean shear flow velocities across the dissipative layer and we estimate the magnitude of these jumps for conditions typical in the solar upper chromosphere and corona. In Sect. VI we draw our conclusions and discuss the results. Once the governing equations of the generated mean shear flow are found, the nonlinear theory of resonant Alfvén waves is complete.

## II. GOVERNING EQUATIONS AND ASSUMPTIONS

To mathematically study the mean shear flow generated at resonance we use the visco-resistive MHD equations. In spite of the presence of dissipation, we use the adiabatic equation as an approximation to the energy equation. Numerical studies by Poedts *et al.* (author?) [31] in linear MHD have shown that dissipation due to viscosity and finite electrical conductivity in the energy equation does not, significantly, alter the behaviour of resonant MHD waves in the driven problem.

When the product of the ion (electron) gyrofrequency,  $\omega_{i(e)}$ , and the ion (electron) collision time,  $\tau_{i(e)}$ , is much greater than unity (as in the solar corona) the viscosity and finite electrical conductivity become anisotropic. The parallel and perpendicular components of anisotropic finite electrical conductivity only differ by a factor of 2, therefore, we will consider only one of them without loss of generality. The anisotropic viscosity is given by Braginskii’s viscosity tensor (see, e.g. Braginskii [32]). The components of the viscosity tensor that remove the Alfvén singularity are the shear components, however these components (in their form) act as the isotropic viscosity. Indeed, Clack *et al.* (author?) [27] showed that one can interchange isotropic viscosity and shear viscosity when studying the Alfvén resonance. Another possible additive to our model might be dispersive effects caused by Hall currents, however Clack *et al.* (author?) [27] showed that this effect does not alter the MHD waves in the vicinity of the Alfvén resonance.

The dynamics of waves in our model is described by the visco-resistive MHD equations

$$\frac{\partial \bar{p}}{\partial t} + \nabla \cdot (\bar{\rho} \mathbf{v}) = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\bar{\rho}} \nabla \bar{P} + \frac{1}{\mu_0 \bar{\rho}} (\mathbf{B} \cdot \nabla) \mathbf{B} + \bar{\nu} \nabla \cdot \mathbf{S}_1, \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \bar{\eta} \nabla^2 \mathbf{B}, \quad (3)$$

$$\frac{\partial}{\partial t} \left( \frac{\bar{p}}{\bar{\rho}^\gamma} \right) + \mathbf{v} \cdot \nabla \left( \frac{\bar{p}}{\bar{\rho}^\gamma} \right) = 0. \quad (4)$$

In Eqs. (1)–(4),  $\mathbf{v}$  and  $\mathbf{B}$  are the velocity and magnetic induction vectors,  $\bar{P}$  the total pressure,  $\bar{p}$  the plasma pressure,  $\bar{\rho}$  the density,  $\bar{\eta}$  the coefficient of magnetic diffusivity,  $\bar{\nu} = \bar{\eta}_1 / \bar{\rho}$  the coefficient of shear viscosity over density,  $\nabla \cdot \mathbf{S}_1$  the shear component of Braginskii’s viscosity tensor,  $\gamma$  the adiabatic exponent, and  $\mu_0$  the magnetic permeability of free space.

We adopt Cartesian coordinates  $x, y, z$  and limit our analysis to a static background equilibrium ( $\mathbf{v}_0 = 0$ ). We assume that all equilibrium quantities (terms with subscript ‘0’) depend on  $x$  only. The equilibrium magnetic field,  $\mathbf{B}_0$ , is unidirectional and lies in the  $yz$ -plane. The equilibrium quantities must satisfy the condition of total pressure balance,  $p_0 + B_0^2 / (2\mu_0) = \text{constant}$ .

For simplicity we assume that the perturbations of all quantities are independent of  $y$  ( $\partial/\partial y = 0$ ). We note that since the magnetic field is not aligned with the  $z$ -axis, an Alfvén resonance can still exist. Even though the Alfvén resonance is governed by a linear equation, we must consider nonlinear effects to obtain the second manifestation of nonlinearity - mean shear flows. In the linear theory of driven waves all perturbed quantities oscillate with the same frequency,  $\omega$ , which means that they can be Fourier-analysed and taken to be proportional to  $\exp(i[kz - \omega t])$ . Solutions are sought in the form of propagating waves and all perturbations in these

solutions depend on the combination  $\theta = z - Vt$ , rather than  $z$  and  $t$  separately, with  $V = \omega/k$ . In the context of resonant absorption the phase velocity,  $V$ , must match the projection of the Alfvén velocity,  $v_A$ , onto the  $z$ -axis when  $x = x_a$  where  $x_a$  is the resonant position. To define the resonant position mathematically it is convenient to introduce the angle,  $\alpha$ , between the  $z$ -axis and the direction of the equilibrium magnetic field, so that the components of the equilibrium magnetic field can be written as  $B_{0y} = B_0 \sin \alpha$ ,  $B_{0z} = B_0 \cos \alpha$ . The definition of the resonant position can now be defined mathematically as

$$V = v_A(x_a) \cos \alpha, \quad (5)$$

where  $v_A$  is the Alfvén speed defined as  $v_A = B_0/(\mu_0 \rho_0)^{1/2}$ . In addition, we introduce the squares of the sound and cusp speeds as  $c_S^2 = \gamma p_0/\rho_0$  and  $c_T^2 = c_S^2 v_A^2/(c_S^2 + v_A^2)$ , respectively. In what follows we can take  $x_a = 0$  without loss of generality. The perturbations of the physical quantities are defined by

$$\begin{aligned} \bar{\rho} &= \rho_0 + \rho, & \bar{p} &= p_0 + p, & \mathbf{B} &= \mathbf{B}_0 + \mathbf{b}, \\ P &= p + \frac{\mathbf{B}_0 \cdot \mathbf{b}}{\mu_0} + \frac{\mathbf{b}^2}{2\mu_0}, \end{aligned} \quad (6)$$

where  $P$  is the perturbation of total pressure.

The dominant dynamics of resonant Alfvén waves, in linear MHD, resides in the components of the perturbed magnetic field and velocity that are perpendicular to the equilibrium magnetic field and to the  $x$ -direction. This dominant behaviour is created by an  $x^{-1}$  singularity in the spatial solution of these quantities at the Alfvén resonance (see, e.g. Sakurai *et al.* [17], Goossens *et al.* [33]); these variables are known as *large variables*. All other variables are known as *small variables*.

To make the mathematical analysis more concise, we define the components of velocity and magnetic field that are in the  $yz$ -plane and are either parallel or perpendicular to the equilibrium magnetic field:

$$\begin{aligned} \begin{pmatrix} v_{\parallel} \\ b_{\parallel} \end{pmatrix} &= \begin{pmatrix} v & w \\ b_y & b_z \end{pmatrix} \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix}, \\ \begin{pmatrix} v_{\perp} \\ b_{\perp} \end{pmatrix} &= \begin{pmatrix} v & -w \\ b_y & -b_z \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \end{aligned} \quad (7)$$

where  $v$ ,  $w$ ,  $b_y$  and  $b_z$  are the  $y$ - and  $z$ -components of the velocity and perturbation of magnetic field, respectively.

