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A Lyapunov approach to control of microgrids with a network-preserved differential-algebraic model

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Abstract— **We provide sufficient conditions for asymptotic stability and optimal resource allocation for a network-preserved microgrid model with active and reactive power loads. The model considers explicitly the presence of constant-power loads as well as the coupling between the phase angle and voltage dynamics. The analysis of the resulting nonlinear differential algebraic equation (DAE) system is conducted by leveraging incremental Lyapunov functions, definiteness of the load flow Jacobian and the implicit function theorem.**

I. Introduction

Driven by considerable societal and political efforts to reduce carbon emissions, the electric energy grid is undergoing a period of unprecedented changes. One major turning point is the replacement of conventional bulk power generation plants by numerous small-scale renewable energy sources (RES). While the former are interfaced to the high-voltage network through synchronous generators (SGs), the latter are usually connected to the distribution networks via power electronic devices called inverters. As inverters have very different physical properties from SGs, RES-dominated networks also exhibit significantly different dynamics than their conventional counterparts. Consequently, the transition to an RES-based energy mix calls for fundamental paradigm shifts in the operation of electric power systems.

In this regard, the microgrid (MG) concept has been identified as a key element of future power networks [1]–[3]. A MG is an electrically connected subset of a distribution network that possesses an own control and energy management infrastructure. Therefore, a MG can also operate in *islanded* mode, i.e., disconnected from the larger utility grid. Hence, by design, MGs offer several promising features such as reduction in losses, smooth integration of RES, and increased network resiliency [1], [2].

Another main feature of MGs compared to conventional distribution networks is that - assuming an adequate MG control architecture is in place - they can provide ancillary services (e.g., frequency, voltage and load-generation control) [1]. These services were traditionally delivered by large SGs via centralized or fully decentralized control architectures [4]. Yet, as outlined above, this is not a feasible option in networks with large share of small-scale RES.

Motivated by this fact, the present paper aims at designing a MG control architecture that enables MGs to provide ancillary services. To this end, we focus on the problems of frequency and voltage stability combined with optimal power injections and frequency restoration. These problems are highly relevant in MGs [1] and have thus recently attracted significant interest. However, existing work is limited by the facts that it is conducted under the assumption of decoupled frequency and voltage dynamics [5]–[8], constant impedance loads [9]–[11] or without explicit representation of electrical network interconnections and loads [12].

Building upon [10], [13], the present paper overcomes the abovementioned limitations by considering a heterogeneous and structure-preserving differential algebraic equation (DAE) model of an inverter-based MG. Such model has several advantages. First, the load buses are explicitly represented, rather than absorbed into the network impedances through Kron reduction [9] Second, it allows to suitably represent constant-power-controlled loads. Third, it does not rely on the prevalent mathematically convenient assumption of decoupling voltage magnitude and phase angle dynamics, (not valid in heavily loaded grids) [5]–[8], [10], [13], [14].

We assume that the grid-forming inverters (i.e., the units responsible for frequency and voltage regulators) are equipped with the standard active and reactive primary droop controllers [3]. Inspired by [5] and recent incremental Lyapunov and passivity-based methods [11], [15], we augment this basic control layer with a distributed-averaging PI controller that permits to restore the grid frequency to its nominal value while minimizing a quadratic criterion for the active power injections. Such a distributed architecture could provide a flexible alternative to the centralized operation of MGs discussed in [1], [3]. For the resulting closed-loop system, we derive a sufficient condition for asymptotic stability that relies on a local definiteness assumption of the load flow Jacobian common in voltage collapse studies [16]–[18]. The analysis is inspired by classic DAE analysis tools in power systems [19], [20], energy function methods [21], and center-of-inertia coordinates [22] blended with with some recent DAE extensions [13] and incremental Lyapunov methods [11] tailored to power systems. The proofs are omitted due to lack of space and will be presented elsewhere.

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Notation For $i \in \{1, 2, \dots, n\}$, by $\mathrm{col}(a_i)$ we denote the vector $\begin{bmatrix} a_1 \ a_2 \ \dots \ a_n \end{bmatrix}^T$. We also use $\mathrm{col}(A, B)$ to denote the matrix $\begin{bmatrix} A^T \ B^T \end{bmatrix}^T$ for given matrices A and B. For a given vector $a \in \mathbb{R}^n$, the diagonal matrix $\mathrm{diag}(a_1, a_2, \dots, a_n)$ is denoted in short by $[a]$. The symbol $\mathbb{1}$ denotes the vector of ones with appropriate dimension.

