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Corresponding Author: Dr. Thomas M Michelitsch, DR CNRS

Corresponding Author's Institution: CNRS

First Author: Thomas M MICHELITSCH, Dr. rer. nat. (PhD)

Order of Authors: Thomas M MICHELITSCH, Dr. rer. nat. (PhD); Bernard A Collet, Prof Emeritus; Alejandro P Riascos, PhD; Andrzej F Nowakowski, PhD; Franck C Nicolleau, PhD

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The fractional lattice Laplacian matrix contains for alpha = 2 the classical local lattice approach with well known continuum limit of classic local standard elasticity, and for other integer powers to gradient elasticity.

We also present a generalization of the fractional Laplacian matrix to n-dimensional cubic periodic (nD tori) and infinite lattices. We show that in the continuum limit the fractional Laplacian matrix yields the well-known kernel of the Riesz fractional Laplacian derivative being the kernel of the fractional power of Laplacian operator. In this way we demonstrate the interlink of the fractional lattice approach with existing continuous fractional calculus. The developed approach appears to be useful to analyze fractional random walks on lattices as well as fractional wave propagation phenomena in lattices.

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Thomas Michelitsch, DR CNRS

Sorbonne Universités

Université Pierre et Marie Curie (Paris 6)

Institut Jean le Rond d'Alembert, CNRS UMR 7190

4 place Jussieu (Case 162)

75252 Paris cedex 05

FRANCE

Tél. +33 144 278720

Fax. +33 144 275259

Email: michel@lmm.jussieu.fr

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A fractional generalization of the classical lattice dynamics approach

T.M. Michelitsch[1]*, *B.A. Collet*[1], A.P. Riascos[2]

A. F. Nowakowski[3], *F.C.G.A. Nicolleau*[3]

[1] Sorbonne Universités
Université Pierre et Marie Curie (Paris 6)
Institut Jean le Rond d'Alembert, CNRS UMR 7190
4 place Jussieu, 75252 Paris cedex 05, France

[2]Instituto de Física, Universidad Nacional Autónoma de México
Apartado Postal 20-364, 01000 México, D.F., México

[3] Sheffield Fluid Mechanics Group
Department of Mechanical Engineering
University of Sheffield
Mappin Street, Sheffield S1 3JD, United Kingdom

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*Corresponding author, e-mail : michel@lmm.jussieu.fr

Abstract

We develop physically admissible lattice models in the harmonic approximation which define by Hamilton's variational principle fractional Laplacian matrices of the forms of power law matrix functions on the n-dimensional periodic and infinite lattice in $n = 1, 2, 3, ..$ dimensions. The present approach can be interpreted as the discrete analogue of the fractional derivative calculus. As continuous fractional calculus generalizes differential operators such as the Laplacian to non-integer powers of Laplacian operators, the fractional lattice approach developed in this paper generalized difference operators such as second difference operators to their fractional (non-integer) powers. Whereas differential operators and difference operators constitute local operations, their fractional generalizations introduce nonlocal long-range features. This is true for discrete and continuous fractional operators. The nonlocality property of the lattice fractional Laplacian matrix allows to describe numerous anomalous transport phenomena such as anomalous fractional diffusion and random walks on lattices. We deduce explicit results for the fractional Laplacian matrix in 1D for finite periodic and infinite linear chains and their Riesz fractional derivative continuum limit kernels.

The fractional lattice Laplacian matrix contains for $\alpha = 2$ the classical local lattice approach with well known continuum limit of classic local standard elasticity, and for other integer powers to gradient elasticity. We also present a generalization of the fractional Laplacian matrix to n-dimensional cubic periodic (nD tori) and infinite lattices. We show that in the continuum limit the fractional Laplacian matrix yields the well-known kernel of the Riesz fractional Laplacian derivative being the kernel of the fractional power of Laplacian operator. In this way we demonstrate the interlink of the fractional lattice approach with existing continuous fractional calculus. The developed approach appears to be useful to analyze fractional random walks on lattices as well as fractional wave propagation phenomena in lattices.

1 Introduction

There are various phenomena in nature including complex, chaotic, turbulent, critical, fractal and anomalous transport phenomena having erratic trajectories with often non-differentiable characteristics. Such 'anomalous' phenomena as a rule cannot be described by standard approaches involving integer order partial differential equations. However, it has been shown that they often can be described by non-integer order, i.e. fractional differential equations [6, 7].

There are many definitions for fractional derivatives and integrals (Riemann, Liouville, Caputo, Grünwald-Letnikow, Marchaud, Weyl, Riesz, Feller, and others), see e.g. [3, 6, 21, 22, 15] and the references therein. This diversity of definitions is due to the fact that fractional operators take different kernel representations in different function spaces which is a consequence of the nonlocal character of fractional kernels.

Whereas fractional operators are well known in the continuous space and obtained as power law convolutional kernels, the fractional calculus on discrete networks and lattices is more involved and much less developed. An approach to define fractional differential operators on lattices was suggested in the paper of Tarasov [23]. In this approach fractional differential operators on the lattice are introduced. In contrast to that approach, the goal of the present work is to introduce *fractional centered difference operators* on the lattice appearing as a natural fractional generalization of Born von Karman's centered symmetric second order difference operator. In the same time all good properties of the classical Born von Karman lattice approach such as translational symmetry and elastic stability is conserved by the fractional lattice model to be developed in the present paper. In this way the present approach opens the door towards a generalization of the classical lattice approaches [14].

In the context of Markovian processes on networks, the concept of 'fractional diffusion on undirected networks' generalizing the 'normal random walk' was recently introduced by Riascos and Mateos [16, 17, 18, 19]. Such random motions on lattices are defined by diffusion equations where instead of discrete Laplacian matrices defined by second order difference operators their fractional generalizations come into play. In these works it has been demonstrated that fractional generalizations of lattice models have a huge interdisciplinar potential as they are able to describe phenomena which account for nonlocal interactions including the emergence of Lévy flights on lattices [17, 18].

Beside the applications on diffusion problems on the lattice, the importance of fractional lattice models appears also for a description of fractional lattice vibrational phenomena, a generalization of crystal lattice dynamics. Some initial steps towards such a fractional generalization generalization of nonlinear classical lattice dynamics has been introduced by Laskin and Zaslavsky [5]. In a lattice dynamics model which defines by Hamilton's variational principle the 'Laplacian matrix' which contains all constitutive information of the harmonic interparticle interactions, it is therefore desirable to develop a 'fractional generalization' of the existing lattice dynamics approach. To this end in the present paper we utilize the methodology of characteristic matrix function which was introduced recently [10].

