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Bearing Rigidity and Almost Global Bearing-Only Formation Stabilization

Shiyu Zhao and Daniel Zelazo

***Abstract*—This paper studies the problem of distributed control of *bearing-constrained* multi-agent formations using *bearing-only* measurements. In order to solve this problem, we first propose a bearing rigidity theory that is applicable to arbitrary dimensions. The proposed bearing rigidity theory is then applied to solve two bearing-only formation control problems. In the first, each agent can measure the relative bearings to their neighbors in a global reference frame, while in the second problem, each agent can only measure the bearings and relative orientations of their neighbors in their local frames. For the two problems, we propose distributed bearing-only control laws and prove almost global formation stability.**

***Index Terms*—Bearing rigidity, formation control, bearing-only measurement, attitude synchronization, almost global input-to-state stability**

I. Introduction

Multi-agent formation control has been studied extensively in recent years with *distance-constrained formation control* taking a prominent role [1]–[6]. In this setting it is assumed that the target formation is specified by inter-agent distances, and each agent is able to measure relative positions of their neighbors. *Bearing-constrained formation control* has also attracted much attention recently [7]–[13]. Instead of distances, the formation is specified by inter-agent bearings, and each agent can measure the relative positions or bearings of their neighbors. Bearing measurements are often cheaper and more accessible than position measurements, spurring interest in cooperative control using bearing-only measurements [8]–[17]. Bearing-based formation control can be potentially applied to vision-based cooperative control of multi-vehicle systems where each vehicle can measure the bearings of their neighbors with a camera.

This paper studies a *bearing-only formation control* problem where the target formation is *bearing-constrained* and each agent has access to the *bearing-only* measurements of their neighbors. Relative position or distance measurements are not available. Moreover, it is noted that while bearing measurements can be used to estimate relative distances or positions [15], [17], [18], such schemes may significantly increase the complexity of the sensing system in terms of both hardware and software. This then motivates our study focusing on a pure bearing-only control scheme, where the bearing measurements are directly applied in the formation control and it is not required to estimate additional quantities (e.g., relative position).

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Although bearing-only formation control has attracted much interest in recent years, many problems on this topic remain unsolved. The studies in [7], [10], [14] considered bearing-constrained formation control in two-dimensional spaces, but required access to position or other measurements in the proposed control laws. The results reported in [15], [17] only require bearing measurements, but the bearing measurements are used to estimate additional relative-state information such as distance ratios or scale-free coordinates. The works in [8], [9], [11], [12] studied formation control with bearing measurements directly applied in the control. However, these results were applied to special formations, such as cyclic formations, and may not be extendable to arbitrary formation shapes. A very recent work reported in [13] solved bearing-only formation control for arbitrary underlying sensing graphs. This result, however, is valid only for two-dimensional formations. Bearing-only formation control in arbitrary dimensions with general underlying sensing graphs still remains an open problem.

It is well known that a central tool in the study of distance-based formation control is *distance rigidity theory*. Similarly, a central tool for analyzing bearing-based problems is *bearing rigidity theory* (also referred to as *parallel rigidity* in some literature). Up to now, the existing works on bearing rigidity mainly focused on frameworks in two-dimensional ambient spaces [7], [9], [18], [19]. The first contribution of our work, therefore, is an extension of the existing bearing rigidity theory to arbitrary dimensions. We also explore connections between bearing rigidity and distance rigidity, and in particular show that a framework in $\mathbb{R}^2$ is infinitesimally bearing rigid if and only if it is also infinitesimally distance rigid.

Based on the proposed bearing rigidity theory, we investigate distributed bearing-only formation control in arbitrary dimensions in the presence of a global reference frame. We propose a distributed bearing-only formation control law and show by a Lyapunov approach that the control law can almost globally stabilize infinitesimally bearing rigid formations. We also provide a sufficient condition ensuring collision avoidance between any pair of agents under the action of the control.

In the third part of the paper, we investigate bearing-only formation control in three dimensions without a global reference frame known to the agents. In this case, each agent can only measure the bearings and relative orientations of their neighbors in their local reference frames. We propose a distributed control law to control both the position and the orientation of each agent. It is shown that the orientation will synchronize, and the target formation is almost globally stable. Formation control of both positions and orientations (also known as formation control in $SE(2)$ or $SE(3)$) has received some attention very recently [18], [20]–[23]. As the position and orientation dynamics usually forms a cascade system, input-to-state stability (ISS) can be used to prove the formation stability [22], [23]. While the conventional ISS is defined for globally stable equilibriums, we employ the recently developed *almost global ISS* [24] to prove the almost global formation stability.

This paper is organized as follows. Section II presents the bearing rigidity theory that is applicable to arbitrary dimensions. Section III studies bearing-only formation control in arbitrary dimensions in the presence of a global reference frame, and Section IV studies the case without a global reference frame. Simulation results are presented in Section V. Conclusions and future works are given in Section VI.

Notations: Given $A_i \in \mathbb{R}^{p\times q}$ for $i = 1, \dots, n$, denote $\mathrm{diag}(A_i) \triangleq \mathrm{blkdiag}\{A_1, \dots, A_n\} \in \mathbb{R}^{np\times nq}$. Let $\mathrm{Null}(\cdot)$ and $\mathrm{Range}(\cdot)$ be the null space and range space of a matrix, respectively. Denote $I_d \in \mathbb{R}^{d\times d}$ as the identity matrix, and $\mathbf{1} \triangleq [1, \dots, 1]^{\mathrm{T}}$. Let $\|\cdot\|$ be the Euclidian norm of a vector or the spectral norm of a matrix, and $\otimes$ the Kronecker product. For any $x = [x_1, x_2, x_3]^{\mathrm{T}} \in \mathbb{R}^3$, the associated skew-symmetric matrix is denoted as

$$[x]_\times \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{1}$$

An undirected graph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consists of a vertex set $\mathcal{V} = \{1, \dots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ with $m = |\mathcal{E}|$. The set of neighbors of vertex i is denoted as $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. An orientation of an undirected graph is the assignment of a direction to each edge. An oriented graph is an undirected graph together with an orientation. The incidence matrix $H \in \mathbb{R}^{m\times n}$ of an oriented graph is the $\{0, \pm 1\}$-matrix with rows indexed by edges and columns by vertices: $[H]_{ki} = 1$ if vertex i is the head of edge k, $[H]_{ki} = -1$ if vertex i is the tail of edge k, and $[H]_{ki} = 0$ otherwise. For a connected graph, one always has $H\mathbf{1} = 0$ and $\mathrm{rank}(H) = n - 1$ [25].

II. Bearing Rigidity in Arbitrary Dimensions

The basic problem that bearing rigidity theory studies is whether a framework can be uniquely determined up to a translation and a scaling factor given the bearings between each pair of neighbors in the framework. This problem can be equivalently stated as whether two frameworks with the same inter-neighbor bearings have the same shape. The existing bearing rigidity theory is developed mainly for two dimensions. In this section, we propose a bearing rigidity theory that is applicable to arbitrary dimensions.

We first define some necessary notations. Given a finite collection of n points $\{p_i\}_{i=1}^n$ in $\mathbb{R}^d$ ($n \geq 2$, $d \geq 2$), a *configuration* is denoted as $p = [p_1^{\mathrm{T}}, \dots, p_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$. A *framework* in $\mathbb{R}^d$, denoted as $\mathcal{G}(p)$, is a combination of an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a configuration p, where vertex $i \in \mathcal{V}$ in the graph is mapped to the point p_i in the configuration. For a framework $\mathcal{G}(p)$, define the *edge vector* and the *bearing*, respectively, as

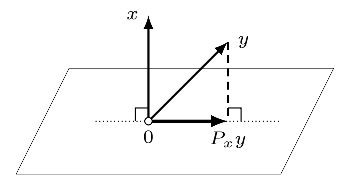


Fig. 1: The geometric interpretation of the orthogonal projection operator.

$$e_{ij} \triangleq p_j - p_i, \quad g_{ij} \triangleq e_{ij}/\|e_{ij}\|, \quad \forall (i, j) \in \mathcal{E}. \tag{2}$$

As defined in (2), the bearing between two points is represented by a unit vector. This is different from the conventional representation where a bearing is described as one angle (azimuth) in two dimensions or two angles (azimuth and altitude) in three dimensions. The unit-vector representation enables us to conveniently describe bearings in arbitrary dimensions. Note also that $e_{ij} = -e_{ji}$ and $g_{ij} = -g_{ji}$.

We now introduce an important orthogonal projection operator that will be widely used in this paper. For any nonzero vector $x \in \mathbb{R}^d$ ($d \geq 2$), define the operator $P : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ as

$$P(x) \triangleq I_d - \frac{x}{\|x\|}\frac{x^{\mathrm{T}}}{\|x\|}.$$

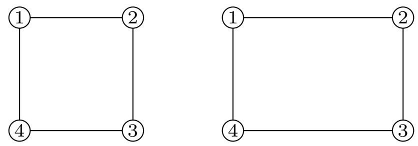
For notational simplicity, denote $P_x = P(x)$. Note P_x is an orthogonal projection matrix which geometrically projects any vector onto the orthogonal compliment of x (see Figure 1). It can be verified that $P_x^{\mathrm{T}} = P_x$, $P_x^2 = P_x$, and P_x is positive semi-definite. Moreover, $\mathrm{Null}(P_x) = \mathrm{span}\{x\}$ and the eigenvalues of P_x are $\{0, 1, \dots, 1\}$, where the zero eigenvalue is simple and the multiplicity of the eigenvalue 1 is $d - 1$.

In the bearing rigidity theory, the relationship between any two frameworks is evaluated by comparing their bearings. The bearings of two vectors are the same only if they are parallel to each other. As a result, the notion of parallel vectors is the core concept for the development of the bearing rigidity theory. The orthogonal projection operator can be used to characterize if two vectors in an arbitrary dimension are parallel.

Lemma 1. *Two nonzero vectors $x, y \in \mathbb{R}^d$ are parallel if and only if $P_x y = 0$ (or equivalently $P_y x = 0$).*

Proof. The result directly follows from the property $\mathrm{Null}(P_x) = \mathrm{span}\{x\}$. □

Remark 1. *The orthogonal projection operator provides a more general way to characterize parallel vectors in arbitrary dimensions. Many existing works use the notion of* normal vectors *to describe parallel vectors in $\mathbb{R}^2$ [7], [9], [19]. Specifically, given a nonzero vector $x \in \mathbb{R}^2$, denote $x^\perp \in \mathbb{R}^2$ as the normal vector satisfying $x^{\mathrm{T}}x^\perp = 0$. Then any vector $y \in \mathbb{R}^2$ is parallel to x if and only if $(x^\perp)^{\mathrm{T}}y = 0$. This approach is applicable to two dimensional cases but difficult to extend to arbitrary dimensions. Furthermore, it is straightforward to show that the use of normal vectors is equivalent to the use of the general orthogonal projection*



(a) (b)

Fig. 2: The two frameworks are bearing equivalent but *not* bearing congruent. That is because the bearings between (p_1, p_3) or (p_2, p_4) of the two frameworks are different.

matrix in $\mathbb{R}^2$. To see that, note

$$P_x = \frac{x^\perp}{\|x^\perp\|}\frac{(x^\perp)^\mathrm{T}}{\|x^\perp\|}, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}. \tag{3}$$

To prove (3)*, consider $B = [x/\|x\|, x^\perp/\|x^\perp\|] \in \mathbb{R}^{2\times 2}$. Note B is an orthogonal matrix. It follows from $BB^\mathrm{T} = I_2$ that $xx^\mathrm{T}/\|x\|^2 + x^\perp(x^\perp)^\mathrm{T}/\|x^\perp\|^2 = I_2$ leading to* (3)*. It then follows from* (3) *that*

$$(x^\perp)^\mathrm{T} y = 0 \;\Leftrightarrow\; x^\perp(x^\perp)^\mathrm{T} y/\|x^\perp\|^2 = 0 \;\Leftrightarrow\; P_x y = 0.$$

We are now ready to define the fundamental concepts in bearing rigidity. These concepts are defined analogously to those in the distance rigidity theory.

Definition 1 (Bearing Equivalency)**.** *Two frameworks $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are* bearing equivalent *if $P_{(p_i-p_j)}(p'_i - p'_j) = 0$ for all $(i,j) \in \mathcal{E}$.*

Definition 2 (Bearing Congruency)**.** *Two frameworks $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are* bearing congruent *if $P_{(p_i-p_j)}(p'_i - p'_j) = 0$ for all $i, j \in \mathcal{V}$.*

By definition, bearing congruency implies bearing equivalency. Figure 2 shows two frameworks that are bearing equivalent but not bearing congruent.

Definition 3 (Bearing Rigidity)**.** *A framework $\mathcal{G}(p)$ is* bearing rigid *if there exists a constant $\epsilon > 0$ such that any framework $\mathcal{G}(p')$ that is bearing equivalent to $\mathcal{G}(p)$ and satisfies $\|p' - p\| < \epsilon$ is also bearing congruent to $\mathcal{G}(p)$.*

Definition 4 (Global Bearing Rigidity)**.** *A framework $\mathcal{G}(p)$ is* globally bearing rigid *if an arbitrary framework that is bearing equivalent to $\mathcal{G}(p)$ is also bearing congruent to $\mathcal{G}(p)$.*

By definition, global bearing rigidity implies bearing rigidity. As will be shown later, the converse is also true.