II. Preliminaries on DAE Systems

In this section, based on [13], [19], we briefly recap some notions from stability theory of differential algebraic systems used to establish part of the results of the present paper. We are concerned with autonomous semi-explicit DAE systems of the form

$$\dot{x} = f(x, y), \tag{1a}$$
$$0 = g(x, y), \tag{1b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are twice continuously differentiable functions. The maximal domain of a solution of (1) is denoted by $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$. Furthermore, we consider only solutions $\big(x(x_0, y_0, t),\ y(x_0, y_0, t)\big)$ of (1) with admissible initial conditions $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying the algebraic constraint

$$0 = g(x_0, y_0). \tag{2}$$

We make the following assumptions on the system (1).

Assumption 1: The system (1) possesses an equilibrium point $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^m$.

Definition 1: Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open connected set. The algebraic equation (1b) is *regular* if the Jacobian of g with respect to y has constant full rank on Ω, that is,

$$\mathrm{rank}\left(\nabla_y g(x, y)\right) = m \quad \forall (x, y) \in \Omega.$$

If (1b) is regular on Ω, then we say that the DAE system (1) is regular on Ω.

Assumption 2: Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open connected set containing (x^*, y^*). The DAE system (1) is regular on Ω.

By [19, Theorem 1], existence and uniqueness of solutions of (1) in Ω over an interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$ for any $\big(x(x_0, y_0, t),$ $y(x_0, y_0, t)\big) \in \Omega$ satisfying (2) is guaranteed by Assumption 2. We refer the reader to [13], [19] for Lyapunov-LaSalle-type stability results of DAE (1).

III. Model of MG, inverters, and loads

A. Modeling of AC MG circuitry

We consider a structure-preserving inverter-based MG model composed of load and generation buses. We restrict our attention to a system-level model, in which distributed power units interfaced to the network via power electronics are represented as controllable voltage sources, and the interconnecting circuitry is modeled by constant impedances. The latter corresponds to the standard quasi-steady model of power lines and transformers employed in most power systems and MG analysis; see [23] for a detailed derivation of this model. The topology of the grid is represented by a connected and undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with vertex set (or buses) $\mathcal{V} = \{1, 2, \dots, n\}$, and the edge set $\mathcal{E}$ is the set of unordered pairs $\{i, j\}$ of distinct vertices i and j. By associating an arbitrary ordering to the edges, the node-edge incidence matrix $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ is defined element wise as $d_{il} = 1$, if node i is the sink of the l^{th} edge, $d_{il} = -1$, if i is the source of the l^{th} edge and $d_{il} = 0$ otherwise. We assume that the line admittances are purely inductive, and two nodes $\{i, j\} \in \mathcal{E}$ are connected by a nonzero real susceptance $B_{ij} < 0$. The set of neighbors of the i^{th} node is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid \{i, j\} \in \mathcal{E}\}$. The voltage phase angle and magnitude at node $i \in \mathcal{V}$ are denoted by $\theta_i \in \mathbb{R}$ and $V_i \in \mathbb{R}_{\geq 0}$, respectively. The relative phase angles are denoted in short by $\theta_{ij} := \theta_i - \theta_j, \{i, j\} \in \mathcal{E}$. The electrical frequency at the i^{th} node is given by $\dot{\theta}_i = \omega_i \in \mathbb{R}$.

With the above notation, the active and reactive power flows at each node $i \in \mathcal{V}$ are given by

$$P_i = \sum_{j \in \mathcal{N}_i} |B_{ij}| V_i V_j \sin(\theta_{ij}), \tag{3a}$$
$$Q_i = |B_{ii}| V_i^2 - \sum_{j \in \mathcal{N}_i} |B_{ij}| V_i V_j \cos(\theta_{ij}), \tag{3b}$$

with $B_{ii} = \sum_{j \in \mathcal{N}_i} B_{ij} + \hat{B}_{ii}$ and where $\hat{B}_{ii} \leq 0$ is the shunt susceptance at the i^{th} node. This shunt susceptance represents either a constant impedance load or the magnetizing susceptance of a transformer. As we are mainly concerned with dynamics of generation units, we express all power flows in generator convention [24]. In this paper we do not make the prevalent decoupling assumption separating active power and phase angles from reactive power and voltage magnitudes [5]–[8], [13]. This mathematically convenient assumption is valid near an operating point with a flat voltage profile but only poorly justified otherwise.

B. Modeling of MG devices

We consider a MG model consisting of inverter-interfaced units at buses $\mathcal{V}_I$ and PQ loads with a constant demand of active and reactive power at buses $\mathcal{V}_L = \mathcal{V} \setminus \mathcal{V}_I$. The inverters follow the standard droop control equations trading off frequency and active power and voltage and reactive power (after filtering power measurements) [9]

$$\begin{aligned} \dot{\theta}_i &= \omega_i, \\ T_{P,i} \dot{\omega}_i &= -(\omega_i - \omega^*) - K_{P,i}(P_i - P_i^*) + u_{P,i}, \\ T_{Q,i} \dot{V}_i &= -(V_i - V_i^*) - K_{Q,i}(Q_i - Q_i^*), \end{aligned} \tag{4}$$