The present paper is organized as follows. In the first part of the paper we deduce from "fractional harmonic lattice potentials" on the cyclically closed linear chain a discrete fractional Laplacian matrix. We do so by applying our recent approach to generate nonlocal lattice models by matrix functions where the generator operator is the discrete centered Born von Karman Laplacian [10]. First we obtain the discrete fractional Laplacian in explicit form for the infinite chain for particle numbers $N \to \infty$, being in accordance with the fractional centered difference models of Ortiguiera [20] and Zoia et al. [24]. Utilizing the discrete infinite chain fractional Laplacian matrix we construct an explicit representation for the *fractional Laplacian matrix on the N-periodic finite 1D lattice* where the particle number N can be arbitrary not necessarily large. Then we analyse continuum limits of the discrete fractional model: The infinite space continuum limit of the fractional Laplacian matrix yields the well known infinite space kernel of the standard fractional Laplacian. The periodic string continuum limit yields an explicit representation for the kernel of the fractional Laplacian (Riesz fractional derivative) which fulfills periodic boundary conditions and is defined on the finite L-periodic string.

In the second part of the paper we suggest an extension of the fractional approach on nD periodic and infinite lattices. We deduce an integral representation for fractional Laplacian on the infinite nD lattice and proof that as asymptotic representation the well known Riesz fractional derivative of the nD infinite space is emerging. More detailed derivations of some of the results of the present paper can be found in recent articles [8, 9]. All these results are fully equivalent and can also be deduced by employing the more general approach of Riascos and Mateos for fractional diffusion problems on networks [17, 18], and see also the references therein.

2 Fractional Laplacian matrix on cylically closed chains

We consider first a periodic, cyclically closed linear chain (1D periodic lattice or ring) with equidistant lattice points $p = 0,..,N-1$ consisting of N identical particles with particle mass μ. Each mass point p has equilibrium position at $0 \leq x_p = ph < L = Nh$ $(p = 0,..,N-1)$ where L denotes the length of the chain and h the interparticle distance (lattice constant). Further we impose periodicity (cyclic closure of the chain). For convenience of our demonstration we introduce the unitary shift operator D defined by $Du_p = u_{p+1}$ and its adjoint $D^\dagger = D^{-1}$ with $D^\dagger u_p = u_{p-1}$. We employ periodic boundary conditions (cyclic closure of the chain) $u_p = u_{p+sN}$ $(s \in \mathbf{Z})$ and equivalently, cyclic index convention $p \to p \ mod\,(N) \in \{0, 1, .., N-1\}$. Any elastic potential in the *harmonic approximation* defined on the 1D periodic lattice can be written in the representation [10]

$$V_f = \frac{\mu}{2} \sum_{p=0}^{N-1} u_p^* f(2\hat{1} - D - D^\dagger) u_p = -\frac{1}{2} \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} u_q^* \Delta_f(|p-q|) u_p, \tag{1}$$

where $\Delta_f(|p-q|) = -\mu f_{|p-q|}$ indicates the (negative-semidefinite) Laplacian $N \times N$-matrix, $\hat{1}$ the identity matrix, and f we refer to as the characteristic function: Physically admissible, elastically stable and translational invariant positive elastic potentials require for the 1D periodic lattice (cyclic ring) that the characteristic function f which is defined as a scalar function to have the following properties $0 < f(\lambda) < \infty$ for $0 < \lambda \leq 4$ (elastic stability) and $f(\lambda = 0) = 0$ (translational invariance, zero elastic energy for uniform translations of the lattice). For the approach to be developed we propose the characteristic function to assume power law form

$$f^{(\alpha)}(\lambda) = \Omega_\alpha^2 \lambda^{\frac{\alpha}{2}}, \qquad \alpha > 0, \tag{2}$$

which fulfills for $\alpha > 0$ and $\Omega_\alpha^2 > 0$ the above required good properties for the characteristic function. Ω_α denotes a dimensional constant of physical dimension sec^{-1}. Note that $2\hat{1} - D - D^\dagger$ is the central symmetric second difference operator which is defined by $(2\hat{1} - D - D^\dagger)u_p = 2u_p - u_{p+1} - u_{p-1}$. The matrix function $f(2\hat{1} - D - D^\dagger)$ is in general a self-adjoint (symmetric) positive semidefinite $N \times N$-matrix function of the simple $N \times N$ generator matrix $[2\hat{1} - D - D^\dagger]_{pq} = 2\delta_{pq} - \delta_{p+1,q} - \delta_{p-1,q}$. It can be easily seen that $f(2\hat{1} - D - D^\dagger)$ has Töplitz structure, i.e. its additional symmetry consists in the form $f_{pq} = f_{qp} = f_{|p-q|}$, $p, q = 0, ..N-1$

giving the fractional generalization of the Born von Karman centered difference operator. The fractional elastic potential has then with (1) the representation

$$V_\alpha = \frac{\mu\Omega_\alpha^2}{2}\sum_{p=0}^{N-1} u_p^*(2-D-D^\dagger)^{\frac{\alpha}{2}}u_p = \frac{\mu}{2}\sum_{p=0}^{N-1}\sum_{q=0}^{N-1} u_q^* f_{|p-q|}^{(\alpha)} u_p, \tag{3}$$

with the matrix elements $f_{|p-q|}^{(\alpha)} = \Omega_\alpha^2[(2\,\hat{1}-D-D^\dagger)^{\frac{\alpha}{2}}]_{|p-q|}$ of the *fractional characteristic matrix function.* In full analogy to the negative semidefinite continuous Laplacian (second derivative operator) we define here the fractional Laplacian matrix as the negative semidefinite matrix defined through Hamilton's variational principle[1]

$$\Delta_\alpha u_p = -\frac{\partial}{\partial u_p}V_\alpha, \qquad \Delta_\alpha = -\mu\Omega_\alpha^2(2-D-D^\dagger)^{\frac{\alpha}{2}} = -\mu f_{|p-q|}^{(\alpha)}. \tag{4}$$

In this relation we have introduced the positive-semidefinite fractional characteristic matrix $f_{|p-q|}^{(\alpha)}$ which is in our convention up to the prefactor $-\mu$ idential with the fractional Laplacian matrix (which we define as negative-semidefinite being the fractional analogue of $\frac{d^2}{dx^2}$. To determine the fractional Laplacian matrix it is useful to consider the spectral representation

$$f_{|p-q|}^{(\alpha)} = \frac{\Omega_\alpha^2}{N}\sum_{\ell=0}^{N-1} e^{i\kappa_\ell(p-q)}\left(4\sin^2\frac{\kappa}{2}\right)^{\frac{\alpha}{2}}, \qquad \kappa_\ell = \frac{2\pi}{N}\ell \tag{5}$$

where we take advantage of the N-periodicity of the chain, i.e. the eigenvectors of the fractional Laplacian matrix are ortho-normal Bloch-vectors having the components $\frac{e^{i\kappa_\ell p}}{\sqrt{N}}$. For the infinite chain in the limit $N\to\infty$ the matrix elements of the fractional Laplacian matrix (4) can be evaluated explicitly.