Let us introduce the characteristic scale of inhomogeneity,  $l_{\text{inh}}$ . The classical viscous Reynolds number,  $R_e$ , and the magnetic Reynolds number,  $R_m$ , are defined as

$$R_e = \frac{V l_{\text{inh}}}{\bar{\nu}}, \quad R_m = \frac{V l_{\text{inh}}}{\bar{\eta}}. \quad (8)$$

These two numbers determine the importance of viscosity and finite electrical conductivity. We introduce the total Reynolds number as

$$\frac{1}{R} = \frac{1}{R_e} + \frac{1}{R_m}. \quad (9)$$

The aim of this paper is to study the generation of a mean shear flow outside the dissipative layer due to the nonlinear behaviour of driven resonant Alfvén waves in the dissipative layer. We are not interested in the effects of MHD waves that have large amplitude everywhere and require a nonlinear description in the whole space. We focus on waves that have small dimensionless amplitude  $\epsilon \ll 1$  far away from the ideal Alfvén resonant point.

In nonlinear theory, when studying resonant behaviour in the dissipative layer, we must re-scale the dissipative coefficients (see, e.g., Ruderman *et al.* [26], Clack *et al.* [27], Ballai *et al.* [34] and Clack and Ballai [35]) so that dissipation is of the same order as nonlinearity,

$$\bar{\nu} = \epsilon^{3/2} \nu, \quad \bar{\eta} = \epsilon^{3/2} \eta. \quad (10)$$

It is easy to estimate the nonlinearity parameter, which is the ratio of the largest nonlinear and dissipative terms and is obtained to be  $\epsilon R^{2/3}$  (see Clack *et al.* [27]). This implies that if  $\epsilon R^{2/3} \ll 1$  linear theory holds, whereas if  $\epsilon R^{2/3} \gtrsim 1$  then nonlinearity is important and linear theory breaks down.

In linear theory all perturbed quantities are harmonic functions of  $\theta$ , therefore their mean values over a period vanish. On the other hand, in nonlinear theory the perturbed variables can have nonzero mean values as a result of nonlinear interaction of different harmonics. Let us introduce the mean value of a function  $f(\theta)$  over a period  $L$  as

$$\langle f \rangle = \frac{1}{L} \int_0^L f(\theta) d\theta. \quad (11)$$

It directly follows from Eq. (1) that

$$\langle \bar{\rho} u \rangle = \langle b_x \rangle = 0. \quad (12)$$

We can always define the background state in such a way that the mean values of density, pressure and magnetic field vanish:

$$\langle \rho \rangle = \langle p \rangle = \langle b_{\parallel} \rangle = \langle b_{\perp} \rangle = 0. \quad (13)$$

This is not possible for the velocity, since we assume a static equilibrium. It is convenient, therefore, to divide  $v_{\parallel}$  and  $v_{\perp}$  into mean and oscillatory parts. Using the Reynolds decomposition we can write

$$\begin{aligned} U_{\parallel} &= \langle v_{\parallel} \rangle, & U_{\perp} &= \langle v_{\perp} \rangle, \\ \tilde{v}_{\parallel} &= v_{\parallel} - U_{\parallel}, & \tilde{v}_{\perp} &= v_{\perp} - U_{\perp}. \end{aligned} \quad (14)$$

The quantities  $U_{\parallel}$  and  $U_{\perp}$  describe the mean flow parallel to the magnetic surfaces. The mean flow is generated by the nonlinear interaction of the harmonics in the Fourier expansion of the perturbed quantities with respect to  $\theta$ .

We can rewrite Eqs. (1)–(4) in the scalar form as

$$V \frac{\partial \rho}{\partial \theta} - \frac{\partial(\rho_0 u)}{\partial x} - \rho_0 \frac{\partial \tilde{w}}{\partial \theta} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho w)}{\partial \theta}, \quad (15)$$

$$\begin{aligned} \rho_0 V \frac{\partial u}{\partial \theta} - \frac{\partial P}{\partial x} + \frac{B_0 \cos \alpha}{\mu_0} \frac{\partial b_x}{\partial \theta} = \\ \bar{\rho} \left( u \frac{\partial u}{\partial x} + w \frac{\partial w}{\partial \theta} \right) - \rho V \frac{\partial u}{\partial \theta} - \frac{b_x}{\mu_0} \frac{\partial b_x}{\partial x} \\ - \frac{b_z}{\mu_0} \frac{\partial b_x}{\partial \theta} - \bar{\rho} \nu (\nabla \cdot \mathbf{S}_1)_x, \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \left( \rho_0 V \tilde{v}_{\parallel} - P \cos \alpha + \frac{B_0 \cos \alpha}{\mu_0} b_{\parallel} \right) + \frac{b_x}{\mu_0} \frac{dB_0}{dx} = \\ \bar{\rho} \left( u \frac{\partial v_{\parallel}}{\partial x} + w \frac{\partial v_{\parallel}}{\partial \theta} \right) - \rho V \frac{\partial v_{\parallel}}{\partial \theta} - \frac{b_x}{\mu_0} \frac{\partial b_{\parallel}}{\partial x} \\ - \frac{b_z}{\mu_0} \frac{\partial b_{\parallel}}{\partial \theta} - \bar{\rho} \nu (\nabla \cdot \mathbf{S}_1)_{\parallel}, \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \left( \rho_0 V \tilde{v}_{\perp} + P \sin \alpha + \frac{B_0 \cos \alpha}{\mu_0} b_{\perp} \right) = \\ \bar{\rho} \left( u \frac{\partial v_{\perp}}{\partial x} + w \frac{\partial v_{\perp}}{\partial \theta} \right) - \rho V \frac{\partial v_{\perp}}{\partial \theta} - \frac{b_x}{\mu_0} \frac{\partial b_{\perp}}{\partial x} \\ - \frac{b_z}{\mu_0} \frac{\partial b_{\perp}}{\partial \theta} - \bar{\rho} \nu (\nabla \cdot \mathbf{S}_1)_{\perp}, \end{aligned} \quad (18)$$

$$V b_x + B_0 u \cos \alpha = w b_x - u b_z + \bar{\eta} \left( \frac{\partial b_x}{\partial \theta} - \frac{\partial b_z}{\partial x} \right), \quad (19)$$

$$\begin{aligned} \frac{\partial}{\partial \theta} (V b_{\parallel} + B_0 \tilde{v}_{\parallel} \cos \alpha) - \frac{\partial (B_0 u)}{\partial x} - B_0 \frac{\partial w}{\partial \theta} \\ = \frac{\partial (u b_{\parallel})}{\partial x} + \frac{\partial (w b_{\parallel})}{\partial \theta} - b_x \frac{\partial v_{\parallel}}{\partial x} - b_z \frac{\partial v_{\parallel}}{\partial \theta} - \bar{\eta} \nabla^2 b_{\parallel}, \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{\partial}{\partial \theta} (V b_{\perp} + B_0 \tilde{v}_{\perp} \cos \alpha) = \frac{\partial (u b_{\perp})}{\partial x} + \frac{\partial (w b_{\perp})}{\partial \theta} \\ - b_x \frac{\partial v_{\perp}}{\partial x} - b_z \frac{\partial v_{\perp}}{\partial \theta} - \bar{\eta} \nabla^2 b_{\perp}, \end{aligned} \quad (21)$$

$$\begin{aligned} V \left( \frac{\partial p}{\partial \theta} - c_s^2 \frac{\partial \rho}{\partial \theta} \right) - u \left( \frac{dp_0}{dx} - c_s^2 \frac{d\rho_0}{dx} \right) \\ = \frac{1}{\rho_0} \left\{ V \left( \gamma p \frac{\partial \rho}{\partial \theta} - \rho \frac{\partial p}{\partial \theta} \right) - w \left[ \gamma \bar{\rho} \frac{\partial \rho}{\partial \theta} - \bar{\rho} \frac{\partial p}{\partial \theta} \right] \right. \\ \left. + u \left[ \rho \frac{dp_0}{dx} - \gamma p \frac{d\rho_0}{dx} + \bar{\rho} \frac{\partial p}{\partial x} - \gamma \bar{\rho} \frac{\partial \rho}{\partial x} \right] \right\} \end{aligned} \quad (22)$$

$$P = p + \frac{1}{2\mu_0} (b_x^2 + b_{\perp}^2 + b_{\parallel}^2 + 2B_0 b_{\parallel}), \quad (23)$$