We next define *infinitesimal bearing rigidity*, which is one of the most important concepts in the bearing rigidity theory. Recall the graph $\mathcal{G}$ is assumed to be undirected. Consider an arbitrary oriented graph of $\mathcal{G}$ and denote

$$e_k \triangleq p_j - p_i, \quad g_k \triangleq e_k/\|e_k\|, \quad \forall k \in \{1, \ldots, m\} \tag{4}$$

as the edge vector and the bearing for the kth directed edge. Denote $e = [e_1^\mathrm{T}, \ldots, e_m^\mathrm{T}]^\mathrm{T}$ and $g = [g_1^\mathrm{T}, \ldots, g_m^\mathrm{T}]^\mathrm{T}$. Note e satisfies $e = \bar{H}p$ where $\bar{H} = H \otimes I_d$ and H is the incidence matrix. Define the *bearing function* $F_B : \mathbb{R}^{dn} \to \mathbb{R}^{dm}$ as

$$F_B(p) \triangleq \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} \in \mathbb{R}^{dm}.$$

The bearing function describes all the bearings in the network. The *bearing rigidity matrix* is defined as the Jacobian of the bearing function,

$$R(p) \triangleq \frac{\partial F_B(p)}{\partial p} \in \mathbb{R}^{dm \times dn}. \tag{5}$$

Let δp be a variation of the configuration p. If $R(p)\delta p = 0$, then δp is called an *infinitesimal bearing motion* of $\mathcal{G}(p)$. This is analogous to infinitesimal motions used in distance-based rigidity. Distance preserving motions of a framework include rigid-body translations and rotations, whereas bearing preserving motions of a framework include translations and scalings. An infinitesimal bearing motion is called *trivial* if it corresponds to a translation and/or a scaling of the entire framework.

Definition 5 (Infinitesimal Bearing Rigidity)**.** *A framework is* infinitesimally bearing rigid *if all the infinitesimal bearing motions are trivial.*

Up to this point, we have introduced all the fundamental concepts in the bearing rigidity theory. We next explore the properties of these concepts. We first derive a useful expression for the bearing rigidity matrix.

Lemma 2. *The bearing rigidity matrix in* (5) *can be expressed as*

$$R(p) = \mathrm{diag}\left(\frac{P_{g_k}}{\|e_k\|}\right)\bar{H}. \tag{6}$$

Proof. It follows from $g_k = e_k/\|e_k\|, \forall k \in \{1, \ldots, m\}$ that

$$\frac{\partial g_k}{\partial e_k} = \frac{1}{\|e_k\|}\left(I_d - \frac{e_k}{\|e_k\|}\frac{e_k^\mathrm{T}}{\|e_k\|}\right) = \frac{1}{\|e_k\|}P_{g_k}.$$

As a result, $\partial F_B(p)/\partial e = \mathrm{diag}\left(P_{g_k}/\|e_k\|\right)$ and consequently

$$R(p) = \frac{\partial F_B(p)}{\partial p} = \frac{\partial F_B(p)}{\partial e}\frac{\partial e}{\partial p} = \mathrm{diag}\left(\frac{P_{g_k}}{\|e_k\|}\right)\bar{H}.$$

□

The expression (6) can be used to characterize the null space and the rank of the bearing rigidity matrix.

Lemma 3. *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ always satisfies* $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R(p))$ *and* $\mathrm{rank}(R(p)) \leq dn - d - 1$.

Proof. First, it is clear that $\mathrm{span}\{\mathbf{1} \otimes I_d\} \subseteq \mathrm{Null}(\bar{H}) \subseteq \mathrm{Null}(R(p))$. Second, since $P_{e_k}e_k = 0$, we have $R(p)p = \mathrm{diag}(P_{e_k}/\|e_k\|)\bar{H}p = \mathrm{diag}(P_{e_k}/\|e_k\|)e = 0$ and hence $p \subseteq \mathrm{Null}(R(p))$. The inequality $\mathrm{rank}(R(p)) \leq dn - d - 1$ follows immediately from $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R(p))$. □

For any undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, denote $\mathcal{G}^\kappa$ as the complete graph over the same vertex set $\mathcal{V}$, and $R^\kappa(p)$ as the bearing rigidity matrix of the framework $G^\kappa(p)$. The next result gives the necessary and sufficient conditions for bearing equivalency and bearing congruency.

Theorem 1. *Two frameworks $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are bearing equivalent if and only if $R(p)p' = 0$, and bearing congruent if and only if $R^\kappa(p)p' = 0$.*

Proof. Since $R(p)p' = \operatorname{diag}(I_d/\|e_k\|)\operatorname{diag}(P_{g_k})\bar{H}p' = \operatorname{diag}(I_d/\|e_k\|)\operatorname{diag}(P_{g_k})e'$, we have

$$R(p)p' = 0 \Leftrightarrow P_{g_k}e'_k = 0, \forall k \in \{1, \ldots, m\}.$$

Therefore, by the definition of bearing equivalency, the two frameworks are bearing equivalent if and only if $R(p)p' = 0$. By the definition of bearing congruency, it can be analogously proved that two frameworks are bearing equivalent if and only if $R^\kappa(p)p' = 0$. □

One implication of Theorem 1 is that *an infinitesimal bearing motion is a motion that preserves the bearing between each pair of neighbors in a framework*. To see that, for any infinitesimal motion $\delta p \in \operatorname{Null}(R(p))$, we have $R(p)\delta p = 0 \Rightarrow R(p)(p + \delta p) = 0$ and hence $\mathcal{G}(p + \delta p)$ is bearing equivalent to $\mathcal{G}(p)$ according to Theorem 1. As a result, $\mathcal{G}(p + \delta p)$ has the same inter-neighbor bearings as $\mathcal{G}(p)$.

We next give a useful lemma and then prove the necessary and sufficient condition for global bearing rigidity.

Lemma 4. *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ always satisfies $\operatorname{span}\{\mathbf{1}\otimes I_d, p\} \subseteq \operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$ and $dn - d - 1 \geq \operatorname{rank}(R^\kappa(p)) \geq \operatorname{rank}(R(p))$.*

Proof. The results that $\operatorname{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \operatorname{Null}(R^\kappa(p))$ and $dn - d - 1 \geq \operatorname{rank}(R^\kappa(p))$ can be proved similarly as Lemma 3. For any $\delta p \in \operatorname{Null}(R^\kappa(p))$, we have $R^\kappa(p)\delta p = 0 \Rightarrow R^\kappa(p)(p + \delta p) = 0$. As a result, $\mathcal{G}(p + \delta p)$ is bearing congruent to $\mathcal{G}(p)$ by Theorem 1. Since bearing congruency implies bearing equivalency, we know $R(p)(p + \delta p) = 0$ and hence $R(p)\delta p = 0$. Therefore, any δp in $\operatorname{Null}(R^\kappa(p))$ is also in $\operatorname{Null}(R(p))$ and thus $\operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$. Since $R(p)$ and $R^\kappa(p)$ have the same column number, it follows immediately that $\operatorname{rank}(R^\kappa(p)) \geq \operatorname{rank}(R(p))$. □

Remark 2. *The intuition behind Lemma 4 is that any motion δp that preserves the bearings between (p_i, p_j) for all $i, j \in \mathcal{V}$ also preserves the bearings between (p_i, p_j) for all $(i, j) \in \mathcal{E}$.*

Theorem 2 (Condition for Global Bearing Rigidity)**.** *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ is globally bearing rigid if and only if $\operatorname{Null}(R^\kappa(p)) = \operatorname{Null}(R(p))$ or equivalently $\operatorname{rank}(R^\kappa(p)) = \operatorname{rank}(R(p))$.*

Proof. (Necessity) Suppose the framework $\mathcal{G}(p)$ is globally bearing rigid. We next show that $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$. For any $\delta p \in \operatorname{Null}(R(p))$, we have $R(p)\delta p = 0 \Rightarrow R(p)(p + \delta p) = 0$. As a result, $\mathcal{G}(p + \delta p)$ is bearing equivalent to $\mathcal{G}(p)$ according to Theorem 1. Since $\mathcal{G}(p)$ is globally bearing rigid, we further know that $\mathcal{G}(p+\delta p)$ is also bearing congruent to $\mathcal{G}(p)$, which means $R^\kappa(p)(p + \delta p) = 0 \Rightarrow R^\kappa(p)\delta p = 0$. Therefore, any δp in $\operatorname{Null}(R(p))$ is in $\operatorname{Null}(R^\kappa(p))$ and thus $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$. Since $\operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$ as shown in Lemma 4, we have $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$.

(Sufficiency) Suppose $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$. Any framework $\mathcal{G}(p')$ that is bearing equivalent to $\mathcal{G}(p)$ satisfies $R(p)p' = 0$. It then follows from $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$ that $R^\kappa(p)p' = 0$, which means $\mathcal{G}(p')$ is also bearing congruent to $\mathcal{G}(p)$. As a result, $\mathcal{G}(p)$ is globally bearing rigid.

Because $R(p)$ and $R^\kappa(p)$ have the same column number, it follows immediately that $\operatorname{Null}(R^\kappa(p)) = \operatorname{Null}(R(p))$ if and only if $\operatorname{rank}(R^\kappa(p)) = \operatorname{rank}(R(p))$. □

The following result shows that bearing rigidity and global bearing rigidity are actually equivalent notions.

Theorem 3. *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ is globally bearing rigid if and only if it is bearing rigid.*

Proof. By definition, global bearing rigidity implies bearing rigidity. We next prove the converse is also true. Suppose the framework $\mathcal{G}(p)$ is bearing rigid. By the definition of bearing rigidity and Theorem 1, any framework satisfying $R(p)p' = 0$ and $\|p' - p\| \leq \epsilon$ also satisfies $R^\kappa(p)p' = 0$. We then have

$$R(p)(p + \delta p) = 0 \Rightarrow R^\kappa(p)(p + \delta p) = 0, \quad \forall \delta p, \|\delta p\| \leq \epsilon,$$

where $\delta p = p' - p$. Then, it follows from $R(p)p = 0$ and $R^\kappa(p)p = 0$ that

$$R(p)\delta p = 0 \Rightarrow R^\kappa(p)\delta p = 0, \quad \forall \delta p, \|\delta p\| \leq \epsilon,$$

which means $\operatorname{Null}(R(p)) \subseteq \operatorname{Null}(R^\kappa(p))$. Since $\operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$ as shown in Lemma 4, we further have $\operatorname{Null}(R(p)) = \operatorname{Null}(R^\kappa(p))$ and consequently $\mathcal{G}(p)$ is global bearing rigid. □

We next give the necessary and sufficient condition for infinitesimal bearing rigidity.

Theorem 4 (Condition for Infinitesimal Bearing Rigidity)**.** *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ is infinitesimally bearing rigid if and only if $\operatorname{rank}(R(p)) = dn - d - 1$ or equivalently*

$$\operatorname{Null}(R(p)) = \operatorname{span}\{\mathbf{1} \otimes I_d, p\} = \operatorname{span}\{\mathbf{1} \otimes I_d, p - \mathbf{1} \otimes \bar{p}\},$$

where $\bar{p} = (\mathbf{1} \otimes I_d)^{\mathrm{T}} p/n$ is the centroid of $\{p_i\}_{i\in\mathcal{V}}$.

Proof. Lemma 3 shows $\operatorname{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \operatorname{Null}(R(p))$. Observe $\mathbf{1}\otimes I_d$ and p correspond to a rigid-body translation and a scaling of the framework, respectively. The stated condition directly follows from the definition of infinitesimal bearing rigidity. Note also that $\{\mathbf{1} \otimes I_d, p - \mathbf{1} \otimes \bar{p}\}$ is an orthogonal basis for $\operatorname{span}\{\mathbf{1} \otimes I_d, p\}$. □

The special cases of $\mathbb{R}^2$ and $\mathbb{R}^3$ are of the most interests to us. A framework $\mathcal{G}(p)$ is infinitesimally bearing rigid in $\mathbb{R}^2$ if and only if $\operatorname{rank}(R(p)) = 2n - 3$, and in $\mathbb{R}^3$ if and only if $\operatorname{rank}(R(p)) = 3n-4$. In addition, Theorem 4 does not require $n \geq d$.

The following result characterizes the relationship between infinitesimal bearing rigidity and global bearing rigidity.

Theorem 5. *Infinitesimal bearing rigidity implies global bearing rigidity.*

Proof. Infinitesimal bearing rigidity implies $\operatorname{Null}(R(p)) = \operatorname{span}\{\mathbf{1} \otimes I_d, p\}$. Since $\operatorname{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \operatorname{Null}(R^\kappa(p)) \subseteq \operatorname{Null}(R(p))$ as shown in Lemma 4, it immediately follows from $\mathrm{Null}(R(p)) = \mathrm{span}\{\mathbf{1} \otimes I_d, p\}$ that $\mathrm{Null}(R^\kappa(p)) = \mathrm{Null}(R(p))$, which means $\mathcal{G}(p)$ is globally bearing rigid according to Theorem 2. □

We have at this point discussed three notions of bearing rigidity: (i) bearing rigidity, (ii) global bearing rigidity, and (iii) infinitesimal bearing rigidity. According to Theorem 3 and Theorem 5, the relationship between the three kinds of bearing rigidity can be summarized as below:

$$\begin{array}{c} \text{bearing rigidity} \\ \Updownarrow \\ \text{global bearing rigidity} \end{array} \quad \Leftarrow \text{infinitesimal bearing rigidity}$$

It is notable that (global) bearing rigidity does not imply infinitesimal bearing rigidity. For example, the collinear framework as shown in Figure 3(a) is globally bearing rigid but not infinitesimally bearing rigid.

We now further explore some features of infinitesimal bearing rigidity. The following theorem shows that infinitesimal bearing rigidity can uniquely determine the shape of a framework.