$$\begin{aligned} f_{|p-q|}^{(\alpha)} &= \Omega_\alpha^2(2-D-D^\dagger)^{\frac{\alpha}{2}}_{|p-q|}, \qquad && p,q\in \mathbb{Z}_0, \\ f_{|p|}^{(\alpha)}, &= \frac{\Omega_\alpha^2}{2\pi}\int_{-\pi}^{\pi} e^{i\kappa p}\left(4\sin^2\frac{\kappa}{2}\right)^{\frac{\alpha}{2}}\mathrm{d}\kappa, \qquad && p\in\mathbb{Z}_0. \end{aligned} \tag{6}$$

This expression can be obtained in explicit form [8, 9, 17, 24]

$$f^{(\alpha)}(|p|) = \Omega_\alpha^2\,\frac{\alpha!}{\frac{\alpha}{2}!(\frac{\alpha}{2}+|p|)!}(-1)^p\prod_{s=0}^{|p|-1}(\frac{\alpha}{2}-s) = \Omega_\alpha^2\,(-1)^p\,\frac{\alpha!}{(\frac{\alpha}{2}-p)!(\frac{\alpha}{2}+p)!}, \tag{7}$$

where we introduced the generalized factorial function $\beta! = \Gamma(\beta+1)$. In view of (7) we observe that for noninteger $\frac{\alpha}{2}$ any matrix element $f^{(\alpha)}(|p-q|)\neq 0$ is non-vanishing indicating the nonlocality of the harmonic fractional interparticle interaction (4). For $\frac{\alpha}{2} = m\in\mathbf{N}$ the matrix elements (7) take the values of the standard binomial coefficients. $(6)_2$ can be read as the Fourier coeffcients of the infinite Fourier series

$$\omega_\alpha^2(\kappa) = \Omega_\alpha^2\left(4\sin^2\frac{\kappa}{2}\right)^{\frac{\alpha}{2}} = \Omega_\alpha^2\left(2-e^{i\kappa}-e^{-i\kappa}\right)^{\frac{\alpha}{2}} = \sum_{p=-\infty}^{\infty} f_{|p|}^{(\alpha)}e^{ip\kappa}. \tag{8}$$

representing the fractional dispersion relation. Putting $\kappa=0$ in the dispersion relation, we can verify directly that the fractional Laplacian matrix conserves translational symmetry which is expressed by

$$\sum_{p=-\infty}^{\infty} f_{|p|}^{(\alpha)} = 0 \tag{9}$$

[1]The sign convention differes in many references, so e.g. in [17, 18] the fractional Laplacian matrix is defined positive semidefinite corresponding to the definition of the characteristic fractional operator $(2\hat{1}-D-D^\dagger)^{\frac{\alpha}{2}}$.

This equation can also be read as $(2 - D - D^{\dagger})^{\frac{\alpha}{2}} 1 = 0$, i.e. the fractional centered difference operator applied to a constant is vanishing. This property appears as the fractional generalization of the same property of second order centered differences when $\alpha = 2$. We further observe in view of (8) the positive semi-definiteness of the fractional characteristic matrix $f^{(\alpha)}_{|p-q|}$ where positiveness of (8) for $0 < \kappa < 2\pi$ indicates elastic stability of the chain.

The fractional dispersion relation (8) leads to the remarkable relation which holds *only* for complex numbers on the unit circle $z = e^{i\kappa}$, namely

$$\left(2 - z - \frac{1}{z}\right)^{\frac{\alpha}{2}} = \sum_{p=-\infty}^{\infty} (-1)^p \frac{\alpha!}{(\frac{\alpha}{2} - p)!(\frac{\alpha}{2} + p)!} z^p, \qquad |z| = 1. \tag{10}$$

This Laurent series converges nowhere except on the unit circle $|z| = 1$. For instance the zero eigenvalue $\omega^2_\alpha(\kappa = 0) = 0$ which corresponds to translational invariance (zero elastic energy for uniform translations) is obtained by putting $z = 1$ in (10). For integer $\frac{\alpha}{2} = m \in \mathbf{N}$ (7) takes the form of the standard binomial coefficients and the series (8), (10) then take the representations of standard binomial series of $\left(2 - z - \frac{1}{z}\right)^{\frac{\alpha}{2}} = (-1)^m(\sqrt{z} - \frac{1}{\sqrt{z}})^{2m}$ breaking at $|p| = m$ corresponding to zero values for the matrix elements for (7) for $|p| > m$. We further observe for noninteger $\frac{\alpha}{2} \notin \mathbf{N}$ the *power law asymptotics* for $|p| >> 1$ which can be obtained by utilizing Stirling's asymptotic formula for the Γ-function [8, 9]

$$f^{(\alpha)}_{|p|>>1} \rightarrow -\Omega^2_\alpha \, \frac{\alpha!}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \, p^{-\alpha-1}. \tag{11}$$

The asymptotic power law (scale free) characteristics of the fractional Laplacian matrix $\Delta_{pq} \sim |p-q|^{-\alpha-1}$ is the essential property which gives rise to many 'anomalous phenomena' such as in 'fractional diffusion' problems on networks such as the emergence of Lévy flights [17, 18] (and references therein). The fractional continuum limit kernels are discussed in the subsequent section. The expressions (6)-(11) hold for the infinite 1D lattice corresponding to $N \rightarrow \infty$. As everything in nature is limited we shall consider now the fractional Laplacian matrix for a finite periodic lattice where the particle number N is arbitrary and not necessarily large.

1D finite periodic lattice - ring

It is only a small step to construct the finite lattice Laplacian matrix in terms of infinite lattice Laplacian matrix. We can perform this step by the following consideration: Let $-\mu f^{(\infty)}_{|p-q|}$ the Laplacian matrix of the infinite lattice, and $\omega^2(\kappa)$ the continuous dispersion relation of the infinite lattice matrix $f_{|p-q|}$ obeying the eigenvalue relation

$$\sum_{q=-\infty}^{\infty} f^{(\infty)}_{|p-q|} e^{iq\kappa} = \omega^2(\kappa) e^{ip\kappa}, \qquad 0 \leq \kappa < 2\pi. \tag{12}$$

This relation holds identically in the entire principal interval $0 \leq \kappa < 2\pi$ and is 2π-periodic in the κ-space. Let us now choose $\kappa = \kappa_\ell = \frac{2\pi}{N}\ell$ with $\ell = 0, .., N-1$ being the Bloch wave number of the *finite* periodic lattice of N lattice points where N is not necessarily large. Since the Bloch wave numbers of the chain are discrete points within the interval $0 \leq \kappa_\ell < 2\pi$, then relation (12) holds as well for these N κ-points, namely [8, 9][2]

$$\begin{aligned} &\sum_{p=-\infty}^{\infty} f^{(\infty)}_{|q|} e^{iq\kappa_\ell} = \omega^2(\kappa_\ell), \qquad 0 \leq \kappa_\ell = \frac{2\pi}{N}\ell < 2\pi, \\ &\sum_{p=0}^{N-1} \sum_{s=-\infty}^{\infty} f^{(\infty)}_{|p+sN|} e^{i(p+sN)\kappa_\ell} = \sum_{p=0}^{N-1} e^{ip\kappa_\ell} \sum_{s=-\infty}^{\infty} f^{(\infty)}_{|p+sN|} = \sum_{p=0}^{N-1} e^{ip\kappa_\ell} f^{finite}_{|p|} = \omega^2(\kappa_\ell). \end{aligned} \tag{13}$$

[2] where $p = 0$ in (12) has been put to zero.