The equations for  $U_{\parallel}$  and  $U_{\perp}$  are obtained by averaging Eqs. (17) and (18), respectively, and then dividing by  $\bar{\nu}$ ,

$$\begin{aligned} \bar{\nu} \frac{d^2 U_{\parallel}}{dx^2} = \left\langle u \frac{\partial v_{\parallel}}{\partial x} \right\rangle + \left\langle v_{\parallel} \frac{\partial v_{\perp}}{\partial \theta} \right\rangle \cos \alpha + \left\langle \frac{1}{\bar{\rho}} \frac{\partial P}{\partial \theta} \right\rangle \cos \alpha \\ - \frac{B_0 \cos \alpha}{\mu_0} \left\langle \frac{1}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \theta} \right\rangle - \frac{1}{\mu_0} \left\langle \frac{b_x}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial x} \right\rangle - \frac{\cos \alpha}{\mu_0} \left\langle \frac{b_{\parallel}}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \theta} \right\rangle \\ + \frac{\sin \alpha}{\mu_0} \left\langle \frac{b_{\perp}}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \theta} \right\rangle - \frac{1}{\mu_0} \frac{dB_0}{dx} \left\langle \frac{b_x}{\bar{\rho}} \right\rangle, \end{aligned} \quad (24)$$

$$\begin{aligned} \bar{\nu} \frac{d^2 U_{\perp}}{dx^2} = \left\langle u \frac{\partial v_{\perp}}{\partial x} \right\rangle + \left\langle v_{\parallel} \frac{\partial v_{\perp}}{\partial \theta} \right\rangle \cos \alpha - \left\langle \frac{1}{\bar{\rho}} \frac{\partial P}{\partial \theta} \right\rangle \sin \alpha \\ - \frac{B_0 \cos \alpha}{\mu_0} \left\langle \frac{1}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \theta} \right\rangle - \frac{1}{\mu_0} \left\langle \frac{b_x}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial x} \right\rangle \\ - \frac{\cos \alpha}{\mu_0} \left\langle \frac{b_{\parallel}}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \theta} \right\rangle + \frac{\sin \alpha}{\mu_0} \left\langle \frac{b_{\perp}}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \theta} \right\rangle. \end{aligned} \quad (25)$$

Here  $\nabla = (\partial/\partial x, 0, \partial/\partial \theta)$ ,  $u$  is the  $x$ -component of the velocity, and  $\tilde{w} = \tilde{v}_{\parallel} \cos \alpha - \tilde{v}_{\perp} \sin \alpha$  is the oscillatory part of  $w$ .

Equations (15)–(25) will be used in the following sections in order to calculate the mean flow that is generated outside the dissipative layer by the resonant waves.

### III. OUTER SOLUTION

To calculate the mean shear flow generated outside the dissipative layer we have to recall some results found by Clack *et al.* (author?) [27]. They used the method of simplified matched asymptotic expansions (see, e.g., Bal-lai *et al.* [34] and Nayfeh [36]) in order to derive the equation governing resonant Alfvén waves in the dissipative layer. The method consists of finding the so-called outer and inner expansions and matching these expansions in overlap regions around the dissipative layer. This nomenclature is well adopted for our scenario. The outer expansion corresponds to the solution outside the dissipative layer and the inner expansion refers to the solution inside the dissipative layer. The solution in the outer region is represented by asymptotic expansions of the form

$$f = \epsilon f^{(1)} + \epsilon^{3/2} f^{(2)} + \dots, \quad (26)$$

where  $\epsilon$  is the dimensionless amplitude of perturbations far from the dissipative layer and  $f$  represents any perturbed quantity with the exception of  $v_{\parallel}$  and  $v_{\perp}$ . We shall show that resonant Alfvén waves create a shear flow with an amplitude proportional to  $\epsilon^{1/2}$  outside the dissipative layer. As a consequence, we expand  $\tilde{v}_{\parallel}$  and  $\tilde{v}_{\perp}$  in the form of Eq. (26), and  $U_{\parallel}$  and  $U_{\perp}$  in the form

$$\begin{aligned} U_{\parallel} = \epsilon^{1/2} U_{\parallel}^{(0)} + \epsilon U_{\parallel}^{(1)} + \epsilon^{3/2} U_{\parallel}^{(2)} + \dots, \\ U_{\perp} = \epsilon^{1/2} U_{\perp}^{(0)} + \epsilon U_{\perp}^{(1)} + \epsilon^{3/2} U_{\perp}^{(2)} + \dots \end{aligned} \quad (27)$$

Clack *et al.* (author?) [27] found that the substitution of Eq. (26) into Eqs. (15)–(23) leads, in the first order approximation, to a system of linear equations for the variables with the superscript ‘1’. All variables can be eliminated in favour of  $u^{(1)}$  and  $P^{(1)}$ , leading to the system

$$V \frac{\partial P^{(1)}}{\partial \theta} = F \frac{\partial u^{(1)}}{\partial x}, \quad V \frac{\partial P^{(1)}}{\partial x} = \rho_0 D_A \frac{\partial u^{(1)}}{\partial \theta}, \quad (28)$$

where

$$F = \frac{\rho_0 D_A D_C}{V^4 - V^2 (v_A^2 + c_s^2) + v_A^2 c_s^2 \cos^2 \alpha}, \quad (29)$$

$$D_A = V^2 - v_A^2 \cos^2 \alpha,$$

$$D_C = (v_A^2 + c_S^2) (V^2 - c_T^2 \cos^2 \alpha). \quad (30)$$

The quantities  $D_A$  and  $D_C$  vanish at the Alfvén and slow resonant positions, respectively. As a result these two positions are regular singular points for the system (28). The remaining variables can be expressed in terms of  $u^{(1)}$  and  $P^{(1)}$  as,

$$\tilde{v}_\perp^{(1)} = -\frac{V \sin \alpha}{\rho_0 D_A} P^{(1)}, \quad \tilde{v}_\parallel^{(1)} = \frac{V c_S^2 \cos \alpha}{\rho_0 D_C} P^{(1)}, \quad (31)$$

$$b_x^{(1)} = -\frac{B_0 \cos \alpha}{V} u^{(1)}, \quad b_\perp^{(1)} = \frac{B_0 \cos \alpha \sin \alpha}{\rho_0 D_A} P^{(1)}, \quad (32)$$

$$\frac{\partial b_\parallel^{(1)}}{\partial \theta} = \frac{B_0 (V^2 - c_S^2 \cos^2 \alpha)}{\rho_0 D_C} \frac{\partial P^{(1)}}{\partial \theta} + \frac{u^{(1)}}{V} \frac{dB_0}{dx}, \quad (33)$$

$$\frac{\partial p^{(1)}}{\partial \theta} = \frac{V^2 c_S^2}{D_C} \frac{\partial P^{(1)}}{\partial \theta} - \frac{u^{(1)} B_0}{\mu_0 V} \frac{dB_0}{dx}, \quad (34)$$

$$\frac{\partial \rho^{(1)}}{\partial \theta} = \frac{V^2}{D_C} \frac{\partial P^{(1)}}{\partial \theta} + \frac{u^{(1)}}{V} \frac{d\rho_0}{dx}. \quad (35)$$

Since Eq. (28) has regular singular points, the solutions can be obtained in terms of Fröbenius series with respect to  $x$  (for details see, e.g., Clack *et al.* [27], Ruderman *et al.* [26], Ballai *et al.* [34]). From Eqs. (31)–(35), we see that the quantity  $\tilde{v}_\parallel^{(1)}$  is regular, while all other quantities are singular. The quantities  $u^{(1)}$ ,  $b_x^{(1)}$ ,  $b_\parallel^{(1)}$ ,  $p^{(1)}$  and  $\rho^{(1)}$  behave as  $\ln|x|$ , while  $\tilde{v}_\perp^{(1)}$  and  $b_\perp^{(1)}$  behave as  $x^{-1}$ , so they are the most singular.