Theorem 6 (Unique Shape)**.** *An infinitesimally bearing rigid framework can be globally and uniquely determined up to a translation and a scaling factor.*

Proof. Suppose $\mathcal{G}(p)$ is an infinitesimally bearing rigid framework in $\mathbb{R}^d$. Consider an arbitrary framework $\mathcal{G}(p')$ that is bearing equivalent to $\mathcal{G}(p)$. Our aim is to prove $\mathcal{G}(p')$ is different from $\mathcal{G}(p)$ only in a translation and a scaling factor. The configuration p' can always be decomposed as

$$p' = cp + \mathbf{1} \otimes \eta + q, \tag{7}$$

where $c \in \mathbb{R} \setminus \{0\}$ stands for a scaling factor, $\eta \in \mathbb{R}^d$ denotes a rigid-body translation of the framework, and $q \in \mathbb{R}^{dn}$, which satisfies $q \perp \mathrm{span}\{\mathbf{1} \otimes I_d, p\}$, represents a transformation other than translation and scaling. We only need to prove $q = 0$. Since infinitesimal bearing rigidity implies that $\mathrm{Null}(R(p)) = \mathrm{span}\{\mathbf{1} \otimes I_d, p\}$, multiplying $R(p)$ on both sides of (7) yields

$$R(p)p' = R(p)q. \tag{8}$$

Since $\mathcal{G}(p')$ is bearing equivalent to $\mathcal{G}(p)$, we have $R(p)p' = 0$ by Theorem 1. Therefore, (8) implies

$$R(p)q = 0.$$

Since $q \perp \mathrm{span}\{\mathbf{1} \otimes I_d, p\} = \mathrm{Null}(R(p))$, the above equation suggests $q = 0$. As a result, p' is different from p only in a scaling factor c and a rigid-body translation η. □

The following theorem shows that if a framework is infinitesimally bearing rigid in a lower dimension, it is still infinitesimally bearing rigid when evaluated in a higher dimensional space.

Theorem 7 (Invariance to Dimension)**.** *Infinitesimal bearing rigidity is invariant to space dimensions.*

Proof. Consider a framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ ($n \geq 2$, $d \geq 2$). Suppose the framework becomes $\mathcal{G}(\tilde{p})$ when the dimension is lifted from d to $\tilde{d}$ ($\tilde{d} > d$). Our goal is to prove that

$$\mathrm{rank}(R(p)) = dn - d - 1 \Leftrightarrow \mathrm{rank}(R(\tilde{p})) = \tilde{d}n - \tilde{d} - 1,$$

and consequently $\mathcal{G}(\tilde{p})$ is infinitesimally bearing rigid in $\mathbb{R}^{\tilde{d}}$ if and only if $\mathcal{G}(p)$ is infinitesimally bearing rigid in $\mathbb{R}^d$.

First, consider an oriented graph and write the bearings of $\mathcal{G}(p)$ and $\mathcal{G}(\tilde{p})$ as $\{g_k\}_{k=1}^m$ and $\{\tilde{g}_k\}_{k=1}^m$, respectively. Since $\tilde{p}_i$ is obtained from p_i by lifting the dimension, without loss of generality, assume $\tilde{p}_i = [p_i^{\mathrm{T}}, 0]^{\mathrm{T}}$ ($\forall i \in \mathcal{V}$) where the zero vector is $(\tilde{d} - d)$-dimensional. Then,

$$\tilde{g}_k = \begin{bmatrix} g_k \\ 0 \end{bmatrix}, \quad P_{\tilde{g}_k} = \begin{bmatrix} P_{g_k} & 0 \\ 0 & I_{\tilde{d}-d} \end{bmatrix}, \quad \forall k = \{1, \ldots, m\}.$$

The bearing rigidity matrix of $\mathcal{G}(\tilde{p})$ is $R(\tilde{p}) = \mathrm{diag}\left(I_{\tilde{d}}/\|e_k\|\right) \mathrm{diag}\left(P_{\tilde{g}_k}\right)(H \otimes I_{\tilde{d}})$, where

$$\begin{aligned} &\mathrm{diag}\left(P_{\tilde{g}_k}\right)(H \otimes I_{\tilde{d}}) \\ &= \mathrm{diag}\left(\begin{bmatrix} P_{g_k} & 0 \\ 0 & I_{\tilde{d}-d} \end{bmatrix}\right) H \otimes \begin{bmatrix} I_d & 0 \\ 0 & I_{\tilde{d}-d} \end{bmatrix}. \end{aligned}$$

Permutate the rows of $\mathrm{diag}\left(P_{\tilde{g}_k}\right)(H \otimes I_{\tilde{d}})$ to obtain

$$A = \begin{bmatrix} \mathrm{diag}\left(P_{g_k}\right) H \otimes \begin{bmatrix} I_d & 0 \end{bmatrix} \\ I_{(\tilde{d}-d)m} H \otimes \begin{bmatrix} 0 & I_{\tilde{d}-d} \end{bmatrix} \end{bmatrix} \triangleq \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

Since the permutation of the rows does not change the matrix rank, we have $\mathrm{rank}(R(\tilde{p})) = \mathrm{rank}(A)$. Because the rows of A_1 are orthogonal to the rows of A_2, we have $\mathrm{rank}(A) = \mathrm{rank}(A_1) + \mathrm{rank}(A_2)$. As a result, considering $\mathrm{rank}(A_1) = \mathrm{rank}(\mathrm{diag}\left(P_{g_k}\right) H \otimes I_d) = \mathrm{rank}(R(p))$ and $\mathrm{rank}(A_2) = \mathrm{rank}(H \otimes I_{\tilde{d}-d}) = (\tilde{d} - d)(n-1)$, we have

$$\mathrm{rank}(R(\tilde{p})) = \mathrm{rank}(R(p)) + (\tilde{d} - d)(n-1).$$

It can be easily verified using the above equation that $\mathrm{rank}(R(\tilde{p})) = \tilde{d}n - \tilde{d} - 1$ if and only if $\mathrm{rank}(R(p)) = dn - d - 1$. □

Some examples are given in Figure 3 and Figure 4 to demonstrate infinitesimally bearing rigid frameworks. Figure 3 shows some non-infinitesimal bearing rigid frameworks. We

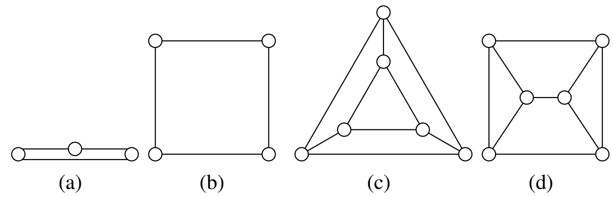


Fig. 3: Examples of *non-infinitesimally bearing rigid* frameworks.

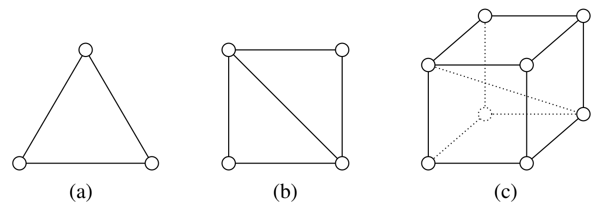


Fig. 4: Examples of *infinitesimally bearing rigid* frameworks.

can intuitively show that the frameworks in Figure 3 are not infinitesimally bearing rigid by identifying the *non-trivial* infinitesimal bearing motions. Specifically, for the collinear framework in Figure 3(a), the middle point can be moved along the line freely without changing any bearings. For the rectangular framework in Figure 3(b), any edge can be moved in the normal direction without changing any bearings. For the framework in Figure 3(c), the inner and outer triangles are concentric. We can change the scale of either the inner or the outer triangle without affecting any bearings. For the framework in Figure 3(d), the three horizontal edges are parallel. We can move either the left or the right triangle in the horizontal direction without changing any bearings. Therefore, all the frameworks in Figure 3 have non-trivial infinitesimal bearing motions and hence they are not infinitesimal bearing rigid. Figure 4 shows some infinitesimally bearing rigid frameworks. It can be verified that each of the frameworks satisfies $\mathrm{rank}(R(p)) = dn - d - 1$.

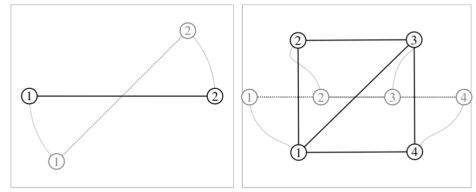
A. Connections to Distance Rigidity Theory

The bearing rigidity theory and the distance rigidity theory study similar problems of whether the shape of a framework can be uniquely determined by the inter-neighbor bearings and inter-neighbor distances, respectively. It is meaningful to study the connections between the two rigidity theories. The following theorem establishes the equivalence between infinitesimal bearing rigidity and infinitesimal distance rigidity in $\mathbb{R}^2$.

Theorem 8. *In $\mathbb{R}^2$, a framework is infinitesimally bearing rigid if and only if it is infinitesimally distance rigid.*

The proof of Theorem 8 requires some preliminaries to distance rigidity theory and is given in Appendix A. We next give two remarks on Theorem 8. Firstly, Theorem 8 cannot be generalized to $\mathbb{R}^3$ or higher dimensions. For example, the coplanar and cubic frameworks as shown in Figure 4(b)-(c) are infinitesimally bearing rigid but not distance rigid in $\mathbb{R}^3$. Secondly, Theorem 8 suggests that we can determine the infinitesimal distance rigidity of a framework by examining its infinitesimal bearing rigidity. For example, it may be tricky to see the frameworks in Figure 3(c)-(d) are not infinitesimally distance rigid while it is straightforward to find the non-trivial infinitesimal bearing motions and hence conclude that the frameworks are not infinitesimally bearing rigid.

To end this section, we briefly compare the proposed bearing rigidity theory with the well-known distance rigidity theory. In the distance rigidity theory, there are three kinds of rigidity: (i) distance rigidity, (ii) global distance rigidity, and (iii) infinitesimal distance rigidity. The relationship between them is (ii)⇒(i) and (iii)⇒(i). Note (ii) and (iii) do not imply each other. The global distance rigidity can uniquely determine the shape of a framework, but it is usually difficult to mathematically examine [26], [27]. Infinitesimal distance rigidity can be conveniently examined by a rank condition (see Lemma 14 in Appendix A), but it is not able to ensure a unique shape. As a comparison, the proposed infinitesimal bearing rigidity not only can be examined by a rank condition (Theorem 4) but also can ensure the unique shape of a framework



(a) (b)

Fig. 5: Target formation: black solid; initial formation: grey dotted. (a) The bearing constraints for the target formation are $g_{12}^* = -g_{21}^* = [1, 0]^{\mathrm{T}}$. (b) The bearing constraints for the target formation are $g_{12}^* = -g_{21}^* = [0, 1]^{\mathrm{T}}$, $g_{23}^* = -g_{32}^* = [1, 0]^{\mathrm{T}}$, $g_{34}^* = -g_{43}^* = [0, -1]^{\mathrm{T}}$, $g_{41}^* = -g_{14}^* = [-1, 0]^{\mathrm{T}}$, and $g_{13}^* = -g_{31}^* = [\sqrt{2}/2, \sqrt{2}/2]^{\mathrm{T}}$.

(Theorem 6). In addition, the rank condition for infinitesimal distance rigidity requires to distinguish the cases of $n \geq d$ and $n < d$ (Lemma 14), while the rank condition for infinitesimal bearing rigidity does not. Finally, an infinitesimally distance rigid framework in a lower dimension may become non-rigid in a higher dimension (see, for example, Figure 4(b)), while infinitesimal bearing rigidity is invariant to dimensions. In summary, the bearing rigidity theory possesses a number of attractive features compared to the distance rigidity theory, and as we will show in the sequel, it is a powerful tool for analyzing problems of distributed control and estimation in multi-agent systems.

III. Bearing-only Formation Control with a Global Reference Frame

In this section, we study bearing-only formation control of multi-agent systems in arbitrary dimensions in the presence of a global reference frame. Consider n agents in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$). Note $n \geq d$ is not required. Assume there is a global reference frame known to each agent. All the vector quantities given in this section are expressed in this global frame. Denote $p_i \in \mathbb{R}^d$ as the position of agent $i \in \{1, \dots, n\}$. The dynamics of agent i is

$$\dot{p}_i(t) = v_i(t),$$

where $v_i(t) \in \mathbb{R}^d$ is the velocity input to be designed. Denote $p = [p_1^{\mathrm{T}}, \dots, p_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$ and $v = [v_1^{\mathrm{T}}, \dots, v_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$. The underlying sensing graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is assumed to be undirected and fixed, and the formation is denoted by $\mathcal{G}(p)$. The edge vector e_{ij} and the bearing g_{ij} are defined as in (2). Considering an arbitrary oriented graph, we can reexpress the edge and bearing vectors as $e = [e_1^{\mathrm{T}}, \dots, e_m^{\mathrm{T}}]^{\mathrm{T}}$ and $g = [g_1^{\mathrm{T}}, \dots, g_m^{\mathrm{T}}]^{\mathrm{T}}$ as defined in (4).

If $(i, j) \in \mathcal{E}$, agent i can measure the relative bearing g_{ij} of agent j. As a result, the *bearing measurements* obtained by agent i at time t are $\{g_{ij}(t)\}_{j \in \mathcal{N}_i}$. The constant *bearing constraints* for the target formation are specified as $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ with $g_{ij}^* = -g_{ji}^*$. Figure 5 gives two examples to illustrate the bearing constraints.

Definition 6 (Feasible Bearing Constraints)**.** *The bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ are* feasible *if there exists a formation $\mathcal{G}(p)$ that satisfies $g_{ij} = g_{ij}^*$ for all $(i, j) \in \mathcal{E}$.*

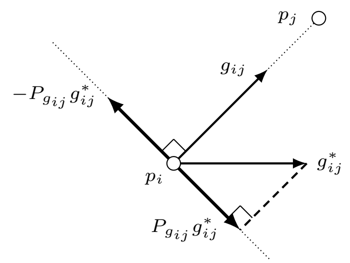


Fig. 6: The geometric interpretation of control law (9). The control term $-P_{g_{ij}} g^*_{ij}$ is perpendicular to the bearing g_{ij}.