In the second relation the N-periodicity of the finite lattice Bloch eigenvector $e^{i(p+sN)\kappa_\ell} = e^{ip\kappa_\ell}$ has been used. The last relation can be read as the eigenvalue relation for the N-periodic lattice matrix of Töplitz structure

$$f_{|p-q|}^{finite} = \sum_{s=-\infty}^{\infty} f_{|p-q+sN|}^{(\infty)} = f_{|p-q|}^{(\infty)} + \sum_{s=1}^{\infty}(f_{|p-q+sN|}^{(\infty)} + f_{|p-q-sN|}^{(\infty)}). \tag{14}$$

It follows that in the limiting case $N \to \infty$ the finite lattice matrix (14) recovers the infinite lattice matrix $f^{finite} \to f^{(\infty)}$. From (14) we read of for the fractional lattice Laplacian of the finite periodic 1D lattice

$$\Delta_{\alpha,N}(|p|) = -\mu f_{|p|}^{(\alpha,finite)}, \qquad 0 \le p \le N-1 \tag{15}$$

with

$$f_{|p|}^{(\alpha,finite)} = \Omega_\alpha^2 \frac{(-1)^p \alpha!}{(\frac{\alpha}{2}-p)!(\frac{\alpha}{2}+p)!} + \Omega_\alpha^2 \sum_{s=1}^{\infty}(-1)^{p+Ns}\alpha! \left(\frac{1}{(\frac{\alpha}{2}-p-sN)!(\frac{\alpha}{2}+p+sN)!} \right.$$
$$\left. + \frac{1}{(\frac{\alpha}{2}-p+sN)!(\frac{\alpha}{2}+p-sN)!} \right). \tag{16}$$

We observe N-periodicity of (16) and furthermore the necessary property that in the limit of infinite chain $N \to \infty$, (16) recovers the infinite lattice expression of eq. (7).

3 Fractional continuum limit kernels

In this section we investigate the interlink between the lattice fractional approach introduced above and continuum fractional derivatives. To this end we introduce the following hypotheses which are to be observed when performing continuum limits. Following [10] we require in the continuum limit that extensive physical quantities, i.e. quantities which scale with the length of the 1D system, such as the total mass $N\mu = M$ and the total elastic energy of the chain remain finite when its length L is kept finite[3], i.e. neither vanish nor diverge. Let $L = Nh$ be the length of the chain and h the lattice constant (distance between two neighbor atoms or lattice points).
We can define two kinds of continuum limits:
(i) The *periodic string continuum limit* where the length of the chain $L = Nh$ is kept finite and $h \to 0$ (i.e. $N(h) = Lh^{-1} \to \infty$).
(ii) The *infinite space continuum limit* where $h \to 0$, however, the length of the chain tends to infinity $N(h)h = L(h) \to \infty$[4]. The kernels of the infinite space limit can be recovered from those of the periodic string limit by letting $L \to \infty$. From the finiteness of the total mass of the chain, it follows that the particle mass $\mu = \frac{M}{N} = \frac{M}{L}h = \rho_0 h$ scales as $\sim h$. Then by employing expression (3) for the fractional elastic potential, the total continuum limit elastic energy $\tilde{V}_\alpha$ can be defined by

$$\tilde{V}_\alpha = \lim_{h\to 0+} V_\alpha = \frac{\mu\Omega_\alpha^2}{2} \sum_{p=0}^{N-1} u^*(x_p) \left(-4\sinh^2 \frac{h}{2}\frac{d}{dx}\right)^{\frac{\alpha}{2}} u(x_p). \tag{17}$$

Putting $D = e^{h\frac{d}{dx}}$ ($ph = x_p \to x$) and accounting for $2 - D(h) - D(-h) = -4\sinh^2 \frac{h}{2}\frac{d}{dx} \approx -h^2\frac{d^2}{dx^2} + O(h^4)$ we get

$$\lim_{h\to 0} \left(-4\sinh^2 \frac{h}{2}\frac{d}{dx}\right)^{\frac{\alpha}{2}} = h^\alpha(-\frac{d^2}{dx^2})^{\frac{\alpha}{2}}. \tag{18}$$

The formal relation (18) shows that the continuum limit kernels to be deduced in explicit forms have the interpretation of the *Fractional Laplacian* or also in the literature referred to as *Riesz Fractional Derivative*.

[3] In the case of infinite string $L \to \infty$ we require the mass per unit length and elastic energy per unit length to remain finite.
[4] which can be realized for instance by chosing by $N(h) \sim h^{-\delta}$ where $\delta > 1$.

To maintain finiteness of the elastic energy in the continuum limit $h \to 0$ the following scaling relations for the characteristic model constants, the mass μ and the frequency Ω_α are required [8, 9]

$$\Omega_\alpha^2(h) = A_\alpha h^{-\alpha}, \qquad \mu(h) = \rho_0 h, \qquad A_\alpha, \rho_0 > 0 \tag{19}$$

where ρ_0 denotes the mass density with dimension $g \times cm^{-1}$ and A_α denotes a positive dimensional constant of dimension $sec^{-2} \times cm^\alpha$, where the new constants ρ_0, A_α are independent of h. Note that the dimensional constant A_α is only defined up to a non-dimensional positive scaling factor as its absolute value does not matter due to the scale-freeness of the power law. We obtain then as continuum limit of the elastic energy by taking into account $\sum_{p=0}^{N-1} hG(x_p) \to \int_0^L G(x)\mathrm{d}x$ and $h \to dx$, $x_p \to x$,

$$\begin{aligned} \tilde{V}_\alpha &= \lim_{h\to 0} \frac{\mu(h)}{2} \sum_{q=0}^{N-1}\sum_{p=0}^{N-1} u_q^* f_N^{(\alpha)}(|p-q|)u_p \\ \tilde{V}_\alpha &= \frac{\rho_0 A_\alpha}{2}\int_0^L u^*(x)\left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}} u(x)\,\mathrm{d}x =: -\frac{1}{2}\int_0^L\int_0^L u^*(x')\tilde{\Delta}_\alpha(|x-x'|)u(x)\mathrm{d}x\mathrm{d}x'. \end{aligned} \tag{20}$$

The continuum limit Laplacian kernel $\tilde{\Delta}_\alpha(|x-x'|)$ can then formally be represented by the distributional kernel representation in the spirit of generalized functions [2]

$$\tilde{\Delta}_{\alpha,L}(|x-x'|) = -\rho_0 A_\alpha \left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}} \delta_L(x-x'). \tag{21}$$

The last relation contains the distributional representation of the fractional Laplacian and is obtained for the infinite space limit (ii) in explicit form as [8, 9]

$$\mathcal{K}_\infty^{(\alpha)}(x) = -\left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}} \delta_L(x-x') = -\frac{\alpha!}{\pi}\lim_{\epsilon\to 0+}\Re\frac{i^{\alpha+1}}{(x+i\epsilon)^{\alpha+1}}, \tag{22}$$

being defined 'under the integral' which yields for noninteger $\frac{\alpha}{2} \notin \mathbf{N}$ for $x \neq 0$ the well known Riesz fractional derivative kernel of the infinite space $\mathcal{K}_\infty^{(\alpha)}(x) = \frac{\alpha!\sin\left(\frac{\alpha\pi}{2}\right)}{\pi}\frac{1}{|x|^{\alpha+1}}$ with a characteristic $|x|^{-\alpha-1}$ power law nonlocality reflecting the asymptotic power law behavior (11) of (7) for sufficiently large $|p| >> 1$.

