



Deposited via The University of Leeds.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/id/eprint/104054/>

Version: Accepted Version

Article:

Agler, J, Lykova, ZA and Young, NJ (2019) Geodesics, retracts, and the norm-preserving extension property in the symmetrized bidisc. *Memoirs of the American Mathematical Society*, 258. ISSN: 0065-9266

<https://doi.org/10.1090/memo/1242>

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

Geodesics, retracts, and the norm-preserving extension property in the symmetrized bidisc

Jim Agler

Zinaida Lykova

Nicholas Young

Author address:

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SAN DIEGO,
CA 92103, USA

SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEW-
CASTLE UPON TYNE NE1 7RU, U.K.

E-mail address: `Zinaida.Lykova@ncl.ac.uk`

SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEW-
CASTLE UPON TYNE NE1 7RU, U.K. *and* SCHOOL OF MATHEMATICS, LEEDS
UNIVERSITY, LEEDS LS2 9JT, U.K.

E-mail address: `Nicholas.Young@ncl.ac.uk`

12.1. Preliminary lemmas	70
12.2. $\sigma : R \rightarrow V$ is analytic on R^-	72
12.3. Degree considerations	73
Chapter 13. Sets in \mathbb{D}^2 with the symmetric extension property	77
Chapter 14. Applications to the theory of spectral sets	83
Chapter 15. Anomalous sets with the norm-preserving extension property in some other domains	87
Appendix A. Some useful facts about the symmetrized bidisc	91
A.1. Basic properties of G and Γ	91
A.2. Complex C -geodesics in G	92
A.3. Automorphisms of G	93
A.4. A trichotomy theorem	94
A.5. Datums for which all Φ_ω are extremal	95
Appendix B. Types of geodesic: a crib and some cartoons	99
B.1. Crib sheet	99
B.2. Cartoons	100
Bibliography	103
Index	105

Abstract

A set V in a domain U in \mathbb{C}^n has the *norm-preserving extension property* if every bounded holomorphic function on V has a holomorphic extension to U with the same supremum norm. We prove that an algebraic subset of the *symmetrized bidisc*

$$G \stackrel{\text{def}}{=} \{(z + w, zw) : |z| < 1, |w| < 1\}$$

has the norm-preserving extension property if and only if it is either a singleton, G itself, a complex geodesic of G , or the union of the set $\{(2z, z^2) : |z| < 1\}$ and a complex geodesic of degree 1 in G . We also prove that the complex geodesics in G coincide with the nontrivial holomorphic retracts in G . Thus, in contrast to the case of the ball or the bidisc, there are sets in G which have the norm-preserving extension property but are not holomorphic retracts of G . In the course of the proof we obtain a detailed classification of the complex geodesics in G modulo automorphisms of G . We give applications to von Neumann-type inequalities for Γ -contractions (that is, commuting pairs of operators for which the closure of G is a spectral set) and for symmetric functions of commuting pairs of contractive operators. We find three other domains that contain sets with the norm-preserving extension property which are not retracts: they are the spectral ball of 2×2 matrices, the tetrablock and the pentablock. We also identify the subsets of the bidisc which have the norm-preserving extension property for symmetric functions.

Received by the editor 14th March 2016, revised 20th August 2016.

2010 *Mathematics Subject Classification*. Primary: 32A07, 53C22, 54C15, 47A57, 32F45; Secondary: 47A25, 30E05.

Key words and phrases. symmetrized bidisc, complex geodesic, norm-preserving extension property, holomorphic retract, spectral set, Kobayashi extremal problem, Carathéodory extremal problem, von Neumann inequality, semialgebraic set.

Partially supported by National Science Foundation Grant DMS 1361720 and by the London Mathematical Society grant 41527.

Partially supported by the Engineering and Physical Sciences grant EP/N03242X/1.