The bearing-only formation control problem to be solved in this section is formally stated as below.

Problem 1. *Given feasible constant bearing constraints $\{g^*_{ij}\}_{(i,j)\in\mathcal{E}}$ and the initial formation $\mathcal{G}(p(0))$, design $v_i(t)$ for agent $i \in \mathcal{V}$ based only on the bearing measurements $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ such that $g_{ij}(t) \to g^*_{ij}$ as $t \to \infty$ for all $(i,j) \in \mathcal{E}$.*

A. A Bearing-Only Control Law

The proposed nonlinear bearing-only formation control law is

$$v_i(t) = -\sum_{j\in\mathcal{N}_i} P_{g_{ij}(t)} g^*_{ij}, \quad \forall i \in \mathcal{V}. \tag{9}$$

The control law is distributed because the control of agent i merely requires the bearing measurements $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ from its neighbors. The control law also has a clear geometric interpretation illustrated in Figure 6. The control term $-P_{g_{ij}} g^*_{ij}$ is perpendicular to g_{ij} since $g_{ij}^{\mathrm{T}} P_{g_{ij}} g^*_{ij} = 0$. As a result, the control law attempts to reduce the bearing error of g_{ij} while preserving the distance between agents i and j. This geometric interpretation can be well demonstrated by the example shown in Figure 5(a), where the bearing error is reduced to zero while the inter-agent distance is preserved. In addition, similar "projective" control laws have been used before in [28], [29] for circular formation coordination control.

In order to analyze the proposed control law, we next rewrite it in a matrix-vector form. Since $g^*_{ij} = -g^*_{ji}$, the bearing constraints $\{g^*_{ij}\}_{(i,j)\in\mathcal{E}}$ can be reexpressed as $\{g^*_k\}_{k=1}^m$ by considering an oriented graph. Let $g^* = [(g^*_1)^{\mathrm{T}}, \ldots, (g^*_m)^{\mathrm{T}}]^{\mathrm{T}}$, then (9) can be written as

$$v = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \triangleq \tilde{R}^{\mathrm{T}}(p) g^*. \tag{10}$$

It should be noted that the oriented graph is merely used to obtain the matrix expression while the underlying sensing graph of the formation is still the undirected graph $\mathcal{G}$. Moreover, it is worth mentioning that control law (10) is a modified gradient control law. If we consider the bearing error $\sum_{k=1}^m \|g_k - g^*_k\|^2$, a short calculation shows the corresponding gradient control law is $u = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}/\|e_k\|) g^*$, which is exactly $u = R^{\mathrm{T}}(p) g^*$, where $R(p)$ is the bearing rigidity matrix. This gradient control law, however, requires the distance measurement $\|e_k\|$. By removing the distance term $\|e_k\|$, we can obtain the proposed control law (10).

We next examine some useful properties of the control law. First of all, we show that both the centroid and scale of the formation are invariant quantities under the action of the control law. In this direction, define

$$\bar{p} \triangleq \frac{1}{n}\sum_{i=1}^n p_i,$$

to be the *centroid* of the formation, and

$$s \triangleq \sqrt{\frac{1}{n}\sum_{i=1}^n \|p_i - \bar{p}\|^2},$$

as the quadratic mean of the distances from the agents to the centroid. The quantity s can be interpreted as the *scale* of the formation.

Lemma 5. *Under control law* (10)*,*

$$\dot{p}(t) \perp \mathrm{span}\{\mathbf{1} \otimes I_d, p(t)\}.$$

Proof. The dynamics $\dot{p} = \tilde{R}^{\mathrm{T}}(p) g^*$ implies $\dot{p} \in \mathrm{Range}(\tilde{R}^{\mathrm{T}}(p))$. Since $\mathrm{Range}(\tilde{R}^{\mathrm{T}}(p)) \perp \mathrm{Null}(\tilde{R}(p))$, we have $\dot{p} \perp \mathrm{Null}(\tilde{R}(p))$. Furthermore, $\mathrm{Null}(\tilde{R}(p)) = \mathrm{Null}(R(p))$ and $\mathrm{span}\{\mathbf{1} \otimes I_d, p\} \subseteq \mathrm{Null}(R(p))$ by Lemma 3 conclude the proof. □

Theorem 9 (Centroid and Scale Invariance)**.** *The centroid $\bar{p}$ and the scale s are invariant under the control law* (10).

Proof. Since $\bar{p} = (\mathbf{1} \otimes I_d)^{\mathrm{T}} p/n$, we have $\dot{\bar{p}} = (\mathbf{1} \otimes I_d)^{\mathrm{T}} \dot{p}/n$. It follows from $\dot{p} \perp \mathrm{Range}(\mathbf{1} \otimes I_d)$ as shown in Lemma 5 that $\dot{\bar{p}} \equiv 0$. Rewrite s as $s = \|p - \mathbf{1} \otimes \bar{p}\|/\sqrt{n}$. Then,

$$\dot{s} = \frac{1}{\sqrt{n}} \frac{(p - \mathbf{1} \otimes \bar{p})^{\mathrm{T}}}{\|p - \mathbf{1} \otimes \bar{p}\|} \dot{p}.$$

It follows from $\dot{p} \perp p$ and $\dot{p} \perp \mathbf{1} \otimes \bar{p}$ as shown in Lemma 5 that $\dot{s} \equiv 0$. □

Theorem 9 can be well demonstrated by the simple simulation example as shown in Figure 5(a). As can be seen, the middle point (i.e., the centroid) and the distance (i.e., the scale) of the two agents are invariant during the formation evolution. The invariance of centroid and scale has also been observed by [13] for bearing-only formation control in two-dimensional cases.

The following results, which can be obtained from Theorem 9, characterize the behavior of the formation trajectories. In particular, we give bounds for the quantities $\max_{i\in\mathcal{V}} \|p_i(t) - \bar{p}\|$ and $\|p_i(t) - p_j(t)\|, \forall i,j \in \mathcal{V}$.

Corollary 1. *The formation trajectory under the control law* (10) *satisfies the following inequalities,*

(a) $s \le \max_{i\in\mathcal{V}} \|p_i(t) - \bar{p}\| \le s\sqrt{n-1}, \quad \forall t \ge 0.$
(b) $\|p_i(t) - p_j(t)\| \le 2s\sqrt{n-1}, \quad \forall i,j \in \mathcal{V}, \ \forall t \ge 0.$

Proof. (a) We first prove $\|p_i - \bar{p}\| \le s\sqrt{n-1}$ for all $i \in \mathcal{V}$. On one hand, the fact that $\sum_{j\in\mathcal{V}}(p_j - \bar{p}) = (p_i - \bar{p}) +$

$\sum_{j\in\mathcal{V},j\neq i}(p_j-\bar{p})=0$ implies

$$\|p_i-\bar{p}\|^2\leq\left(\sum_{\substack{j\in\mathcal{V}\\ j\neq i}}\|p_j-\bar{p}\|\right)^2\leq(n-1)\sum_{\substack{j\in\mathcal{V},\\ j\neq i}}\|p_j-\bar{p}\|^2. \quad (11)$$

On the other hand, scale invariance implies that $\|p_i-\bar{p}\|^2+\sum_{j\in\mathcal{V},j\neq i}\|p_j-\bar{p}\|^2=ns^2$. Substituting this expression into (11) gives $\|p_i-\bar{p}\|^2\leq(n-1)(ns^2-\|p_i-\bar{p}\|^2)$, which implies $\|p_i-\bar{p}\|\leq s\sqrt{n-1}$.

We secondly prove $s\leq\max_{i\in\mathcal{V}}\|p_i-\bar{p}\|$. Since $\max_{i\in\mathcal{V}}\|p_i-\bar{p}\|^2\geq\|p_j-\bar{p}\|^2$, we have $n(\max_{i\in\mathcal{V}}\|p_i-\bar{p}\|^2)\geq\sum_{i=1}^n\|p_i-\bar{p}\|^2=ns^2$, which implies $\max_{i\in\mathcal{V}}\|p_i-\bar{p}\|\geq s$.

(b) The inequality in (b) is obtained from $\|p_i(t)-p_j(t)\|=\|(p_i(t)-\bar{p})-(p_j(t)-\bar{p})\|\leq\|p_i(t)-\bar{p}\|+\|p_j(t)-\bar{p}\|\leq 2s\sqrt{n-1}$. □

B. Formation Stability Analysis

In order to prove the formation stability, we adopt the following rigidity assumption.

Assumption 1. *Any formation that satisfies the bearing constraints $\{g^*_{ij}\}_{(i,j)\in\mathcal{E}}$ is infinitesimally bearing rigid.*

Assumption 1 gives two conditions that will be useful for the formation stability analysis. The first condition is that the shape of any formation that satisfies the bearing constraints is unique according to Theorem 6. The second condition is a mathematical condition. More specifically, suppose $\mathcal{G}(p)$ is a formation that satisfies the bearing constraints, then Assumption 1 indicates that the bearing rigidity matrix $R(p)$ satisfies $\mathrm{rank}(R(p))=dn-d-1$ and $\mathrm{Null}(R(p))=\mathrm{span}\{\mathbf{1}\otimes I_d,p\}$ according to Theorem 4.

The basic idea of the formation stability proof is to show that the formation converges from an initial formation $\mathcal{G}(p(0))$ to a *target formation* $\mathcal{G}(p^*)$ as defined below.

Definition 7 (Target Formation)**.** *Let $\mathcal{G}(p^*)$ be a* target formation *satisfying*

(a) Centroid: $\bar{p}^*=\bar{p}(0)$.
(b) Scale: $s^*=s(0)$.
(c) Bearing: $(p^*_j-p^*_i)/\|p^*_j-p^*_i\|=g^*_{ij}$ for all $(i,j)\in\mathcal{E}$.

Lemma 6 (Existence and Uniqueness)**.** *The target formation $\mathcal{G}(p^*)$ in Definition 7 always exists and is unique under Assumption 1.*

Proof. Since the bearing constraints are feasible, there exist formations that satisfy the bearings. Due to the infinitesimal bearing rigidity in Assumption 1, these formations including $\mathcal{G}(p^*)$ can be uniquely determined up to translations and scaling factors. Since $\mathcal{G}(p^*)$ additionally has the centroid and the scale as $\bar{p}(0)$ and $s(0)$, the translation and the scale of $\mathcal{G}(p^*)$ can be uniquely determined. Therefore, the target formation $\mathcal{G}(p^*)$ exists and is unique. □

Remark 3. *In fact, we are able to calculate the unique value of p^*. Since $\mathcal{G}(p^*)$ is infinitesimally bearing rigid, the bearing rigid matrix $R(p^*)=\mathrm{diag}(P_{g_k}/\|e_k\|)\bar{H}$ satisfies $\mathrm{Null}(R(p^*))=\mathrm{span}\{\mathbf{1}\otimes I_d,p^*\}$. From the bearing constraints, construct $\tilde{R}\triangleq\mathrm{diag}(P_{g^*_k})\bar{H}$, which has the same null space as $R(p^*)$. We can calculate an orthogonal basis of $\mathrm{Null}(\tilde{R})$ as $\mathrm{span}\{\mathbf{1}\otimes I_d,q\}$ where $q\perp\mathrm{Range}(\mathbf{1}\otimes I_d)$. Since $p^*\in\mathrm{Null}(\tilde{R})$, we can express p^* as a linear combination of $\mathbf{1}\otimes I_d$ and q:*

$$p^*=\mathbf{1}\otimes x+\alpha q,$$

where $x\in\mathbb{R}^d$ and $\alpha\in\mathbb{R}$ are the coefficients to be calculated. Since $\bar{p}^*=(\mathbf{1}\otimes I_d)^{\mathrm{T}}p^*/n=\bar{p}(0)$ and $s^*=\|p^*-\mathbf{1}\otimes\bar{p}^*\|/\sqrt{n}=s(0)$, a short calculation shows that $x=\bar{p}(0)$ and $\alpha=\pm s(0)\sqrt{n}/\|q\|$. The correct sign of α can be determined by comparing the signs of q_j-q_i and g^*_{ij}. The calculation of p^* actually is a bearing-only network localization problem (see [30] and the reference therein). It is noted that the specific value of p^* is not required for the formation stability proof.

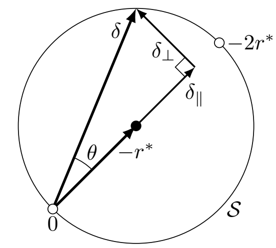


Fig. 7: Geometric interpretation of δ which satisfies $\|\delta+r^*\|=\|r^*\|$.

The target formation $\mathcal{G}(p^*)$ has the same centroid and scale as the initial formation. More importantly, the target formation satisfies all the bearing constraints. Our stability proof is to show that the formation converges to the target formation and consequently the bearing errors converge to zero. This idea was originally proposed by [13] to solve bearing-only formation control in two dimensions. In this direction, let $\delta_i=p_i-p^*_i$ and then $\dot{\delta}_i=f_i(\delta)=\dot{p}_i$. Denote $\delta=[\delta_1^{\mathrm{T}},\dots,\delta_n^{\mathrm{T}}]^{\mathrm{T}}$ and $f(\delta)=[f_1^{\mathrm{T}}(\delta),\dots,f_n^{\mathrm{T}}(\delta)]^{\mathrm{T}}$. With control law (10), the δ-dynamics is expressed as

$$\dot{\delta}(t)=f(\delta)=\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^*. \quad (12)$$

Our aim is to show $\delta(t)$ converges to zero. We next identify the equilibriums of the δ-dynamics. Denote

$$r(t)\triangleq p(t)-(\mathbf{1}\otimes\bar{p}),\quad r^*\triangleq p^*-(\mathbf{1}\otimes\bar{p}^*).$$

Note $r(t)$ is obtained by moving the centroid of $p(t)$ to the origin. Due to the scale invariance, it can be verified that $\|r(t)\|\equiv\|r^*\|=\sqrt{n}s$ for all $t\geq 0$. Moreover, since $\bar{p}=\bar{p}^*$, we have $\delta(t)=r(t)-r^*$.