3.1 (i) Periodic string continuum limit

The continuum procedure of L-periodic string where L is kept finite is then obtained as [8, 9][5]

$$\begin{aligned} -\left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}}\delta_L(x) = K_L^{(\alpha)}(|x|) &= \frac{\alpha!\sin\left(\frac{\alpha\pi}{2}\right)}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{|x-nL|^{\alpha+1}}, \qquad \xi = \frac{x}{L}, \\ K_L^{(\alpha)}(|x|) &= \frac{\alpha!\sin\left(\frac{\alpha\pi}{2}\right)}{\pi L^{\alpha+1}}\left\{-\frac{1}{|\xi|^{\alpha+1}} + \tilde{\zeta}(\alpha+1,\xi) + \tilde{\zeta}(\alpha+1,-\xi)\right\} \\ K_L^{(\alpha)}(|x|) &= -\frac{\alpha!}{\pi}\lim_{\epsilon\to 0+}\Re\left\{\sum_{n=-\infty}^{\infty}\frac{i^{\alpha+1}}{(x-nL+i\epsilon)^{\alpha+1}}\right\} \\ &= \frac{\alpha!}{\pi L^{\alpha+1}}\lim_{\epsilon\to 0+}\Re\left\{i^{\alpha+1}\left(\frac{1}{(\xi+i\epsilon)^{\alpha+1}} - \zeta(\alpha+1,\xi+i\epsilon) - \zeta(\alpha+1,-\xi+i\epsilon)\right)\right\}. \end{aligned} \tag{23}$$

[5] where $\Re(..)$ denotes the real part of a quantity (..)

This kernel can be conceived as the explicit representation of the fractional Laplacian (Riesz fractional derivative) on the L-periodic string. The last relation is the distributional representation and is expressed by standard Hurwitz ζ-functions denoted by $\zeta(..)$. The two variants of ζ- functions which occur in above relation are defined by

$$\tilde{\zeta}(\beta,x) = \sum_{n=0}^{\infty} \frac{1}{|x+n|^{\beta}}, \qquad \zeta(\beta,x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^{\beta}}, \qquad \Re\beta > 1. \tag{24}$$

We see for $\alpha > 0$ and $x \neq 0$ that the series in (23) are absolutely convergent as good as the power function integral $\int_1^{\infty} \xi^{-\alpha-1} \mathrm{d}\xi$. For integer powers $\frac{\alpha}{2} \in \mathbf{N}$ the distributional representations $(23)_{3,4}$ take the (distributional) forms of the (negative semi-definite) 1D integer power Laplacian operators, namely

$$\begin{aligned} K_L^{(\alpha=2m)}(|x|) &= (-1)^{m+1} \frac{d^{2m}}{dx^{2m}} \sum_{n=-\infty}^{\infty} \lim_{\epsilon\to 0+} \frac{1}{\pi} \frac{\epsilon}{((x-nL)^2+\epsilon^2)}, \qquad \frac{\alpha}{2} = m \in \mathbf{N_0}, \\ &= (-1)^{m+1} \frac{d^{2m}}{dx^{2m}} \sum_{n=-\infty}^{\infty} \delta_{\infty}(x-nL) = -\left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}=m} \delta_L(x), \end{aligned} \tag{25}$$

where $\delta_{\infty}(..)$ and δ_L indicate the Dirac's δ-functions of the infinite and the L-periodic string, respectively. We further observe in full correspondence to the discrete fractional Laplacian matrix, the necessary property that in the limit of an infinite string $\lim_{L\to\infty} K_L^{(\alpha)}(|x|) = \mathcal{K}_{\infty}^{(\alpha)}(x)$ (23) recovers the expression of the standard 1D infinite space fractional Laplacian kernel (22) known from the literture (see for a further discussion [8, 9] and references therein).

4 Fractional Laplacian matrix on n-dimensional cubic lattices

In this section we deduce the nD counterpart of the fractional Laplacian matrix introduced above. With that approach the fundamentals of '*fractional lattice dynamics*' can be deduced as a generalization of conventional lattice dynamics.

In this section our goal is to generalize the above 1D lattice approach to cubic periodic lattices in $n = 1, 2, 3, ..$ dimensions of the physical space where the 1D lattice case is contained. We assume the lattice contains $N = N_1 .. \times N_n$ lattice points, each covered by identical atoms with mass μ. Each mass point is characterized by $\vec{p} = (p_1, p_2, .., p_n)$ $(p_j = 0, ..N_j - 1)$ and $n = 1, 2, 3, ..$ denotes the dimension of the physical space embedding the lattice. In order to define the lattice fractional Laplacian matrix, it is sufficient to consider a *scalar* generalized displacement field $u_{\vec{p}}$ (one field degree of freedom) associated to each mass point $\vec{p}$ only. The physical nature of this scalar field can be any scalar field, such as for instance a one degree of freedom displacement field, an electric potential or, in a stochastic context a probablitity density function (pdf) or in a fractional quantum mechanics context a Schrödinger wave function. This demonstrates the interdisciplinary character of the present fractional lattice approach.

The fractional Laplacian matrix for general networks was only recently and to our knowledge for the first time introduced by Riascos and Mateos [17, 18] in the framework of fractional diffusion analysis on networks which include nD periodic lattices (nD tori) as special cases being subject of the present analysis. For cubic nD lattices the fractional Laplacian matrix can be written as [8, 9, 17, 18]

$$\Delta_{\alpha,n} = -\mu\Omega_{\alpha,n}^2 L_n^{\frac{\alpha}{2}}, \qquad L_n^{\frac{\alpha}{2}} = \left(2n\hat{1} - A_n\right)^{\frac{\alpha}{2}}, \quad \alpha > 0, \tag{26}$$

where $\hat{1}$ denotes the identity matrix, n indicates the dimension of the physical space and $2n$ indicates the connectivity, i.e. the number of next neighbors of a lattice point in the nD cubic lattice. In (26) we introduced the adjacency matrix A_n which has for the cubic lattice with next neighbor connections the form