Preface

The symmetrized bidisc is just one domain in \mathbb{C}^2 , but it is a fascinating one which amply repays a detailed study. In the absence of a Riemann mapping theorem in dimensions greater than one there is a choice between studying broad classes of domains and choosing some special domains. From the work of Oka onwards there has developed a beautiful and extensive theory of general domains of holomorphy; however, in such generality there can be no expectation of a detailed analytic function theory along the lines of the spectacularly successful theory of functions on the unit disc, which is one of the pinnacles of twentieth century analysis. For this reason the general theory is complemented by the study of functions on special domains such as the ball and the polydisc [42, 43], or more generally Cartan domains [30, 26]. Thereafter it seems natural to investigate *inhomogeneous* domains for which a detailed function theory is possible. The symmetrized bidisc G in \mathbb{C}^2 is emphatically one such. It is inhomogeneous, but it has a 3-parameter group of automorphisms. It is closely related to the bidisc, being the image of the bidisc under the polynomial map $(\lambda^1 + \lambda^2, \lambda^1 \lambda^2)$. Its geometry and function theory have many resemblances to, but also many differences from, those of the bidisc. It transpires that many quantities of interest for G , for example the Kobayashi and Carathéodory distances, can be calculated in a fairly explicit form.

We originally started studying the symmetrized bidisc in connection with the *spectral Nevanlinna-Pick problem*, an interpolation problem for analytic 2×2 -matrix-valued functions subject to a bound on the spectral radius. This problem is of interest in certain engineering problems. We can claim only partial success in resolving the interpolation problem, but we happened on a rich vein of function theory, which was of particular interest to specialists in the theories of invariant distances [32] and models of multioperators [18].

In this paper we determine the holomorphic retractions of G and the subsets of G which permit the extension of holomorphic functions without increase of supremum norm. The methods we use are of independent interest: we analyse the complex geodesics of G , and show that there are *five* qualitatively different types thereof. The type of a complex geodesic \mathcal{D} can be characterized in two quite different ways. One way is in terms of the intersection of \mathcal{D} with the special variety

$$\mathcal{R} \stackrel{\text{def}}{=} \{(2z, z^2) : |z| < 1\}.$$

\mathcal{R} is itself a complex geodesic of G , uniquely specified by the property that it is invariant under every automorphism of G . Remarkably, the type of \mathcal{D} can also be characterized by the number of Carathéodory extremal functions for \mathcal{D} from a certain one-parameter family of extremals. In contrast, for the bidisc, one can define types of complex geodesics in an analogous way, and one finds that there are only *two* types.

CHAPTER 1

Introduction

A celebrated theorem of H. Cartan states the following [21].

THEOREM 1.1. *If V is an analytic variety in a domain of holomorphy U and if f is an analytic function on V then there exists an analytic function on U whose restriction to V is f .*

In the case that the function f is bounded, Cartan's theorem gives no information on the supremum norm of any holomorphic extension of f . We are interested in the case that an extension exists with the *same* supremum norm as f .

DEFINITION 1.2. A subset V of a domain U has the *norm-preserving extension property* if, for every bounded analytic function f on V , there exists an analytic function g on U such that

$$g|_V = f \quad \text{and} \quad \sup_U |g| = \sup_V |f|.$$

Here V is not assumed to be a variety; to say that f is analytic on V simply means that, for every $z \in V$ there exists a neighborhood W of z in U and an analytic function h on W such that h agrees with f on $W \cap V$.

There are at least two reasons to be interested in the norm-preserving extension property, one function-theoretic and one operator-theoretic. In the case that U is the bidisc, the description of the solutions of an interpolation problem for holomorphic functions on U leads to norm-preserving extensions, while knowledge of the sets having the norm-preserving extension property in the bidisc also leads to an improvement of Ando's inequality for pairs of commuting contractions [2, Section 1].