Carrying out calculations on Eqs. (24) and (25), and utilising Eqs. (28), (31)–(35), we find that in the first and second order approximations we have

$$\frac{d^2 U_\parallel^{(0)}}{dx^2} = \frac{d^2 U_\parallel^{(1)}}{dx^2} = \frac{d^2 U_\perp^{(0)}}{dx^2} = \frac{d^2 U_\perp^{(1)}}{dx^2} = 0. \quad (36)$$

The functions  $U_\parallel^{(0)}(x)$ ,  $U_\perp^{(0)}(x)$ ,  $U_\parallel^{(1)}(x)$  and  $U_\perp^{(1)}(x)$  are all continuous at  $x = 0$ . Since Eq. (36) implies that  $U_\parallel^{(0)}(x)$ ,  $U_\perp^{(0)}(x)$ ,  $U_\parallel^{(1)}(x)$  and  $U_\perp^{(1)}(x)$  are linear functions of  $x$ , we can include  $U_\parallel^{(1)}(x)$  and  $U_\perp^{(1)}(x)$  into  $U_\parallel^{(0)}(x)$  and  $U_\perp^{(0)}(x)$  and take  $U_\parallel^{(1)}(x) = U_\perp^{(1)}(x) = 0$ , without loss of generality. We choose a mobile coordinate system such that  $U_\parallel^{(0)}(0) = U_\perp^{(0)}(0) = 0$ . Using this assumption in conjunction with Eq. (36) it follows that

$$U_\parallel^{(0)} = V_\parallel^\pm x, \quad U_\perp^{(0)} = V_\perp^\pm x, \quad (37)$$

where  $V_\parallel^\pm$  and  $V_\perp^\pm$  are constants and the superscripts ‘ $-$ ’ and ‘ $+$ ’ refer to  $x < 0$  and  $x > 0$ , respectively.

Continuing in the same way we can find the solutions of the subsequent higher order approximations. At each step in the scheme of approximations we obtain equations with the left-hand sides equal to the left-hand sides of the equations found in the first order approximation. The right-hand sides of the equations are expressed in terms of variables of lower order approximations. In carrying out this process we obtain that

$$U_\parallel^{(n)} = O(x^{-n+1}), \quad U_\perp^{(n)} = O(x^{-n+1}), \quad n \geq 2. \quad (38)$$

This means that the mean velocity starts to behave singularly from the third order approximation.

Taking into account Eqs. (37) and (38) we write the expansion for the mean velocity in the form

$$U_i = \epsilon^{1/2} V_i x + \sum_{n=1}^{\infty} \epsilon^{(n+2)/2} V_i^{(n)}(x) x^{-n}, \quad (39)$$

where subscript ‘ $i$ ’ represents either the subscript ‘ $\parallel$ ’ or ‘ $\perp$ ’ and the functions  $V_i(x)$  and  $V_i^{(n)}(x)$  have finite limits at  $|x| \rightarrow 0$ . The most important property of expansion (39) is that the term of the lowest order approximation (proportional to  $\epsilon^{1/2}$ ) is very small inside the dissipative layer, but becomes large far away from the resonance. It is also interesting to note that the remaining terms tend to zero far from resonance. This result is in complete agreement with the studies by Ruderman *et al.* (author?) [26] and Ballai *et al.* (author?) [37], where they found the mean flow generated by resonant slow waves outside the dissipative layer to be proportional to  $\epsilon^{1/2}$ .

#### IV. INNER SOLUTION

In this section we determine the inner expansion, which is the solution inside the dissipative layer. The thickness of the dissipative layer is of the order of  $l_{\text{inh}} R^{-1/3}$ , where  $l_{\text{inh}}$  is the characteristic scale of inhomogeneity. Since we assume that  $R = \mathcal{O}(\epsilon^{-3/2})$  we have  $l_{\text{inh}} R^{-1/3} = \mathcal{O}(\epsilon^{1/2} l_{\text{inh}})$ . As a consequence it is convenient to introduce a stretching variable  $\xi = \epsilon^{-1/2} x$  inside the dissipative layer.

We can rewrite Eqs. (15)–(23) using the stretching variable, however, to avoid repetition of previous studies and for brevity, we only display the equations we use explicitly, which are the perpendicular component of momentum and normal component of induction,

$$\begin{aligned} & \frac{1}{\bar{\rho}} \left[ \epsilon^{1/2} \frac{\partial P}{\partial \theta} \sin \alpha + \frac{b_x}{\mu_0} \frac{\partial b_\perp}{\partial \xi} + \frac{\epsilon^{1/2}}{\mu_0} (B_0 \cos \alpha + b_z) \frac{\partial b_\perp}{\partial \theta} \right] \\ & = -\epsilon^{1/2} (V - w) \frac{\partial v_\perp}{\partial \theta} + u \frac{\partial v_\perp}{\partial \xi} - \epsilon \nu \left( \frac{\partial^2 v_\perp}{\partial \xi^2} + \epsilon \frac{\partial^2 v_\perp}{\partial \theta^2} \right), \end{aligned} \quad (40)$$

$$\epsilon^{1/2} (V - w) \frac{\partial b_x}{\partial \theta} + \epsilon^{1/2} (B_0 \cos \alpha + b_z) \frac{\partial u}{\partial \theta} + \epsilon \eta \left( \frac{\partial^2 b_x}{\partial \xi^2} + \epsilon \frac{\partial^2 b_x}{\partial \theta^2} \right) = 0, \quad (41)$$

We use the stretched versions of Eqs. (15)–(23) to find the relationships between variables in each successive order of approximation (for full details we refer to Clack *et al.* (author?) [27]). Additionally, Eqs. (40) and (41) are used to help derive the equations governing the mean shear flow inside the dissipative layer. Equations (24) and (25) for the generated mean flow are transformed to

$$\begin{aligned} \epsilon \nu \frac{d^2 U_{\parallel}}{d\xi^2} &= \left\langle u \frac{\partial v_{\parallel}}{\partial \xi} \right\rangle - \frac{1}{\mu_0} \left\langle \frac{b_x}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \xi} \right\rangle \\ &+ \epsilon^{1/2} \left[ \left\langle v_{\parallel} \frac{\partial v_{\perp}}{\partial \theta} \right\rangle \cos \alpha + \left\langle \frac{1}{\bar{\rho}} \frac{\partial P}{\partial \theta} \right\rangle \cos \alpha \right. \\ &- \frac{B_0 \cos \alpha}{\mu_0} \left\langle \frac{1}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \theta} \right\rangle - \frac{\cos \alpha}{\mu_0} \left\langle \frac{b_{\parallel}}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \theta} \right\rangle \\ &\left. + \frac{\sin \alpha}{\mu_0} \left\langle \frac{b_{\perp}}{\bar{\rho}} \frac{\partial b_{\parallel}}{\partial \theta} \right\rangle - \frac{1}{\mu_0} \frac{dB_0}{dx} \left\langle \frac{b_x}{\bar{\rho}} \right\rangle \right], \quad (42) \end{aligned}$$