Lemma 7. *System* (12) *evolves on the surface of the sphere*

$$\mathcal{S}=\{\delta\in\mathbb{R}^{dn}:\|\delta+r^*\|=\|r^*\|\}.$$

Proof. It follows from $\delta(t)=r(t)-r^*$ that $\|\delta(t)+r^*\|=\|r(t)\|=\|r^*\|$, where $\|r(t)\|=\|r^*\|$ is due to the scale invariance. □

The state manifold $\mathcal{S}$ is illustrated by Figure 7. We next introduce a useful lemma and then prove that system (12) has *two isolated equilibriums* on $\mathcal{S}$.

Lemma 8. *Any two unit vectors $g_1, g_2 \in \mathbb{R}^d$ always satisfy $g_1^{\mathrm{T}} P_{g_2} g_1 = g_2^{\mathrm{T}} P_{g_1} g_2$.*

Proof. Since $g_1^{\mathrm{T}} g_1 = g_2^{\mathrm{T}} g_2 = 1$, we have $g_1^{\mathrm{T}} P_{g_2} g_1 = g_1^{\mathrm{T}}(I_d - g_2 g_2^{\mathrm{T}}) g_1 = g_1^{\mathrm{T}} g_1 - g_1^{\mathrm{T}} g_2 g_2^{\mathrm{T}} g_1 = g_2^{\mathrm{T}} g_2 - g_2^{\mathrm{T}} g_1 g_1^{\mathrm{T}} g_2 = g_2^{\mathrm{T}}(I_d - g_1 g_1^{\mathrm{T}}) g_2 = g_2^{\mathrm{T}} P_{g_1} g_2$. □

Theorem 10 (Equilibrium)**.** *Under Assumption 1, system* (12) *has two isolated equilibriums on $\mathcal{S}$,*

(a) $\delta = 0$,

(b) $\delta = -2r^*$.

Proof. Any equilibrium $\delta \in \mathcal{S}$ must satisfy $f(\delta) = \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* = 0$, which implies

$$\begin{aligned} 0 &= (p^*)^{\mathrm{T}} \bar{H}^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* = (e^*)^{\mathrm{T}} \mathrm{diag}(P_{g_k}) g^* \\ &= \sum_{k=1}^{m} (e_k^*)^{\mathrm{T}} P_{g_k} g_k^* = \sum_{k=1}^{m} \|e_k^*\| (g_k^*)^{\mathrm{T}} P_{g_k} g_k^*. \end{aligned}$$

Since $(g_k^*)^{\mathrm{T}} P_{g_k} g_k^* \geq 0$, the above equation implies $(g_k^*)^{\mathrm{T}} P_{g_k} g_k^* = 0$ for all k. As a result, by Lemma 8, we have $g_k^{\mathrm{T}} P_{g_k^*} g_k = 0 \Rightarrow e_k^{\mathrm{T}} P_{g_k^*} e_k = 0$ for all k and thus

$$0 = e^{\mathrm{T}} \mathrm{diag}\left(P_{g_k^*}\right) e = p^{\mathrm{T}} \underbrace{\bar{H}^{\mathrm{T}} \mathrm{diag}\left(P_{g_k^*}\right)}_{\tilde{R}^{\mathrm{T}}(p^*)} \underbrace{\mathrm{diag}\left(P_{g_k^*}\right) \bar{H}}_{\tilde{R}(p^*)} p,$$

where the last equality is due to the facts that $P_{g_k^*} = P_{g_k^*}^2$ and $e = \bar{H} p$. The above equation indicates

$$\tilde{R}(p^*) p = 0.$$

Observe $\tilde{R}(p^*) = \mathrm{diag}(P_{g_k^*}) \bar{H}$ has the same null space as the bearing rigidity matrix $R(p^*) = \mathrm{diag}(P_{g_k^*}/\|e_k^*\|) \bar{H}$. Since $\mathcal{G}(p^*)$ is infinitesimally bearing rigid by Assumption 1, it follows from Theorem 4 that $\mathrm{Null}(\tilde{R}(p^*)) = \mathrm{span}\{\mathbf{1} \otimes I_d, p^* - \mathbf{1} \otimes \bar{p}^*\}$. Considering $\tilde{R}(p^*) p = 0 \Leftrightarrow \tilde{R}(p^*)(p - \mathbf{1} \otimes \bar{p}) = 0$, we have

$$p - \mathbf{1} \otimes \bar{p} \in \mathrm{span}\{\mathbf{1} \otimes I_d, p^* - \mathbf{1} \otimes \bar{p}^*\}.$$

Because $p - \mathbf{1} \otimes \bar{p} \perp \mathrm{Range}(\mathbf{1} \otimes I_d)$, we further know $p - \mathbf{1} \otimes \bar{p} \in \mathrm{span}\{p^* - \mathbf{1} \otimes \bar{p}^*\}$. Moreover, since $\|p - \mathbf{1} \otimes \bar{p}\| = \|p^* - \mathbf{1} \otimes \bar{p}^*\|$ due to the scale invariance, we have

$$p - \mathbf{1} \otimes \bar{p} = \pm(p^* - \mathbf{1} \otimes \bar{p}^*).$$

(i) In the case of $p - \mathbf{1} \otimes \bar{p} = p^* - \mathbf{1} \otimes \bar{p}^*$, we have $p = p^* \Leftrightarrow \delta = 0$ and consequently $g_{ij} = g_{ij}^*$ for all $(i,j) \in \mathcal{E}$. (ii) In the case of $p - \mathbf{1} \otimes \bar{p} = -(p^* - \mathbf{1} \otimes \bar{p}^*)$, we have $p = -p^* + 2(\mathbf{1} \otimes \bar{p}^*) \Leftrightarrow \delta = -2(p^* - \mathbf{1} \otimes \bar{p}^*)$, and consequently $g_{ij} = -g_{ij}^*$ for all $(i,j) \in \mathcal{E}$. □

The equilibrium $\delta = 0$ is desired, while the other one $\delta = -2r^*$ is undesired. As shown in the proof, the formation at the undesired equilibrium is geometrically a *point reflection* of the target formation about the *centroid* (see Figure 8 for an illustration). As a result, the two formations at the two equilibriums have the same centroid, scale, and shape, but they have the opposite bearings.

Although we will present a nonlinear stability analysis of the two equilibriums later, it is still meaningful to examine the Jacobian matrices at the two equilibriums. Based on the

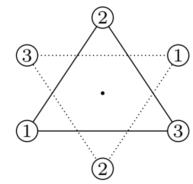


Fig. 8: An illustration of the two equilibriums. Solid line: target formation where $\delta = 0$ and $g_{ij} = g_{ij}^*, \forall (i,j) \in \mathcal{E}$. Dashed line: point reflection of the target formation about the centroid where $\delta = -2r^*$ and $g_{ij} = -g_{ij}^*, \forall (i,j) \in \mathcal{E}$.

Jacobian matrices, we are able to conclude by Lyapunov's indirect method that the undesired equilibrium $\delta = -2r^*$ is unstable.

Proposition 1. *Let*

$$A = \frac{\partial f(\delta)}{\partial \delta}$$

be the Jacobian of $f(\delta)$. At the desired equilibrium $\delta = 0$, the Jacobian matrix $A|_{\delta=0}$ is symmetric negative semi-definite. At the undesired equilibrium $\delta = -2r^*$, the Jacobian matrix $A|_{\delta=-2r^*}$ is symmetric positive semi-definite and at least one eigenvalue is positive. As a result, the undesired equilibrium $\delta = -2r^*$ is unstable.

Proof. Recall $f_i(\delta) = -\sum_{j \in \mathcal{N}_i} P_{g_{ij}} g_{ij}^*, \forall i \in \mathcal{V}$. For any $j \notin \mathcal{N}_i$, we have $A_{ij} = \partial f_i / \partial \delta_j = 0$. For any $j \in \mathcal{N}_i$, we have

$$\begin{aligned} A_{ij} &= \frac{\partial f_i}{\partial \delta_j} = -\frac{\partial P_{g_{ij}}}{\partial \delta_j} g_{ij}^* = \left(\frac{\partial g_{ij}}{\partial \delta_j} g_{ij}^{\mathrm{T}} + g_{ij} \left(\frac{\partial g_{ij}}{\partial \delta_j} \right)^{\mathrm{T}} \right) g_{ij}^* \\ &= \underbrace{\left(g_{ij}^{\mathrm{T}} g_{ij}^* I_d + g_{ij} g_{ij}^{*\mathrm{T}} \right)}_{G_{ij}} \frac{\partial g_{ij}}{\partial \delta_j} = G_{ij} \frac{P_{g_{ij}}}{\|e_{ij}\|}. \end{aligned}$$

For any $i \in \mathcal{V}$, we have

$$A_{ii} = -\sum_{j \in \mathcal{N}_i} \frac{\partial P_{g_{ij}}}{\partial \delta_i} g_{ij}^* = \sum_{j \in \mathcal{N}_i} G_{ij} \frac{\partial g_{ij}}{\partial \delta_i} = -\sum_{j \in \mathcal{N}_i} G_{ij} \frac{P_{g_{ij}}}{\|e_{ij}\|}.$$

Observe $A_{ii} = -\sum_{j \in \mathcal{N}_i} A_{ij}$ and $A_{ij} = A_{ji}$. Therefore, A has a similar structure as graph Laplacian [25].

At the undesired equilibrium $\delta = -2r^*$ where $g_{ij} = -g_{ij}^*$ for all $(i,j) \in \mathcal{E}$, we have

$$A_{ij}|_{\delta=-2r^*} = -\left(I_d + g_{ij}^* g_{ij}^{*\mathrm{T}} \right) \frac{P_{g_{ij}^*}}{\|e_{ij}\|} = -\frac{P_{g_{ij}^*}}{\|e_{ij}^*\|} \leq 0$$

for all $j \in \mathcal{N}_i$. Similarly, we obtain

$$A_{ii}|_{\delta=-2r^*} = \sum_{j \in \mathcal{N}_i} \frac{P_{g_{ij}^*}}{\|e_{ij}^*\|} \geq 0, \quad \forall i \in \mathcal{V}.$$

Note $A|_{\delta=-2r^*}$ is positive semi-definite definite. To see that, consider any vector $y = [y_1^{\mathrm{T}}, \ldots, y_n^{\mathrm{T}}]^{\mathrm{T}}$ where $y_i \in \mathbb{R}^d$. Then, $y^{\mathrm{T}}(A|_{\delta=-2r^*}) y = \sum_{(i,j) \in \mathcal{E}} (y_i - y_j)^{\mathrm{T}} P_{g_{ij}^*} (y_i - y_j) / \|e_{ij}^*\| \geq 0$. Thus, $A|_{\delta=-2r^*}$ has at least one positive eigenvalue and consequently the undesired equilibrium $\delta = -2r^*$ is unstable by Lyapunov's indirect method. Similarly, it can be shown that $A|_{\delta=0} = -A|_{\delta=-2r^*} \leq 0$. But the stability of the desired equilibrium $\delta = 0$ cannot be straightforwardly determined based on $A|_{\delta=0}$. $\square$

We next present a nonlinear stability analysis of the two equilibriums of system (12). Choose the Lyapunov function as

$$V = \frac{1}{2}\|\delta\|^2.$$

The next is the main stability result.

Theorem 11 (Almost Global Exponential Stability)**.** *Under Assumption 1, the system trajectory $\delta(t)$ of* (12) *exponentially converges to $\delta = 0$ from any $\delta(0) \in \mathcal{S}$ except $\delta(0) = -2r^*$.*

Remark 4. *In terms of bearings, Theorem 11 indicates that $g_{ij}(t)$ converges to g_{ij}^* for all $(i,j) \in \mathcal{E}$ from any initial conditions except $g_{ij}(0) = -g_{ij}^*, \forall (i,j) \in \mathcal{E}$.*

Proof. The derivative of V is $\dot{V} = \delta^{\mathrm{T}}\dot{\delta} = (p-p^*)^{\mathrm{T}}\dot{p} = -(p^*)^{\mathrm{T}}\dot{p}$. Substituting control law (10) into $\dot{V}$ yields

$$\begin{aligned}\dot{V} &= -(p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^* = -(e^*)^{\mathrm{T}}\mathrm{diag}(P_{g_k})g^* \\ &= -\sum_{k=1}^{m}(e_k^*)^{\mathrm{T}}P_{g_k}g_k^* = -\sum_{k=1}^{m}\|e_k^*\|(g_k^*)^{\mathrm{T}}P_{g_k}g_k^* \leq 0. \quad (13)\end{aligned}$$