The subsets of the bidisc \mathbb{D}^2 having the norm-preserving extension property were identified in [2]. In this memoir we shall do the same for the symmetrized bidisc. Let \mathbb{C} denote the complex field and \mathbb{D} the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. The *symmetrization map* is the map $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ given by

$$\pi(\lambda^1, \lambda^2) = (\lambda^1 + \lambda^2, \lambda^1 \lambda^2), \quad \lambda^1, \lambda^2 \in \mathbb{C},$$

and the *symmetrized bidisc* is the set

$$\begin{aligned} G &= \pi(\mathbb{D}^2) \\ &= \{(z + w, zw) : z, w \in \mathbb{D}\} \subset \mathbb{C}^2. \end{aligned}$$

Retracts of a domain have the norm-preserving extension property. A (holomorphic) *retraction* of U is a holomorphic map of $\rho : U \rightarrow U$ such that $\rho \circ \rho = \rho$, and a *retract* in U is a set which is the range of a retraction of U . Evidently, for any retraction ρ , a bounded analytic function f on $\text{ran } \rho$ has the analytic norm-preserving extension $f \circ \rho$ to U . Two simple types of retracts of U are singleton

sets and the whole of U ; these retracts are *trivial*. Hyperbolic geometry in the sense of Kobayashi [34] provides an approach to the construction of nontrivial retracts. A *complex geodesic* of U is the range V of a holomorphic map $k : \mathbb{D} \rightarrow U$ which has a holomorphic left inverse C . Here, since $C \circ k$ is the identity map $\text{id}_{\mathbb{D}}$ on \mathbb{D} , clearly $k \circ C$ is a retraction of U with range V .

To summarize, for a subset V of a domain U , the statements

- (1) V is a singleton, a complex geodesic or all of U ,
- (2) V is a retract in U ,
- (3) V has the norm-preserving extension property in U

satisfy (1) implies (2) implies (3). It was shown in [2] that, in the case that U is the bidisc \mathbb{D}^2 , the converse implications also hold. One might conjecture that the same is true for a general domain in \mathbb{D}^2 ; however, the two main results of the paper show that, for the symmetrized bidisc, (2) implies (1) but (3) does not imply (2).

THEOREM 1.3. *A subset V of G is a nontrivial retract of G if and only if V is a complex geodesic of G .*

Since complex geodesics in G are algebraic sets, as we prove in Theorem 5.4 below, it follows that retracts in G are algebraic sets.

A *flat geodesic* of G is a complex geodesic of G which is the intersection of G with a complex line: see Proposition 4.10.

THEOREM 1.4. *V is an algebraic subset of G having the norm-preserving extension property if and only if either V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$, where $\mathcal{R} = \{(2z, z^2) : z \in \mathbb{D}\}$ and \mathcal{D} is a flat geodesic in G .*

Sets of the form $\mathcal{R} \cup \mathcal{D}$ are not retracts of G , but nevertheless have the norm-preserving extension property (see Theorem 10.1).

Theorems 1.3 and 1.4 have applications to the function theory of the bidisc, the theory of von Neumann inequalities for commuting pairs of operators and the complex geometry of certain domains in dimensions three and four. These applications are given in Chapters 13 to 15. They are briefly described in the next chapter.

It should be mentioned that there is a substantial theory of holomorphic extensions that refines Cartan's Theorem 1.1. However, as far as we know the results in the literature barely overlap the theorems in this paper. Previous authors (with the exception of [2]) do not consider such a stringent condition as the preservation of the supremum norm, and they typically consider either other special domains, such as polydiscs or polyhedra [13, 40, 24], or strictly pseudoconvex domains [29] (note that G is not smoothly embedded in \mathbb{C}^2).

CHAPTER 2

An overview

Much of this memoir is devoted to the proof of the main result, Theorem 1.4 from the introduction.

Neither necessity nor sufficiency is trivial. Sufficiency is proved in Chapters 5 and 10. In the former it is shown that complex geodesics in G are algebraic sets and that nontrivial retracts are complex geodesics. In the latter the Herglotz Representation Theorem is used to show that $\mathcal{R} \cup \mathcal{D}$ has the norm-preserving extension property. The proof of necessity in Theorem 1.4 takes up the remainder of Chapters 3 to 12.