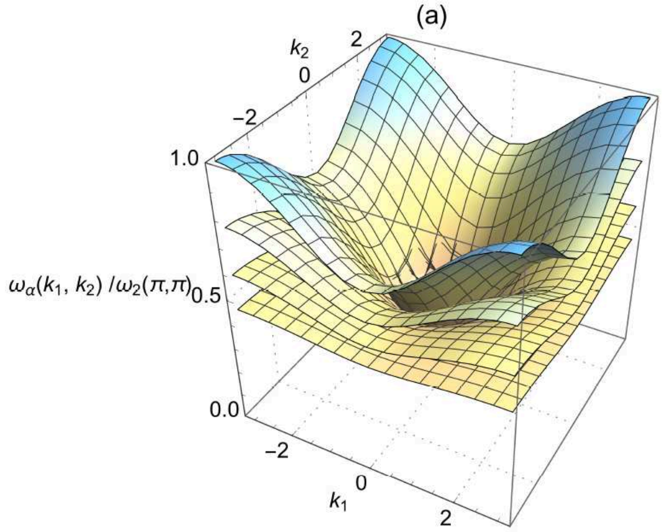
$$A_n = \sum_{j=1}^{n} (D_j + D_j^{\dagger}), \tag{27}$$

where then D_j and $D_j^\dagger = D_j^{-1}$ denote the next neighbor shift operators in the $j = 1,..,n$-directions defined by $D_j u_{p_1,..p_j,..,p_n} = \vec{u}_{p_1,..p_j+1,..,p_n}$ and $D_j^\dagger \vec{u}_{p_1,..p_j,..,p_n} = u_{p_1,..p_j-1,..,p_n}$, i.e. D_j shifts the field associated to lattice point $\vec{p} = (..,p_j,..$ to the field associated with the adjacent lattice point in the positive j-direction $(..,p_{j+1},..)$, and the inverse (adjoint) shift operator $D_j^\dagger = D_j^{-1}$ to the adjacent lattice point in the negative j-direction $(..,p_{j-1},..)$. All matrices introduced in (26) and (27) are defined on the nD lattice being $N \times N$ matrices $(N = N_1 \times .. \times N_n)$. As in the case of 1D lattice the so defined fractional Laplacian matrix (26) describes for non-integer powers $\frac{\alpha}{2}, \notin \mathbf{N}$ nonlocal elastic interactions, whereas they are generated by the 'local' next neighbor Born von Karman Laplacian which is in our definition up to a negative dimension factor $-\mu\Omega_2$ equal to L_n. We therefore refer to L_n as 'generator matrix'. We emphasize that the sign convention of what we call '(fractional) Laplacian matrix' varies in the literature (e.g. by denoting the positive semidefinite matrix $L_n^{\frac{\alpha}{2}}$ as 'fractional Laplacian matrix', this convention is chosen, e.g. in [17, 18]). We have chosen to refer to as 'fractional Laplacian matrix' the negative-semidefinite matrix $-\mu\Omega_\alpha^2 L_n^{\frac{\alpha}{2}}$ to be in accordance with the negative definiteness of continuum limit fractional Laplacian (25) containing as a special case $\frac{\alpha}{2} = 1$ the negative semidefinite conventional Laplacian $\frac{d^2}{dx^2}\delta_L(x - x')$). For a discussion of some general properties of the fractional Laplacian (26) well defined on general networks including nD lattices, we refer to [17, 18]. In the periodic and infinite lattice the shift operators are unitary. Assuming N_j-periodicity in each direction j, the fractional Laplacian matrix is defined by the spectral properties of the L_n-matrix, namely by

$$[L_n^{\frac{\alpha}{2}}]_{(\vec{p}-\vec{q})} = \frac{1}{N}\sum_{\vec{\ell}} e^{i\vec{\kappa}_{\vec{\ell}}\cdot(\vec{p}-\vec{q})}\lambda_{\vec{\ell}}^{\frac{\alpha}{2}}, \qquad \lambda_{\vec{\ell}} = \left(2n - 2\sum_{j=1}^{n}\cos(\kappa_{\ell_j})\right), \; \alpha > 0, \tag{28}$$

where we denoted $\sum_{\vec{\ell}}(..) = \sum_{\ell_1=0}^{N_1-1}(..)..\sum_{\ell_n=0}^{N_n-1}(..)$ and $\vec{\kappa}_{\vec{\ell}} = (\kappa_{\ell_1},..\kappa_{\ell_n})$ denotes the Bloch wave vectors of the Brillouin zone where their components can the values $\kappa_{\ell_j} = \frac{2\pi}{N_j}\ell_j$ $(\ell_j = 0,..,N_j - 1)$. It can be seen that (28) has Töplitz structure depending only on $|p_1 - q_1|,..,|p_j - q_j|,..|p_n - q_n|)$. For the infinite lattice when all $N_j \to \infty$ in (28), the summation over the reciprocal lattice points assumes asymptotically the form of an integral $\frac{1}{N}\sum_{\vec{\ell}} g(\vec{\kappa}_\ell) \sim \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}..\int_{-\pi}^{\pi} d\kappa_1..d\kappa_n g(\vec{\kappa})$, where the integration intervals $[-\pi, \pi]$ can be chosen instead of $[0, 2\pi]$ for 2π-periodic functions $g(\kappa_j) = g(\kappa_j + 2\pi)$.

(a)