$$\begin{aligned} \epsilon \nu \frac{d^2 U_{\perp}}{d\xi^2} &= \left\langle u \frac{\partial v_{\perp}}{\partial \xi} \right\rangle - \frac{1}{\mu_0} \left\langle \frac{b_x}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \xi} \right\rangle \\ &+ \epsilon^{1/2} \left[ \left\langle v_{\parallel} \frac{\partial v_{\perp}}{\partial \theta} \right\rangle \cos \alpha - \frac{B_0 \cos \alpha}{\mu_0} \left\langle \frac{1}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \theta} \right\rangle \right. \\ &- \left\langle \frac{1}{\bar{\rho}} \frac{\partial P}{\partial \theta} \right\rangle \sin \alpha - \frac{\cos \alpha}{\mu_0} \left\langle \frac{b_{\parallel}}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \theta} \right\rangle \\ &\left. + \frac{\sin \alpha}{\mu_0} \left\langle \frac{b_{\perp}}{\bar{\rho}} \frac{\partial b_{\perp}}{\partial \theta} \right\rangle \right]. \quad (43) \end{aligned}$$

Following a similar procedure as introduced by Ruderman *et al.* (author?) [26] we find that variables inside the dissipative layer expand in a different way. The large variables ( $\tilde{v}_{\perp}$ ,  $b_{\perp}$ ) have expansion of the form

$$g = \epsilon^{1/2} g^{(1)} + \epsilon g^{(2)} + \dots, \quad (44)$$

whereas small variables ( $P$ ,  $p$ ,  $\rho$ ,  $u$ ,  $\tilde{v}_{\parallel}$ ,  $b_x$ ,  $b_{\parallel}$ ) expand as

$$h = \epsilon h^{(1)} + \epsilon^{3/2} h^{(2)} + \dots \quad (45)$$

The variables  $v_{\parallel}$  and  $v_{\perp}$  are expanded in series of the same form as  $\tilde{v}_{\parallel}$  and  $\tilde{v}_{\perp}$ , hence we expand  $U_{\parallel}$  and  $U_{\perp}$  as

$$U_{\parallel} = \epsilon U_{\parallel}^{(1)} + \dots, \quad U_{\perp} = \epsilon U_{\perp}^{(2)} + \dots \quad (46)$$

We have chosen the superscripts in Eq. (46) in such a way that

$$v_{\parallel}^{(n)} = \tilde{v}_{\parallel}^{(n)} + U_{\parallel}^{(n)}, \quad v_{\perp}^{(n)} = \tilde{v}_{\perp}^{(n)} + U_{\perp}^{(n)}. \quad (47)$$

On substituting these expansions into the stretched versions of Eqs. (15)–(23) we find a linear homogeneous system of equations for the variables with superscript ‘1’

in the first order approximation. This system parallels the results of linear theory (see, e.g., Clack *et al.* [27]). For the purposes of this paper, we are only interested in the expressions of variables found in terms of  $u^{(1)}$ ,  $\tilde{v}_{\perp}^{(1)}$  and  $P^{(1)}$ :

$$\tilde{v}_{\parallel}^{(1)} = \frac{c_{S_a}^2 \cos \alpha}{v_{A_a}^2 \rho_{0a} V} P^{(1)}, \quad (48)$$

$$b_{\perp}^{(1)} = -\frac{B_{0a} V}{v_{A_a}^2 \cos \alpha} \tilde{v}_{\perp}^{(1)}, \quad b_x^{(1)} = -\frac{B_{0a} \cos \alpha}{V} u^{(1)}, \quad (49)$$

$$\frac{\partial b_{\parallel}^{(1)}}{\partial \theta} = \frac{B_{0a} (v_{A_a}^2 - c_{S_a}^2)}{\rho_{0a} v_{A_a}^4} \frac{dP^{(1)}}{d\theta} + \frac{u^{(1)}}{V} \left( \frac{dB_0}{dx} \right)_a, \quad (50)$$

$$\frac{\partial p^{(1)}}{\partial \theta} = \frac{c_{S_a}^2}{v_{A_a}^2} \frac{dP^{(1)}}{d\theta} - \frac{u^{(1)} B_{0a}}{V \mu_0} \left( \frac{dB_0}{dx} \right)_a, \quad (51)$$

$$\frac{\partial \rho^{(1)}}{\partial \theta} = \frac{1}{v_{A_a}^2} \frac{dP^{(1)}}{d\theta} + \frac{u^{(1)}}{V} \left( \frac{d\rho_0}{dx} \right)_a. \quad (52)$$

In addition the equation that relates  $u^{(1)}$  and  $\tilde{v}_{\perp}^{(1)}$  is,

$$\frac{\partial u^{(1)}}{\partial \xi} = \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \theta} \sin \alpha. \quad (53)$$

Here the subscript ‘a’ means that the equilibrium quantities have been evaluated at the resonant position. In the second order approximation, after eliminating all variables with superscript ‘1’ in favour of  $\tilde{v}_{\perp}^{(1)}$  and  $P^{(1)}$  using Eqs. (48)–(53), and satisfying the compatibility condition, we derive the equation governing resonant Alfvén waves inside the dissipative layer (author?) [27]

$$\Delta_a \xi \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \theta} + V(\nu + \eta) \frac{\partial^2 \tilde{v}_{\perp}^{(1)}}{\partial \xi^2} = -\frac{V \sin \alpha}{\rho_{0a}} \frac{dP^{(1)}}{d\theta}, \quad (54)$$

where  $\Delta_a = -\left( \frac{dv_A^2}{dx} \right)_a \cos^2 \alpha$ , is the gradient of the square of the Alfvén speed. This governing equation is linear despite it being derived by using the full nonlinear MHD equations, a result explained fully by Clack *et al.* (author?) [27]. Let us summarise their conclusions needed here for the sake of clarity. If all perturbations are expanded asymptotically, one can see that the second order terms describe magnetoacoustic modes and are, therefore, not resonant at the Alfvén resonance. In addition, these terms act to cancel the small perturbations produced by the first order modes. This is in stark contrast to resonant slow waves, where the second order terms act to enhance the perturbations from the first order.