The principal tool is the theory of hyperbolic complex spaces in the sense of Kobayashi [34]. The Carathéodory and Kobayashi extremal problems are described in Section 3.1. There is a substantial theory of these two problems in the special case of the symmetrized bidisc; the relevant parts of it are described in Chapters 3 and 4. More of the known function theory of G , with citations, is given in Appendix A.

There are two versions of the Carathéodory and Kobayashi extremal problems, one for pairs of points and one for tangent vectors. In order to treat both versions of the problems simultaneously we introduce the terminology of *datums*. A datum in a domain U is either a pair of points in U or an element of the tangent bundle of U ; the former is a *discrete datum*, the latter an *infinitesimal datum*. If V is a subset of U and δ is a datum in U then δ *contacts* V if either

- (1) $\delta = (s_1, s_2)$ is discrete and both s_1 and s_2 lie in V , or
- (2) $\delta = (s, v)$ is infinitesimal and either $v = 0$ and $s \in V$, or there exist two sequences of points $\{s_n\}$ and $\{t_n\}$ in V such that $s_n \neq t_n$ for all n , $s_n \rightarrow s$, $t_n \rightarrow s$ and

$$\frac{t_n - s_n}{\|t_n - s_n\|} \rightarrow v_0,$$

for some unit vector v_0 collinear with v .

A datum δ is *degenerate* if either $\delta = (s_1, s_2)$ is discrete and $s_1 = s_2$ or $\delta = (s, v)$ is infinitesimal and $v = 0$. An important property of G is that every nondegenerate datum δ in G contacts a unique complex geodesic of G , denoted by \mathcal{D}_δ .

The proof of necessity in Theorem 1.4 is based on an analysis of the datums that contact a subset V of G having the norm-preserving extension property. We divide the nondegenerate datums in G into five types, called *purely unbalanced*, *exceptional*, *purely balanced*, *royal* and *flat*. The definitions of these types (Definition 3.4 in Section 3.3) for a datum δ are in terms of the Carathéodory extremal functions for the datum δ . We use another special feature of G : for every nondegenerate datum in G there is a solution Φ_ω of the Carathéodory extremal problem having

the form

$$\Phi_\omega(s) = \frac{2\omega s^2 - s^1}{2 - \omega s^1}, \quad \text{for } s = (s^1, s^2),$$

for some $\omega \in \mathbb{T}$, where \mathbb{T} denotes the unit circle in \mathbb{C} (Theorem A.4). For royal and flat datums, *every* Φ_ω is a Carathéodory extremal for δ . If there are exactly two values of $\omega \in \mathbb{T}$ for which Φ_ω is extremal for δ , then δ is said to be *purely balanced*. If there is a unique $\omega \in \mathbb{T}$ for which Φ_ω is extremal for δ , then we say that δ is either *purely unbalanced* or *exceptional*, depending on whether a certain second derivative is zero. The strategy is to consider the consequences for V of the hypothesis that a datum of each of the five types in turn contacts V .

To put this plan into effect we have to establish several results about types of datum. Firstly, according to Theorem 3.6, every nondegenerate datum in G is of exactly one of the five types. A second important fact is that the type of a nondegenerate datum δ can be characterized in terms of the geometry of the complex geodesic \mathcal{D}_δ ; this is the gist of Chapters 6 and 7, and it requires some detailed calculations. The upshot is (Theorem 7.1) that the type of a datum δ is determined by the number and location of the points of intersection of \mathcal{D}_δ^- and the royal variety $\mathcal{R}^- = \{(2z, z^2) : |z| \leq 1\}$ (that is, whether these points belong to G or ∂G). This result has two significant consequences: (1) the type of a datum is preserved by automorphisms of G , and (2) if two nondegenerate datums contact the same complex geodesic then they have the same type. We may therefore define the type of a complex geodesic \mathcal{D} to be the type of any nondegenerate datum that contacts \mathcal{D} , whereupon the complex geodesics of G are themselves partitioned into five types. In particular, \mathcal{R} is the sole geodesic of royal type, and the geodesics of flat type are precisely the complex lines

