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Pareto Optimality and Existence of Quasi-Equilibrium in Exchange Economies with an Indefinite Future

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Abstract

We study the attainability of Pareto optimal allocations and existence of quasi-equilibrium in exchange economies where agents have utility functions that value consumption in an indefinite future. These utility functions allow for fairly general discounting of consumption over finite time horizons, but add a utility weight to the bulk of the consumption sequence, which we identify with the indefinite future. As our commodity space we use the space of all convergent sequences with the limit of the sequence representing consumption in the indefinite future. We derive a necessary and sufficient condition for the attainability of the Pareto optimal allocations. This condition implies that efficiency can only be attained if consumers' valuations of time are very similar. Our proof relies on the existence of an interior solution to certain infinite dimensional optimization problems. If the condition is not met, no interior quasi-equilibria exist. We extend the model to include consumers with Rawlsian-like maximin utility.

Keywords: Infinite horizon exchange economy, Pareto optimality, non-discounting preferences
JEL classification: D51

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1 Introduction

In many fields of economics there is a need to describe economic situations with an open-ended time-line. Following Bewley (1972) this has mainly been modelled by assuming an infinite time horizon. As has been clarified by Araujo (1985) and Brown and Lewis (1981), the commonly used commodity–price duality of bounded and summable sequences in such models, necessitates the assumption of myopic preferences. So, in a sense, models in the tradition of Bewley (1972) analyse economies that “fizzle out”, emphasizing the interpretation of infinity as representing time so far away that it is of no concern to economic agents.

In applications of such models, the description of an agent’s intertemporal choice is a crucial ingredient. For example, in the economics of climate change, typically, intertemporal choice is assumed to be independent and stationary (see, for example, Heal, 2005), which implies that the standard exponential discounting model can be used. Even so, there is a substantial debate about what the appropriate discount rate should be.[1] Notwithstanding this debate, in the existing literature, the utility value of consumption far into the future is assumed to vanish sufficiently fast.

In this paper we want to present a different approach to modelling an open-ended future, using the notion of *indefinite future*. By this we mean that we wish, in an infinite horizon setting, to separate consumption at individual time periods from consumption of the bulk of the sequence in the far future. Such a setting allows for preferences that are consistent with the behaviour of agents who care about a “steady state” of consumption, but are less able to distinguish between individual time periods in the far future than in the near future. From this point of view, infinity does not literally represent the end of times, but, rather, the “indefinite future”. For example, people may recycle goods at some (low) cost to themselves out of environmental concerns for the indefinite future. Also, many people give money to organizations that conserve our cultural heritage for future generations.

As a first step in this research programme we wish to establish under what conditions the Pareto frontier of such an economy is attainable and under what conditions existence of (quasi-) equilibrium can be guaranteed. It is immediately clear that such models can not use the usual commodity–price duality, because on the space of bounded sequences behaviour in the indefinite future is not well-defined. Instead, we are naturally led to the

[1] See Stern (2007), Nordhaus (2007), Weitzman (2007), Dasgupta (2008), and Heal and Millner (2013).

space of convergent sequences, with the limit representing consumption in the indefinite future. On this commodity space we then define utility functions that are a mixture of discounting over finite time periods and a utility assigned to the indefinite future. We allow for fairly general modes of discounting, including exponential or hyperbolic discounting.

Our main result is to provide a necessary and sufficient condition for the attainability of the Pareto frontier for such utility functions in an infinite horizon exchange economy with one consumption good. This necessary and sufficient condition, which we call *time value consistency*, requires that for each pair of consumers its ratio of utility weights on consumption far into the future is consistent with the ratio of the weights they put on consumption at infinity. Since our utility functions are continuous in an appropriate topology and we assume that initial endowments are in the interior of the positive cone, standard arguments now imply that time value consistency is a sufficient condition for the existence of quasi-equilibrium. If time-value consistency is violated, then some (but not necessarily all) Pareto optima can not be allocated. It turns out that this implies that any quasi-equilibrium in such an economy (assuming existence) has the property that at least one consumer gets allocated nothing in the indefinite future.

Since time value consistency requires that there is strong agreement among all consumers about the value of time, our result hints at potential inter-generational issues in economies where agents have different senses of how far away the indefinite future is. This is very different from the analysis of conventional (finite and infinite horizon) models, where, as long as all agents are myopic, agents can discount in any way they like. This, however, rules out any far-sighted (presbyopic) element to preferences. In particular, such models can not deal with consumers who have, for example, maximin preferences. It turns out that our main result can easily be extended to economies where some consumers have maximin preferences. Their presence does not affect the attainability of Pareto optimal allocations: Pareto optimal allocations are attainable as long as every pair of non-maximin preferences is time value consistent.

We prove our results using infinite dimensional versions of the implicit function and Lagrange theorems. This approach is different from the standard approach, which tends to rely on the Alaoglu theorem. We, therefore, also make a contribution to the mathematical toolbox that can be used for analysing infinite dimensional economies.

Unlike in finite dimensional economies, modelling an infinite horizon economy always involves a trade-off between the commodity and price spaces and the topology that puts these two spaces in duality. In order to derive results beyond the standard Bewley (1972) approach one has carefully recalibrate this duality. We present one such possibility in this paper. Another approach is studied by Chichilnisky (2012a,b), who extends the price space to the space of all boundedly additive sequences. That way she can allow for preferences that value infinity *and* ensure equilibrium existence. A disadvantage of this approach, however, is that non-summable price sequences are very difficult to interpret economically. For example, there is no algorithm that allows a social planner or a Walrasian auctioneer to construct such prices. In fact, the existence of such price functionals depends crucially on the axiom of choice, i.e. such prices can only exist by making an uncountable number of arbitrary choices. In our approach, the use of convergent sequences to represent commodity bundles avoids this problem, because the dual space of prices consists of summable sequences. While this restriction to the space of convergent sequences is not innocuous, it has both economic and mathematical appeal, as we explain in Section 2.

Section 3, then, describes the main ingredients of the infinite-dimensional exchange economy, followed by a description of time value consistency and the main theorem in Section 4. The proof of the main theorem is given in Section 5. Results on existence of quasi-equilibria are presented in Section 6, together with some examples to illustrate the main issues. Section 7 presents some extensions and special cases. In Section 8 we discuss the case where several consumers have Rawlsian-like maximin preferences. These are preferences where a consumer values a consumption sequence solely through the utility provided by the period in which she is worst-off, much like an intertemporal version of the famous Rawlsian social welfare function where individual agents' preferences are aggregated by looking at the utility of the agent who is worst-off (see, for example, Moulin, 1988). Such preferences provide an example of utility functions that cannot be dealt with in the standard commodity–price duality (because they are not continuous in the Mackey topology), but do not present any problems in our set-up. Finally, Section 9 provides some concluding remarks.

2 Choosing A Commodity Space to Deal with the Infinite Future

As was explained in Section 1, preferences that value the indefinite future can not be used in an economy with the $\left\langle \ell^\infty, \ell^1 \right\rangle$ duality, because such preferences are not continuous in the Mackey topology for this duality. In order to remedy this, we will have to change the commodity space or the price space (or both). Bewley (1972) has given convincing arguments why, when the commodities consist of sequences, the price space should be the space of summable sequences, ℓ^1. One of the arguments is that sequences in the topological dual of ℓ^∞, the space of boundedly additive sequences, *ba*, have no clear economic interpretation. In fact, many functionals in *ba* cannot be constructed in any meaningful way.

The same argument, however, can be levelled at the commonly-used commodity space of the bounded sequences, ℓ^∞. Some of the sequences in this space exhibit such erratic behaviour over time, that they cannot be constructed in any meaningful way. It can be argued that such sequences should not be used as potential commodities, especially since allowing their presence heavily restricts the types of preferences that can be accommodated. In fact, it is precisely this erratic behaviour that requires continuous preferences – in the Mackey topology on $\left\langle \ell^\infty, \ell^1 \right\rangle$ – to exhibit sufficiently fast discounting; thereby precluding any behaviour that values consumption in the far future. This provides a strong argument to question the choice of ℓ^∞ as a commodity space.

Another economic disadvantage of focussing on Mackey continuous preferences on $\left\langle \ell^\infty, \ell^1 \right\rangle$ is that, if a sequence has a long-run average, this long-run average is not valued by the consumer *per se*. The consumer cannot "see" that she is consuming a long-run average with variations over time. In fact, when thinking about economies over a long time horizon, it may be quite natural to consider sequences (possibly after de-trending to account for economic growth) that have a long-run average. This naturally leads to the consideration of the space of convergent sequences, c, as the limit of a sequence can be thought of as the long-run average.

In addition, the space of convergent sequences is "close" to the space of bounded sequences in the sense that c is dense in ℓ^∞. Also, every sequence in c can be constructed via finite dimensional approximations. This has great advantages for computational implementations of infinite horizon models. In fact, our main proofs rely heavily on this type of approximation.

Finally, our results can fairly easily be extended to more general spaces, for example those of the periodic sequences. The main ideas of the paper, however, are most easily explained in the context of the convergent sequences, which is why we focus on infinite horizon economies where the commodity space is given by c.

3 An Infinite-Dimensional Exchange Economy with Convergent Commodities

An exchange economy with N consumers is a collection

$$\mathscr{E} = \left(\langle X, P\rangle, \tau, (X^i, u^i, \omega^i)_{i=1}^N\right). \tag{1}$$

where $\langle X, P\rangle$ is a commodity-price duality, τ is a topology that is consistent with the duality $\langle X, P\rangle$, and $X^i \subseteq X_+$, u^i and $\omega^i \in X$ are the consumption set, utility function on X and initial endowments of consumer i, $i = 1, \ldots, N$, respectively.[2]

For our purposes, we must be able to construct prices for consumption at each individual time as well as in the indefinite future. Mathematically, these are the functions $x \mapsto x_t$ for $t \in \mathbb{N} \cup \{\infty\}$, where x_∞ is shorthand for $\lim_{t\to\infty} x_t$. The largest commodity space on which we can do this is the space of convergent sequences, c.

The price functionals must then be built from the functionals $x \mapsto x_t$. As remarked by Bewley (1972), there are many non-intuitive ways of doing this (relying on the axiom of choice). However, the only meaningful ones lead to a value of consumption bundles equal to:

$$\langle x, p\rangle = \sum_{t=1}^{\infty} p_t x_t + p_\infty x_\infty, \tag{2}$$

where $p_t \geq 0$ for all t. It is important to realise that p_∞ is not a limit, but, rather, a freely chosen price "at infinity". For this to make sense we must have that $\sum_{t=1}^{\infty} p_t < \infty$; conversely, if $p_t \geq 0$ and $\sum_{t=1}^{\infty} p_t < \infty$ then (2) makes sense for all $x \in c$. The price space is then the linear span of these positive functionals, i.e. the space of all functionals of the form (2) with $\sum_{t=1}^{\infty} |p_t| < \infty$. Denote this space $\ell^1 \oplus \mathbb{R}$; this is now in duality with c via the bilinear form (2).

To allow for the widest possible range of utility functions, we should place the strongest possible topology on c, subject to the dual space being (via the bilinear form (2)) exactly

[2]Here X_+ denotes the positive cone of the commodity space X, i.e. $x \leq y$ iff $y - x \in X_+$ for all $x, y \in X$.

$\ell^1 \oplus \mathbb{R}$; that is, we require the Mackey topology on c induced by the duality $\left\langle c, \ell^1 \oplus \mathbb{R} \right\rangle$. In the uniform norm $\|\cdot\|_\infty$, c is a Banach space whose dual is $\ell^1 \oplus \mathbb{R}$ (Dunford and Schwartz, 1958, IV.6.3); in this case (Schaefer, 1999, IV.3.4) the Mackey topology coincides with the norm topology. The required topology on c is therefore that induced by $\|\cdot\|_\infty$. Note that the model itself does not require that we stipulate a topology on the price space. According to convenience, we might use the dual norm $\|p\|_1 = \sum_{t=1}^{\infty} |p_t| + |p_\infty|$, or the weak* topology induced by c and (2).

For N consumers, we work in a Banach space which we denote c^N, consisting of convergent sequences of N-dimensional vectors. An element x of this space is defined by the real numbers x_t^i, where $1 \le i \le N$ and $t \in \mathbb{N}$, representing an allocation of x_t^i to consumer i at time t. We require x_t^i to converge to a limit as $t \to \infty$, and denote this limit by x_∞^i. We want to develop a natural vectorization of $\left\langle c, \ell^1 \oplus \mathbb{R} \right\rangle$ and, therefore, use the norm on this space given by $\|x\|_\infty = \sup_{1 \le i \le N, t \in \mathbb{N}} |x_t^i|$. There are natural projections of this space onto c and $\mathbb{R}^N$: for any given i, x^i will denote the convergent real sequence, $(x_t^i)_{t \in \mathbb{N}}$; for any given t, x_t will denote the vector in $\mathbb{R}^N$, $(x_t^i)_{i=1}^N$. On $\mathbb{R}^N$ we use the ∞-norm

$$\|y\|_\infty = \max_{1 \le i \le N} |y^i|.$$

An element of the dual space $(c^N)^*$ can be represented as a sequence $(\mu_t^i)_{t \in \mathbb{N}, 1 \le i \le N}$ and a vector $(\nu^i)_{i=1}^N$, where for each i, $\sum_{t=1}^{\infty} \mu_t^i$ converges, i.e. $(\mu_t^i)_{t \in \mathbb{N}} \in \ell^1$. The bilinear form expressing the duality is

$$\langle x, (\mu, \nu) \rangle = \sum_{i=1}^{N} \left[\sum_{t=1}^{\infty} \mu_t^i x_t^i + \nu^i \lim_{t \to \infty} x_t^i \right]. \tag{3}$$

Let

$$Z = \left\{ x = (x^1, \ldots, x^N) \in X^1 \times \cdots \times X^N \;\middle|\; \sum_{i=1}^{N} x^i \le \omega \right\},$$

be the set of *attainable allocations*, where $\omega = \sum_{i=1}^{N} \omega^i$. It is assumed here that consumers can freely dispose of goods. The total endowment at time t is denoted by ω_t. Since $\omega^i \in c$, all $i = 1, \ldots, N$, it follows that the sequence of total endowments converges to a limit ω_∞. In addition we assume that this limit is strictly positive limit (the case $\omega_\infty = 0$ is different in character, and considerably simpler; see Section 7).