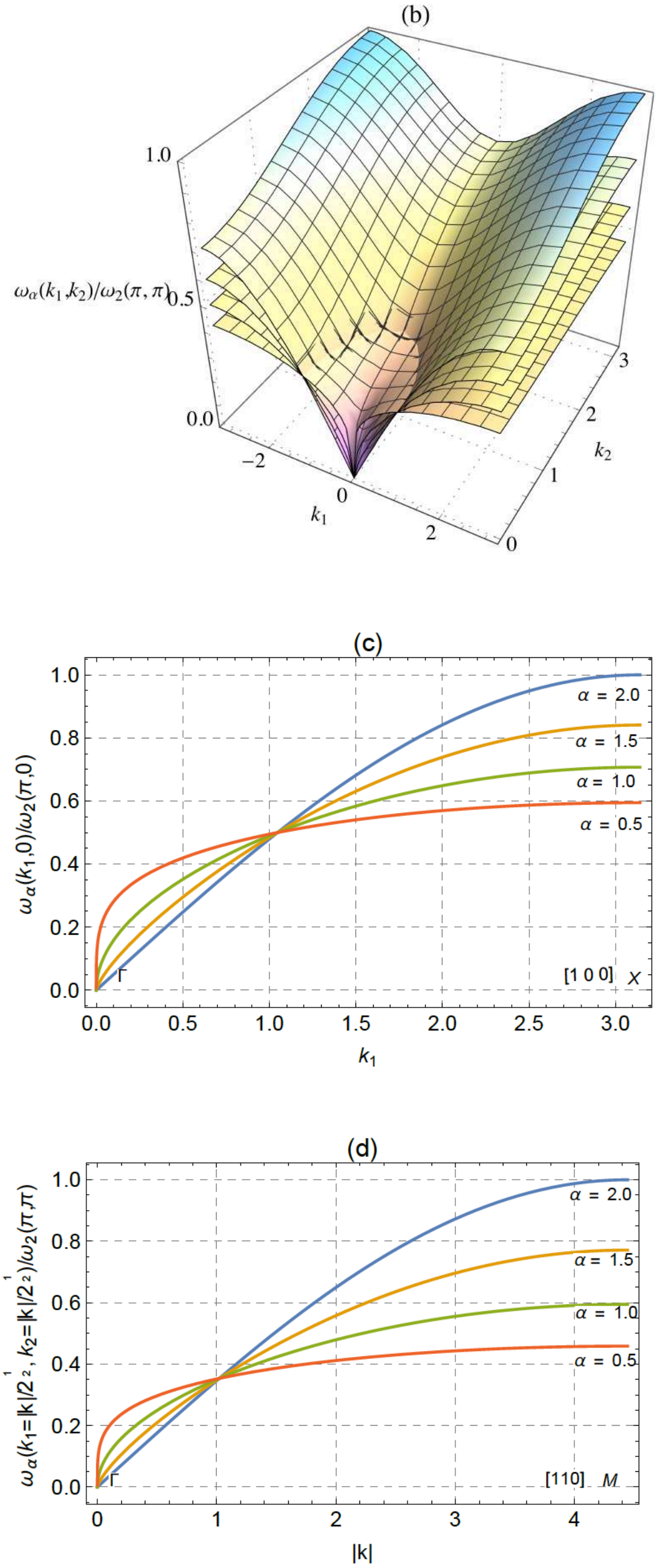


Fig.1 *(a-b) Show the dispersion surfaces* $\omega_\alpha(\kappa_1,\kappa_2)/\omega_2(\pi,\pi) = \lambda^{\frac{\alpha}{4}}(\kappa_1,\kappa_2)/2^{\frac{3}{2}} = 2^{\frac{\alpha-3}{2}}(sin^2(\kappa_1/2)+sin^2(\kappa_2/2))^{\frac{\alpha}{4}}$ *for the 2D cubic lattice* ($n=2$) *of (28) for four values of* α, *while (c-d) illustrate cross-sections of these dispersion sheets with the planes (0 1 0) and (1 1 0), respectively.*

For α fixed, the circular frequency is given by $\omega_\alpha(\kappa_1,\kappa_2)=\lambda^{\frac{\alpha}{4}}$. The linear frequency spectra (a,b,d), for $n=2$, are normalized by the maximum frequency $\omega_{\alpha=2}(\pi,\pi)=\lambda^{\frac{1}{2}}(\pi,\pi)=2^{\frac{3}{2}}$ obtained for a wave vector located in (001) plane. It will be noted that the sheets cut at dimensionless frequency $\omega_\alpha(\kappa_1,\kappa_2)/\omega_{\alpha=2}(\pi,\pi)\approx 0.351$ and the dispersion relations of the classical next neighbor Born von Karman lattice are recovered (indicated by $\omega_\alpha(\kappa_1,\kappa_2)/\omega_{\alpha=2}(\pi,\pi)\to 1$ for $\alpha=2$ and $\kappa_{1,2}\to\pi$, $\omega_\alpha(\kappa_1,0)/\omega_{\alpha=2}(\pi,0)\to 1$ for $\alpha=2$ and $\kappa_1\to\pi$). When the value of α decreases, one observes in agreement with another work [8], namely a decrease of the maximum dimensionless frequency in end of the first Brillouin zone.

The goal is now to deduce a more convenient integral representation of (28). To this end we utilize the following observation: Let in the following $\mathcal{L}$ be a positive semidefinite[6] matrix and $\alpha>0$ like (26) with the spectral representation

$$\mathcal{L}=\sum_{\vec{\ell}}\lambda_{\vec{\ell}}|\vec{\ell}><\vec{\ell}|, \qquad \mathcal{L}_{pq}=<p|\mathcal{L}q>, \tag{29}$$

where we have to put for the periodic nD lattice of (28) the Bloch-eigenvectors $<\vec{p}|\vec{\ell}>=N^{-\frac{1}{2}}e^{i\vec{\kappa}_{\vec{\ell}}\vec{p}}$. Then it will be useful to define the matrix Dirac δ-function by

$$\delta(\mathcal{L}-\tau\hat{1})=\sum_{\vec{\ell}}|\vec{\ell}><\vec{\ell}|\,\delta(\tau-\lambda_{\vec{\ell}}), \tag{30}$$

where τ is a scalar parameter and $\hat{1}$ the identity matrix and $\delta(\tau-\lambda_{\vec{\ell}})$ the conventional scalar Dirac δ-function. Then with the matrix δ-function defined in (30) we can write

$$\mathcal{L}^{\frac{\alpha}{2}}=\int_{-\infty}^{\infty}\delta(L-\tau\hat{1})|\tau|^{\frac{\alpha}{2}}\mathrm{d}\tau \tag{31}$$

and by utilizing $\delta(\tau-\lambda_{\vec{\ell}})=\frac{1}{(2\pi)}\int_{-\infty}^{\infty}e^{ik(\tau-\lambda_{\vec{\ell}})}\mathrm{d}k$ together with the kernel $-\mathcal{D}_{\frac{\alpha}{2}}$ of the 1D fractional Laplacian (Riesz fractional derivative) of order $\frac{\alpha}{2}$ in its distributional form [12]

$$\begin{aligned}\mathcal{D}_{\frac{\alpha}{2}}(k-\xi)=\left(-\frac{d^2}{dk^2}\right)^{\frac{\alpha}{4}}\delta(k-\xi)=:\frac{1}{(2\pi)}\int_{-\infty}^{\infty}e^{i(k-\xi)\tau}|\tau|^{\frac{\alpha}{2}}\mathrm{d}\tau=\\ \lim_{\epsilon\to 0+}\frac{1}{\pi}\Re\int_0^{\infty}e^{-\tau(\epsilon-i(k-\xi))}|\tau|^{\frac{\alpha}{2}}\mathrm{d}\tau=\lim_{\epsilon\to 0+}\Re\frac{\Gamma(\frac{\alpha}{2}+1)}{\pi(\epsilon-i(k-\xi))^{\frac{\alpha}{2}}}.\end{aligned} \tag{32}$$