Inserting the expansions (44)–(46) into Eqs. (42)–(43) and collecting terms proportional to  $\epsilon$ , we have

$$\nu \frac{d^2 U_{\parallel}^{(1)}}{d\xi^2} = \frac{\sin \alpha}{\mu_0 \rho_{0a}} \left\langle b_{\perp}^{(1)} \frac{\partial b_{\parallel}^{(1)}}{\partial \theta} \right\rangle + \sin \alpha \left\langle \tilde{v}_{\parallel}^{(1)} \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \theta} \right\rangle - \frac{1}{\mu_0 \rho_{0a}} \left\langle b_x^{(1)} \frac{\partial b_{\parallel}^{(1)}}{\partial \xi} \right\rangle, \quad (55)$$

$$\begin{aligned} \nu \frac{d^2 U_{\perp}^{(2)}}{d\xi^2} &= \left\langle u^{(1)} \frac{\partial \tilde{v}_{\perp}^{(2)}}{\partial \xi} + u^{(2)} \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} \right\rangle \\ &- \frac{1}{\mu_0 \rho_{0a}} \left\langle b_x^{(1)} \frac{\partial b_{\perp}^{(2)}}{\partial \xi} + b_x^{(2)} \frac{\partial b_{\perp}^{(1)}}{\partial \xi} \right\rangle + \cos \alpha \left\langle \tilde{v}_{\parallel}^{(1)} \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \theta} \right\rangle \\ &- \frac{\cos \alpha}{\mu_0 \rho_{0a}} \left\langle b_{\parallel}^{(1)} \frac{\partial b_{\perp}^{(1)}}{\partial \theta} \right\rangle + \frac{B_{0a} \cos \alpha}{\mu_0 \rho_{0a}^2} \left\langle \rho^{(1)} \frac{\partial b_{\perp}^{(1)}}{\partial \theta} \right\rangle \\ &+ \frac{\xi}{\mu_0 \rho_{0a}^2} \left( \frac{d\rho_0}{dx} \right)_a \left\langle b_x^{(1)} \frac{\partial b_{\perp}^{(1)}}{\partial \xi} \right\rangle. \quad (56) \end{aligned}$$

In deriving Eqs. (55) and (56) we have utilized the fact that

$$\left\langle \frac{\partial f}{\partial \theta} \right\rangle = \left\langle f \frac{\partial}{\partial \theta} \right\rangle = \left\langle \frac{\partial g}{\partial \theta} \right\rangle = \left\langle g \frac{\partial}{\partial \theta} \right\rangle = 0,$$

(which follows directly from Eq. (11) and Eqs. (12) and (13). Now we use Eqs. (48)–(53) to eliminate terms on the right-hand sides of Eqs. (55) and (56) in favour of  $\tilde{v}_{\perp}^{(1)}$  and  $P^{(1)}$  to leave

$$\nu \frac{d^2 U_{\parallel}^{(1)}}{d\xi^2} = -\frac{V \sin \alpha}{\rho_{0a} v_{Aa}^2 \cos \alpha} \left\langle \tilde{v}_{\perp}^{(1)} \frac{dP^{(1)}}{d\theta} \right\rangle, \quad (57)$$

$$\begin{aligned} \nu \frac{d^2 U_{\perp}^{(2)}}{d\xi^2} &= \left\langle u^{(1)} \frac{\partial \tilde{v}_{\perp}^{(2)}}{\partial \xi} + u^{(2)} \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} \right\rangle \\ &- \frac{1}{\mu_0 \rho_{0a}} \left\langle b_x^{(1)} \frac{\partial b_{\perp}^{(2)}}{\partial \xi} + b_x^{(2)} \frac{\partial b_{\perp}^{(1)}}{\partial \xi} \right\rangle \\ &+ \frac{1}{\rho_{0a}} \left( \frac{d\rho_0}{dx} \right)_a \left\langle u^{(1)} \frac{\partial \left( \xi \tilde{v}_{\perp}^{(1)} \right)}{\partial \xi} \right\rangle \\ &- \frac{1}{B_{0a}} \left( \frac{dB_0}{dx} \right)_a \left\langle u^{(1)} \tilde{v}_{\perp}^{(1)} \right\rangle. \quad (58) \end{aligned}$$

Averaging the governing equation (54) and substituting it into Eq. (57) we obtain

$$\frac{d^2 U_{\parallel}^{(1)}}{d\xi^2} = -\frac{V(\nu + \eta)}{\nu v_{Aa}^2 \cos \alpha} \left\langle \tilde{v}_{\perp}^{(1)} \frac{\partial^2 \tilde{v}_{\perp}^{(1)}}{\partial \xi^2} \right\rangle, \quad (59)$$

which constitutes the equation that governs the generated mean flow inside the dissipative layer parallel to the

magnetic field lines. In order to derive the equivalent equation for the generated mean flow perpendicular to the magnetic field lines we note that Eqs. (40) and (41) in the second order approximation lead to

$$\begin{aligned} \frac{\partial \tilde{v}_{\perp}^{(2)}}{\partial \theta} + \frac{B_{0a} \cos \alpha}{\mu_0 \rho_{0a} V} \frac{\partial b_{\perp}^{(2)}}{\partial \theta} &= -\frac{\sin \alpha}{\rho_{0a} V} \frac{dP^{(1)}}{d\theta} - \frac{\nu}{V} \frac{\partial^2 \tilde{v}_{\perp}^{(1)}}{\partial \xi^2} \\ &+ \left[ \frac{B_{0a}}{\mu_0 \rho_{0a} v_{Aa}^2} \left( \frac{dB_0}{dx} \right)_a - \frac{1}{\rho_{0a}} \left( \frac{d\rho_0}{dx} \right)_a \right] \xi \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \theta}, \quad (60) \end{aligned}$$

$$u^{(2)} + \frac{V b_x^{(2)}}{B_{0a} \cos \alpha} = \frac{\eta \sin \alpha}{V} \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} - \frac{\xi u^{(1)}}{B_{0a}} \left( \frac{dB_0}{dx} \right)_a. \quad (61)$$

Substituting Eq. (54) into Eq. (60) we obtain

$$\begin{aligned} \frac{\partial \tilde{v}_{\perp}^{(2)}}{\partial \xi} + \frac{B_{0a} \cos \alpha}{\mu_0 \rho_{0a} V} \frac{\partial b_{\perp}^{(2)}}{\partial \xi} &= \left[ \frac{\nu \Delta}{V^2(\nu + \eta)} \right. \\ &+ \left. \frac{1}{B_{0a}} \left( \frac{dB_0}{dx} \right)_a - \frac{1}{\rho_{0a}} \left( \frac{d\rho_0}{dx} \right)_a \right] \frac{\partial \left( \xi \tilde{v}_{\perp}^{(1)} \right)}{\partial \xi}. \quad (62) \end{aligned}$$

On substitution of Eqs. (61) and (62) into Eq. (58) we produce

$$\begin{aligned} \nu \frac{d^2 U_{\perp}^{(2)}}{d\xi^2} &= \frac{\eta \sin \alpha}{V} \left\langle \left( \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} \right)^2 \right\rangle \\ &+ \frac{\nu}{V^2(\nu + \eta)} \frac{\partial}{\partial \xi} \left( \Delta \xi \left\langle u^{(1)} \tilde{v}_{\perp}^{(1)} \right\rangle \right). \quad (63) \end{aligned}$$

It follows directly from Eqs. (53) and (54) that

$$\begin{aligned} \frac{\Delta \xi}{V^2} \left\langle u^{(1)} \tilde{v}_{\perp}^{(1)} \right\rangle &= -\frac{\sin \alpha}{\rho_{0a} V} \left\langle u^{(1)} P^{(1)} \right\rangle \\ &- \frac{(\nu + \eta)}{V} u^{(1)} \int \left\langle \frac{\partial^2 \tilde{v}_{\perp}^{(1)}}{\partial \xi^2} \right\rangle d\theta \quad (64) \end{aligned}$$

and

$$\left\langle P^{(1)} \frac{\partial u^{(1)}}{\partial \xi} \right\rangle = \rho_{0a} (\nu + \eta) \left\langle \tilde{v}_{\perp}^{(1)} \frac{\partial^2 \tilde{v}_{\perp}^{(1)}}{\partial \xi^2} \right\rangle. \quad (65)$$

Using Eq. (63) along with Eqs. (64)–(65) we find the equation governing the generated mean shear flow inside the dissipative layer perpendicular to the magnetic field lines is

$$\begin{aligned} \frac{d^2 U_{\perp}^{(2)}}{d\xi^2} &= \frac{\eta \sin \alpha}{\nu V} \left\langle \left( \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} \right)^2 \right\rangle \\ &- \frac{\sin \alpha}{V} \left\langle \tilde{v}_{\perp}^{(1)} \frac{\partial^2 \tilde{v}_{\perp}^{(1)}}{\partial \xi^2} \right\rangle - \frac{1}{V} \left\langle \int \frac{\partial^3 \tilde{v}_{\perp}^{(1)}}{\partial \xi^3} d\theta \right\rangle. \quad (66) \end{aligned}$$

In the next section we will use Eqs. (59) and (66) along with the compatibility condition to derive the jumps in the derivatives of the parallel and perpendicular components of the mean velocity across the dissipative layer.