Since $\dot{V} \leq 0$, we have $\|\delta(t)\| \leq \|\delta(0)\|$ for all $t \geq 0$. Furthermore, it follows from Lemma 8 that

$$(g_k^*)^{\mathrm{T}}P_{g_k}g_k^* = g_k^{\mathrm{T}}P_{g_k^*}g_k,$$

substituting which into (13) gives

$$\begin{aligned}\dot{V} &= -\sum_{k=1}^{m}\|e_k^*\|g_k^{\mathrm{T}}P_{g_k^*}g_k = -\sum_{k=1}^{m}\frac{\|e_k^*\|}{\|e_k\|^2}e_k^{\mathrm{T}}P_{g_k^*}e_k \\ &\leq -\underbrace{\frac{\min_{k=1,\dots,m}\|e_k^*\|}{4(n-1)s^2}}_{\alpha}\sum_{k=1}^{m}e_k^{\mathrm{T}}P_{g_k^*}e_k, \quad (14)\end{aligned}$$

where the inequality uses the fact that $\|e_k\| \leq 2\sqrt{n-1}s$ as given in Corollary 1(b). Inequality (14) can be further written as

$$\begin{aligned}\dot{V} &\leq -\alpha e^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})e = -\alpha p^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}p \\ &= -\alpha\delta^{\mathrm{T}}\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})\bar{H}\delta \quad \big(\text{due to } \mathrm{diag}(P_{g_k^*})\bar{H}p^* = 0\big) \\ &= -\alpha\delta^{\mathrm{T}}\underbrace{\bar{H}^{\mathrm{T}}\mathrm{diag}(P_{g_k^*})}_{\tilde{R}^{\mathrm{T}}(p^*)}\underbrace{\mathrm{diag}(P_{g_k^*})\bar{H}}_{\tilde{R}(p^*)}\delta. \quad (15)\end{aligned}$$

Observe $\tilde{R}(p^*)$ has the same rank and null space as the bearing rigidity matrix $R(p^*)$. Under the assumption of infinitesimal bearing rigidity, it follows from Theorem 4 that $\mathrm{Null}(\tilde{R}(p^*)) = \mathrm{span}\{\mathbf{1}\otimes I_d, p^*\}$ and $\mathrm{rank}(\tilde{R}(p^*)) = dn-d-1$. As a result, the smallest $d+1$ eigenvalues of $\tilde{R}^{\mathrm{T}}(p^*)\tilde{R}(p^*)$ are zero. Let the minimum positive eigenvalue of $\tilde{R}^{\mathrm{T}}(p^*)\tilde{R}(p^*)$ be λ_{d+2}. Decompose δ to $\delta = \delta_\perp + \delta_\parallel$, where $\delta_\perp \perp \mathrm{Null}(\tilde{R}(p^*))$ and $\delta_\parallel \in \mathrm{Null}(\tilde{R}(p^*))$. Then (15) implies

$$\dot{V} \leq -\alpha\lambda_{d+2}\|\delta_\perp\|^2. \quad (16)$$

Note $\delta_\parallel$ is the orthogonal projection of δ on $\mathrm{Null}(\tilde{R}(p^*)) = \mathrm{span}\{\mathbf{1}\otimes I_d, r^*\}$. Since $\delta \perp \mathrm{span}\{\mathbf{1}\otimes I_d\}$, we further know that $\delta_\parallel$ is the orthogonal projection of δ on r^* (see Figure 7). Let θ be the angle between δ and $-r^*$. Thus, $\|\delta_\perp\| = \|\delta\|\sin\theta$, and (16) becomes

$$\dot{V} \leq -\alpha\lambda_{d+2}\sin^2\theta\|\delta\|^2. \quad (17)$$

It can be seen from Figure 7 that $\theta \in [0, \pi/2)$. Let θ_0 be the value of θ at time $t = 0$. Since $\|\delta(t)\| \leq \|\delta(0)\|$ for all t, it is clear from Figure 7 that $\theta(t) \geq \theta_0$. Then, (17) becomes

$$\dot{V} \leq -\underbrace{2\alpha\lambda_{d+2}\sin^2\theta_0}_{K}V.$$

(i) If $\theta_0 > 0$, then $K > 0$. As a result, the error $\|\delta(t)\|$ decreases to zero exponentially fast. (ii) If $\theta_0 = 0$, it can be seen from Figure 7 that $\delta(0) = -2r^*$ which is the undesired equilibrium. In summary, the system trajectory $\delta(t)$ converges to $\delta = 0$ exponentially fast from any initial points except $\delta = -2r^*$. $\square$

The behavior of the δ-dynamics is intuitively similar to an inverse pendulum, which has one instable equilibrium at the top and one stable equilibrium at the bottom. Moreover, as shown in the proof, the eigenvalue λ_{d+2} of $\tilde{R}^{\mathrm{T}}(p^*)\tilde{R}(p^*)$ affects the convergence rate of the system. Since $\lambda_{d+2} > 0$ if and only if $\mathcal{G}(p^*)$ is infinitesimally bearing rigid, the eigenvalue λ_{d+2} can be viewed as a measure of the "degree of infinitesimal bearing rigidity". As shown in another work of ours [30], the matrix $\tilde{R}^{\mathrm{T}}(p^*)\tilde{R}(p^*)$ is a matrix-weighted Laplacian (called *bearing Laplacian*), which plays important roles in bearing-only network localization problems.

C. Collision Avoidance

It is worth noting that there is an implicit assumption in the stability analysis in Theorem 11. That is, no two neighbors collide with each other during the formation evolution. If two neighbors collide, the bearing between them will be mathematically invalid. As a result, without this assumption, the stability result in Theorem 11 is merely valid until collision happens. In fact, control law (10) is not able to globally guarantee collision avoidance (see, for example, Figure 9). In practice, the proposed control law may be implemented together with some other mechanisms like artificial potentials to guarantee collision avoidance. In this paper, we merely give a sufficient theoretical condition to show that a minimum distance between any agents (even if they are not neighbors) can be ensured if the initial formation is sufficiently close to the target formation.

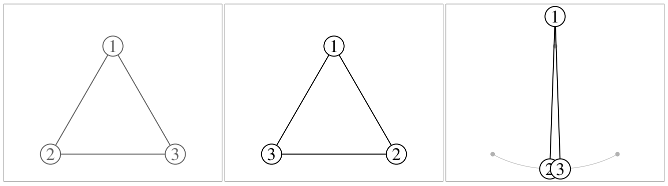
Theorem 12. *Under Assumption 1, given a minimum distance γ satisfying $0 \leq \gamma < \min_{i,j\in\mathcal{V}}\|p_i(0)-p_j(0)\|$, it can be guaranteed that*

$$\|p_i(t)-p_j(t)\| > \gamma, \quad \forall i,j \in \mathcal{V}, \forall t \geq 0 \quad (18)$$

if $\delta(0)$ satisfies

$$\|\delta(0)\| < \frac{1}{2\sqrt{n}}\left(\min_{i,j\in\mathcal{V}}\|p_i(0)-p_j(0)\| - \gamma\right). \quad (19)$$

Proof. For any $i,j \in \mathcal{V}$, since $p_i(t)-p_j(t) \equiv [p_i(t)-p_i(0)]-$



(a) Initial formation (b) Target formation (c) Collision

Fig. 9: Control law (10) is not able to globally guarantee collision avoidance. As can be seen, with the initial formation given in (a) and the target formation given in (b), agents 2 and 3 will collide as shown in (c).

$[p_j(t) - p_j(0)] + [p_i(0) - p_j(0)]$, we have

$$\begin{aligned}
&\|p_i(t) - p_j(t)\| \\
&\geq \|p_i(0) - p_j(0)\| - \|p_i(t) - p_i(0)\| - \|p_j(t) - p_j(0)\| \\
&\geq \|p_i(0) - p_j(0)\| - \sum_{\ell=1}^{n} \|p_\ell(t) - p_\ell(0)\| \\
&\geq \|p_i(0) - p_j(0)\| - \sqrt{n}\|p(t) - p(0)\|, \quad \forall t \geq 0. \qquad (20)
\end{aligned}$$

Since $\delta(t) = p(t) - p^* = [p(t) - p(0)] - [p^* - p(0)]$, we have

$$\begin{aligned}
\|p(t) - p(0)\| &\leq \|\delta(t)\| + \|p^* - p(0)\| \\
&\leq \|\delta(0)\| + \|p(0) - p^*\| = 2\|\delta(0)\|, \quad \forall t \geq 0.
\end{aligned}$$

Substituting the above inequality into (20) yields

$$\|p_i(t) - p_j(t)\| \geq \|p_i(0) - p_j(0)\| - 2\sqrt{n}\|\delta(0)\|, \quad \forall t \geq 0.$$

As a result, if (19) holds, we have (18). □

The upper bound for $\|\delta(0)\|$ given Theorem 12 is inversely proportional to $\sqrt{n}$. This is intuitively reasonable since the chance for two agents colliding is high when the number of the agents is large and consequently the initial error must be small to avoid collision. In addition, the condition given in Theorem 12 is conservative. However, extensive simulations have shown that the proposed controller can avoid collisions even if the above condition is not satisfied.

IV. Bearing-Only Formation Control without a Global Reference Frame

In the previous section, we assumed a global reference frame whose orientation is known to all agents. In this section, we study the case where the global reference frame is unknown to the agents and each agent can only measure the bearings of their neighbors in their local reference frames.

Consider $n \geq 2$ agents in $\mathbb{R}^3$. Denote $p_i \in \mathbb{R}^3$, $v_i \in \mathbb{R}^3$, and $w_i \in \mathbb{R}^3$ as the position, linear velocity, and angular velocity of agent $i \in \mathcal{V}$ expressed in a global reference frame which is unknown to each agent. There is a local reference frame fixed on the body of each agent. We use the superscript b to indicate a vector expressed in the local body frame. A vector quantity without the superscript is expressed in the global frame. In particular, v_i^b and w_i^b represent the linear velocity and angular velocity of agent i expressed in its own body frame. Let $Q_i \in SO(3)$ be the rotation form the body frame of agent i to the global frame. Then, $v_i = Q_i v_i^b$ and $w_i = Q_i w_i^b$. The position and orientation dynamics of agent i is

$$\begin{aligned}
\dot{p}_i &= Q_i v_i^b, \\
\dot{Q}_i &= Q_i \left[w_i^b\right]_\times, \qquad (21)
\end{aligned}$$

where $[\,\cdot\,]_\times$ is the skew-symmetric matrix operator defined in (1), and v_i^b and w_i^b are the inputs to be designed.

Denote, as before, $e_{ij} \triangleq p_j - p_i$ and $g_{ij} \triangleq e_{ij}/\|e_{ij}\|$ for $(i,j) \in \mathcal{E}$. Agent i can measure the bearings of its neighbors in its local frame, $\{g_{ij}^b\}_{j\in\mathcal{N}_i}$, where $g_{ij}^b = Q_i^{\mathrm{T}} g_{ij}$. Moreover, assume agent i can also measure the *relative orientation* of its neighbors, $\{Q_i^{\mathrm{T}} Q_j\}_{j\in\mathcal{N}_i}$. The bearing-only formation control problem to be solved in this section is stated as below.

Problem 2. *Given feasible constant bearing constraints $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ and an initial formation $\mathcal{G}(p(0))$ with agent orientations as $\{Q_i(0)\}_{i\in\mathcal{V}}$, design $v_i^b(t)$ and $w_i^b(t)$ for agent $i \in \mathcal{V}$ based only on the local bearing measurements $\{g_{ij}^b(t)\}_{j\in\mathcal{N}_i}$ and relative orientation measurements $\{Q_i^{\mathrm{T}}(t)Q_j(t)\}_{j\in\mathcal{N}_i}$ such that $\{Q_i(t)\}_{i\in\mathcal{V}}$ converge to a common value and $g_{ij}^b(t) \to g_{ij}^*$ as $t \to \infty$ for all $(i,j) \in \mathcal{E}$.*

It is notable that there is an orientation synchronization problem embedded in Problem 2. This scheme is inspired by the works on formation control based on orientation alignment [21], [22]. Once the orientations of the agents have synchronized, the synchronized local frames can be viewed as a common frame where the bearing constraints should be satisfied. It is worth mentioning that the value of the finally synchronized orientation is not of our interest, and we only care about the shape of the formation. If the final orientation of the formation is desired in practice, one may introduce a leader to control the value of the synchronized orientation and the formation stability analysis presented in this section is still valid in this case.

A. A Bearing-Only Control Law

The proposed position and orientation control laws are

$$v_i^b = -\sum_{j\in\mathcal{N}_i} P_{g_{ij}^b}(I_3 + Q_i^{\mathrm{T}} Q_j) g_{ij}^*, \qquad (22a)$$

$$\left[w_i^b\right]_\times = -\sum_{j\in\mathcal{N}_i} \left(Q_j^{\mathrm{T}} Q_i - Q_i^{\mathrm{T}} Q_j\right). \qquad (22b)$$

The proposed control law is distributed and can be implemented without the knowledge of the global frame. It only requires local bearing measurements $\{g_{ij}^b\}_{j\in\mathcal{N}_i}$ and relative orientation measurements $\{Q_i^{\mathrm{T}} Q_j\}_{j\in\mathcal{N}_i}$. Control law (22b) actually is the orientation synchronization control proposed in [31]. Substituting control law (22) into (21) gives the closed-loop system dynamics with all vector quantities expressed in the global frame as

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i} P_{g_{ij}}(Q_i + Q_j) g_{ij}^*, \qquad (23a)$$

$$\dot{Q}_i = -\sum_{j\in\mathcal{N}_i} Q_i \left(Q_j^{\mathrm{T}} Q_i - Q_i^{\mathrm{T}} Q_j\right). \qquad (23b)$$

While deriving (23a), we use the fact that $g_{ij} = Q_i g_{ij}^b$ and $Q_i P_{g_{ij}^b} Q_i^{\mathrm{T}} = P_{g_{ij}}$.

We next show that the centroid and the scale of the formation are invariant under control law (22).

Lemma 9. *Under control law* (22),

$$\dot{p} \perp \operatorname{span}\{\mathbf{1}\otimes I_3, p\}. \tag{24}$$

Proof. Let $Q_{ij} \triangleq Q_i + Q_j$. Then, $\dot{p}_i = -\sum_{j\in\mathcal{N}_i} P_{g_{ij}} Q_{ij} g_{ij}^*$. Consider an arbitrary oriented graph, the position dynamics (23a) can be written in a matrix form as $\dot{p} = \bar{H}^{\mathrm{T}} \operatorname{diag}(P_{g_k}) \operatorname{diag}(Q_k) g^*$. Because $\mathbf{1}\otimes I_3$ and p are all in the left null space of $\bar{H}^{\mathrm{T}}\operatorname{diag}(P_{g_k})$, we obtain (24). □

Theorem 13 (Centroid and Scale Invariance)**.** *The centroid* $\bar{p}$ *and the scale* s *are invariant under control law* (22).