Our interest is in the study of preferences that reflect concerns about the indefinite future. In order to stay close to the Bewley world and the recent literature, in particular

Araujo et al. (2011), we consider utility functions of the form

$$u^i(x^i) = \sum_{t=1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i \lim_{t\to\infty} v^i(x_t^i), \quad x^i \in X^i = c_+ \tag{4}$$

defined on the positive cone of the space c. Here, for each i, $(\delta_t^i)_{t\in\mathbb{N}}$ is a strictly positive, summable sequence, $\zeta^i > 0$ is the weight that the consumer places on consumption at infinity (see Section 7 for some observations about the cases $\delta_t^i = 0$ for some t and $\zeta^i = 0$) and v^i is defined on an open set containing $[0,\infty)$, i.e. on $(-\varepsilon,\infty)$ for some $\varepsilon > 0$, and is twice continuously differentiable. We also assume that $v^i(0) = 0$ and that for $x \in [0,\infty)$, $(v^i)'(x) > 0$, and $(v^i)''(x) < 0$. The *utility possibility set* of the economy $\mathscr{E}$ is then given by

$$\begin{aligned} U &= \{u \in \mathbb{R}^N \mid u \le u(x) = (u^1(x^1),\dots,u^N(x^N)), \text{ for some } x \in Z\} \\ &= u(Z) - \mathbb{R}_+^N. \end{aligned}$$

The utility vector $u \in U$ is a *weak (Pareto) optimum* if there is no $\hat{u} \in U$ such that $\hat{u}^i \ge u^i$ for all i with strict inequality for at least one i. The set of Pareto optimal allocations is essentially the positive boundary of the utility possibility set, $\partial U \cap \mathbb{R}_+^N$.

Assuming that ω is in the interior of X_+, Mas–Colell and Zame (1991) show that closedness of U is sufficient for the existence of a quasi-equilibrium.[3] Closedness of the utility possibility set also means that every Pareto optimum is attainable. In the Bewley-world closedness of U follows immediately from the Alaoglu theorem. There, however, non-discounting preferences are not continuous in the Mackey topology on $\langle \ell^\infty, \ell^1 \rangle$. In our setting, since we are working in the norm topology continuity of utility functions of the form (4) is easily obtained. Closedness of U, however, cannot be obtained using the Alaoglu theorem and is, therefore, the main focus of the paper.

Preferences of the form (4) value both individual time periods and the indefinite future. One way to think about such preferences is to reinterpret limit consumption x_∞ as long-run *average* consumption. The parameter ζ^i measures the weight consumer i places on average consumption relative to deviations from the average at each individual point in time. These deviations are discounted over time.

The "discounting sequence", $(\delta_t^i)_{t\in\mathbb{N}}$ can take various forms, as long as $(\delta_t^i)_{t\in\mathbb{N}} \in \ell^1$. For example, the agent could use exponential discounting: $\delta_t = \delta^t$, for some sufficiently small

[3] A pair $(x,p) \in X \times P$ is a *quasi-equilibrium* if $\langle p,\omega\rangle \ne 0$ and if, for all i, $\langle p,\bar{x}^i\rangle \ge \langle p,\omega^i\rangle$ whenever $u^i(\bar{x}^i) > u^i(x^i)$.

δ. Hyperbolic discounting in the sense of Laibson (1997), i.e. setting

$$\delta_t = \frac{\gamma}{1+\kappa t},$$

with γ and κ fixed, does not work, because the speed of decay of this sequence is too low. However, for every $\varepsilon > 0$, the discounting sequence $(\delta_t)_{t\in\mathbb{N}}$ with

$$\delta_t = \frac{\gamma}{1+\kappa t^{1+\varepsilon}},$$

is admissible. Alternatively, the agent could use quasi-hyperbolic discounting: $\delta_t = (\beta, \delta, \delta^2, \ldots)$, for $\beta, \delta \in (0,1)$. In fact, agents could even have arbitrary discount factors for any finite number of periods:

$$(\delta_t)_{t\in\mathbb{N}} = (\delta_1, \delta_2, \ldots, \delta_T, \delta^{T+1}, \delta^{T+2}, \ldots),$$

for arbitrary $(\delta_1, \ldots, \delta_T)$ and T, and δ small enough.

4 Time Value Consistency, Pareto Optimality and the Main Theorem

We begin with some terminology and notation about the utility possibility set U and some ways in which it can be decomposed. As defined, U contains non-positive vectors which, because of our normalization $v^i(0) = 0$, do not represent feasible allocations. Since we are more concerned with allocatable, i.e. non-negative, elements of U and of its boundary, we make the following definition.

Definition 1. The *positive part* of the utility possibility set is defined by $U^+ = U \cap \mathbb{R}^N_+$ or, more constructively,

$$U^+ = \{ (u^i(x^i))_{i=1}^N \mid x_t^i \geq 0, x_t^i + \cdots + x_t^N \leq \omega_t \ (1 \leq i \leq N, t \in \mathbb{N}) \}.$$

Similarly, the *positive boundary* of U is defined by $\partial^+ U = (\partial U) \cap \mathbb{R}^N_+$.

The *pointwise* or *Minkowski* sum of sets of vectors in $\mathbb{R}^N$ is defined by

$$A + B = \{ a + b \mid a \in A, b \in B \}.$$

Because of the time-separable nature of our utility functions, there are various ways of decomposing U^+ into Minkowski sums. The most important is, for some $T \in \mathbb{N}$,

$$\begin{aligned} U_{T-} &= \left\{ \left(\sum_{t=1}^{T} \delta_t^i v^i(x_t^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, 1 \leq t \leq T) \right\} \\ U_{T+} &= \left\{ \left(\sum_{t=T+1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i v^i(x_\infty^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, t > T) \right\} \\ U^+ &= U_{T-} + U_{T+}. \end{aligned} \tag{5}$$

Here we decompose U^+ into the utilities attained up to time period T — an essentially finite-dimensional object — plus the utilities attained from time $T+1$ onwards, including utility attained at ∞. Another occasionally useful decomposition is

$$\begin{aligned} U_{\mathrm{F}} &= \left\{ \left(\sum_{t=1}^{\infty} \delta_t^i v^i(x_t^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, t \in \mathbb{N}) \right\} \\ U_\infty &= \left\{ \left(\zeta^i v^i(x_\infty^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq i \leq N, t \in \mathbb{N}) \right\} \\ U^+ &= U_{\mathrm{F}} + U_\infty \end{aligned} \tag{6}$$

which we can think of a decomposition of U^+ into the utilities attained over all finite time times, plus the utilities attained at ∞. Here U_∞ is essentially finite-dimensional.

At this point, it is helpful to give a concrete description of the closure of U^+ as an infinite Minkowski sum, and to mention an important strict convexity property.

Lemma 1. *For $t \in \mathbb{N}$, let*

$$U_t = \{(\delta_t^i v^i(x_t^i))_{i=1}^{N} \mid x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t\},$$

and let

$$\check{U} = \left\{ \left(\zeta^i v^i(\check{x}^i) \right)_{i=1}^{N} \,\middle|\, \check{x}^i \geq 0, \check{x}^1 + \cdots + \check{x}^N \leq \omega_\infty \right\}.$$

Then the closure of the positive part of the utility possibility set is given by

$$\bar{U}^+ = \left\{ \left(\sum_{t=1}^{\infty} y_t \right) + \check{y} \,\middle|\, y_t \in U_t \quad (t \in \mathbb{N}), \check{y} \in \check{U} \right\}.$$

If $y \in \partial^+ U$, then any supporting hyperplane for $\bar{U}$ through y has no other points of intersection with $\bar{U}$, i.e. y is an exposed point *of $\bar{U}$.*

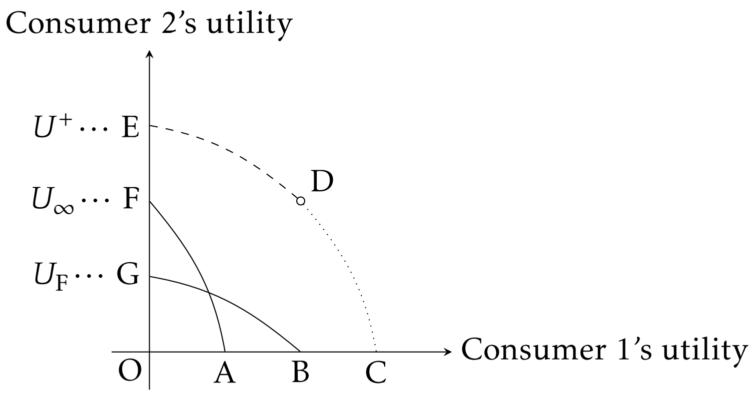

Figure 1: U^+, U_∞ and U_F in a two-consumer case.

The proof of this lemma is in Appendix A. In the finite-dimensional setting, the hyperplane property follows directly from strict convexity of the utility functions but, in the infinite-dimensional setting, it is a little more delicate: even though $(v^i)''$ is bounded away from zero, $\delta_t^i (v^i)''$ becomes arbitrarily small for large enough t, making it more difficult to deduce strict convexity results about the closure.

To illustrate why U^+ might not be closed, consider the following example.

Example 1. Consider two consumers who value time in different ways. Consumer 1 values finite time periods more highly than the indefinite future; Consumer 2 has exactly the opposite view. For a concrete example, suppose ω_t is constant and that the two utility functions are

$$u^1(x^1) = \frac{2}{3}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x_t^1) + \frac{1}{3}v(x_\infty^1)$$
$$u^2(x^2) = \frac{1}{3}\sum_{t=1}^{\infty}\frac{1}{2^t}v(x_t^2) + \frac{2}{3}v(x_\infty^2)$$

Then the possible utilities can be represented on a diagram as in Figure 1, where we are using the decomposition described in (6). The convex region OBGO represents U_F, while OAFO represents U_∞. According to Lemma 1, the closure of the positive part of the utility possibility set is the sum of these two regions, shown as OCDEO. The positive boundary, $\partial^+ U$, is in two parts: CD is parallel to AF, while DE is parallel to BG. We can see from this diagram, without any calculation, that the utility possibility set is not closed. The simplest observation is that point D is not included: this point can be represented as the sum of an element of U_F (OBGO) and an element of U_∞ (OAFO) in only one way, namely as the sum of B and F. This represents an allocation where all endowments are given to Consumer 1 in all finite time periods (B), and all endowments are given to Consumer 2 at infinity (F); because allocations are convergent sequences, this is not possible and the utility possibility set is not closed. The crucial point is that we cannot treat infinity as just another time period: consumption at infinity is determined by consumption in the far, but finite, future.

More generally, any point on the open arc CD can be represented in only one way as the sum of an element of U_{F} (OBGO) and an element of U_∞ (OAFO): namely, the point B plus a point on the open arc AF. But point B represents the allocation of all endowments in all time periods to Consumer 1; in such a case, Consumer 2 has no utility in any time period, and hence no utility in the limit at infinity. The consumers' utilities at infinity thus lie on OA, not on AF. Points on the open arc CD thus do not represent allocatable utilities.

In fact, in an example of this type, we should expect the whole of the open arc from C through D to E to be missing from the utility possibility set, but this cannot so easily be seen from the diagram. We return to this point in Section 6. $\triangleleft$

The problem in the example above is an inconsistency between the values placed by the consumers on the far, but finite, future, and the indefinite future. This leads us towards an important concept related to attainability of Pareto optimal allocations: *time value consistency*.

Definition 2. Let u^i and u^j be utility functions of the form (4). The pair of utility functions (u^i, u^j) is *time value consistent* if

$$\frac{\delta_t^i}{\delta_t^j} \to \frac{\zeta^i}{\zeta^j}, \quad \text{as} \quad t \to \infty. \tag{7}$$

This condition holds if a pair of consumers value consumption in the far future consistently with consumption in the indefinite future. This condition is very strong: requiring the ratio of these sequences to be convergent means that the two consumers' time value weighting sequences have very similar decay rates.[4]

Note that in the case of exponential discounting, $\delta_t^i = (\delta^i)^t$, time value consistency is equivalent to $\delta^i = \delta^j$ and $\zeta^i = \zeta^j$, for all i and j. For quasi-hyperbolic-like discounting,

$$(\delta_t^i)_{t\in\mathbb{N}} = (\delta_1^i, \dots, \delta_{T_i}^i, \delta_i^{T_i+1}, \delta_i^{T_i+2}, \dots),$$

[4]Note that since both $(\delta_t^i)_{t\in\mathbb{N}}$ and $(\delta_t^j)_{t\in\mathbb{N}}$ are summable sequences, they converge to zero.

a similar reasoning shows that two consumers are time value consistent if and only if $\delta_i = \delta_j$ and $\zeta_i = \zeta_j$. For hyperbolic discounting, $\delta_t^i = \gamma_i/(1+\kappa_i t^{1+\varepsilon})$, two consumers are time value consistent if

$$\frac{\gamma_i}{\zeta_i \kappa_i} = \frac{\gamma_j}{\zeta_j \kappa_j}.$$

Note that by rescaling the v^i, the same consistent functions u^i can be represented with weights such that $\delta_t^i/\delta_t^j \to 1$ as $t \to \infty$ and $\zeta^i = \zeta^j$. Also, if we have N consumers, then they are all time value consistent if and only if they are all consistent with some chosen one: for example, if $\delta_t^1/\delta_t^j \to \zeta^1/\zeta^j$ as $t \to \infty$ for all j, then $\delta_t^i/\delta_t^j \to \zeta^i/\zeta^j$ as $t \to \infty$ for all i and j.