for each $i \in \mathcal{V}_I := \{1, 2, \dots, n_I\}$. Here, $\omega^* \in \mathbb{R}_{>0}$ is the nominal (synchronous) frequency and P_i and Q_i, given by (3a), (3b), are the active, respectively reactive, power drawn from node i. Similarly, P_i^* and Q_i^* denote the (positive) active and reactive power setpoints. The term $u_{P,i}$ accounts for a secondary control input to be designed later in Section V. The parameters $K_{P,i}$, $K_{Q,i}$, $T_{P,i}$, and $T_{Q,i}$ are strictly positive gains. Consider constant power loads that satisfy the following algebraic power balance equations

$$0 = P_i - P_i^*, \tag{5}$$
$$0 = Q_i - Q_i^*, \tag{6}$$

where P_i and Q_i, given by (3a), (3b), denote the active and reactive power demand at node $i \in \mathcal{V}_L = \mathcal{V} \setminus \mathcal{V}_I$. Again P_i^* and Q_i^* are generally nonzero constant setpoints for active and reactive power demand; e.g., a resistive-inductive load has $P^*, Q^* < 0$. The cardinality of $\mathcal{V}_L$ is denoted by n_L.

C. Specifications on optimal synchronous motion

We are interested in a motion of a MG evolving exactly at nominal frequency and possessing an optimal resource allocation with regards to active power generation. We define a *synchronous motion* of the i^{th} bus by

$$\begin{aligned} \theta_i(t) &= \overline{\theta}_i = \theta_i^0 + \omega^* t\,, && i \in \mathcal{V}, \\ \omega_i(t) &= \overline{\omega}_i = \omega^*\,, && i \in \mathcal{V}_I, \\ V_i(t) &= \overline{V}_i\,, && i \in \mathcal{V}, \end{aligned}$$

where $\theta_i^0 \in \mathbb{R}$ is the initial condition of $\overline{\theta}_i$ and $\overline{V}_i \in \mathbb{R}_{>0}$ is the constant voltage magnitude. Recall that $\omega^* \in \mathbb{R}_{>0}$ is the synchronous frequency. Along any synchronous motion, the dynamics of the inverter at the i^{th} node, $i \in \mathcal{V}_I$, satisfy

$$\dot{\overline{\theta}}_i = \omega^*\,, \tag{7}$$

$$0 = -K_{P,i}(\overline{P}_i - P_i^*) + \overline{u}_{P,i}\,, \tag{8}$$

$$0 = -(\overline{V}_i - V_i^*) - K_{Q,i}(\overline{Q}_i - Q_i^*)\,, \tag{9}$$

where $\overline{P}_i = \sum_{j\in\mathcal{N}_i} |B_{ij}|\overline{V}_i\overline{V}_j \sin(\overline{\theta}_{ij})$. In addition to an operation at the nominal frequency ω^*, it also is desirable to allocate the synchronized power injections $\overline{P}_i$ in an optimal manner, e.g., to ensure a cost-efficient system operation. From (8), it is clear that the latter can be achieved via a suitable choice of $\overline{u}_{P,i}$. Observe that by summing over all equations (5) for $i \in \mathcal{V}_L$ and (8) for $i \in \mathcal{V}_I$, and leveraging $\sum_{i\in\mathcal{V}} \overline{P}_i = 0$, we obtain the supply-demand balancing condition

$$0 = \sum_{i\in\mathcal{V}_I} \frac{\overline{u}_{P,i}}{K_{P,i}} + \sum_{i\in\mathcal{V}_I} P_i^* + \sum_{i\in\mathcal{V}_L} P_i^*\,. \tag{10}$$

Clearly, from (10), along any synchronized motion the secondary control inputs $u_{P,i}$ have to balance the mismatch between power injections P_i^*, $i \in \mathcal{V}_I$, and load demands P_j^*, $j \in \mathcal{V}_L$. Observe that, as soon as $|\mathcal{V}_I| \geq 2$, there is no unique assignment of source injections $\overline{u}_{P,i}$ to achieve this objective. Here we aim for an optimal resource allocation such that the synchronized control signals $\overline{u}_P = \mathrm{col}(\overline{u}_{P,i})$ minimize the quadratic cost function

$$u_P = \arg\min_{u\in\mathbb{R}^{n_I}} \sum_{i\in\mathcal{V}_I} \frac{1}{2} r_i u_i^2, \tag{11}$$

subject to the power balance constraint given by (10) and where $r_i \in \mathbb{R}_{>0}$ is the cost coefficient of the i^{th} inverter.