Proof. With Lemma 5, the proof is similar to Theorem 9. □

Remark 5. *In fact, it can be easily verified that Lemma 9 and Theorem 13 hold for any position control law that has the form of* $\dot{p}_i = -\sum_{i=1}^n P_{g_{ij}} y_{ij}$ *where* $y_{ij}\in\mathbb{R}^3$ *and* $y_{ij} = -y_{ji}$.

The following results, which can be obtained from Theorem 13, give bounds for $\max_{i\in\mathcal{V}}\|p_i(t)-\bar{p}\|$ and $\|p_i(t)-p_j(t)\|, \forall i,j\in\mathcal{V}$.

Corollary 2. *The formation trajectory under the control law* (22) *satisfies the following inequalities,*

(a) $s \le \max_{i\in\mathcal{V}}\|p_i(t)-\bar{p}\| \le s\sqrt{n-1}, \quad \forall t\ge 0.$
(b) $\|p_i(t)-p_j(t)\| \le 2s\sqrt{n-1}, \quad \forall i,j\in\mathcal{V},\ \forall t\ge 0.$

Proof. The proof is similar to Corollary 1. □

B. Formation Stability Analysis

In order to prove the formation stability, we adopt Assumption 1 as well as the following assumption.

Assumption 2. *In the initial formation, there exists* $Q_0 \in SO(3)$ *such that* $Q_0^{\mathrm{T}} Q_i$ *is (non-symmetric) positive definite for all* $i\in\mathcal{V}$.

Remark 6. *Assumption 2 has been widely adopted for attitude synchronization control [31], [32]. Based on axis-angle representation, a rotation matrix is positive definite if and only if the rotation angle is in* $(-\pi/2, \pi/2)$*. The existence of* Q_0 *in Assumption 2 means there is a coordinate transformation of the world frame such that all orientation matrices become positive definite. Another interpretation is that there is a point* $Q_0\in SO(3)$ *such that* $\{Q_i\}_{i\in\mathcal{V}}$ *are contained within a ball of radius less than* $\pi/2$ *in the* $SO(3)$ *manifold [32].*

The closed-loop system (23) is a cascade system: the dynamics of the orientation is independent to the dynamics of the position, whereas the converse is not true. It has been proved by [31] that the orientation of the agents will synchronize under control law (22b).

Lemma 10 ([31, Thm 1])**.** *Under Assumption 2, if the interconnection graph is fixed and strongly connected, the orientation control law* (22b) *guarantees orientation synchronization in the sense that*

$$\lim_{t\to\infty} Q_i^{\mathrm{T}} Q_j = I_3, \quad \forall i,j\in\mathcal{V}.$$

Although the value of the final converged orientation is not given, Lemma 10 indicates that there exists a unique $Q^*\in SO(3)$ such that Q_i ($i\in\mathcal{V}$) converges to Q^* asymptotically. The specific value of Q^* is not of our interest and it is not required to prove the formation stability. The idea of the stability proof is similar to the case with a global reference frame. We will prove that the formation converges to a target formation as defined below.

Definition 8 (Target Formation)**.** *Let* $\mathcal{G}(p^*)$ *be the* target formation *that satisfies*

(a) Centroid: $\bar{p}^* = \bar{p}(0)$.
(b) Scale: $s^* = s(0)$.
(c) Bearing: $(p_j^*-p_i^*)/\|p_j^*-p_i^*\| = Q^* g_{ij}^*$ *for all* $(i,j)\in\mathcal{E}$.

Lemma 11 (Existence and Uniqueness)**.** *The target formation* $\mathcal{G}(p^*)$ *in Definition 8 always exists and is unique under Assumptions 1 and 2.*

Proof. The proof is similar to Lemma 6. But it should be noted that the bearings of $\mathcal{G}(p^*)$ in Definition 8 are $\{Q^* g_{ij}^*\}_{(i,j)\in\mathcal{E}}$ instead of $\{g_{ij}^*\}_{(i,j)\in\mathcal{E}}$. □

Let $\delta_i \triangleq p_i - p_i^*$. It follows from the closed-loop position dynamics (23a) that

$$\begin{aligned}\dot{\delta}_i &= -\sum_{j\in\mathcal{N}_i} P_{g_{ij}}(Q_i+Q_j) g_{ij}^* \\ &= \underbrace{-2\sum_{j\in\mathcal{N}_i} P_{g_{ij}} Q^* g_{ij}^*}_{f_i(\delta)} + \underbrace{\sum_{j\in\mathcal{N}_i} P_{g_{ij}}(2Q^*-Q_i-Q_j) g_{ij}^*}_{h_i(t)}.\end{aligned}$$

Denote $\delta = [\delta_1^{\mathrm{T}},\dots,\delta_n^{\mathrm{T}}]^{\mathrm{T}}$, $f(\delta) = [f_1^{\mathrm{T}}(\delta),\dots,f_n^{\mathrm{T}}(\delta)]^{\mathrm{T}}$, and $h(t) = [h_1^{\mathrm{T}}(t),\dots,h_n^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then, the δ-dynamics is

$$\dot{\delta} = f(\delta) + h(t), \tag{25}$$

where $h(t)$ can be viewed as an input. It should be noted that the autonomous system (i.e., system (25) with $h(t)\equiv 0$)

$$\dot{\delta} = f(\delta)$$

has already been well studied in Section III. For this autonomous system, we can conclude based on Section III that $\delta = 0$ is an almost globally stable equilibrium and $g_{ij}(t)\to Q^* g_{ij}^*$ almost globally as $t\to\infty$.

Lemma 12. *The input* $h(t)$ *converges to zero asymptotically.*

Proof. Since $\|h(t)\| \le \sum_{i=1}^n \|h_i(t)\| \le \sum_{i=1}^n \sum_{j\in\mathcal{N}_i} \|P_{g_{ij}}\| \|2Q^*-Q_i-Q_j\| \|g_{ij}^*\|$, where $\|P_{g_{ij}}\| = \|g_{ij}^*\| = 1$ and $\|2Q^*-Q_i-Q_j\| \le \|Q_i-Q^*\| + \|Q_j-Q^*\|$, we have $\|h(t)\| \le 2\sum_{i=1}^n \|Q_i(t)-Q^*\|$. Since $Q_i(t)\to Q^*$ asymptotically by Lemma 10, we have $\|h(t)\|\to 0$ as $t\to\infty$. □

We next identify the state manifold and the equilibriums of the δ-dynamics (25). Denote, as before, $r(t) = p(t) - \mathbf{1}\otimes\bar{p}$ and $r^* = p^* - \mathbf{1}\otimes\bar{p}^*$.

Lemma 13. *System* (25) *evolves on the surface of the sphere*

$$\mathcal{S} = \{\delta\in\mathbb{R}^{3n} : \|\delta + r^*\| = \|r^*\|\}.$$

Proof. It follows from $\delta(t) = r(t) - r^*$ that $\|\delta(t) + r^*\| = \|r(t)\| = \|r^*\|$, where $\|r(t)\| = \|r^*\|$ is due to the scale invariance. □

Theorem 14 (Equilibrium)**.** *Under Assumptions 1 and 2, the closed-loop system* (23) *(i.e., the δ-dynamics together with the orientation dynamics) has two equilibrium points,*

(a) $\delta = 0$ and $Q_i = Q^*, \forall i \in \mathcal{V}$,
(b) $\delta = -2r^*$ and $Q_i = Q^*, \forall i \in \mathcal{V}$.

Proof. Any equilibrium must satisfy

$$\sum_{j\in\mathcal{N}_i} P_{g_{ij}}(Q_i + Q_j)g^*_{ij} = 0, \quad \forall i \in \mathcal{V}. \tag{26}$$

It follows from Lemma 10 that $Q_i = Q^*$ ($\forall i \in \mathcal{V}$) is the equilibrium for the orientation dynamics (23b) under Assumption 2. Then, (26) becomes

$$\sum_{j\in\mathcal{N}_i} P_{g_{ij}} Q^* g^*_{ij} = 0, \quad \forall i \in \mathcal{V}.$$

Similar to the proof of Theorem 10, it can be shown that the above equation suggests two equilibriums: $\delta = 0$ and $\delta = -2r^*$. The bearings at the two equilibriums are $g_{ij} = Q^* g^*_{ij}, \forall (i,j) \in \mathcal{E}$ and $g_{ij} = -Q^* g^*_{ij}, \forall (i,j) \in \mathcal{E}$, respectively. □

The equilibrium $\delta = 0$ is desired while the other one $\delta = -2r^*$ is undesired. The formations at the two equilibriums have the same centroid, scale, and shape, but they have the opposite bearings. We next present the main stability result and show that the desired equilibrium $\delta = 0$ is almost globally stable. The idea of the proof is to show system (25) is almost globally ISS [24]. Then, the almost global stability can be concluded by $\lim_{t\to\infty} h(t) = 0$. The conventional ISS is not applicable because it is defined for globally stable equilibriums while the equilibrium $\delta = 0$ of $\dot{\delta} = f(\delta)$ is almost globally stable. A review of the almost global ISS is presented in Appendix B.

Theorem 15 (Almost Global Asymptotical Stability)**.** *Under Assumptions 1 and 2, the system trajectory $\delta(t)$ of* (25) *asymptotically converges to $\delta = 0$ from any $\delta(0) \in \mathcal{S}$ except a set of measure zero.*

Remark 7. *In terms of bearings, Theorem 15 indicates that $g_{ij}(t)$ almost globally converges to $Q^* g^*_{ij}$ for all $(i,j) \in \mathcal{E}$. Consequently, $g^i_{ij}(t) = Q_i^{\mathrm{T}}(t) g_{ij}(t) \to (Q^*)^{\mathrm{T}} Q^* g^*_{ij} = g^*_{ij}$ as $t \to \infty$. Therefore, control law* (22) *solves Problem 2.*

Proof. We first prove system (25) fulfills the ultimate boundedness property with Lemma 15 (see Appendix B). Consider the Lyapunov function $V = \|\delta\|^2/2$. For the autonomous system $\dot{\delta} = f(\delta)$, we already know from the proof of Theorem 11 that there exists a positive constant κ such that

$$\frac{\partial V}{\partial \delta} f(\delta) \le -\kappa \sin^2\theta \|\delta\|^2 = -\kappa \left(1 - \frac{\|\delta\|^2}{4\|r^*\|^2}\right) \|\delta\|^2.$$

The derivative of V along the trajectory of system (25) is

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial \delta}(f(\delta) + h(t)) \le -\kappa \left(1 - \frac{\|\delta\|^2}{4\|r^*\|^2}\right) \|\delta\|^2 + \|\delta\|\|h(t)\| \\ &= -\kappa\|\delta\|^2 + \frac{\kappa\|\delta\|^4}{4\|r^*\|^2} + \|\delta\|\|h(t)\| \\ &\le -2\kappa V + 4\kappa\|r^*\|^2 + 2\|r^*\|\|h(t)\|,\end{aligned}$$

where the last inequality is due to $\|\delta\| \le 2\|r^*\|$. By Lemma 15, system (25) fulfills the ultimate boundedness property.

We next show system (25) satisfies all the three conditions (a)-(c) in Lemma 16 (see Appendix B). First, the state of (25) evolves on the sphere $\mathcal{S}$ which satisfies condition (a) in Lemma 16. Second, consider $V = \|\delta\|^2/2$. For the autonomous system $\dot{\delta} = f(\delta)$, we have $(\partial V/\partial \delta) f(\delta) \le -\kappa \sin^2\theta \|\delta\|^2 < 0$ for all $\delta \in \mathcal{S}$ except the equilibriums $\delta = 0$ and $\delta = -2r^*$. Thus, condition (b) is fulfilled. Third, the unstable equilibrium of the autonomous system $\dot{\delta} = f(\delta)$ is $\delta = -2r^*$. It is isolated. Similar to the proof of Proposition 1, it can be shown that the Jacobian $A = \partial f/\partial \delta$ at $\delta = -2r^*$ is positive semi-definite and at least one eigenvalue is positive. As a result, condition (c) is fulfilled.

Thus, it can be concluded from Lemma 16 that system (25) is almost globally ISS. Furthermore, since the input $h(t)$ converges to zero as shown in Lemma 12, the equilibrium $\delta = 0$ is almost globally asymptotically stable. The trajectory of (25) asymptotically converges to $\delta = 0$ from any $x(0) \in \mathcal{S}$ except a set of zero measure. □

Remark 8. *In Theorem 15, the set of measure zero, starting from which $\delta(t)$ will converge to the undesired equilibrium $\delta = -2r^*$, is affected by the initial values of the agent positions and orientations. This set of measure zero is not specifically identified in Theorem 15. In addition, the formation at the equilibrium $\delta = -2r^*$ may be desirable in practical tasks where we only care about the shape of the formation. In this case, the equilibriums $\delta = 0$ and $\delta = -2r^*$ are both desired and the formation becomes globally stable.*

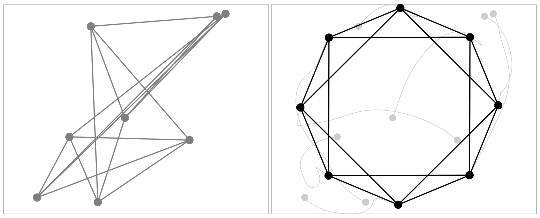
V. Simulation Examples

A. Formation Control with a Global Reference Frame

We have already presented two simulation examples in Figure 5. It is worth noting that collinear initial formations may cause troubles for distance-based formation control, but as shown in Figure 5(b) it is not a problem for bearing-only formation control. Two more simulation examples are shown in Figures 10 and 11, respectively. The initial formations are generated randomly. It is shown that control law (10) can steer the agents to a formation that satisfies the bearing constraints.