Our main result is that time value consistency is a necessary and sufficient condition for the attainability of Pareto efficient allocations.

Main Theorem. *Consider an economy $\mathscr{E}$ which consists of N consumers with utility functions of the form (4) with $\zeta^i > 0$, $\delta_t^i > 0$, $t \in \mathbb{N}$, and $\sum_{t=1}^{\infty} \delta_t^i < \infty$, for each $i = 1,\ldots,N$. Assume that the sequence of total endowments lies in the interior of the cone c_+, so $\omega_t > 0$ for all t and $\omega_\infty > 0$. Then the utility possibility set is closed if and only if for each i and j, the utility functions u^i and u^j are time value consistent.*

5 Proof of the Main Theorem

Before we prove the Theorem, we introduce some notation and technical results needed in the proof. Let ι represent the constant sequence $(1)_{t\in\mathbb{N}}$. The constant sequence $(\xi)_{t\in\mathbb{N}}$ can thus be denoted $\xi\iota$. In proving that the utility possibility set generated by utility functions of the form (4) is closed, our basic technical tool is the following result, which is proved in Appendix A.

Lemma 2. *The utility possibility set is closed if and only if for any allocation $x \in c^N$, with $u^j(x^j) = y^j > 0$ $(1 \le j \le N)$ and for any i $(1 \le i \le N)$, we can find an allocation which maximizes u^i subject to the constraints $u^j(x^j) = y^j$ $(1 \le j \le N, j \ne i)$, $x^j \ge 0$ $(1 \le j \le N)$ and $x_t^1 + \cdots + x_t^N = \omega_t$ $(t \in \mathbb{N})$.*

The following result, also proved in Appendix A, essentially states that, in showing that the utility possibility set is closed, we can discard the first T time periods and work only with the tail of the economy.

Lemma 3. *Consider the positive part U^+ of the utility possibility set and, for $T \in \mathbb{N}$, the sets U_{T-} and U_{T+} described in* (5)*, so $U^+ = U_{T-} + U_{T+}$. Then U^+ is closed if and only if U_{T+} is closed; equivalently, U is closed.*

The proof of the Theorem proceeds in four stages. Part (i) makes use of the infinite-dimensional analogue of Lagrange multipliers described in Appendix C. The closedness of the utility possibility set guarantees the existence of certain constrained extrema of the utility functions, which in turns leads to the first-order conditions from Lagrange's method holding true. These fall into two types: those associated with finite time periods, and those associated with infinity. Provided the utility functions are sufficiently smooth, we can take a limit as time tends to infinity to link the limiting behaviour of the finite-time conditions with the infinite-time conditions and hence derive a necessary condition for the utility possibility set to be closed; this is the time value consistency condition.

In parts (ii)–(iv), we start [(ii)] with the simplest possible scenario, in which all the sequences describing the economy are constant. Not too surprisingly, in this very special case we can explicitly solve the equations arising from the Lagrange conditions and deduce that, assuming time value consistency, the utility possibility set is closed. We then [(iii)] use the implicit function theorem to see that, in a small neighbourhood of each of these solutions, the maximisation problem can still be solved; this shows that the utility possibility set is closed when the sequences describing the economy are sufficiently close to being constant. Finally [(iv)], we break the economy into two pieces around some time point T in the far future; times before T give a closed set because of the topological nature of finite-dimensional space, and those beyond T give a closed set because the sequences beyond T are sufficiently close to constant. This is the point where we crucially use the fact that our sequences are convergent: they have approximately constant tails. We can, in fact, obtain similar results for sequences with approximately periodic tails, but the consistency conditions and the associated arguments all become more complicated.

(i) Proof of necessity. Suppose the utility possibility set is closed and choose $T \in \mathbb{N}$ so that for all i,

$$\sum_{t=T+1}^{\infty} \delta_t^i v^i(\omega_t) < \zeta^i v^i(\omega_\infty/N). \tag{8}$$

By Lemma 3, the set

$$U_{T+} = \left\{ \left(\sum_{t=T+1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i v^i(x_\infty^i) \right)_{i=1}^{N} \;\middle|\; x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \right\}$$

is closed. For $1 \le i \le N$, let

$$y^i = \sum_{t=T+1}^{\infty} \delta_t^i v^i(\omega_t/N) + \zeta^i v^i(\omega_\infty/N).$$

Clearly, $y \in U_{T+}$. Now consider the maximization problem

$$\begin{aligned} \max_x \sum_{t=T+1}^{\infty} \delta_t^1 v^1(x_t^1) + \zeta^1 v^1(x_\infty^1) \\ \text{s.t. } x_t^i \ge 0, \quad \sum_{i=1}^{N} x_t^i = \omega_t \quad (t > T), \\ \sum_{t=T+1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i v^i(x_\infty^i) = y^i \quad (2 \le i \le N), \end{aligned} \tag{9}$$

which, by Lemma 2, has a solution. Inequality (8) shows that any allocation with $x_\infty^i = 0$ for some $i \ge 2$ cannot meet these constraints, so we must have $x_\infty^i > 0$ for all $i \ge 2$. Moreover,

$$\sum_{t=T+1}^{\infty} \delta_t^1 v^1(x_t^1) + \zeta^1 v^1(x_\infty^1) \ge y^1,$$

so (8) shows that $x_\infty^1 > 0$. It follows that there exists T' such that $x_t^i > 0$ for all i with $1 \le i \le N$ and all $t > T'$. Now fix x_t for $t \le T'$ and consider the same maximization problem (9) as a function only of $\{x_t \mid t > T'\}$. Of course, we have the same solution; but, as $x_t^i > 0$ for $t > T'$ and $x_\infty^i > 0$, the solution as a function of $\{x_t \mid t > T'\}$ is in the interior of the cone $\{(z_t)_{t>T'} \mid z_t \ge 0\}$. The Lagrangian $L : c^N \times \mathbb{R}^{N-1} \times c^* \to \mathbb{R}$ associated with this maximization problem is given by (see Appendix C)

$$L(x, \lambda^2, \dots, \lambda^N, \mu, \nu) = u^1(x^1) - \sum_{i=2}^{N} \lambda^i (u^i(x^i) - y^i) - \sum_{t=1}^{\infty} \mu_t \left(\sum_{i=1}^{N} x_t^i - \omega_t \right) - \nu \left(\sum_{i=1}^{N} x_\infty^i - \omega_\infty \right). \tag{10}$$

Note that the constraints can be written as $G(x) = (y^2, y^3, \dots, y^N, \omega)$, where $G : c^N \to \mathbb{R}^{N-1} \times c$ is defined by

$$G(x) = (u^2(x^2), u^3(x^3), \dots, u^N(x^N), x^1 + x^2 + \cdots + x^N).$$

Differentiating equation (10) with respect to $(x_t)_{t>T'}$ (see Lemma 5 in Appendix B) gives the following first-order conditions, which must be satisfied at an interior maximum (see

Appendix C where we have evaluated at a point h and expanded the definition of the adjoint from the formulation by Deimling (1985, Theorem 26.1); surjectivity of the derivative is easy to check):

$$\sum_{t=T'+1}^{\infty} \delta_t^1 (v^1)'(x_t^1) h_t^1 + \zeta^1 (v^1)'(x_\infty^1) h_\infty^1 - \sum_{i=2}^{N} \left[\lambda^i \sum_{t=T'+1}^{\infty} \delta_t^1 (v^i)'(x_t^i) h_t^i + \zeta^i (v^i)'(x_\infty^i) h_\infty^i \right] - \sum_{t=T'+1}^{\infty} \mu_t \sum_{i=1}^{N} h_t^i - \nu \sum_{i=1}^{N} h_\infty^i = 0 \qquad (h \in c_{T'}^N).$$

where $c_{T'}^N$ is the space of all sequences $(h_t^i)_{i=1,\dots,N,t>T'}$ which converge for all i as $t \to \infty$. Writing these in the same form as (3) we have

$$\sum_{i=1}^{N} \left[\sum_{t=T'+1}^{\infty} \left(\left\{ {1 \atop -\lambda^i} \right\} \delta_t^i (v^i)'(x_t^i) - \mu_t \right) h_t^i + \left(\left\{ {1 \atop -\lambda^i} \right\} \zeta^i (v^i)'(x_\infty^i) - \nu \right) h_\infty^i \right] = 0 \quad (h \in c_{T'}^N),$$

where $\left\{ {1 \atop -\lambda^i} \right\}$ is 1 if $i = 1$ or $-\lambda^i$ if $i > 1$. For this to be zero for all $h \in C_{T'}^N$, all the coefficients of the h_t^i must be zero. This gives the equations

$$\delta_t^1 (v^1)'(x_t^1) - \mu_t = 0 \qquad (t > T') \tag{11}$$
$$-\lambda^i \delta_t^i (v^i)'(x_t^i) - \mu_t = 0 \qquad (2 \le i \le N) \tag{12}$$
$$\zeta^1 (v^1)'(x_\infty^1) - \nu = 0 \qquad (t > T') \tag{13}$$
$$-\lambda^i \zeta^i (v^i)'(x_\infty^i) - \nu = 0 \qquad (2 \le i \le N). \tag{14}$$

We can now eliminate μ_t from the first two equations and ν from the second two (note that this step is reversible; $\mu_t = \delta_t^1 (v^1)'(x_t^1)$ defines a summable series because δ_t^1 is summable and $(v^1)'(x_t^1)$ converges to a non-zero limit):

$$\delta_t^1 (v^1)'(x_t^1) = -\lambda^i (v^i)'(x_t^i) \delta_t^i \qquad (t > T') \tag{15}$$
$$\zeta^1 (v^1)'(x_\infty^1) = -\lambda^i \zeta^i (v^i)'(x_\infty^i) \qquad (2 \le i \le N). \tag{16}$$

Because $\zeta^i, \delta_t^i, (v^i)' > 0$, it follows from these equations that $\lambda^i < 0$ for all i. We can rearrange to give

$$\frac{\delta_t^1}{\delta_t^i} = -\lambda^i \frac{(v^i)'(x_t^i)}{(v^1)'(x_t^1)} \qquad (t > T')$$
$$\frac{\zeta^1}{\zeta^i} = -\lambda^i \frac{(v^i)'(x_\infty^i)}{(v^1)'(x_\infty^1)} \qquad (2 \le i \le N).$$

Letting $t \to \infty$ in the first equation and comparing with the second, we have $\delta_t^1/\delta_t^i \to \zeta^1/\zeta^i$ as $t \to \infty$. Taking two different values of i and dividing, we must have for all i, j:

$$\frac{\delta_t^i}{\delta_t^j} \to \frac{\zeta^i}{\zeta^j} \qquad (t \to \infty),$$

as claimed (this could also be established by maximizing u^j subject to the other u^i being fixed).

We also note at this point that any solution of the Lagrange equations (15) and (16) with $T' = 0$, i.e. for all $t \in \mathbb{N}$, leads to a global maximum of u^1, subject to the given constraints. To see this, suppose x, λ^i, μ and ν are a solution. Suppressing the dependency on λ, μ, ν, which are now fixed, if $x+h$ satisfies the constraints, then

$$u^1(x+h) = L(x+h) = L(x) + L'(x)h + \frac{1}{2}L''(x+\theta h)(h,h) = u^1(x) + \frac{1}{2}L''(x+\theta h)(h,h),$$

for some $\theta \in (0,1)$, because x satisfies the constraints and $L'(x) = 0$. It is therefore enough to show that $L''(x+\theta h)(h,h) \le 0$. This follows easily from Lemma 5 in Appendix B:

$$L''(x+\theta h)(h,h) = \sum_{t=1}^{\infty} \delta_t^1 (v^1)''(x_t^1 + \theta h_t^1)(h_t^1)^2 + \zeta^1 (v^1)''(x_\infty^1 + \theta h_\infty^1)(h_\infty^1)^2 - \sum_{i=2}^{N} \lambda^i \sum_{t=1}^{\infty} \delta_t^i (v^i)''(x_t^i + \theta h_t^i)(h_t^i)^2 + \zeta^i (v^i)''(x_\infty^i + \theta h_\infty^i)(h_\infty^i)^2, \quad (17)$$

which is negative because $(v^i)'' < 0$, $\lambda^i < 0$, $\delta_t^i > 0$ and $\zeta^i > 0$. ■

(ii) Proof of sufficiency: constant total allocations and equal weighting. We now consider the special case where ω is constant, say $\omega = \omega_0 \iota$, and for each i and j, δ_t^i/δ_t^j is constant in t. By rescaling the v^i, we can assume that all the δ_t^i are equal, say $\delta_t^i = \delta_t$; in accordance with time value consistency, all the ζ^i must also be equal, say $\zeta^i = \zeta$. We shall show that the Lagrange equations derived in stage (i) have a unique solution in this case; it will then follow from Lemma 2 that the utility possibility set is closed. In fact, apart from the original constraint equations, we need only solve equation (15): equation (16) follows from that and the hypotheses that $\delta_t^i = \delta_t^j$ and $\zeta^i = \zeta^j$; equations (11)–(14) then follow from these, as remarked in stage (i). After cancelling δ_t, equation (15) reads

$$(v^1)'(x_t^1) = -\lambda^i (v^i)'(x_t^i). \quad (18)$$

Notice that this is independent of t: each $x_t \in \mathbb{R}^N$ satisfies the same system of equations. The same is true of the constraint $x_t^1 + \cdots + x_t^N = \omega_0$. We know from the previous stage that in any solution to these equations we have $\lambda^i < 0$ for all i, so $-\lambda^i (v^i)'(x_t^i)$ is a strictly decreasing function of x_t^i; similarly, $(v^1)'(x_t^1)$ is a strictly decreasing function of x_t^1. It follows from Lemma 6 in Appendix B that, for any fixed $(\lambda^i)_{i=1}^N$, these equations have at most one solution; because each x_t satisfies them, any solution to equation (18) must be constant in t.