$$(2.1) \quad \mathcal{F}_\beta \stackrel{\text{def}}{=} \{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\}$$

for $\beta \in \mathbb{D}$. These flat geodesics constitute a foliation of G . Theorem 7.8 gives concrete formulae for each of the five types of geodesics modulo automorphisms of G .

Here is a rough outline of the proof of necessity in Theorem 1.4. Let V be an algebraic set in G having the norm-preserving extension property. We may assume that V is neither a singleton set nor G . We must show that either V is a complex geodesic in G or $V = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic \mathcal{D} .

Chapter 9 proves some consequences of the assumption that V has contact with certain types of datum. Lemma 9.5 states that if some flat, royal or purely balanced datum δ contacts V then $\mathcal{D}_\delta \subseteq V$. According to Lemma 9.10, if a flat datum δ contacts V then V is the flat geodesic \mathcal{D}_δ (which is a retract in G) or $V = \mathcal{R} \cup \mathcal{D}_\delta$, while Lemma 9.11 states that if a royal datum contacts V then either $V = \mathcal{R}$ or $V = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic \mathcal{D} . Hence Chapter 9 reduces the problem to the case that neither a flat nor a royal datum contacts V . By Lemma 12.2, if a purely balanced datum δ contacts V then V is the purely balanced geodesic \mathcal{D}_δ .

In Chapter 11 and in Chapter 12 it is shown that if no flat datum contacts V then V is a properly embedded planar Riemann surface of finite type, meaning that the boundary of V has finitely many connected components. Finite connectedness is proved with the aid of real algebraic geometry, specifically, from the permanence properties of semialgebraic sets, which follow from the Tarski-Seidenberg Theorem. By a classical theorem on finitely connected domains [22, Theorem 7.9], V is conformally equivalent to a finitely-circled domain R in the complex plane, meaning

that the boundary ∂R is a union of finitely many disjoint circles. In Lemma 12.3 we prove that if ∂R has a single component then V is a nontrivial retract in G and so is a complex geodesic of G . To complete the proof it remains to show that, if V has no contact with any flat, royal or purely balanced datum, then ∂R has a single component. This task is achieved by a winding number argument that depends on known properties of Ahlfors functions for the domain R ; this argument takes up Sections 12.1 and 12.2.

Finally, some applications of the main results are presented in the last three chapters. In Chapter 13 we identify the symmetric subsets of the bidisc which have the *symmetric extension property* (roughly, the norm-preserving extension property for symmetric functions). In Theorem 13.5 we show that there are six mutually exclusive classes of such sets and we give explicit formulae for them.

In Chapter 14 we give two applications of Theorem 1.4 to the theory of spectral sets. Theorem 14.4 states that the sets V of Theorem 13.5 are the only symmetric algebraic sets of \mathbb{D}^2 for which the inequality

$$\|f(T)\| \leq \sup_V |f|$$

holds for all bounded symmetric holomorphic functions f on V and all pairs of commuting contractions T subordinate to V (subordination is the condition needed to ensure that $f(T)$ be well defined).

For any set A of bounded holomorphic functions on a set $V \subseteq \mathbb{C}^2$, we say that V is an *A-von Neumann set* if

$$\|f(T)\| \leq \sup_V |f|$$

for all $f \in A$ and all pairs T of commuting contractions which are subordinate to V . Here subordination is the natural notion that ensures that the operator $f(T)$ be well defined. In Theorem 14.4 we show that if V is a symmetric algebraic subset of \mathbb{D}^2 and A is the algebra of all bounded symmetric holomorphic functions on V then V is an *A-von Neumann set* if and only if V belongs to one of the six classes of sets which are described by explicit formulae in Theorem 13.5.