Following the standard Lagrange multipliers method, the optimal control $\overline{u}_{P,i}$ that minimizes (11) subject to the constraint (10) is computed as

$$\overline{u}_{P,i}^\star = -\left(\frac{1}{r_i K_{P,i}}\right) \frac{\sum_{j\in\mathcal{V}} P_j^*}{\sum_{j\in\mathcal{V}_I} \frac{1}{r_j K_{P,j}^2}}.$$

By substituting the above expression into (8), we obtain the steady-state injection as

$$\overline{P}_i - P_i^* = \frac{\overline{u}_{P,i}^\star}{K_{P,i}} = -\left(\frac{1}{r_i K_{P,i}^2}\right) \frac{\sum_{j\in\mathcal{V}} P_j^*}{\sum_{j\in\mathcal{V}_I} \frac{1}{r_j K_{P,j}^2}}\,, \quad i \in \mathcal{V}_I\,.$$

To simplify the notation in the forthcoming analysis, we select the droop gains as $K_{P,i} = \frac{1}{r_i}$ for all $i \in \mathcal{V}_I$ so that

$$\overline{u}_{P,i}^\star = \overline{u}_P^\star = -\frac{\sum_{i\in\mathcal{V}} P_i^*}{\sum_{i\in\mathcal{V}_I} \frac{1}{K_{P,i}}} \quad \forall i \in \mathcal{V}_I\,. \tag{12}$$

Under this choice of gains, we observe that all steady-state secondary control inputs need to be identical: $\overline{u}_{P,i}^\star = \overline{u}_P^\star$ for all $\in \mathcal{V}_I$. We will later on make explicit use of this criterion in our design of a dynamic feedback controller for $u_{P,i}(t)$.

IV. Choice of coordinates, regularity, & Lyapunov function candidates

A. Compact model formulation

The MG model (3), (4), (5) and (6) reads compactly as

$$\dot{\theta}_I = \omega_I, \tag{13a}$$

$$T_P\dot{\omega}_I = -(\omega_I - \omega_I^*) - K_P(P_I - P_I^*) + u_P, \tag{13b}$$

$$T_Q\dot{V}_I = -(V_I - V_I^*) - K_Q(Q_I - Q_I^*), \tag{13c}$$

$$0 = P_L - P_L^*, \tag{13d}$$

$$0 = Q_L - Q_L^*, \tag{13e}$$

where $\theta_I = \mathrm{col}(\theta_i)$, $\omega_I = \mathrm{col}(\omega_i)$, $P_I = \mathrm{col}(P_i)$, $Q_I = \mathrm{col}(Q_i)$, $P_I^* = \mathrm{col}(P_i^*)$, $Q_I^* = \mathrm{col}(Q_i^*)$, $V_I = \mathrm{col}(V_i)$, $u_P = \mathrm{col}(u_{P,i})$, $T_Q = \mathrm{diag}(T_{Q,i})$, $T_P = \mathrm{diag}(T_{P,i})$, and $K_P = \mathrm{diag}(K_{P,i})$ for $i \in \mathcal{V}_I$. In addition, $P_L = \mathrm{col}(P_i)$, $Q_L = \mathrm{col}(Q_i)$, $P_L^* = \mathrm{col}(P_i^*)$, and $Q_L^* = \mathrm{col}(Q_i^*)$ for $i \in \mathcal{V}_L$. Finally, $\theta = \mathrm{col}(\theta_i)$ for $i \in \mathcal{V}$. For simplicity, in the sequel, we set T_P and T_Q to the identity matrix.

For the subsequent analysis, it is useful to derive compact representations for both the active and reactive power flows P and Q. To this end, we set $\Gamma(V) = \mathrm{diag}(\gamma_1(V), \ldots, \gamma_m(V))$, $\gamma_k(V) = |B_{ij}|V_iV_j$, with $k \in \{1, 2, \ldots, m\}$ being the index corresponding to the edge $\{i, j\}$. Then, the vector of the active power flows reads as

$$P = D\Gamma(V)\mathbf{sin}(D^T\theta), \tag{14}$$

where $D = [d_{ij}]$ is the incidence matrix of $\mathcal{G}$ (see Subsection III-A), and $\mathbf{sin}(\cdot)$ is defined element-wise. By partitioning the incidence matrix as $D = \mathrm{col}(D_I, D_L)$ we obtain from (14)

$$P_I = D_I\Gamma(V)\mathbf{sin}(D^T\theta), \quad P_L = D_L\Gamma(V)\mathbf{sin}(D^T\theta).$$

To write the reactive power in a compact form, let the matrix A be defined as

$$A_{ij} = \begin{cases} -|B_{ij}|\cos(\theta_{ij}) & i \neq j \\ \mathrm{diag}(|B_{ii}|) & i = j. \end{cases}$$

For clarity, we use the more informative notation $A(\mathbf{cos}(D^T\theta))$ rather than A, where $\mathbf{cos}(\cdot)$ is defined element-wise. Then it is easy to observe that

$$Q = [V]A(\mathbf{cos}(D^T\theta))V, \tag{15}$$

The vector Q of reactive injections can be partitioned as

$$\begin{bmatrix} Q_I \\ Q_L \end{bmatrix} = \begin{bmatrix} [V_I] & 0 \\ 0 & [V_L] \end{bmatrix} \begin{bmatrix} A_{II}(\cdot) & A_{IL}(\cdot) \\ A_{LI}(\cdot) & A_{LL}(\cdot) \end{bmatrix} \begin{bmatrix} V_I \\ V_L \end{bmatrix}.$$