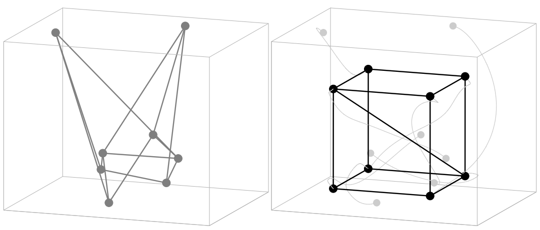
B. Formation Control without a Global Reference Frame

Three simulation examples are shown in Figures 12, 13, and 14, respectively. The local frame for each agent is represented by the line segments in red, green, and blue in the figures. The initial positions and orientations of the agents are generated randomly. The target formations in Figures 12, 13, and 14 have the same shape as those in Figures 5(b), 10 and 11, respectively. As can be seen, the orientations of the agents



(a) Initial formation (b) Final formation

Fig. 10: The case with a global reference frame: a two-dimensional formation with $n = 8$, $m = 16$.



(a) Initial formation (b) Final formation

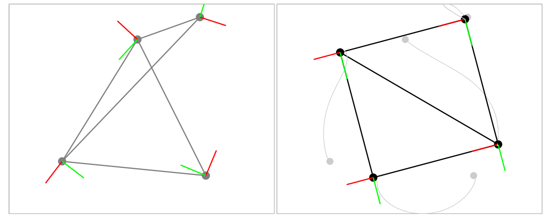
Fig. 11: The case with a global reference frame: a three-dimensional formation with $n = 8$, $m = 13$.

finally synchronize, and the bearing constraints are satisfied in the synchronized frames.

VI. Conclusions and Future Works

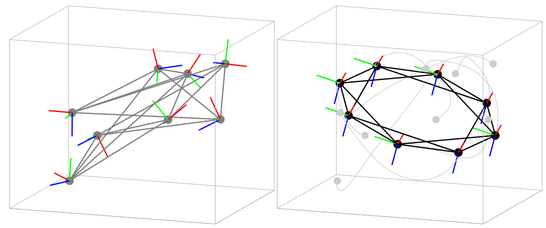
The first contribution of this paper is to propose a bearing rigidity theory that is applicable to arbitrary dimensions. We showed that the infinitesimal bearing rigidity not only can ensure the unique shape of a framework and but also can be conveniently examined by a mathematical condition. We also explored the connection between the proposed bearing rigidity and the well-known distance rigidity, and showed that a framework in $\mathbb{R}^2$ is infinitesimally bearing rigid if and only if it is also infinitesimally distance rigid. Based on the bearing rigidity theory, we studied two bearing-only formation control problems with and without global reference frames, respectively. We proposed two distributed control laws to solve the two problems, respectively. It has been proved that the control laws can almost globally stabilize infinitesimally bearing rigid target formations.

Bearing-only formation control is a research topic highly motivated by practical vision-based cooperative control tasks. There exist many future research directions from both of theoretical and practical perspectives. For example, this paper only considered undirected and fixed underlying sensing graphs. It is meaningful to investigate the case with directed and switching graphs. Second, vision-based identification of a group of agents usually requires visual tagging which may make the vision system complicated. Motivated by that, formation control with anonymous bearing measurements is a meaningful topic for future research. Third, bearing-only formation control with leaders and followers or with human-agent interaction control [13], [15] should also be studied.



(a) Initial formation (b) Final formation

Fig. 12: The case without a global reference frame: a two-dimensional formation with $n = 4$, $m = 5$.



(a) Initial formation (b) Final formation

Fig. 13: The case without a global reference frame: a three-dimensional formation with $n = 8$, $m = 18$. The target formation is coplanar.

Appendix

A. Proof of Theorem 8

In order to prove Theorem 8, we need introduce some concepts and results in the distance rigidity theory [26], [27]. Define the *distance function* for a framework $\mathcal{G}(p)$ as

$$F_D(p) \triangleq \frac{1}{2}\begin{bmatrix} \cdots & \|p_j - p_i\|^2 & \cdots \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^m. \tag{27}$$

Each entry of $F_D(p)$ corresponds to the length of an edge of the framework. The *distance rigidity matrix* is defined as the Jacobian of the distance function,

$$R_D(p) \triangleq \frac{\partial F_D(p)}{\partial p} \in \mathbb{R}^{m \times dn}.$$

We use the subscript D to distinguish the distance rigidity matrix $R_D(p)$ from the bearing rigidity matrix $R(p)$. Let δp be a variation of p. If $R_D(p)\delta p = 0$, then δp is called an *infinitesimal distance motion* of $\mathcal{G}(p)$. A framework is *infinitesimally distance rigid* if the infinitesimal motion only corresponds to rigid-body rotations and translations.

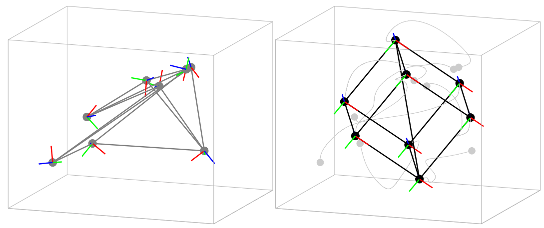
Lemma 14 ([26]). *A framework $\mathcal{G}(p)$ in $\mathbb{R}^d$ is infinitesimally distance rigid if and only if*

$$\operatorname{rank}(R_D(p)) = \begin{cases} dn - d(d+1)/2 & \text{if } n \geq d, \\ n(n-1)/2 & \text{if } n < d. \end{cases}$$

By Lemma 14, in the case of $n \geq d$, the framework $\mathcal{G}(p)$ is infinitesimally distance rigid in $\mathbb{R}^2$ if and only if $\operatorname{rank}(R_D(p)) = 2n - 3$, and in $\mathbb{R}^3$ if and only if $\operatorname{rank}(R_D(p)) = 3n - 6$.

To prove Theorem 8, we first prove the following result.

Proposition 2. *A framework $\mathcal{G}(p)$ in $\mathbb{R}^2$ always satisfies* $\operatorname{rank}(R(p)) = \operatorname{rank}(R_D(p))$.



(a) Initial formation (b) Final formation

Fig. 14: The case without a global reference frame: a three-dimensional formation with $n = 8$, $m = 13$.

Proof. Consider an oriented graph and write the bearings of the framework as $\{g_k\}_{k=1}^m$. Let $Q_{\pi/2}$ be a 2×2 rotation matrix that rotates any vector $\pi/2$ counterclockwise. Denote $g_k^\perp \triangleq Q_{\pi/2} g_k$. Then, $g_k^\perp \perp g_k$ and $\|g_k^\perp\| = \|g_k\| = 1$. Since $P_{g_k} = g_k^\perp (g_k^\perp)^{\mathrm{T}}$ by (3), the bearing rigidity matrix can be rewritten as

$$R(p) = \mathrm{diag}\left(\frac{P_{g_k}}{\|e_k\|}\right)\bar{H} = \mathrm{diag}\left(\frac{g_k^\perp}{\|e_k\|}\right)\mathrm{diag}\left((g_k^\perp)^{\mathrm{T}}\right)\bar{H}.$$

The matrix $\mathrm{diag}\left((g_k^\perp)^{\mathrm{T}}\right)\bar{H}$ can be further written as

$$\begin{aligned}
&\mathrm{diag}\left((g_k^\perp)^{\mathrm{T}}\right)\bar{H} = \mathrm{diag}\left(g_k^{\mathrm{T}} Q_{\pi/2}^{\mathrm{T}}\right)\bar{H} \\
&= \mathrm{diag}\left(g_k^{\mathrm{T}}\right)(I_m \otimes Q_{\pi/2}^{\mathrm{T}})(H \otimes I_2) \\
&= \mathrm{diag}\left(g_k^{\mathrm{T}}\right)(H \otimes Q_{\pi/2}^{\mathrm{T}}) = \mathrm{diag}\left(g_k^{\mathrm{T}}\right)\bar{H}(I_n \otimes Q_{\pi/2}^{\mathrm{T}}).
\end{aligned}$$

Furthermore, the distance rigidity matrix can be expressed as $R_D(p) = \mathrm{diag}\left(e_k^{\mathrm{T}}\right)\bar{H}$ (this expression can be obtained by calculating the Jacobian of the distance function (27)). As a result, we have $\mathrm{diag}\left(g_k^{\mathrm{T}}\right)\bar{H} = \mathrm{diag}\left(1/\|e_k\|\right)R_D(p)$. Therefore, $R(p)$ can be expressed by

$$R(p) = \mathrm{diag}\left(\frac{g_k^\perp}{\|e_k\|^2}\right)R_D(p)\left(I_n \otimes Q_{\pi/2}^{\mathrm{T}}\right).$$

Since $\mathrm{diag}\left(g_k^\perp/\|e_k\|^2\right)$ has full column rank and $I_n \otimes Q_{\pi/2}^{\mathrm{T}}$ is invertible, we have $\mathrm{rank}(R(p)) = \mathrm{rank}(R_D(p))$. □

Proof of Theorem 8. By Theorem 4, a framework $\mathcal{G}(p)$ in $\mathbb{R}^2$ is *infinitesimally bearing rigid* if and only if $\mathrm{rank}(R(p)) = 2n-3$. By Lemma 14, a framework is *infinitesimally distance rigid* if and only if $\mathrm{rank}(R_D(p)) = 2n-3$. Since $\mathrm{rank}(R(p)) = \mathrm{rank}(R_D(p))$ as proved in Proposition 2, we know $\mathrm{rank}(R(p)) = 2n-3$ if and only if $\mathrm{rank}(R_D(p)) = 2n-3$, which concludes the theorem. □

B. Preliminaries to Almost Global Input-to-State Stability

We review some results on almost global ISS [24]. These results are used to prove the almost global stability of bearing-only formation control without a global reference frame.

Consider a nonlinear system evolving on a smooth manifold $\mathcal{M}$ and subject to input disturbance:

$$\dot{x} = f(x,u), \tag{28}$$

where $x \in \mathcal{M}$ is the state, $u \in \mathbb{U}$ is the input, and $f : \mathcal{M}\times\mathbb{U} \to T\mathcal{M}$ is a locally Lipschitz manifold map satisfying $f(x,u) \in T_x\mathcal{M}$ for all $x \in \mathcal{M}$ and all $u \in \mathbb{U}$ ($T\mathcal{M}$ and $T_x\mathcal{M}$ denotes the tangent space of $\mathcal{M}$ and the tangent space of $\mathcal{M}$ at x, respectively).

Definition 9 (Almost Global ISS)**.** *System* (28) *is almost globally ISS with respect to an equilibrium point x_e if x_e is locally asymptotically stable for $u \equiv 0$, and for all u and almost all $x(t_0) \in \mathcal{M}$ the following inequality holds,*

$$\limsup_{t\to\infty} |x(t,t_0,u)|_{x_e} \le \gamma(\|u\|_\infty), \tag{29}$$

where γ is a class $\mathcal{K}$ function, $\|u\|_\infty \triangleq \sup_{t_0\le\tau\le\infty}\|u(\tau)\|$, and $|\cdot|_{x_e}$ denotes the distance to x_e.

Remark 9. *Since inequality* (29) *holds for all t_0, it is easy to see almost global ISS implies almost global asymptotic stability when $u(t)$ converges to zero as $t\to\infty$.*

Definition 10 (Ultimate Boundedness)**.** *System* (28) *fulfills the* ultimate boundedness *property if there exists a point $\xi \in \mathcal{M}$ and for all $u \in \mathbb{U}$ and all $x(t_0) \in \mathcal{M}$, the system trajectory $x(t,x(t_0),u)$ is defined on $[t_0,\infty)$ and eventually confined to*

$$\{z \in M : |z|_\xi \le \gamma(\|u\|_\infty) + c\},$$

where γ is a class $\mathcal{K}$ function and $c \in \mathbb{R}$ is a constant, and $|\cdot|_\xi$ denotes the distance to ξ.

Lemma 15 ([24])**.** *For system* (28)*, if there exists a nonnegative and proper $\mathcal{C}^1$ function $V : M \to \mathbb{R}_{\ge 0}$ such that the derivative of V along the trajectory of system* (28) *satisfies*

$$\forall u,\ \forall x,\ \dot{V} \le -\beta(V) + c + \gamma(\|u\|),$$

where β and γ are class $\mathcal{K}$ functions, and $c \in \mathbb{R}$ is a constant, then system (28) *fulfills the ultimate boundedness property.*

Lemma 16 ([24])**.** *Assume* (28) *satisfies the following properties:*

(a) $\mathcal{M}$ is a $\mathcal{C}^2$ connected, orientable manifold without boundary.
(b) There exists a nonnegative and proper $\mathcal{C}^1$ function $V : \mathcal{M} \to \mathbb{R}_{\ge 0}$ such that the derivative of V along the system trajectory of $\dot{x} = f(x,0)$ satisfies $\dot{V} < 0$ for all $x \in \mathcal{M}$ and $f(x,0) \neq 0$.
(c) Any equilibrium x_ℓ which is not asymptotically stable, is isolated and at least one eigenvalue of the Jacobian $\partial f(x,0)/\partial x|_{x_\ell}$ has strictly positive real part.

Assume the equilibrium x_e for $\dot{x} = f(x,0)$ is asymptotically stable. If ultimate boundedness holds, then, (28) *is almost globally ISS with respect to x_e.*

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