We may therefore consider a reduced problem involving only constant sequences: maximize $u^1(\xi^1 \iota)$ ($\xi^1 \in \mathbb{R}_+$) subject to $u^i(\xi^i \iota) = y^i$ ($\xi^i \in \mathbb{R}_+$) and $\xi^1 + \cdots + \xi^N = \omega_0$. If we let $\Delta = \sum_{t=1}^\infty \delta_t + \zeta$, then we wish to maximize $\Delta v^1(\xi^1)$ subject to $\Delta v^i(\xi^i) = y^i$. This is essentially trivial: because v^i is strictly increasing, the equation $\Delta v^i(\xi^i) = y^i$ uniquely determines ξ^i for $2 \le i \le N$; ξ^1 is then uniquely determined by $\xi^1 + \cdots + \xi^N = \omega_0$ ($0 \le \xi^1 \le \omega_0$ because the y^i can be allocated). Finally, we let $\lambda^i = -(v^1)'(\xi^1)/(v^i)'(\xi^i)$.

The constant sequences $\xi^i \iota$ now satisfy the constraints $u^i(\xi^i \iota) = y^i$ ($2 \le i \le N$), $\xi^i \iota \ge 0$, $\xi^1 \iota + \cdots + \xi^N \iota = \omega_0 \iota$ and the Lagrange equation (18); that is, we have a critical point of the Lagrangian which is allocatable and satisfies all constraints. As observed at the end of stage (i), this is a global maximum of u^1.

We chose to maximize u^1 for notational convenience; we could equally have maximized any other u^i. It now follows from Lemma 2 that the utility possibility set is closed. ■

(iii) Proof of sufficiency: near-constant total allocations and near-equal weighting. Suppose the consumers are time value consistent, so $\delta_t^i/\delta_t^1 \to \zeta^i/\zeta^1$ as $t \to \infty$. As before, by rescaling v^i, we can assume that $\delta_t^i/\delta_t^1 \to 1$ and that $\zeta^i = \zeta^1$.

We shall now perturb the solution from the previous result, to show that if ω is close to a constant sequence and each δ^i is close to a constant multiple of δ^1 then the utility possibility set is closed. More precisely, we shall show that, given ω_0 and δ^1, there exists $r > 0$ such that if for all t, $|\omega_t - \omega_0| < r$ and for all t and i, $|\delta_t^i/\delta_t^1 - 1| < r$, then the utility possibility set is closed.

For notational convenience, we shall write $\delta_t^1 = \delta_t$, $\delta_t^i = (1 + \varepsilon_t^i)\delta_t$ for $2 \le i \le N$ (so $\varepsilon_t^i \to 0$ as $t \to \infty$) and $\zeta^i = \zeta$ for all i. Equation (15) now has the form

$$(v^1)'(x_t^1) = -\lambda^i (1 + \varepsilon_t^i)(v^i)'(x_t^i) \qquad (t \in \mathbb{N}),$$

and equation (16) follows on letting $t \to \infty$. Given ω and ε^i ($2 \le i \le N$), we need to solve this for x and λ in combination with the original constraint equations

$$\sum_{t=1}^{\infty} \delta_t (1 + \varepsilon_t^i) v^i(x_t^i) + \zeta v^i(x_\infty^i) = y^i \qquad (2 \le i \le N),$$

and

$$\sum_{i=1}^{N} x_t^i = \omega_t \qquad (t \in \mathbb{N}).$$

We know from the above that we can do this if ω is constant, say $\omega = \omega_0 \iota$, and $\varepsilon^i = 0$ for all i; the solution is of the form $x^i = \xi^i \iota$, and each λ^i some negative real number. We now start from these solutions and use the Implicit Function Theorem in a Banach space context to show that for any sequence $(\omega_t)_{t\in\mathbb{N}}$ which is sufficiently close to being constant, and any sequences $(\varepsilon_t^i)_{t\in\mathbb{N}}$ which are sufficiently small, we can solve the Lagrange equations.

The Banach spaces are set up as follows: consider the function

$$G : c \times c_0^{N-1} \times \mathbb{R}^{N-1} \times c^N \to \mathbb{R}^{N-1} \times c \times c^{N-1},$$

where c_0^{N-1} is the space of all sequences in $\mathbb{R}^{N-1}$ converging to zero, and for each

$$(\omega, \varepsilon^2, \dots, \varepsilon^N, \lambda^2, \dots, \lambda^N, x) \in c \times c_0^{N-1} \times \mathbb{R}^{N-1} \times c^N,$$

we define,

$$\begin{aligned} &G(\omega, \varepsilon^2, \dots, \varepsilon^N, \lambda^2, \dots, \lambda^N, x) = \\ &\qquad \Big(\underbrace{(u^2(x^2), \dots, u^N(x^N))}_{\in \mathbb{R}^{N-1}}, \underbrace{x^1 + \cdots + x^N - \omega}_{\in c}, \underbrace{((v^1)'(x_t^1) + \lambda^i (v^i)'(x_t^i))_{2\le i\le N, t\in\mathbb{N}}}_{\in c^{N-1}} \Big). \end{aligned} \tag{19}$$

It follows from Lemma 4 in Appendix B that G is continuously differentiable. We wish to solve (given ω and ε, find λ and x) the equation

$$G(\omega, \varepsilon, \lambda, x) = (y^2, \dots, y^N, (0)_{t\in\mathbb{N}}, (0)_{2\le i\le N, t\in\mathbb{N}}).$$

We know that we have a solution when $\omega = \omega_0 \iota$ is a constant sequence, $\varepsilon^i = 0$ for all i and $y^2, \dots, y^N$ are allocatable. According to the Implicit Function Theorem (Deimling, 1985, Theorem 15.2), there will be a ball in $c \times c_0^{N-1}$ centred around $(\omega_0 \iota, 0)$ in which the problem has a unique solution, provided the partial derivative of G with respect to (λ, x) at the established solution defines an invertible mapping from $\mathbb{R}^{N-1} \times c^N$ to $\mathbb{R}^{N-1} \times c \times c^{N-1}$. The radius of this ball gives us the required $r > 0$. We calculate the derivative as follows:

$$\begin{aligned} G_{\lambda,x}(\omega, \varepsilon, \lambda, x)(\mu, h) = \Bigg(&\Big(\sum_{t=1}^{\infty} \delta_t (1 + \varepsilon_t^i)(v^i)'(x_t^i) h_t + \zeta (v^i)'(x_\infty^i) h_\infty^i \Big)_{i=2}^{N}, \\ &h^1 + h^2 + \cdots + h^N, \Big((v^1)''(x_t^1) h_t^1 + (v^i)'(x_t^i) \mu^i + \lambda^i (v^i)''(x_t^i) h_t^i \Big)_{2\le i\le N, t\in\mathbb{N}} \Bigg). \end{aligned} \tag{20}$$

The essential structure of this operator from $\mathbb{R}^{N-1} \times c \times c^{N-1}$ to $\mathbb{R}^{N-1} \times c^N$ is

$$T(\mu^2, \ldots, \mu^N, h) = \left(\phi^2(h^2), \ldots, \phi^N(h^N), h^1 + \cdots + h^N, (M^1 h^1 + M^i h^i + \mu^i a)_{i=2}^N\right),$$

where $\phi^i \in c^*$ is a strictly positive functional, M^1 is an operator of multiplication by a negative sequence, bounded and bounded away from zero, M^i for $2 \leq i \leq N$ is (because $\lambda^i < 0$), an operator of multiplication by a positive sequence, bounded and bounded away from zero, and a is a fixed, positive element of c. We can explicitly calculate the inverse of T by solving the equations:

$$\phi^i(h^i) = k^i \qquad (k^i \in \mathbb{R}, 2 \leq i \leq N) \tag{21}$$

$$h^1 + \cdots + h^N = s \qquad (s \in c) \tag{22}$$

$$M^1 h^1 + M^i h^i + \mu^i a = b^i \qquad (b^i \in c, 2 \leq i \leq N). \tag{23}$$

We first find h^1. Because the multiplier sequences are bounded away from zero, the multiplication operators M^i are all invertible. We can therefore multiply (23) by $(M^i)^{-1}$, $2 \leq i \leq N$, and sum to give

$$((M^2)^{-1} + \cdots + (M^N)^{-1})M^1 h^1 + (h^2 + \cdots + h^N) = (M^2)^{-1}(b^2 - \mu^2 a) + \cdots + (M^N)^{-1}(b^N - \mu^N a).$$

Using (22), this becomes

$$((M^2)^{-1} + \cdots + (M^N)^{-1})M^1 h^1 + (s - h^1) = (M^2)^{-1}(b^2 - \mu^2 a) + \cdots + (M^N)^{-1}(b^N - \mu^N a),$$

or, with I representing the identity operator,

$$[((M^2)^{-1} + \cdots + (M^N)^{-1})M^1 - I]h^1 = (M^2)^{-1}(b^2 - \mu^2 a) + \cdots + (M^N)^{-1}(b^N - \mu^N a) - s.$$

Now, M^1 represents multiplication by a negative sequence and the other M^i multiplication by positive sequences, all bounded away from zero; it follows that $[((M^2)^{-1} + \cdots + (M^N)^{-1})M^1 - I]$ represents multiplication by a negative sequence, bounded away from zero, and hence invertible. This gives us an explicit formula for h^1. Next, we find μ^i by applying $(M^i)^{-1}$ followed by ϕ^i to (23) and substituting from (21):

$$\phi^i((M^i)^{-1} M^1 h^1) + k^i + \mu^i \phi^i((M^i)^{-1} a) = \phi^i((M^i)^{-1} b^i)$$

This gives us an explicit formula for μ^i, provided $\phi^i((M^i)^{-1} a) \neq 0$, which holds because a is a strictly positive sequence, M^i is a strictly positive multiplier, and ϕ^i is a strictly positive functional. Finally, we can find all the remaining h^i by applying $(M^i)^{-1}$ to (23) and rearranging. The inverse mapping can be shown to be continuous by a straightforward but messy calculation, or by the general observation that any continuous invertible linear map between Banach spaces has a continuous inverse (an immediate consequence of the closed graph theorem).

This shows that, if ω is sufficiently close to being constant and ε^i is sufficiently small then the Lagrange equations have a unique solution. As observed at the end of stage (i), this gives us a global maximum. We also need to check that the allocations in the solution are positive: this is true for sufficiently small $\omega - \omega_0 \iota$ and ε, because the unperturbed solution $\xi^1 \iota$ lies in the interior of the positive cone and the perturbed solution depends continuously on ω and ε.

We chose to maximize u^1 for notational convenience; we could equally have maximized any other u^i. It now follows from Lemma 2 that the utility possibility set is closed. ■

(iv) Proof of sufficiency: general case. From Lemma 1, we know that any point y on the positive boundary $\partial^+ U = \partial U \cap \mathbb{R}_+^N$ is an exposed point of $\bar{U}$: that is, any supporting hyperplane for $\bar{U}$ which passes through y does not intersect $\bar{U}$ at any other point. Consider arbitrary $\omega \in c_+ \setminus \partial c_+$ (i.e., such that $\omega_t > 0$ and $\omega_t \to \omega_\infty > 0$) and arbitrary utility functions of the form (4), satisfying the time value consistency condition (7), i.e. $\delta_t^i / \delta_t^1 \to \zeta^i / \zeta^1$. As in the earlier stages, rescale the v^i so that $\delta_t^i / \delta_t^1 \to 1$ and $\zeta^i = \zeta$ for all i. Choose $T \in \mathbb{N}$ such that for $2 \le i \le N$ and $t > T$ we have $|\delta_t^i / \delta_t^1 - 1| < r$ and $|\omega_t - \omega_\infty| < r$, where r is the radius obtained in stage (iii). Consider a perturbed economy with total endowments and utility functions

$$\tilde{\omega}_t = \begin{cases} \omega_\infty & (t \le T) \\ \omega_t & (t > T), \end{cases}$$

$$\tilde{u}^i(x^i) = \sum_{t=1}^{T} \delta_t^1 v^i(x_t^i) + \sum_{t=T+1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta v^i(x_\infty^i).$$

In this economy, by the results of the previous stage, the utility possibility set $\tilde{U}$ is closed.