Next, we write the synchronous motion with the optimal injections (12) compactly as $\overline{\theta} = \mathbb{1}\omega^* + \theta^0$, $\overline{\omega}_I = \mathbb{1}\omega^* = \omega_I^*$, $\overline{V} = \mathrm{col}(\overline{V}_i)$, and $\overline{u}_P = -\mathbb{1}\frac{\mathbb{1}^T P^*}{\mathbb{1}^T K_P^{-1}\mathbb{1}}$. We also call the solution $((\overline{\theta}, \overline{\omega}_I, \overline{V}), \overline{u}_P)$ with $\overline{u}_P$ given by (12) the *optimal synchronous motion* which together with (13) satisfies

$$\begin{aligned} \dot{\overline{\theta}}_I &= \mathbb{1}\omega^*, && (16a)\\ 0 &= -K_P(\overline{P}_I - P_I^*) - \mathbb{1}\frac{\mathbb{1}^T P^*}{\mathbb{1}^T K_P^{-1}\mathbb{1}}, && (16b)\\ 0 &= -(\overline{V}_I - V_I^*) - K_Q(\overline{Q}_I - Q_I^*), && (16c)\\ 0 &= \overline{P}_L - P_L^*, && (16d)\\ 0 &= \overline{Q}_L - Q_L^*, && (16e) \end{aligned}$$

where $\overline{V} = \mathrm{col}(\overline{V}_I, \overline{V}_L)$ and

$$\overline{P} = D\Gamma(\overline{V})\mathbf{sin}(D^T\theta^0), \quad \overline{Q} = [\overline{V}]A(\mathbf{cos}(D^T\theta^0))\overline{V},$$

with $\overline{P} = \mathrm{col}(\overline{P}_I, \overline{P}_L)$, and $\overline{Q} = \mathrm{col}(\overline{Q}_I, \overline{Q}_L)$. We recall that the (optimal) synchronous motion is identified by any solution $(\overline{\theta}, \overline{\omega}_I = \mathbb{1}\omega^*, \overline{V})$ satisfying (16). For the existence of such motion we refer the reader to Assumption 3 below.

B. Choice of coordinates

For our analysis it is convenient to map the synchronous motion to an equilibrium of the system thereby transforming the synchronization problem into a standard stability problem. Inspired by the *center-of-inertia coordinates* in classic power system multi-machine stability studies [22], we define the average of the phase angles of the inverters as the reference, i.e., $\theta_{\mathrm{ref}} = \frac{1}{n_I}\mathbb{1}^T\theta_I$. Let $\delta_i := \theta_i - \theta_{\mathrm{ref}}$ for each $i \in \mathcal{V}$. In addition, let δ_I, δ_L, and δ denote the vector notation of δ_is with $i \in \mathcal{V}_I$, $i \in \mathcal{V}_L$, and $i \in \mathcal{V}$, respectively. Equivalently, in compact formulation

$$\begin{aligned} \delta_I &= \theta_I - \mathbb{1}\theta_{\mathrm{ref}} = \Pi\theta_I,\\ \delta_L &= \theta_L - \mathbb{1}\theta_{\mathrm{ref}} = \theta_L - \frac{1}{n_I}\mathbb{1}\mathbb{1}^T\theta_I, \end{aligned}$$

where $\Pi := (I - \frac{1}{n_I}\mathbb{1}\mathbb{1}^T)$. Hence, we have that

$$\dot{\delta}_I = \Pi\omega_I. \tag{17}$$

Note that the expressions for active and reactive power only depend on the relative phase angles, namely $D^T\theta$. Since

$$D^T\theta = D^T(\delta + \mathbb{1}\theta_{\mathrm{ref}}) = D^T\delta, \tag{18}$$

equations (14) and (15) can be equivalently expressed as

$$\begin{aligned} P &= \mathrm{col}(P_I, P_L) = D\Gamma(V)\mathbf{sin}(D^T\delta), && (19a)\\ Q &= \mathrm{col}(Q_I, Q_L) = [V]A(\mathbf{cos}(D^T\delta))V. && (19b) \end{aligned}$$

By replacing (13a) with (17), the system (13) becomes in the new coordinates

$$\begin{aligned} \dot{\delta}_I &= \Pi\omega_I, && (20a)\\ \dot{\omega}_I &= -(\omega_I - \omega_I^*) - K_P(P_I - P_I^*) + u_P, && (20b)\\ \dot{V}_I &= -(V_I - V_I^*) - K_Q(Q_I - Q_I^*), && (20c)\\ 0 &= P_L - P_L^*, && (20d)\\ 0 &= Q_L - Q_L^*, && (20e) \end{aligned}$$

where the power injections P and Q are given by (19).