## V. JUMP CONDITIONS

To derive the jump in the derivative of the parallel component of the mean velocity across the dissipative layer we remember from Eq. (37) that  $V_{\parallel}^{\pm}$  are constants and the superscripts ‘-’ and ‘+’ refer to  $x < 0$  and  $x > 0$ , respectively. This means that

$$\begin{aligned} V_{\parallel}^+ - V_{\parallel}^- &= \lim_{\xi \rightarrow \infty} \frac{dU_{\parallel}^{(1)}}{d\xi} - \lim_{\xi \rightarrow -\infty} \frac{dU_{\parallel}^{(1)}}{d\xi} \\ &= -\frac{V(\nu + \eta)}{\nu v_{Aa}^2 \cos \alpha} \int_{-\infty}^{\infty} \left\langle \left( \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} \right)^2 \right\rangle d\xi. \end{aligned} \quad (67)$$

In a similar fashion, we find the jump in the derivative of the perpendicular component of the mean velocity across the dissipative layer to be

$$\begin{aligned} V_{\perp}^+ - V_{\perp}^- &= \lim_{\xi \rightarrow \infty} \frac{dU_{\perp}^{(1)}}{d\xi} - \lim_{\xi \rightarrow -\infty} \frac{dU_{\perp}^{(1)}}{d\xi} \\ &= \frac{(\nu + \eta) \sin \alpha}{\nu V} \int_{-\infty}^{\infty} \left\langle \left( \frac{\partial \tilde{v}_{\perp}^{(1)}}{\partial \xi} \right)^2 \right\rangle d\xi. \end{aligned} \quad (68)$$

Outside the dissipative layer we have the following approximate equalities:

$$U_{\parallel} \simeq \epsilon^{1/2} U_{\parallel}^{(0)}(x), \quad U_{\perp} \simeq \epsilon^{1/2} U_{\perp}^{(0)}(x). \quad (69)$$

Let us introduce the new dimensionless variables

$$\sigma = \delta_a^{-1} x = \epsilon^{1/2} \delta_a^{-1} \xi, \quad q = \frac{\epsilon^{1/2} k \delta_a \tilde{v}_{\perp}^{(1)}}{V},$$

where  $\delta_a = [V(\bar{\nu} + \bar{\eta}) / (k|\Delta_a|)]^{1/3}$ , is the width of the dissipative layer and  $k = 2\pi/L$ . Equations (67) and (68) can be rewritten in these new variables as

$$\left[ \frac{dU_{\parallel}}{dx} \right] = -\frac{|\Delta_a| \cos \alpha}{2\pi \bar{\nu}} \int_0^L d\theta \int_{-\infty}^{\infty} \left( \frac{\partial q}{\partial \sigma} \right)^2 d\sigma, \quad (70)$$

$$\left[ \frac{dU_{\perp}}{dx} \right] = \frac{|\Delta_a| \sin \alpha}{2\pi \bar{\nu}} \int_0^L d\theta \int_{-\infty}^{\infty} \left( \frac{\partial q}{\partial \sigma} \right)^2 d\sigma. \quad (71)$$

These equations are implicit connection formulae, and as such, they must be solved in conjunction with Eq. (54) (to determine the dimensionless variable  $q$ ). We note that Eqs. (70) and (71) imply that at  $\alpha = 0, \pi/2$  the jumps are zero, since at these values of  $\alpha$  there is no Alfvén resonance present.

In fact, Eqs. (70) and (71) can be solved explicitly, since the governing equation (54) is linear. It has been shown by, e.g. Goossens *et al.* (author?) [33] and Erdélyi (author?) [38], that the solution of the governing equation can be found in terms of the so-called  $F$  and  $G$  functions. In order to find the solutions to Eqs. (70) and (71) we need to find  $\tilde{v}_{\perp}^{(1)}$  in order to obtain  $q$ . In

cartesian coordinates, this was recently accomplished by Ruderman (author?) [39] where he found that

$$\tilde{v}_{\perp} = \frac{iVP \sin \alpha}{\rho_{0a} \delta_a |\Delta_a|} F(\sigma), \quad (72)$$

where

$$F(\sigma) = \int_0^{\infty} \exp(i\phi \sigma \operatorname{sgn}(\Delta_a) - \phi^3/3) d\phi. \quad (73)$$

Equation (72) is the cartesian version of Eq. (66) found by Goossens *et al.* (author?) [33].

Substituting Eq. (72) into Eqs. (70) and (71) leads to

$$\left[ \frac{dU_{\parallel}}{dx} \right] = -A \cos \alpha \int_0^L P^2 d\theta \int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma, \quad (74)$$

$$\left[ \frac{dU_{\perp}}{dx} \right] = A \sin \alpha \int_0^L P^2 d\theta \int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma, \quad (75)$$

with  $A = k^2 \sin^2 \alpha / (2\pi \bar{\nu} \rho_{0a}^2 |\Delta_a|)$ . It is shown in the Appendix that

$$\int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma = \pi, \quad (76)$$

implying that Eqs. (74) and (75) reduce to

$$\left[ \frac{dU_{\parallel}}{dx} \right] = -\frac{k^2 \sin^2 \alpha \cos \alpha}{2\bar{\nu} \rho_{0a}^2 |\Delta_a|} \int_0^L P^2 d\theta, \quad (77)$$

$$\left[ \frac{dU_{\perp}}{dx} \right] = \frac{k^2 \sin^3 \alpha}{2\bar{\nu} \rho_{0a}^2 |\Delta_a|} \int_0^L P^2 d\theta. \quad (78)$$

Equations (77) and (78) are the explicit connection formulae for the jumps in the derivatives of the mean shear flow across the dissipative layer. They are explicit because we are considering a driven problem, and hence  $P$  is assumed to be known.

If we take  $\alpha = \pi/4$  we have the following approximation

$$\left[ \frac{dU_{\parallel}}{dx} \right] = -\left[ \frac{dU_{\perp}}{dx} \right] \simeq \frac{\epsilon^{1/2} V}{l_{\text{inh}}}, \quad (79)$$

and these values can be seen as jumps in vorticity. Here we have used the obvious estimates  $|\Delta_a| = \mathcal{O}(V^2/l_{\text{inh}})$ ,  $P = \mathcal{O}(\epsilon V^2 k l_{\text{inh}})$  and  $\bar{\nu} = \mathcal{O}(\epsilon^{3/2} V l_{\text{inh}})$ . In order to find the profiles of the components of the generated mean shear flow we need to impose boundary conditions far away from the dissipative layer. For example, if there are rigid walls at  $x = \pm a$  where the condition of adhesion has to be satisfied, then the components of the generated mean flow take the simple form

$$U_{\parallel} = \begin{cases} \left[ \frac{dU_{\parallel}}{dx} \right] \frac{x-a}{2}, & x > 0, \\ -\left[ \frac{dU_{\parallel}}{dx} \right] \frac{x+a}{2}, & x < 0, \end{cases} \quad (80)$$

$$U_{\perp} = \begin{cases} \left[ \frac{dU_{\perp}}{dx} \right] \frac{x-a}{2}, & x > 0, \\ - \left[ \frac{dU_{\perp}}{dx} \right] \frac{x+a}{2}, & x < 0. \end{cases} \quad (81)$$