To establish the corresponding result for the unperturbed economy, we consider three different sets of partial utility allocations:

$$U_1 = \left\{ \left(\sum_{t=1}^{T} \delta_t^1 v^i(x_t^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_\infty \ (1 \leq t \leq T, 1 \leq i \leq N) \right\}$$

$$U_2 = \left\{ \left(\sum_{t=T+1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta v^i(x_\infty^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (t > T, 1 \leq i \leq N) \right\}$$

$$U_3 = \left\{ \left(\sum_{t=1}^{T} \delta_t^i v^i(x_t^i) \right)_{i=1}^{N} \,\middle|\, x_t^i \geq 0, \sum_{i=1}^{N} x_t^i \leq \omega_t \ (1 \leq t \leq T, 1 \leq i \leq N) \right\}.$$

Note that $\tilde{U}^+ = U_1 + U_2$, and $U^+ = U_2 + U_3$. By Lemma 3,

$$U_1 + U_2 \text{ is closed} \iff U_2 \text{ is closed} \iff U_3 + U_2 \text{ is closed}.$$

The property of closedness of the utility possibility set is thus equivalent in the two economies; since it is closed in the perturbed economy, it is closed in the unperturbed economy. ■

6 Existence of Quasi-Equilibria

A *quasi-equilibrium* of the economy $\mathscr{E}$ is a pair $(\bar{p}, \bar{x}) \in (\ell^1 \oplus \mathbb{R}) \times c^N$, such that

1. $\langle \bar{p}, \omega \rangle \neq 0$, and
2. for all i it holds that $\langle \bar{p}, x^i \rangle \geq \langle \bar{p}, \bar{x}^i \rangle$ whenever $u^i(x^i) > u^i(\bar{x}^i)$.

The following result follows immediately from Mas–Colell and Zame (1991, Theorem 8.1) and the main theorem.

Corollary 1. *Consider an economy $\mathscr{E}$ which consists of N consumers with utility functions of the form (4) with $\zeta^i > 0$, $\delta_t^i > 0$, $t \in \mathbb{N}$, and $\sum_{t=1}^{\infty} \delta_t^i < \infty$, for each $i = 1, \ldots, N$. Assume that the sequence of total endowments lies in the interior of the cone c_+, so $\omega_t > 0$ for all t and $\omega_\infty > 0$. If for each i and j, the utility functions u^i and u^j are time value consistent, then the economy $\mathscr{E}$ has a quasi-equilibrium.*

A converse result can also be obtained from the proof of the main theorem.

Corollary 2. *Consider an economy $\mathscr{E}$ which consists of N consumers with utility functions of the form (4) with $\zeta^i > 0$, $\delta_t^i > 0$, $t \in \mathbb{N}$, and $\sum_{t=1}^{\infty} \delta_t^i < \infty$, for each $i = 1, \ldots, N$. Assume that the* *sequence of total endowments lies in the interior of the cone c_+, so $\omega_t > 0$ for all t and $\omega_\infty > 0$. If for some i and j, the utility functions u^i and u^j are not time value consistent, then the economy $\mathscr{E}$ does not have a quasi-equilibrium with the property that each $\bar{x}^i$ is in the interior of the cone c_+.*

Proof. Looking again at part (i) of the proof of the main theorem (Section 5), we can see that, whatever the initial endowments, in the absence of time value consistency the maximization problem (9) never has a solution which is an interior point of the cone $\{(z_t)_{t>T'} | z_t \geq 0\}$ for any T'; equivalently, any solution must involve one consumer's consumption tending to zero at ∞, making that consumer's consumption stream lie on the boundary of c_+. ■

So, it might be that there are (quasi-) equilibria in the case that preferences are not time value consistent, but in any such equilibrium it must be the case that at least one consumer consumes nothing in the indefinite future. This is illustrated in Example 3 and Figure 2, in which the allocatable boundary segments CH and IJ correspond to consumers 2 and 1 respectively having zero consumption in time period 1.

The following example revisits Example 1 in somewhat more detail. The example also shows how the model could be applied in a computational setting, by working explicitly through the approximation of an economy into a finite component and an eventually constant component.

Example 2. Recall Example 1, in which total endowments are constant at ω_0 and the two consumers' utility functions are of the form

$$u^1(x^1) = \frac{2}{3}\sum_{t=1}^{\infty} \frac{1}{2^t} v(x^1_t) + \frac{1}{3} v(x^1_\infty)$$

$$u^2(x^2) = \frac{1}{3}\sum_{t=1}^{\infty} \frac{1}{2^t} v(x^2_t) + \frac{2}{3} v(x^2_\infty).$$

We saw earlier that, in this case, U is not closed. We now give a more detailed analysis, which exactly describes U. Let

$$U_0 = \left\{ \frac{2}{3} v(\xi^1), \frac{1}{3} v(\xi^2) \;\middle|\; \xi^1, \xi^2 \geq 0,\ \xi^1 + \xi^2 \leq \omega_0 \right\}.$$

The set of possible utilities at any finite time t is given by $2^{-t} U_0$, and the set U_{F} of possible utilities at all finite times, OBGO in Figure 1 is (see (6)):

$$U_{\mathrm{F}} = \sum_{t=1}^{\infty} 2^{-t} U_0 = U_0.$$

Suppose y_0 lies in the positive boundary of U_0. Then there is a supporting hyperplane of U_0 passing through y_0; that is, a linear functional ϕ whose maximum value over U_0 is attained at y_0 and, because of the strict concavity of v, at no other point of U_0. We can write $y_0 = \sum_{t=1}^{\infty} 2^{-t} y_0$, and this is the only way of decomposing y_0 as the sum over $t \in \mathbb{N}$ of elements of $2^{-t} U_0$: any other decomposition $y_0 = \sum_{t=1}^{\infty} 2^{-t} y_t$ would lead to the contradiction

$$\phi(y_0) = \sum_{t=1}^{\infty} 2^{-t} \phi(y_t) < \sum_{t=1}^{\infty} 2^{-t} \phi(y_0) = \phi(y_0).$$

Moreover, the strict monotonicity of v shows that there is only one allocation $(\xi, \omega_0 - \xi)$ such that $((2/3)v(\xi), (1/3)v(\omega_0 - \xi)) = y_0$. This shows that the only allocations leading to utilities on the arc BG in Figure 1 are constant. Now, any point on the open arc DE must be the sum of the point F with a point on the open arc BG; this corresponds to constant allocations in which Consumer 2 receives all the endowments at ∞. Consumer 2 thus receives all endowments in all time periods, leading to point E. Points on the open arc DE therefore cannot be allocated.

We can now see that U^+ consists of the figure OCEO, including the closed lines OC and EO but excluding the open arc CE. $\triangleleft$

The crucial feature of this example is that we can cannot independently allocate U_∞ and U_F, because consumption at large finite times determines consumptions at ∞. The inconsistency of the utility functions leads, in this case, to the whole of the interior of ∂U^+ not being allocable. The following example shows that this is not always the case (although the absence of a large subset of ∂U^+ is a general feature of inconsistent utility functions).

Example 3. Consider a two-agent economy with prefenences

$$u^1(x^1) = v(x_1^1) + \frac{2}{3} \sum_{t=2}^{\infty} \frac{1}{2^t} v(x_t^1) + \frac{1}{3} v(x_\infty^1)$$
$$u^2(x^2) = v(x_1^2) + \frac{1}{3} \sum_{t=2}^{\infty} \frac{1}{2^t} v(x_t^2) + \frac{2}{3} v(x_\infty^2)$$

and total endowments

$$\omega_t = \begin{cases} \omega_1 & \text{if } t = 1 \\ \omega_0 & \text{if } t > 1. \end{cases}$$

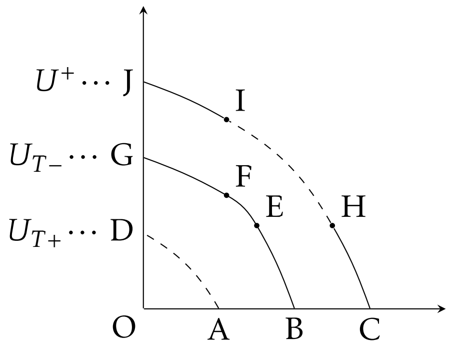

Figure 2: Utility possibility set for two inconsistent consumers

We can decompose the utility possibility set into two parts, $U = U_{T-} + U_{T+}$, with $T = 1$ (so U_{T-} represents utilities at time $t = 1$ and U_{T+} represents utilities at times $t \geq 2$, including $t = \infty$). Apart from a scale factor, the set U_{T+} is exactly as described in Example 2. The gradients of the boundary curve of U_{T-} as it crosses the vertical and horizontal axes are respectively $-v'(\omega_1)/v'(0)$ and $-v'(0)/v'(\omega_1)$; the corresponding gradients for U_{T+} are $-v'(\omega_0)/(2v'(0))$ and $-2v'(0)/v'(\omega_0)$. Under the additional hypothesis that $v'(x) \to 0$ as $x \to \infty$, we can choose ω_1 to be large enough that we have $v'(\omega_1) < v'(\omega_0)/2$. The boundary of U_{T-} therefore meets the vertical axis at a shallower angle than the boundary of U_{T+}, and the horizontal axis at a steeper angle. The possible utilities are illustrated in Figure 2. Here, the positive boundaries of U_{T-}, U_{T+} and U are the arcs BG, AD and CJ, respectively. Points E and F are those at which the positive boundary of U_{T-} is parallel to the positive boundary of U_{T+} at A and D, respectively. The arcs CH and IJ are parallel to BE and FG.

Now, CH is the sum of the point A, which is included in U_{T+}, with the arc BE, which is included in U_{T-}. All of these points are therefore included in U. Similarly, IJ is the sum of FG with D, and is included in U^+. However, the points on the open arc HI can only be represented as sums of points from the open arcs AD and EF; since EF is excluded, these points are excluded.

In summary, the positive boundary of the utility possibility set is the arc CJ; the closed arcs CH and IJ can be allocated, but the open arc HI cannot. $\triangleleft$

For completeness, we briefly discuss an example to illustrates what happens in an economy with more than two consumers, where some consumers are time value consistent and some are not.

Example 4. Suppose we have N consumers with utility functions of the form (4), which have the same weights at each finite time, but possibly different weights at ∞. Total endowments are constant with $\omega_t = \omega_0$. Any strictly positive point in the boundary of U^+ decomposes uniquely into the sum of two strictly positive points in the boundaries of U_{F} and U_∞.[5] In the same way as Example 2, we can introduce the set

$$U_0 = \left\{ (v^i(\xi^i))_{i=1}^N \,\middle|\, \xi^i \geq 0 \ (1 \leq i \leq N),\ \xi^1 + \cdots + \xi^N \leq \omega_0 \right\}$$

so the set of possible utilities at any finite time t is given by $\delta_t U_0$, and the set of possible utilities summed over all finite times is $U_{\mathrm{F}} = (\sum_{t=1}^\infty \delta_t)\, U_0$. For exactly the same reasons as in Example 2, any strictly positive point y_0 in the boundary of U_F can be written as $\sum_{t=1}^\infty \delta_t y_0$ and in no other way as a sum of elements of $\delta_t U_0$. There is a unique (Lemma 6 in Appendix B) allocation $(\xi^1, \dots, \xi^N)$ such that $v^i(\xi^i) = y^i$, $\xi^i \geq 0$ and $\xi^1 + \cdots + \xi^N = \omega_0$, so any allocation leading to a strictly positive boundary point of U, and hence of U_1, must be constant.

If the ζ^i are not all equal then any positive boundary point must be associated with an allocation in which at least one consumer has zero allocation in every time period. This can be thought of as a separate, smaller economy, excluding the consumers with no allocations; the above result can then be applied iteratively, to give a kind of simplicial decomposition of the positive boundary, in which some simplices are included and some excluded.

For example, suppose we have three consumers, where Consumers 1 and 2 are time value consistent but Consumer 3 is not. Then the utility possibility set will look something like Figure 3. Because the three consumers are not compatible with each other, the open face on the positive boundary cannot be allocated. One dimension down, Consumers 1 and 2 are compatible, so the arc joining their axes can be allocated. Consumer 3, on the other hand, is not compatible with either of the other two consumers, so the two boundary arcs from Consumer 3's axis cannot be allocated. Finally, each consumer can be allocated all endowments at all time, so the three boundary points on the axes can be allocated. $\triangleleft$

In general, our approach suggests a way of approximating (locally stable) equilibria in infinite economies by means of finite-dimensional truncations, as follows. Fix a truncation time T and consider the subspace of sequences that are constant for $t \geq T$. This is

[5] In general, an extreme point of a Minkowski sum is the sum of uniquely-determined extreme points of the summands.

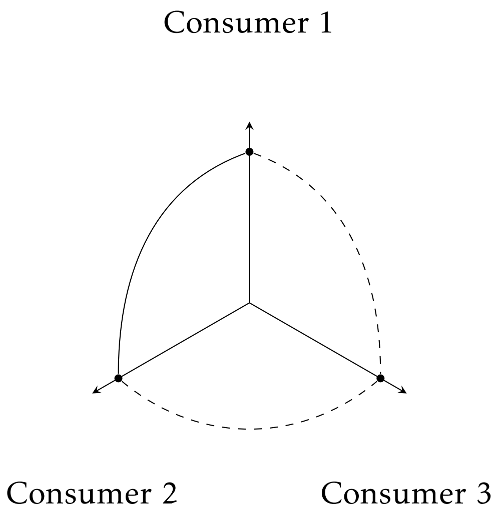

Figure 3: Utility possibility set, when consumers 1 and 2 are time value consistent, but consumer 3 is not.

a T-dimensional subspace (with $T - 1$ standard basis vectors that are 1 in a single place and 0 elsewhere and one basis vector that is 0 before time T and 1 from time T onwards). Our utility functions are easily calculated on this space, and give rise to a finite, time-separable utility function on ordinary T-dimensional real space. In effect, we have collapsed the time periods from T onwards, including infinity, into a single time period. We can now apply known computational methods [Scarf (1982, §3), Kehoe (1991, §§ 2.2, 2.3)] to this finite-dimensional economy.