Furthermore, a synchronous motion $(\overline{\theta}, \overline{\omega}_I, \overline{V})$ will be then mapped to the point $(\overline{\delta}, \overline{\omega}_I, \overline{V})$ with a constant phase angle vector $\overline{\delta} = \mathrm{col}(\overline{\delta}_I, \overline{\delta}_L)$ satisfying

$$\begin{aligned} \overline{\delta}_I &= \Pi\overline{\theta}_I = \Pi\theta_I^0,\\ \overline{\delta}_L &= \overline{\theta}_L - \frac{1}{n_I}\mathbb{1}\mathbb{1}^T\overline{\theta}_I = \theta_L^0 - \frac{1}{n}\mathbb{1}\mathbb{1}^T\theta_I^0. \end{aligned}$$

The phase angle vector $\overline{\delta}$ satisfies (due to (18))

$$D^T\overline{\delta} = D^T\overline{\theta} = D^T\theta^0\,. \tag{21}$$

Thus, in the new coordinates the desired synchronous motion $(\overline{\theta}, \overline{\omega}_I, \overline{V})$ is mapped to the point $(\overline{\delta}, \overline{\omega}_I, \overline{V})$ where $\overline{\omega}_I = \mathbb{1}\omega^*$, and $\overline{\delta}$ as well as $\overline{V}$ are constant vectors.

Clearly, for $u_P = \overline{u}_P$, $(\overline{\delta}, \overline{\omega}_I, \overline{V})$ is an equilibrium point of the system (20). Henceforth we focus on the stability of $(\overline{\delta}, \overline{\omega}_I, \overline{V})$ for system (20). To this end, we introduce:

Assumption 3: Fix ω^*. There exist $D^T\overline{\delta} \in (-\frac{\pi}{2}, \frac{\pi}{2})^m$ and $\overline{V} \in \mathbb{R}^n_{>0}$ such that (16b)–(16e) hold with

$$\begin{aligned} \overline{P} &= \mathrm{col}(\overline{P}_I, \overline{P}_L) = D\Gamma(\overline{V})\mathbf{sin}(D^T\overline{\delta}), && (22a)\\ \overline{Q} &= \mathrm{col}(\overline{Q}_I, \overline{Q}_L) = [\overline{V}]A(\mathbf{cos}(D^T\overline{\delta}))\overline{V} && (22b) \end{aligned}$$

C. Regularity of the algebraic equations

In this section, we investigate the regularity of the algebraic equations (20d)-(20e) of the system (20), where the active and reactive power injections are expressed as in (19). As discussed in Section II, regularity of the algebraic constraints (20d)-(20e) is a crucial property both to investigate the existence/uniqueness of solutions and to study stability properties of the overall system.

Let $\delta_{ij} := \delta_i - \delta_j = \theta_{ij}$ and define the function

$$Z(\delta, V) = \sum_{i\in\mathcal{V}} Q_i = V^T A(\mathbf{cos}(D^T\delta))V,$$

and observe that its gradient satisfies

$$\frac{\partial Z}{\partial \delta_L} = P_L, \quad \frac{\partial Z}{\partial V_L} = [V_L]^{-1}Q_L.$$

As we are interested in solutions satisfying $\overline{V}_i > 0$ in steady state, the latter identity above is well-defined as long as $V_{L,i}(t) > 0$. Under this positivity condition, the algebraic equations (13d) and (13e) can be written as

$$\left.\begin{aligned} 0 &= \frac{\partial Z}{\partial \delta_L} - P_L^*,\\ 0 &= \frac{\partial Z}{\partial V_L} - [V_L]^{-1}Q_L^*, \end{aligned}\right. \quad \Leftrightarrow \quad 0 = g(x, y). \tag{23}$$

with $x = (\delta_I, \omega_I, V_I)$ and $y = (\delta_L, V_L)$. Now let $|D|$ denote the matrix obtained from D by replacing all the elements d_{ik} of D with $|d_{ik}|$ [15]. Also let $|D|$ be partitioned as $\text{col}(|D_I|, |D_L|)$. Then, we have the following lemma.

Lemma 1: Consider the system (20). The algebraic equations (13d)-(13e) are regular in a neighbourhood of $(\overline{\delta}, \overline{\omega}_I, \overline{V})$ if

$$G(D^T\overline{\delta}, \overline{V}) > 0, \tag{24}$$

where $G(D^T\delta, V)$ is given by

$$\begin{bmatrix} \Gamma(V)[\mathbf{cos}(D^T\delta)] & [\mathbf{sin}(D^T\delta)]\Gamma(V)|D_L|^T[V_L]^{-1} \\ [V_L]^{-1}|D_L|\Gamma(V)[\mathbf{sin}(D^T\delta)] & A_{LL}(\mathbf{cos}(D^T\delta)) + [V_L]^{-2}[Q_L^*] \end{bmatrix}.$$

For the remaining analysis, we assume the following.