With the estimate given by Eq. (79) and the simple mean flow profiles in Eqs. (80) and (81) we can calculate the expected mean flow generated outside the dissipative layer in the solar upper chromosphere and solar corona. For example, if the incoming wave has a dimensionless amplitude of  $\epsilon = \mathcal{O}(10^{-4})$ , then the predicted mean shear flow is of the order of  $10\text{kms}^{-1}$  in both the upper chromosphere and corona. Here we have assumed that the characteristic scale of inhomogeneity ( $l_{\text{inh}}$ ) is  $10^2\text{m}$  in the upper chromosphere and  $10^3\text{m}$  in the corona. This generated flow can be superimposed on existing flow, so it is difficult to observe. These results should be used with caution, when applied to the solar atmosphere. We have assumed that outside the dissipative layer the plasma is homogeneous and infinite, and clearly the solar atmosphere is neither, so, since all flows are entirely governed by the boundary conditions, the flow generated may be less than predicted or may even be stopped entirely.

## VI. CONCLUSIONS

In the present paper we have completed the nonlinear theory of resonant Alfvén waves in dissipative layers in a one-dimensional (1-D) planar geometry. Clack *et al.* (author?) [27] showed that even though nonlinearity and dispersion are considered the equation governing resonant Alfvén waves in the dissipative layer is always linear (provided  $\epsilon \ll R^{-1/3}$ ). However, they neglected the second manifestation of nonlinearity at resonance; the generation of mean shear flows outside the dissipative layer. This flow is produced by the nonlinear interaction of harmonics inside the dissipative layer and may still exist even though the governing equation inside the dissipative layer is linear.

We have shown that outside the dissipative layer a mean flow is generated parallel to the magnetic surfaces. The flow has an amplitude proportional to  $\epsilon^{1/2}$ , and depends linearly on  $x$ . The derivatives of the velocity of the generated mean flow have a nonzero jump across the dissipative layer determined by Eqs. (70) and (71). When  $\alpha = \pi/4$  the magnitude of the jumps can be estimated as  $\epsilon^{1/2}Vl_{\text{inh}}^{-1}$ , where  $V$  is the phase speed of the incoming wave and  $l_{\text{inh}}$  is the characteristic scale of inhomogeneity. For typical conditions in the solar upper chromosphere and corona the magnitude of the jumps in the derivatives of mean shear flow velocity would be of the order of  $10\text{s}^{-1}$ . From this, and the simple flow profiles given by Eqs. (80) and (81), we predict a mean flow outside the dissipative layer (generated by resonant absorption) with an amplitude of the order of  $10\text{kms}^{-1}$  in the solar upper chromosphere and corona.

The magnitude of the jumps in the derivatives of the mean flow were found to depend on the phase speed ( $V$ ), the characteristic scale of inhomogeneity ( $l_{\text{inh}}$ ) and the dimensionless amplitude of oscillation ( $\epsilon$ ) far away from the dissipative layer. Even though simple estimations allowed us to approximate the magnitude of the shear flows, the results in the present paper should be used with caution. The present analysis has been carried out for magnetic configurations that are homogeneous and infinite in the direction of wave propagation outside the dissipative layer. This situation can only take place in laboratory devices (such as tokamaks). In the solar atmosphere, magnetic configurations are bounded and/or inhomogeneous in the direction of wave propagation. These additional boundary conditions may reduce (or even prevent) the generation of mean shear flows by resonant absorption.

It is worth mentioning that when an Alfvén resonance is present there may exist a slow resonance coupled to it. If this happens, both resonances will generate their own mean shear flows which could become superimposed. This may enhance the flow through constructive superposition, but equally, they may interact destructively leaving a reduced overall combined generated mean shear flow. In other words, it is possible to see a more efficient coupled dissipative layer create more heating, but actually generate a smaller amplitude mean shear flow compared with two separate dissipative layers.

If the boundary conditions allow the generation of a mean shear flows by resonant absorption in the solar atmosphere, we would expect to see these flows superimposed on existing large scale motions. It is speculated that these mean shear flows (if set up) superimposed on existing bulk motion of the plasma would produce turbulent motions and create a Kelvin–Helmholtz instability (KHI) along the dissipative layer. A KHI instability may already exist due to the shear velocity field of Alfvén waves as was shown by Terradas *et al.*(author?) [40]. The generation of mean shear flow would survive the instability due to the velocity shear and, hence, will still contribute to the bulk motion causing larger shear, and perhaps creating a secondary instability. Either way, the KHI would distort the dissipative layer, locally enhancing the dissipation in (due to enhanced transport coefficients), which in turn would produce greater amplitude mean shear flows. This process of feedback would continue until either; (i) the wave is dissipated, or (ii) the turbulence destroys the dissipative layer. Either way, the turbulence would allow greater absorption and achieve heating over a greater area (here the development of turbulent eddies can be seen as a means of heat transport).

The resonant absorption itself may not produce the heating we see in the solar corona, but we believe it is likely that these dissipative layers are created and KHI are formed. The KHI might be able to trigger small scale reconnection events (nanoflaring) if the mean shear flows produced by resonant absorption become strong enough. However, further study and observations are required be-

fore we can make more definitive investigations.

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### APPENDIX: THE CALCULATION OF THE INTEGRAL OF THE $F$ FUNCTION

In this Appendix, we will derive Eq. (76). We know the form of  $F(\sigma)$  from Eq. (73), so we can rewrite Eq. (76) as

$$\int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma = \int_{-\infty}^{\infty} d\sigma \left( \int_0^{\infty} \tilde{\phi} e^{i\tilde{\phi}\sigma - \tilde{\phi}^3/3} d\tilde{\phi} \times \int_0^{\infty} \tilde{\lambda} e^{-i\tilde{\lambda}\sigma - \tilde{\lambda}^3/3} d\tilde{\lambda} \right), \quad (\text{A1})$$

where the tilde denotes the inclusion of the  $\text{sgn}(\Delta_a)$ , from this point onwards we drop the tilde notation. Here we

are calculating the absolute value of the derivative, and so we use complex conjugates. If we change the order the integration we obtain

$$\int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma = \int_0^{\infty} \phi e^{-\phi^3/3} d\phi \times \int_0^{\infty} \lambda e^{-\lambda^3/3} d\lambda \times \int_{-\infty}^{\infty} e^{-i\sigma(\lambda-\phi)} d\sigma. \quad (\text{A2})$$

The third integral in Eq. (A2) is the definition of the delta function,  $\delta(\lambda - \phi)$ , so we can write Eq. (A2) as

$$\int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma = \int_0^{\infty} \phi e^{-\phi^3/3} d\phi \times \int_0^{\infty} \lambda e^{-\lambda^3/3} \cdot 2\pi\delta(\lambda - \phi) d\lambda. \quad (\text{A3})$$

Since the integral of the delta function is always unity, we arrive at

$$\int_{-\infty}^{\infty} \left| \frac{dF}{d\sigma} \right|^2 d\sigma = 2\pi \int_0^{\infty} \phi^2 e^{-2\phi^3/3} d\phi = -\pi e^{-2\phi^3/3} \Big|_0^{\infty} = \pi. \quad (\text{A4})$$

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