Such methods effectively seek a fixed point $\hat{p}$ of some function ϕ and actually find some p such that $\|p - \phi(p)\| < \varepsilon$ where ε is some tolerance; an allocation vector arising from such a solution can be interpreted in the infinite-dimensional setting by repeating the value in the last time period *ad infinitum*. Of course, the relationship between the statements "p is close to $\phi(p)$" and "p is close to a fixed point of ϕ" is delicate even in finitely many dimensions [Scarf (1982, §3), Kehoe (1991, §2.4)] and further analysis along the lines of Kehoe (1991, §2.4) will be required to make this final step.

7 Some Extensions and Variations

In this section we discuss some extensions and variations on the theorem that were hinted at earlier in the paper, to show that similar results hold in some more general contexts. Some of these extensions are needed in Section 8, where Maximin utility functions are introduced.

Endowments Equalling or Tending to Zero

The case where $\omega_t \to 0$ as $t \to \infty$ is somewhat different in character. Here, because $0 \leq x_t^i \leq \omega_t$, the set of possible allocations forms a closed, bounded and equiconvergent family of sequences, and is, hence, compact in the norm topology on c^N (see Dunford and Schwartz (1958), IV.13.9). Any norm continuous utility functions therefore lead to a closed utility possibility set. Utility functions of the form (4) reduce to a myopic form: we necessarily have $x_t^i \to 0$, so the values of ζ^i are irrelevant.

We can also consider the case where $\omega_t = 0$ for some values of t. Such time periods make no contribution to any utility function, so they can be removed to give an economy with the same utility possibility set and total endowments $\tilde{\omega}_t > 0$ for all t. Assuming this economy has infinitely many time periods, the time value consistency condition works much as above: if $\omega_\infty = 0$, no further condition is needed for the utility possibility set to be closed, and if $\omega_\infty > 0$ then we require $\delta_t^i/\delta_t^j \to \zeta^i/\zeta^j$ as $t \to \infty$ through those t for which $\omega_t \neq 0$. In the extreme case where $\omega_t > 0$ for only finitely many t, the economy is essentially finite-dimensional and closedness follows from the Heine-Borel Theorem.

Purely Myopic Preferences

Suppose some of the ζ^i are zero and some non-zero; for definiteness, say $\zeta^1 > 0$ and $\zeta^i = 0$ for some i. Then (16) cannot be satisfied, so the utility possibility set is not closed. If, however, we have $\zeta^i = 0$ for all i then (16) is trivially satisfied. The consistency condition $\delta_t^i/\delta_t^j \to \zeta^i/\zeta^j$ is needed precisely to ensure that (16) holds; in the event that $\zeta^i = 0$ for all i it can therefore be abandoned, with the rest of the proof of the main theorem showing that the utility possibility set is closed. This is reminiscent of Bewley (1972), where we have an equilibrium provided all consumers are myopic.

8 Maximin Preferences

In this section we present a corollary to the main theorem, in which we consider a mixture of time-separable utility functions of the form (4) and utility functions, in which utility depends only on the time period in which the consumer is worst-off. Inspired by Rawlsian social welfare functions that take a similar approach to measuring the welfare of a cross-section of consumers (see, for example, Moulin, 1988), we call such preferences *maximin*. Precisely, we consider utility functions u^i where

$$u^i(x^i) = \begin{cases} \sum_{t=1}^{\infty} \delta_t^i v^i(x_t^i) + \zeta^i v^i(x_\infty^i) & (1 \le i \le N) \\ v^i(\inf_{t\in\mathbb{N}} x_t^i) = \inf_{t\in\mathbb{N}} v^i(x_t^i) & (N+1 \le i \le M) \end{cases}$$

and δ_t^i, ζ^i and v^i satisfy the conditions stated after (4). As in the main theorem, we assume that the total allocations $(\omega_t)_{t\in\mathbb{N}}$ satisfy $\omega_t > 0$ and $\omega_\infty > 0$. Let $\omega_{\min} = \inf_{t\in\mathbb{N}} \omega_t$, so $\omega_{\min} > 0$.

Corollary. *For the economy described above, the utility possibility set is closed if and only if the time-separable utility functions satisfy the time value consistency condition* (7).

Given the main theorem, the proof that this condition is necessary is straightforward. We prove sufficiency in the same way as in the main theorem, showing that if we fix the utilities of all but one consumer, then we can maximize the utility of the remaining consumer; closedness then follows from Lemma 2. Because there are two forms of utility function, there are two maximization arguments: one for a time-separable utility function, one for a Rawlsian utility function.

Proof of necessity. Suppose the utility possibility set U is closed. Then its positive part U^+ is also closed and hence so is

$$U' = \{(y^1, \dots, Y^N) : y \in U^+ \text{ and } y^{N+1} = \cdots = y^M = 0\}.$$

But this is the positive part of the utility possibility set of the reduced economy consisting only of consumers $1, \dots, N$. This economy has only time-separable utility functions which, by the main theorem, must satisfy the time value consistency condition. ■

Proof of sufficiency: maximizing a time-separable utility function. Suppose the time-separable consumers satisfy the time value consistency condition. We wish to maximize one of the time-separable utilities, say $u^1(x^1)$, subject to the attainable constraints $u^i(x^i) = y^i$ for $2 \le i \le M$.

For $N+1 \le i \le M$ (the maximin consumers), let $\xi^i = (v^i)^{-1}(y^i)$. Any allocation meeting the constraints $u^i(x^i) = y^i$ for $N+1 \le i \le M$ must satisfy $x_t^i \ge \xi^i$ for $N+1 \le i \le M$ and all $t \in \mathbb{N}$; it follows that $\xi^{N+1} + \cdots + \xi^M \le \omega_{\min}$, otherwise no feasible allocation would meet these constraints.

Now consider the reduced economy consisting of N consumers with utility functions $u^1, \ldots, u^N$ and total endowments $\tilde{\omega}_t = \omega_t - \xi_{N+1} - \cdots - \xi_M$; note that $\tilde{\omega}_t \ge 0$ because $\xi^{N+1} + \cdots + \xi^M \le \omega_{\min} \le \omega_t$.

There is an allocation x in the original economy which satisfies the constraints $u^i(x^i) = y^i$ $(2 \le i \le M)$; in this allocation, we have, for $N+1 \le i \le M$ and $t \in \mathbb{N}$, $x_t^i \ge \xi^i$ and hence $x_t^{N+1} + \cdots + x_t^M \le \omega_t - \xi_{N+1} - \cdots - \xi_M = \tilde{\omega}_t$; it follows that $(x^1, \ldots, x^N)$ is a feasible allocation in the reduced economy, satisfying $u^i(x^i) = y^i$ for $2 \le i \le N$. Because the consumers in the reduced economy satisfy the time value consistency condition, it is possible to maximize $u^1(x^1)$ subject to the constraints $u^i(x^i) = y^i$ for $2 \le i \le N$.

If we now let $x_t^i = \xi^i$ for $N+1 \le i \le M$ and all $t \in \mathbb{N}$, then $(x^1, \ldots, x^M)$ is a feasible allocation in the original economy and satisfies the constraints $u^i(x^i) = y^i$ for $2 \le i \le M$. We shall now show that this allocation maximizes $u^1(x^1)$ subject to the given constraints.

Suppose w is any other feasible allocation such that $u^i(w^i) = y^i$ for $2 \le i \le M$. Then, for $N+1 \le i \le M$, $u^i(w^i) = y^i$ so $\inf_{t \in \mathbb{N}} w_t^i = \xi^i$. Define a new allocation z by

$$z_t^i = \begin{cases} w_t^1 + (w_t^{N+1} - \xi^{N+1}) + \cdots + (w_t^M - \xi^M) & \text{if } i = 1 \\ w_t^i & \text{if } 2 \le i \le N \\ \xi^i & \text{if } N+1 \le i \le M. \end{cases}$$

This preserves the total allocation in each time period, so it is a feasible allocation, and it satisfies $u^i(z^i) = y^i$ for $2 \le i \le M$; we also have $z_t^1 \ge w_t^1$ for all t, so $u^1(z^1) \ge u^1(w^1)$. Now, $z_t^1 + \cdots + z_t^N = w_t^1 + \cdots + w_t^M - \xi_{N+1} - \cdots - \xi^M \le \omega_t - \xi_{N+1} - \cdots - \xi^M = \tilde{\omega}_t$, so $(z^1, \ldots, z^N)$ is a feasible allocation in the reduced economy, and for $2 \le i \le N$ we have $u^i(z^i) = u^i(w^i) = y^i$. By construction of x, $u^1(z^1) \le u^1(x^1)$. We already know that $u^1(w^1) \le u^1(z^1)$, so we have $u^1(w^1) \le u^1(x^1)$, showing that x does indeed maximize $u^1(x^1)$ subject to the given constraints. ■

Proof of sufficiency: maximizing a maximin utility function. Assuming again that time-separable consumers satisfy the time value consistency condition, we wish to maximize a maximin utility, say $u^M(x^M)$, subject to the attainable constraints $u^i(x^i) = y^i$ for $1 \le i \le M-1$. Let S be the set of all non-negative real numbers ξ such that there is a feasible allocation x with $u^i(x^i) = y^i$ for $1 \le i \le N$ and $x_t^1 + \cdots + x_t^N \le \omega_t - \xi$ for all t.

For $N+1 \leq i \leq M-1$, let $\xi^i = (v^i)^{-1}(y^i)$. Because the constraints are attainable, there is a feasible allocation x such that $u^i(x^i) = y^i$ for $N+1 \leq i \leq M-1$. In this allocation, $\inf_{t\in\mathbb{N}} x_t^i = \xi^i$ for $N+1 \leq i \leq M-1$, so $x_t^1 + \cdots + x_t^N + \xi^{N+1} + \ldots \xi^{M-1} \leq \omega_t$ for all t; this shows that $\xi_{N+1} + \cdots + \xi_{M-1} \in S$. Also, $\omega_{\min}$ is an upper bound for S, because if $\xi > \omega_{\min}$ then $\omega_t - \xi < 0$ for some t. We can therefore let $\Xi = \sup(S)$; necessarily, $\xi^{N+1} + \cdots + \xi^{M-1} \leq \Xi \leq \omega_{\min}$, and $[0, \Xi) \subseteq S$.

We shall now show that $\Xi \in S$. If $\Xi = 0$ then this is trivial, so assume not.

Let $\tilde{U}$ be the positive part of the utility possibility set in the reduced economy comprising N consumers with utility functions $u^1, \ldots, u^N$ and total endowments $\tilde{\omega}_t = \omega_t - \Xi$. This involves only weighted consumers satisfying the time value consistency condition, so $\tilde{U}$ is closed.

Suppose $\varepsilon > 0$. It follows from the differentiability hypotheses on the v^i that each u^i is uniformly continuous on any bounded subset of c_+^N. Hence, there exists $\delta > 0$ such that if x and z are feasible allocations (in the original economy) such that $\|x - z\|_\infty \leq \delta$ (i.e. $|x_t^i - z_t^i| \leq \delta$ for all i and t) then, for all i, $|u^i(x^i) - u^i(z^i)| < \varepsilon$. Without loss of generality, we may assume that $\delta < \Xi$, so $\Xi - \delta \in S$. By definition of S, there exists an allocation w such that $u^i(w^i) = y^i$ $(1 \leq i \leq N)$ and $w_t^1 + \cdots + w_t^N \leq \omega_t - \Xi + \delta$. Let

$$z_t^i = \begin{cases} \frac{\omega_t - \Xi}{\omega_t - \Xi + \delta} w_t^i & (1 \leq i \leq N) \\ w_t^i & (N+1 \leq i \leq M) \end{cases}$$

Clearly, this is a feasible allocation and $z_t^1 + \cdots + z_t^N \leq \omega_t - \Xi$ for all t. Moreover, if $N+1 \leq i \leq M$ then $|w_t^i - z_t^i| = 0$ and if $1 \leq i \leq N$ then

$$|w_t^i - z_t^i| = \left(1 - \frac{\omega_t - \Xi}{\omega_t - \Xi + \delta}\right) w_t^i = \frac{\delta}{\omega_t - \Xi + \delta} w_t^i \leq \delta$$

because $w_t^i \leq w_t^1 + \cdots + w_t^N \leq \omega_t - \Xi + \delta$. It follows that, for all i, $|u^i(w) - u^i(z)| < \varepsilon$, i.e. $|y^i - u^i(z)| < \varepsilon$.

Now, $(z^1, \ldots, z^N)$ is a feasible allocation in the reduced economy, so $(u^1(z^1), \ldots, u^N(z^N)) \in \tilde{U}$. We have therefore shown that for any $\varepsilon > 0$ there is an element of $\tilde{U}$ closer than ε to $(y^1, \ldots, y^N)$. Because $\tilde{U}$ is closed, we have $(y^1, \ldots, y^N) \in \tilde{U}$, so there is an allocation $(x^1, \ldots, x^N) \geq 0$ such that $u^i(x^i) = y^i$ $(1 \leq i \leq N)$ and $x_t^1 + \cdots + x_t^N \leq \omega_t - \Xi$ for all t; that is, $\Xi \in S$.