Assumption 4: Consider the system (20) with Assumption 3. The desired synchronous motion is such that (24) is satisfied.

In view of (18), (21), condition (24) is the same as condition $G(D^T\overline{\theta}, \overline{V}) > 0$, which implies regularity of (13d), (13e) in a neighborhood of $(\overline{\theta}, \overline{\omega}_I, \overline{V})$. Hence, recalling from Section II, Assumption 4 implies that the DAE system (13), and therefore the DAE system (20), admits a unique solution over an interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$.

D. Storage function candidate

In this section, we carry out a dissipativity analysis of the system (20). The dissipativity of (20), besides being an interesting property *per se*, is used in Section V to show that the solutions of (20) interconnected with a suitable controller converge to the optimal synchronous motion. For this purpose, we introduce the function

$$U(\delta, \omega_I, V) = \frac{1}{2}\omega_I^T K_P^{-1}\omega_I + Z(\delta, V).$$

Observe that the first term is associated with a virtual kinetic energy, and the second term is equal to Z given by $V^T A(\mathbf{cos}(D^T\delta))V$ corresponding to the electromagnetic energy stored in the lines. For these reasons, this type of storage function candidate is typically referred to as *energy function* in the literature [21]. Note that its gradient satisfies

$$\frac{\partial U}{\partial \omega_I} = K_P^{-1}\omega_I, \quad \frac{\partial U}{\partial \delta} = P, \quad \frac{\partial U}{\partial V} = [V]^{-1}Q.$$

For the later convergence/stability analysis of an optimal synchronized motion, it is convenient to shift the critical points of U to $(\overline{\delta}, \overline{\omega}_I, \overline{V})$. Hence, an appropriate incremental extension of U can be constructed as the Bregman distance between a point (δ, ω_I, V) and an optimal synchronous point $(\overline{\delta}, \overline{\omega}_I, \overline{V})$ [11]. To this end, we define the incremental storage function $\mathcal{U}$ as follows

$$\begin{aligned} \mathcal{U}(\delta, \omega_I, V) &= U(\delta, \omega_I, V) - U(\overline{\delta}, \overline{\omega}_I, \overline{V}) \\ &- \left.\frac{\partial U}{\partial \delta}\right|_{-}^{T}(\delta - \overline{\delta}) - \left.\frac{\partial U}{\partial \omega_I}\right|_{-}^{T}(\omega_I - \overline{\omega}_I) - (Q_I^*)^T\mathbf{ln}(V_I) \\ &+ \mathbb{1}^T K_Q^{-1}V_I - (V_I^*)^T K_Q^{-1}\mathbf{ln}(V_I) - (Q_L^*)^T\mathbf{ln}(V_L). \end{aligned} \tag{25}$$

The following result establishes a crucial dissipation property of the system (20).

Lemma 2: Consider the MG model (20), (19a), (19b) with Assumption 4. The time derivative of $\mathcal{U}$ along the solution (δ, ω_I, V), initialized in a neighborhood of $(\overline{\delta}, \overline{\omega}_I, \overline{V})$, satisfies the following dissipation equality

$$\begin{aligned} \dot{\mathcal{U}}(\delta, \omega_I, V) &= -\frac{\partial \mathcal{U}}{\partial \omega_I}^T K_P \frac{\partial \mathcal{U}}{\partial \omega_I} - \frac{\partial \mathcal{U}}{\partial V_I}^T [V_I] K_Q \frac{\partial \mathcal{U}}{\partial V_I} \\ &+ \frac{\partial \mathcal{U}}{\partial \omega_I}^T (u_P - \overline{u}_P), \end{aligned} \tag{26}$$

for a nonzero interval of time $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$.

Note that the above dissipation inequality (26) as an algebraic identity is independent of the choice of $u_P(t)$, it is oblivious regarding the regularity of the algebraic equations, and can be stated whether trajectories actually remain bounded or not. In Section V, we leverage the dissipation inequality (26), with the input-output pair $(u_P - \overline{u}_P, \frac{\partial \mathcal{U}}{\partial \omega}) = (u_P - \overline{u}_P, \omega_I - \overline{\omega}_I)$, to design an optimal frequency controller.