Then we can write for the matrix power function (31) the representation

$$\mathcal{L}^{\frac{\alpha}{2}}=\int_{-\infty}^{\infty}e^{ik\mathcal{L}}\mathcal{D}_{\frac{\alpha}{2}}(k)\mathrm{d}k, \tag{33}$$

where the exponential $e^{ik\mathcal{L}}$ of the matrix $\mathcal{L}=L_n$ can be determined more easily for the generator $L_n=\sum_{j=1}^n L_j$ ($L_j=2-D_j-D_j^{\dagger}$) being the sum of the 1D generator matrices of the N_j-periodic 1D lattices and having therefore the eigenvalues $\lambda(\ell_j)=2-2\cos\kappa_{\ell_j}$ and as a consequence having a Cartesian product space spanned by the periodic Bloch eigenvectors $\frac{e^{i\vec{\kappa}_{\vec{\ell}}\vec{p}}}{\sqrt{N}}=\prod_{j=1}^n\frac{e^{ip_j\kappa_{\ell_j}}}{\sqrt{N_j}}$. The matrix elements of the spectral representation of the exponential of L_n can hence be written as

$$[e^{i\xi L_n}]_{\vec{p}-\vec{q}}=\sum_{\vec{\ell}}\frac{e^{i\vec{\kappa}_{\vec{\ell}}(\vec{p}-\vec{q})}}{N}e^{i\xi\lambda_{\vec{\ell}}}=\prod_{j=1}^n\sum_{\ell_j=1}^{N_j-1}\frac{e^{i(p_j-q_j)\kappa_{\ell_j}}}{N_j}e^{i2k(1-\cos\kappa_{\ell_j})}. \tag{34}$$

Infinite nD lattice
In the limiting case of an infinite nD lattice when all $N_j\to\infty$ we can write by using $\frac{1}{N}\sum_{\vec{\ell}}g(\vec{\kappa}_{\vec{\ell}})\sim\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}..\int_{-\pi}^{\pi}\mathrm{d}\kappa_1..\mathrm{d}\kappa_n g(\vec{\kappa})$ to arrive at

[6] i.e. all eigenvalues λ_ℓ of this matrix are non-negative.

$$[e^{i\xi L_n}]_{\vec{p}-\vec{q}} = [e^{i\xi L_n}]_{|p_1-q_1|,..,|p_n-q_n|} = \prod_{j=1}^{n} \frac{1}{(2\pi)} \int_{-\pi}^{\pi} e^{i(p_j-q_j)\kappa} e^{i2\xi(1-\cos\kappa)} \mathrm{d}\kappa. \tag{35}$$

Taking into account the definition of the modified Bessel functions of the first kind $I_p(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\varphi}\cos p\varphi \mathrm{d}\varphi$ where $p = \mathbf{N}_0$ denotes non-negative integers [1], we can write the exponential matrix (35) in the form

$$[e^{i\xi L_n}]_{|p_1-q_1|,..,|p_n-q_n|} = e^{i2n\xi} \prod_{j=1}^{n} I_{|p_j-q_j|}(-2i\xi). \tag{36}$$

Applying now the matrix relation (31) and plugging in the exponential (36) yields an integral representation of the (negative semidefinite) fractional Laplacian matrix (26) in terms of a product of modified Bessel functions of the first kind, namely

$$\begin{aligned} [\Delta_{\alpha,n}]_{|p_1-q_1|,..,|p_n-q_n|} &= -\mu\Omega_{\alpha,n}^2 L^{\frac{\alpha}{2}}_{|p_1-q_1|,..,|p_n-q_n|} \\ &= -\mu\Omega_{\alpha,n}^2 \int_{-\infty}^{\infty} \mathrm{d}\xi\, e^{i2n\xi} \mathcal{D}_{\frac{\alpha}{2}}(\xi) \prod_{j=1}^{n} I_{|p_j-q_j|}(-2i\xi), \end{aligned} \tag{37}$$

with $-\mathcal{D}_{\frac{\alpha}{2}}(\xi)$ indicating the Riesz fractional derivative kernel of (32).

Asymptotic behavior

Introducing the new vector valued integration variable $\vec{\xi} = \vec{\kappa}p$ ($\xi_j = p\kappa_j, \forall j = 1,..,n$) we can write for the infinite lattice integral of (28) by utilizing spherical polar coordinates $\vec{p} = p\vec{e}_{\vec{p}}$ ($\vec{e}_{\vec{p}} \cdot \vec{e}_{\vec{p}} = 1$, $p^2 = \sum_j^n p_j^2$)

$$L_n^{\frac{\alpha}{2}}(\mathbf{p}) = \frac{1}{(2\pi)^n} \int_{-\pi p}^{\pi p} .. \int_{-\pi p}^{\pi p} \frac{\mathrm{d}\xi_1..\mathrm{d}\xi_n}{p^n} \left(4\sum_{j=1}^{n} \sin^2\frac{\xi_j}{2p}\right)^{\frac{\alpha}{2}} \cos(\vec{\xi}\cdot\vec{e}_{\vec{p}}). \tag{38}$$

The dominating term for $p >> 1$ becomes

$$L_n^{\frac{\alpha}{2}}(\mathbf{p}) \approx \frac{1}{p^{n+\alpha}} \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} \mathrm{d}\xi_1..\mathrm{d}\xi_n \left(\sum_{j=1}^{n} \xi_j^2\right)^{\frac{\alpha}{2}} \cos(\vec{\xi}\cdot\vec{e}_{\vec{p}}), \tag{39}$$

having the form

$$L_n^{\frac{\alpha}{2}}(\vec{p})_{p>>1} \approx -\frac{C_{n,\alpha}}{p^{n+\alpha}}, \tag{40}$$

where the positive normalization constant is obtained explicitly as $C_{n,\alpha} = \frac{2^{\alpha-1}\alpha\Gamma(\frac{\alpha+n}{2})}{\pi^{\frac{n}{2}}\Gamma(1-\frac{\alpha}{2})}$, e.g. [11, 12]. We can identify the asymptotic representation (39), (40) with the kernel of Riesz fractional derivative (fractional Laplacian) of the nD infinite space. For a more detailed discussion of properties we refer to [11, 12].

5 Conclusions

In the present paper we have developed a fractional lattice approach on nD periodic and infinite lattices. The fractional Laplacian matrices conserve the 'good' properties of the Laplacian matrices (translational symmetry and in our sign convention negative semi-definiteness). The fractional lattice approach generalizes the concept of second order centered difference operator appearing in the context of classical lattice models [14] to the concept of centered fractional order difference operator. For $\alpha = 2$ the fractional lattice approach contains the classical lattice approach, and for integer orders $\frac{\alpha}{2} \in \mathbb{N}$ finite centered differences of integer orders of the second difference operator are generated. For a discussion of properties of fractional Laplacian matrix on the cyclic chain, we refer to [8]. In the infinite space and periodic lattice continuum limits these fractional Laplacian matrices take the representations of the well known respective Riesz fractional derivative kernels, i.e.

the convolutional kernels of the (continuous) fractional Laplacians. The approach allows to model 'anomalous diffusion' phenomena on lattices with fractional transport phenomena including asymptotic emergence of Lévy flights. In such a fractional lattice diffusion model, the conventional Laplacian matrix is generalized by its fractional power law matrix function counterpart. The formulation of our approach is consistent with recent works on the fractional approach developed on undirected networks by Riascos and Mateos [17, 18]. The present fractional lattice approach represents a point of departure to investigate anomalous diffusion and fractional random walk phenomena on lattices. Such problems are defined by master equations involving fractional Laplacian matrices such as deduced in the present work as generator matrices for the random dynamics. Fractional random walks on lattices and undirected networks open currently a huge interdisciplinary research field [13, 17, 18].

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