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Complexity of a knot invariant

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The algebraic structures called quandles constitute a complete invariant for tame knots. However, determining when two quandles are isomorphic is an empirically hard problem, so there is some dissatisfaction with quandles as knot invariants. We have confirmed this apparent difficulty, showing within the framework of Borel reducibility that the general isomorphism problem for quandles is as complex as possible.

1 Background

1.1 Quandles

Quandles may be viewed as an algebraic encapsulation of the properties of knots. We give here the basic definition and overview of their relation to knots; for more on quandles see the recent expository article [1] or [2].

Definition 1.1 A *quandle* is a set Q endowed with a binary operation $*$ satisfying the following axioms.

1. The operation $*$ is idempotent: for all q in Q, $q * q = q$.
2. For all q in Q, left multiplication by q is invertible: for all s in Q there is a unique r such that $q * r = s$.
3. For all q in Q, left multiplication by q is a homomorphism: for all r and s in Q, $q * (r * s) = (q * r) * (q * s)$.

(Note that in much of the literature the handed-ness of the axioms is reversed.) Joyce [3] showed how to associate a quandle to any tame knot, with the arcs of the knot as generators and identities dictated by the crossings. In this framework, the three axioms of Definition 1.1 correspond precisely to the familiar Reidemeister moves. Joyce showed that these quandles constitute *complete invariants* for tame knots — two knots are equivalent if and only if their associated quandles are isomorphic. It is thus natural to ask how complex is the problem of determining whether two quandles are isomorphic; empirically this appeared to be a difficult problem.

1.2 Borel reducibility

In recent years the analysis of the complexity of classification problems and the invariants used has become a major topic in set theory. A classification can be formalised as a reasonably definable mapping from one class X of mathematical objects (in our example, tame knots), up to some equivalence relation E (knot equivalence), to another class Y of mathematical objects (quandles), again up to some equivalence relation F (isomorphism). Often the objects in question can be coded up in such a way that we may consider X and Y to be complete separable metric spaces, and in this context an appropriate formalisation of a "reasonably definable mapping" is a Borel function. This motivates the following definition.

Definition 1.2 Given equivalence relations E and F on complete separable metric spaces X and Y respectively, we say that E is *Borel reducible* to F, written $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that for all x_1 and x_2 in X,

$$x_1 \; E \; x_2 \Leftrightarrow f(x_1) \; F \; f(x_2).$$

A first order class of countable structures with isomorphism relation F is *Borel complete* if every first order class of countable structures has isomorphism relation Borel reducible to F.

Thus, being Borel complete means that an isomorphism relation is as complex as possible in this framework. For more on Borel reducibility see, for example, [4] or [5].

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2 Results

In a forthcoming paper [6], we show the following.

Theorem 2.1 *The class of countable quandles is Borel complete.*

Note that this also immediately implies that several more general classes are Borel complete. Structures with a binary operation satisfying (2) and (3) of Definition 1.1 are referred to as *racks*, and even more generally those satisfying (3) of Definition 1.1 are called *left distributive algebras* or *shelves*. We thus have as immediate corollaries of Theorem 2.1 that the class of countable racks is Borel complete, and that the class of countable left distributive algebras is Borel complete.

The proof of Theorem 2.1 goes by way of the folklore result that the class of countable graphs is Borel complete. With this in hand, it suffices to exhibit a Borel reduction of the isomorphism relation on graphs to the isomorphism relation on quandles. We define the quandle $Q(\Gamma)$ associated to a graph Γ, in such a way that two graphs Γ_1 and Γ_2 are isomorphic as graphs if and only $Q(\Gamma_1)$ is isomorphic to $Q(\Gamma_2)$. The idea is to use the same underlying set and to encode the edge relation $x\ E\ y$ of the graph into the relation $x * y = y$ of the quandle; with a slight adaptation this works. For this we use the *dynamical quandle* construction introduced by Kamada [7]. The fact that with this construction we have $\Gamma_1 \cong \Gamma_2 \Rightarrow Q(\Gamma_1) \cong Q(\Gamma_2)$ is immediate from the definition; the converse requires more work but can also be shown to hold. Finally, with the standard topology on the spaces of countable graphs and of countable quandles, the mapping $Q : \Gamma \mapsto Q(\Gamma)$ is in fact continuous, and so is certainly Borel.

3 Concluding comments

Our result confirms the impression amongst knot theorists that determining whether quandles are isomorphic is a complex problem. Whilst we work with more general quandles than arise from knot theory (for example, the quandle associated with a tame knot will always be finitely presented), this at least shows that good techniques for comparing such quandles will necessarily use some of the extra properties associated with being a knot quandle.

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