V. Secondary control and convergence to the optimal synchronous motion

The optimal control $\overline{u}_P^*$ given in (12) requires each inverter to know all the active power setpoints P_i^*, $i \in \mathcal{V}_I$, active power demands P_i^*, $i \in \mathcal{V}_L$, and gains $K_{P,i}$, $i \in \mathcal{V}_I$. This global knowledge of parameters is unpractical and one would like to design controllers that converges to the optimal control in spite of a lack of knowledge of these parameters. Integral controllers are known to provide feedforward control actions in spite of unknown constant terms, but are typically not able to guarantee convergence to a specific optimal solution, as it is of interest here. To asymptotically provide the optimal control $\overline{u}_P$ given in (12), we consider a modified integral control law [15], [5], [10]

$$\dot{\xi} = -L_c\xi - K_P^{-1}(\omega_I - \omega_I^*), \quad u_P = \xi, \tag{27}$$

where L_c is the Laplacian matrix of a connected communication graph, say $G_c = (\mathcal{V}, \mathcal{E}_c)$. The term $\omega_I - \omega_I^*$ regulates the frequency to the nominal frequency, while the consensus based algorithm $-L_c\xi$ aims at steering the input to the optimal one given by (12). As a matter of fact, the consensus part of the controller enforces an equilibrium where all the components of the state ξ, and hence of u are the same, in accordance with the optimal control (12).

Inspired by classic energy functions in power systems [21] and their incremental interpretations in [10], [11], [15], we propose the following incremental Lyapunov function

$$\begin{aligned} \mathcal{W}(\delta, \omega_I, V, \xi) &= \mathcal{U}(\delta, \omega_I, V) + \frac{1}{2}(\xi - \overline{\xi})^T(\xi - \overline{\xi}) \\ &+ \frac{1}{2}(\delta_I - \overline{\delta}_I)\mathbb{1}\mathbb{1}^T(\delta_I - \overline{\delta}_I) \end{aligned} \tag{28}$$

where $\mathcal{U}$ is given by (25), the second term of $\mathcal{W}$ accounts for the controller dynamics with $\overline{\xi} = \overline{u}_P^*$ given by (12), and the third term is added to render $\mathcal{W}$ strictly convex in a neighborhood of the equilibrium point. In order to establish convergence results, some suitable properties of $\mathcal{W}$ are shown next.

First, following the calculations in the previous section, it is easy to see that the partial derivatives of $\mathcal{W}$ vanishes along the optimal synchronous motion $(\overline{\delta}, \overline{\omega}_I, \overline{V}, \overline{\xi})$. Next, the following lemma investigates strict convexity of $\mathcal{W}$.

Lemma 3: Let $\mathcal{W}$ be given by (28). Then, we have

$$\left.\frac{\partial^2 \mathcal{W}}{\partial(\delta, \omega_I, V, \xi)}\right|_{(\overline{\delta}, \overline{\omega}_I, \overline{V}, \overline{\xi})} > 0 \qquad (29)$$

if and only if the matrix

$$\begin{bmatrix} \Gamma(\overline{V})[\mathbf{cos}(D^T\overline{\delta})] & [\mathbf{sin}(D^T\overline{\delta})]\Gamma(\overline{V})|D|^T[\overline{V}]^{-1} \\ [\overline{V}]^{-1}|D|\Gamma(\overline{V})[\mathbf{sin}(D^T\overline{\delta})] & A(\mathbf{cos}(D^T\overline{\delta})) + h(\overline{V}) \end{bmatrix} \qquad (30)$$

is positive definite, where

$$h(V) = \begin{bmatrix} [V_I]^{-2}[Q_I^* + K_Q^{-1}V_I^*] & 0 \\ 0 & [V_L]^{-2}[Q_L^*] \end{bmatrix}. \qquad (31)$$

Notice that so far we have made two explicit assumptions on the vectors $\overline{V}$ and $D^T\overline{\delta} = D^T\overline{\theta} = D^T\theta^0$. The first one is given by (24) which guarantees the regularity of the algebraic equations, and the second one is provided by Lemma 3 and implies strict convexity of the Lyapunov function $\mathcal{W}$. We now remark the following important implication relating the two conditions:

Lemma 4: The inequality (29) implies the inequality (24).

Now the main result of this section is stated in the following:

*Theorem 1 (***Main result***):* Consider the system (20), (19a), (19b), in closed-loop with (27). Suppose that Assumptions 3 holds, and the matrix in (30) is positive definite. Then any solution $(\delta, \omega_I, V, \xi)$ with $\delta_I(0) \in \mathrm{im}(\Pi)$ locally converges to $(\overline{\delta}, \overline{\omega}_I, \overline{V}, \overline{\xi})$.

By the theorem above, the controller (27) regulates the frequency to ω_I^*, and provides the optimal secondary control inputs (12) at steady-state .

VI. Conclusions

The paper proposed a dissipativity-inspired Lyapunov based analysis of inverter-based MGs with constant active and reactive power loads. The inverters follow the standard droop control equations and secondary controllers are added to achieve zero steady state frequency deviation jointly with optimal resource allocation at steady state. We envision that the method can be used to analyze and design other closed-loop differential-algebraic systems covering novel voltage regulation algorithms for heterogeneous and nonlinear microgrid models and also power transmission systems.

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