Now let $\xi^M = \Xi - \xi^{N+1} - \cdots - \xi^{M-1}$ and for $t \in \mathbb{N}$ and $N+1 \leq i \leq M$ let $x_t^i = \xi^i$, so $(x^1, \ldots, x^M)$ is a feasible allocation in the original economy and satisfies the constraints

$u^i(x^i) = y^i$ $(1 \le i \le M-1)$. We claim that this allocation maximizes $u^M(x^M)$ subject to the constraints on $u^i(x^i)$. To see this, notice that any larger value of $u^M(x^M)$ would require a value of x^M with a larger infimum, so we would have $h > 0$ such that $x_t^M \ge \xi^M + h$ for all t. For $N+1 \le i \le M-1$, to meet the constraints $u^i(x^i) = y^i$ we must have $x_t^i \ge \xi^i$ for all t; we therefore have $x_t^{N+1} + \cdots + x_t^M \ge \xi^{N+1} + \cdots + \xi^M + h = \Xi + h$; correspondingly, $x_t^1 + \cdots + x_t^N \le \omega_t - (\Xi + h)$ for all t. But $\Xi + h > \Xi = \sup(S)$, so $\Xi + h \notin S$ and by definition of S no such $x^1, \ldots, x^N$ can satisfy $u^i(x^i) = y^i$. This shows that no allocation with larger $u^M(x^M)$ can meet all the constraints, so we have indeed maximized $u^M(x^M)$. ∎

Note that the reduced economies used in the maximization arguments could specialize into various non-generic forms: specifically, we could have $\tilde{\omega}_\infty = 0$, or $\tilde{\omega}_t = 0$ for some t. As discussed in Section 7, these extensions do not cause any difficulties.

If all consumers are maximin, we can easily adapt the maximin maximization argument to show that the utility possibility set is closed: replace the construction of Ξ with the definition $\Xi = \omega_{\min}$, define $\xi^1, \ldots, \xi^N$ in exactly the same way and allocate $x_t^i = \xi^i$ for all i and t. Any larger value of $u^1(x^1)$ would lead to $x_t^1 + \cdots + x_t^N > \omega_t$ for some t.

9 Concluding Remarks

In this paper we have built a model of an infinite-dimensional exchange economy where consumers care about the indefinite future. We restrict attention to consumption bundles that are convergent. These can be interpreted as bundles which consist of a long-run average component and, for each individual period of time, a deviation from that average. The novelty of this paper is that this long-term average consumption, or "consumption at infinity", needs to be priced. Since limit consumption depends on the tail of the consumption sequence, this "price at infinity" is related to the prices at finite time periods. We find that closedness of the utility possibility set (a sufficient condition for the existence of quasi-equilibrium) can be guaranteed if and only if the preferences of all consumers are time value consistent. This implies that consumers' (utility) valuation of the indefinite future should be closely aligned, which, in turn, means that a completely atomistic view of decentralized market economies can not be combined with claims that such market interactions necessarily lead to efficient allocations.

From a mathematical point of view, the paper shows that infinite-dimensional economic models can be analyzed using the infinite-dimensional versions of techniques that are well-known to economists schooled in finite-dimensional analysis; in particular the implicit function theorem and the theorem of Lagrange. The advantage of using this toolbox as opposed to the more abstract and indirect route that is usually taken (via Alaoglu's theorem) is that the model presented here opens up the possibility of developing a computational variant that can be used in applied economic analysis.

In addition, the model presented here may open up an avenue for alternative general equilibrium approaches, not requiring myopic preferences, of branches of economics that are naturally formulated in the language of the indefinite future. We think, in particular, about possible applications in environmental economics, the theory of economic growth, and financial economics.

Appendix

A Proofs of Lemmas

Lemma 1. *For $t \in \mathbb{N}$, let*

$$U_t = \{(\delta_t^i v^i(x_t^i))_{i=1}^N \mid x_t^i \geq 0, \sum_{i=1}^N x_t^i \leq \omega_t\},$$

and let

$$\check{U} = \left\{ \left(\zeta^i v^i(\check{x}^i)\right)_{i=1}^N \,\middle|\, \check{x}^i \geq 0, \check{x}^1 + \cdots + \check{x}^N \leq \omega_\infty \right\}.$$

Then the closure of the positive part of the utility possibility set is given by

$$\bar{U}^+ = \left\{ \left(\sum_{t=1}^\infty y_t \right) + \check{y} \,\middle|\, y_t \in U_t \quad (t \in \mathbb{N}), \check{y} \in \check{U} \right\}.$$

If $y \in \partial^+ U$, then any supporting hyperplane for $\bar{U}$ through y has no other points of intersection with $\bar{U}$, i.e. y is an exposed point *of $\bar{U}$.*

Proof of Lemma 1. Because $(v^i)'' < 0$ and $\{x_t \mid x_t^i \geq 0, x_t^1 + \cdots + x_t^N \leq \omega_t\}$ is compact, each U_t has the property that a supporting hyperplane intersecting U_t at a strictly positive point intersects U_t at no other point.

Let $\sigma_t = \sup\{\|y\| \mid y \in U_t\}$; because the δ_t are summable and the ω_t are bounded, the σ_t are summable. Let

$$U' = \left\{ \left(\sum_{t=1}^\infty y_t \right) + \check{y} \,\middle|\, y_t \in U_t \quad (t \in \mathbb{N}),\ \check{y} \in \check{U} \right\}$$

(these series all converge because the σ_t are summable). Any element of U^+ certainly lies in U'; however, elements of U' do not obviously lie in U^+: roughly speaking, because the associated sequence $(y_t)_{t\in\mathbb{N}}$ might not converge to the associated $\check{y}$. We shall show that, in fact, U' is the closure of U^+. We begin by showing that U' is compact. Consider a sequence

$$\left(\left(\sum_{t=1}^{\infty} y_{t,n}\right) + \check{y}_n\right)_{n\in\mathbb{N}}$$

in U'. By a Cantor diagonal argument, we can extract subsequences such that as $k \to \infty$, $y_{t,n_k} \to y_t$ for all $t \in \mathbb{N}$ and $\check{y}_{n_k} \to \check{y}$. Since $y_{t,n_k} \in U_t$ and U_t is compact, $y_t \in U_t$; similarly, $\check{y} \in \check{U}$. We also have for all n and t, that $\|y_{t,n_k}\| \le \sigma_t$ and $\sum_{t=1}^{\infty} \sigma_t < \infty$. It now follows from the Dominated Convergence Theorem (often known as Tannery's Theorem in the case of infinite sums, rather than more general integrals), that

$$\left(\sum_{t=1}^{\infty} y_{t,n_k}\right) + \check{y}_{n_k} \to \left(\sum_{t=1}^{\infty} y_t\right) + \check{y} \in U'$$

as $k \to \infty$, showing that U' is compact.

We now show that U^+ is dense in U'. Fix some $z \in U'$, say

$$z = \left(\sum_{t=1}^{\infty} z_t\right) + \check{z}.$$

Given $\varepsilon > 0$, choose T such that $\sum_{t=T+1}^{\infty} \sigma_t < \varepsilon/2$. Now define $y_t = z_t$ for $1 \le t \le T$ and $\check{y} = \check{z}$. Choose any allocation at ∞ giving rise to utility $\check{y}$ and any allocations at times t with $t > T$ which do not exceed the total endowments and converge to the chosen allocation at ∞; these will give rise to utilities y_t for $t > T$ such that $y_\infty = \check{y}$, so we have

$$y = \left(\sum_{t=1}^{\infty} y_t\right) + y_\infty \in U^+.$$

Now,

$$\|y - z\| \le \sum_{t=T+1}^{\infty} \|y_t - z_t\| \le \sum_{t=T+1}^{\infty} 2\sigma_t < \varepsilon.$$

This shows that U^+ is dense in U'; since U' is closed, U' is the closure of U^+.

Suppose z lies in the positive boundary of U'. Then, because U' is compact and convex, there is a supporting functional ϕ such that $\phi(y) \le \phi(z)$ for all $y \in \bar{U}$. The problem of maximizing $\phi(y)$ over U' has a unique solution, namely $z = (\sum_{t=1}^{\infty} z_t) + \check{z}$ where z_t $(t \in \mathbb{N})$ is the unique point of U_t at which ϕ attains its maximum over U_t, and $\check{z}$ is the corresponding point for $\check{U}$. There is thus no other point $y \in U'$ for which $\phi(y) = \phi(z)$, so z is an exposed point of $U' = \bar{U}$. ■

Lemma 2. *The utility possibility set is closed if and only if for any allocation $x \in c^N$, with $u^j(x^j) = y^j > 0$ $(1 \le j \le N)$ and for any i $(1 \le i \le N)$, we can find an allocation which maximizes u^i subject to the constraints $u^j(x^j) = y^j$ $(1 \le j \le N, j \ne i)$, $x^j \ge 0$ $(1 \le j \le N)$ and $x_t^1 + \cdots + x_t^N = \omega_t$ $(t \in \mathbb{N})$.*

The proof of Lemma 2 depends on the following result about convex sets. The crucial property (*) means, essentially, that any line parallel to a coordinate axis intersects the set K in a closed line segment.

Lemma. *Suppose $K \subseteq \mathbb{R}_+^N$ is a non-empty and comprehensive set (i.e., if $y \in K$ and $0 \le z \le y$, then $z \in K$). Then K is closed if and only if:*

(*) for each $y \in K$ and $1 \le i \le N$, $K \cap \left\{ z \in \mathbb{R}^N \;\middle|\; z^j = y^j \ (j \ne i) \right\}$ is closed.

Proof. One direction is trivial: if K is closed then its intersection with any closed set, in particular any line, is closed. Suppose, then, that (*) holds and that y lies in the closure of K; we need to show that $y \in K$. This is trivial if $y = 0$, so assume $y \ne 0$.

Suppose $z \in \mathbb{R}_+^N$ is such that

$$z^i \begin{cases} < y^i & \text{if } y^i > 0 \\ = 0 & \text{if } y^i = 0. \end{cases}$$

Let $r = \min_{y^i \ne 0} y^i - z^i$, so for each i we have either $z^i = y^i = 0$ or $z^i \le y^i - r$. Now, since y lies in the closure of K, we can choose $w \in K$ such that $\|y - w\|_\infty < r$, i.e. $|y^i - w^i| < r$ for all i. We can assume that $w^i = 0$ whenever $y^i = 0$ (this will make w smaller, so still in K, and will if anything make $\|y - w\|_\infty$ smaller). The inequality $|y^i - w^i| < r$ can be rewritten as $y^i - r < w^i < y^i + r$, which leads to $z^i < w^i$ for all i such that $y^i \ne 0$ and $z^i = w^i = 0$ for all i such that $y^i = 0$. We now have $0 \le z \le w \in K$, so $z \in K$.

For simplicity, we initially consider the case where $y^i > 0$ for all i, so if $0 < h^i < y^i$ for all i then $y - h = (y^1 - h^1, y^2 - h^2, \ldots, y^N - h^N) \in K$. Since this lies in K for all $h^1 \in (0, y^1)$, it follows from (*) with $i = 1$ that $(y^1, y^2 - h^2, y^3 - h^3, \ldots, y^N - h^N) \in K$. Since this lies in K for all $h^2 \in (0, y^2)$, it follows from (*) with $i = 2$ that $(y^1, y^2, y^3 - h^3, \ldots, y^N - h^N) \in K$. Continuing in this way, we see that $y \in K$. For a point y with $y^i = 0$ for some i, choose h

such that $h^i = 0$ if $y^i = 0$ and $0 < h^i < y^i$ if $y^i > 0$, and argue in the same way for each i such that $y^i \neq 0$. ∎

Proof of Lemma 2. The utility possibility set is, by definition, $U^+ - \mathbb{R}^N_+$, where

$$U^+ = \{(u^1(x^1), \ldots, u^N(x^N)) : x^i_t \geq 0, \sum_{i=1}^N x^i_t \leq \omega_t\}.$$

It is clear from this that closedness of the utility possibility set and of U^+ are equivalent.

The set U^+ has, by continuity and monotonicity of the utility functions and the normalization $v^i(0) = 0$, the property that if $y \in U^+$ and $0 \leq z \leq y$ then $z \in U^+$. For any point $y \in U^+$, the line through y parallel to the ith coordinate axis intersects U^+ in a line segment. One end of this has the ith coordinate equal to zero; this lies in U^+ because it is less than or equal to y. The intersection is therefore closed if and only if the other end point lies in U^+. This corresponds to maximizing $u^i(x^i)$ subject to the constraints $u^j(x^j) = y^j$ $(1 \leq j \leq N, j \neq i)$, $x^j_t \geq 0$ $(1 \leq j \leq N, t \in \mathbb{N})$ and $x^1_t + \cdots + x^N_t \leq \omega_t$ $(t \in \mathbb{N})$. By strict monotonicity, a maximum cannot occur if $x^1_t + \cdots + x^N_t < \omega_t$ for some t, so we can replace the constraint $x^1_t + \cdots + x^N_t \leq \omega_t$ with $x^1_t + \cdots + x^N_t = \omega_t$, as claimed. Closedness now follows from the preceding lemma.

We need only consider $y^j > 0$, because $y^j = 0$ is exactly equivalent to an allocation of 0 to consumer j in all time periods; we can therefore consider the lower-dimensional problem involving only those consumers for which $y^j > 0$ and then, where $y^j = 0$, assign $x^j_t = 0$ for all t. The resulting set of utility values is closed if and only if the set of utility values in the lower-dimensional problem is closed. ∎

Lemma 3. *Consider the positive part U^+ of the utility possibility set and, for $T \in \mathbb{N}$, the sets U_{T-} and U_{T+} described in* (5)*, so $U^+ = U_{T-} + U_{T+}$. Then U^+ is closed if and only if U_{T+} is closed; equivalently, U is closed.*

Proof of Lemma 3. Note first that U_{T-} is compact, because it is the image of a compact set under a continuous mapping. If U_{T+} is closed, then (Heine-Borel) it is compact, and the sum of two compact sets is easily seen to be compact, and hence closed.

Now suppose that U^+ is closed, and hence compact, and for a contradiction that U_{T+} is not closed. Then there is a point y_0 in the positive boundary of U_{T+} which does not lie in U_{T+} itself. By Lemma 1, y_0 is an exposed point of the closure $\bar{U}_{T+}$ of U_{T+}, so there is functional ϕ such that $\phi(y) \leq \phi(y_0)$ for all $y \in \bar{U}_{T+}$ and $\phi(y) < \phi(y_0)$ for all $y \in \bar{U}_{T+}$ with $y \neq y_0$; in particular, $\phi(y) < \phi(y_0)$ for all $y \in U_{T+}$.

Choose $z_0 \in U_{T-}$ such that $\phi(z_0) = \max_{z\in U_{T-}} \phi(z)$ (such z_0 exists because U_{T-} is compact), and note that for any $y \in U^+$ we have $y = \hat{y} + \check{y}$ for some $\hat{y} \in U_{T-}$, $\check{y} \in U_{T+}$, so

$$\phi(y) = \phi(\hat{y}) + \phi(\check{y}) < \phi(z_0) + \phi(y_0) \tag{24}$$

Because $y_0 \in \bar{U}_{T+}$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in U_{T+} converging to y_0. The sequence $(z_0 + y_n)_{n\in\mathbb{N}}$ lies in $U^+ = U_{T-} + U_{T+}$ and converges to $z_0 + y_0$ so, because U^+ is closed, we have $z_0 + y_0 \in U^+$. But $\phi(z_0 + y_0) = \phi(z_0) + \phi(y_0)$, contradicting (24). ∎

B Some Supporting Technical Material

For a map between Banach spaces, there are various non-equivalent ideas of differentiability. We need only the idea of differentiability in the sense of Fréchet: briefly, if A is an open subset of a Banach space X, then $F : A \to Y$ is differentiable at $x \in A$ if there is a continuous linear mapping from X to Y, denoted $F'(x)$, with the property

$$F(x+h) = F(x) + F'(x)h + r_x(h)$$

where $\|r_x(h)\|/\|h\| \to 0$ as $h \to 0$. We say that F is continuously differentiable on A if it is differentiable at each point of A and the mapping $x \mapsto F'(x)$ is continuous. See, for example, (Deimling (1985), §7.7) for a much fuller description. For completeness, we now find the Fréchet derivatives of the functions used most frequently in our calculations.

Lemma 4. *Let c^N be the space of all convergent sequences in $\mathbb{R}^N$. Suppose A is an open subset of $\mathbb{R}^N$ and $f : A \to \mathbb{R}^M$ is k times differentiable, and therefore has a Taylor expansion*

$$f(\xi + \eta) = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(\xi)(\eta, \ldots, \beta) + r_\xi(\eta).$$

(Here $f^{(j)}$ is a j-linear map from $(\mathbb{R}^N)^j$ to $\mathbb{R}^M$ and $r_x(h) = o(|h|^{k-1})$ as $h \to 0$). Define a subset $\mathcal{A}$ of c^N by $\mathcal{A} = \{x \in c^N \mid x_t \in A \text{ for all } t\}$, and a mapping $F : \mathcal{A} \to c^M$ by $[F(x)]_t = f(x_t)$. Then F is k times differentiable and has a Taylor expansion

$$[F(x+h)]_t = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(x_t)(h_t, \ldots, h_t) + r_{x_t}(h_t). \tag{25}$$

If $f^{(k)}$ is bounded in some neighbourhood of $\lim_{t\to\infty} x_t$, then the error term is $O(\|h\|^k)$ as $h \to 0$. If $f^{(k)}$ is bounded on A, then the error is $O(\|h\|^k)$ uniformly for $x \in \mathcal{A}$.

Proof. Although it is clear that (25) is a valid identity, we need to check that its elements correspond to bounded multilinear forms between c^N and c^M and that the error term has the correct decay, in the norm on c^N. For simplicity, we work with the case $M = 1$; the M-dimensional case is then a direct sum of M 1-dimensional cases. Fix $x \in \mathcal{A}$. For $1 \leq j \leq k-1$, define a mapping from $(c^N)^j$ to the space of all real sequences by

$$[(F^{(j)}(x))(z)]_t = (f^{(j)}(x_t))(z_t).$$

It is clear that this is j-linear. Because $f^{(j)}$ is continuous and $x \in c^N$, $f^{(j)}(x_t)$ converges as $t \to \infty$; we also know that z_t converges as $t \to \infty$. It is now a straightforward consequence of multilinearity that $(f^{(j)}(x_t))(z_t)$ converges as $t \to \infty$, so $F^{(j)}$ maps $(c^N)^j$ to c. We also see that $\|F^{(j)}\| \leq \sup_{t \in \mathbb{N}} \|f^{(j)}(x_t)\|$; this is finite because $f^{(j)}(x_t)$ converges as $t \to \infty$. This shows that the multilinear forms in (25) map continuously between the correct spaces.

It remains to show that the error term has the correct order. We have for some $\theta \in (0,1)$,

$$r_{x_t}(h_t) = \frac{1}{k!} f^{(k)}(x_t)(\theta h_t, \ldots, \theta h_t),$$

from which we have

$$|r_{x_t}(h_t)| \leq \frac{1}{k!} \|f^{(k)}(x_t)\| \|h\|^k. \tag{26}$$

If $f^{(k)}$ is bounded in a neighbourhood of $\lim_{t\to\infty} x_t$, then clearly there is an upper bound for all the $\|f^{(k)}(x_t)\|$ terms, showing that the error is $O(\|h\|^k)$ as $h \to 0$. If there is a bound for $\|f^{(k)}\|$ on A, then this gives a uniform $O(\|h\|^k)$ estimate for the whole of $\mathcal{A}$. ■

Note that the Taylor approximation to the original function f has a remainder which is $O(|h|^k)$ at each point of A, provided $f^{(k)}$ exists on A. For the infinite-dimensional remainder to be $O(\|h\|^k)$, we also require local boundedness of $f^{(k)}$. This is because the remainder involves $f^{(k)}(x_t)$ at every point x_t of a convergent sequence, not just at a single point x. This boundedness hypothesis is not redundant: even in one dimension, everywhere differentiable functions can have locally unbounded derivatives, e.g. $x^2 \sin(1/x^2)$.

Lemma 5. *Suppose A is an open subset of $\mathbb{R}^N$, $v : A \to \mathbb{R}$ is twice differentiable with bounded second derivative, $(\delta_t)_{t\in\mathbb{N}}$ is a positive, summable sequence and $\zeta \in \mathbb{R}$. Define $\mathcal{A} \subseteq c$ as in Lemma 4 and a mapping from $u : \mathcal{A} \to \mathbb{R}$ by*

$$u(x) = \sum_{t=1}^{\infty} \delta_t v(x_t) + \zeta \lim_{t\to\infty} v(x_t).$$

Then u is continuously differentiable on $\mathcal{A}$ and

$$(F'(x))h = \sum_{t=1}^{\infty} \delta_t v'(x_t)h_t + \zeta \lim_{t\to\infty} v'(x_t) \lim_{t\to\infty} h_t.$$

Proof. Define V in the same way as F in Lemma 4: $[V(x)]_t = v(x_t)$, and let

$$\phi(x) = \sum_{t=1}^{\infty} \delta_t x_t + \zeta \lim_{t\to\infty} x_t,$$

so $\phi \in c^*$ and $u = \phi \circ V$. Because ϕ is linear, $\phi'(x) = \phi$ for all x. By the chain rule and Lemma 4, u is differentiable on $\mathcal{A}$ and $u'(x) = \phi \circ V'(x)$, i.e.

$$(F'(x))h = \sum_{t=1}^{\infty} \delta_t v'(x_t)h_t + \zeta \lim_{t\to\infty}(v'(x_t)h_t).$$

This is the result claimed, except that $\lim_{t\to\infty}(v'(x_t)h_t)$ has been rewritten as

$$\lim_{t\to\infty} v'(x_t) \lim_{t\to\infty}(h_t),$$

to fit with the usual way of describing elements of c^*. ■

Lemma 6. *Suppose $f^1,\ldots,f^N$ are strictly decreasing functions on an interval $[0,\omega] \subseteq \mathbb{R}$. Then the equations*

$$f^1(x^1) = f^2(x^2) = \cdots = f^N(x^N); \quad x^1 + x^2 + \cdots + x^N = \omega,$$

($x^i \in [0,\omega]$) have at most one solution.

Proof. Let $R^i = f([0,\omega])$. Because f^i is strictly decreasing, there is a well-defined, strictly decreasing inverse mapping $(f^i)^{-1} : R^i \to [0,\omega]$. Let $R = \cap_{i=1}^N R^i$, so each $(f^i)^{-1}$ is defined on R. Suppose we have two solutions to the stated equations, one with $f^i(x^i) = a$ for all i and one with $f^i(y^i) = b$ for all i. Then $a, b \in R$ and we have $(f^1)^{-1}(a) + \cdots + (f^N)^{-1}(a) = (f^1)^{-1}(b) + \cdots + (f^N)^{-1}(b) = \omega$. But $(f^1)^{-1} + \cdots + (f^N)^{-1}$ is strictly decreasing, so we must have $a = b$. This gives $f^i(y^i) = f^i(x^i)$ for all i; because f is strictly decreasing, $x^i = y^i$ for all i. ■

C Lagrange multipliers in Banach spaces

The well-known method of Lagrange multipliers generalizes without great difficulty from the finite-dimensional to the infinite-dimensional world. We give here a brief description of the main result; for details see, for example, (Deimling, 1985, Theorem 26.1).

In the finite-dimensional setting, we often think of a finite number, say d, constraints, each one of which has an associated scalar, the "Lagrange multiplier." Another way of thinking of this is to regard each constraint equation as one component of a single, composite, vector-valued constraint function mapping to $\mathbb{R}^d$, and the multipliers as a vector in $\mathbb{R}^d$. In this form, we can generalize to infinitely many dimensions: we have a single constraint function mapping to some Banach space Y and the analogue of the Lagrange multiplier is a vector in the dual of Y (the distinction between a space and its dual is not so readily visible in finitely many dimensions).

Suppose A is an open subset of a real Banach space X and that we wish to maximize or minimize $F : A \to \mathbb{R}$, subject to the constraint $G(x) = y_0$, where y_0 is an element of another real Banach space Y and $G : A \to Y$.

As in the finite-dimensional case, we assume that F and G are continuously differentiable; this is in the sense of Frećhet, so $F'(x) \in B(X, \mathbb{R}) = X^*$ and $G'(x) \in B(X, Y)$ (here $B(X, Y)$ denotes the space of continuous linear mappings from X to Y). The Fréchet derivative of a function assigns to each point in its domain a locally good linear approximation to that function (see Appendix B for some more details). In the finite-dimensional setting, where such a derivative exists, its representation in the standard basis is the Jacobian matrix; as a partial converse, if all the partial derivatives exist and are continuous, then the Fréchet derivative exists and is continuous [Apostol (1974, §§12.8, 12.12)].

The infinite-dimensional Lagrange theorem now states that if $x_0 \in A$ is a constrained maximum or minimum and $G'(x_0)$ is surjective (the analogue of the full-rank condition in the finite-dimensional setting), then there is a Lagrange multiplier $\lambda \in Y^*$ such that

$$F'(x_0) + (G'(x_0))^* \lambda = 0. \tag{27}$$

Here $(G'(x_0))^* : Y^* \to X^*$ is the Banach space adjoint operator of the derivative $G'(x_0)$. The multiplier equation can be rewritten by expanding out the definition of the adjoint, leading to the alternative and often more directly useful form

$$F'(x_0)h + \lambda(G'(x_0)h) = 0 \qquad (h \in X). \tag{28}$$

The main difference from the finite-dimensional setting is that the multiplier is now a vector in Y^*. This effectively allows us to work with infinitely many scalar constraints, something which is meaningless in the finite-dimensional world but perfectly sensible in infinitely many dimensions. If Y is finite-dimensional then so is Y^*, and we can think of λ as a finite vector of multipliers, just as in the finite-dimensional case.

Although the theorem has been stated with a single constraint, it easily accommodates more. For example, if we have N constraint functions, say $G_n : A \to Y_n$, then we let $Y = Y_1 \times \cdots \times Y_N$, so $Y^* = Y_1^* \times \cdots \times Y_N^*$ and combine the functions into a single mapping $G : A \to Y$ given by $G(x) = (G_1(x), \ldots, G_N(x))$. The derivative of this map is given by $G'(x_0)h = (G_1'(x_0)h, \ldots, G_N'(x_0)h)$, and a Lagrange multiplier is an element of Y^*, i.e. a vector $(\lambda_1, \ldots, \lambda_N)$ where $\lambda_n \in Y_n^*$. The Lagrange equation becomes (in the formulation of (28))

$$F'(x_0)h + \sum_{n=1}^{N} \lambda_n (G_n'(x_0)h) = 0 \qquad (h \in X).$$

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