For the second application we need to introduce a variant of the notion of *A-von Neumann set* that is adapted to G . Consider a subset V of G and let A be a set of bounded holomorphic functions on V . We say that V is a *(G, A)-von Neumann set* if V is a spectral set for every Γ -contraction T subordinate to V . Here a Γ -contraction is a pair of commuting operators for which Γ is a spectral set. Let A be the algebra of all bounded holomorphic functions on V . In Theorem 14.7 we show that if V is algebraic then V is a *(G, A)-von Neumann set* if and only if either V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$, where \mathcal{D} is a flat geodesic in G (as in equation (2.1) above).

In Chapter 15 we observe that in any domain which contains G as a holomorphic retract there are sets that have the norm-preserving extension property but are not retracts. In particular this observation applies to the 2×2 spectral ball (which comprises the 2×2 matrices of spectral radius less than one) and two domains in \mathbb{C}^3 known as the tetrablock and the pentablock.

In a digression in Section 5.2 we show that our geometric method based on uniqueness of solutions of certain Carathéodory problems can be used to give an alternative proof of a result of Heath and Suffridge [28] on the retracts of the bidisc (they all have the form of either a singleton, \mathbb{D}^2 itself or a complex geodesic).

Appendix A gathers together the relevant known facts about the geometry and function theory of G , with references and a few proofs. Appendix B is intended to help the reader absorb the notions of the five types of complex geodesic by means of a table of their main properties and a series of cartoons.

CHAPTER 3

Extremal problems in the symmetrized bidisc G

In this chapter we shall set out our notation and give a brief exposition of some known properties of the Carathéodory and Kobayashi extremal problems on the symmetrized bidisc. We introduce the notion of a *datum*, which permits the simultaneous analysis of the discrete and infinitesimal versions of the Kobayashi and Carathéodory problems.

If U is an open set in \mathbb{C}^n , then by a *datum in U* we mean an ordered pair δ where either δ is *discrete*, that is, has the form

$$\delta = (s_1, s_2)$$

where $s_1, s_2 \in U$, or δ is *infinitesimal*, that is, δ is an element of the tangent bundle TU of U , and so has the form

$$\delta = (s, v)$$

where $s \in U$ and v is a tangent vector to U at s . We identify the tangent vector

$$v^1 \frac{\partial}{\partial s^1} + \cdots + v^n \frac{\partial}{\partial s^n} \quad \text{with} \quad (v^1, \dots, v^n) \in \mathbb{C}^n.$$

If δ is a datum, we say that δ is *degenerate* if either δ is discrete and $s_1 = s_2$ or δ is infinitesimal and $v = 0$, and *nondegenerate* otherwise. If $U \subseteq \mathbb{C}^{n_1}$ and $\Omega \subseteq \mathbb{C}^{n_2}$ are two open sets we denote by $\Omega(U)$ the set of holomorphic mappings from U into Ω . Furthermore, if $F \in \Omega(U)$, $s \in U$, and $v \in \mathbb{C}^{n_1}$, we denote by $D_v F(s)$ the directional derivative at s of F in the direction v .

For any set X we denote by id_X the identity map on X .

If U and Ω are domains, $F \in \Omega(U)$, and δ is a datum in U , we define a datum $F(\delta)$ in Ω by

$$F(\delta) = (F(s_1), F(s_2))$$

when δ is discrete and by

$$F(\delta) = F_* \delta = (F(s), D_v F(s))$$

when δ is infinitesimal.

Note that if U, V and W are domains, δ is a datum in U and $F \in V(U)$, $G \in W(V)$, then

$$(3.1) \quad (G \circ F)(\delta) = G(F(\delta)).$$

Let δ be a datum in \mathbb{D} . We define the *modulus* of δ , denoted by $|\delta|$, by

$$|\delta| = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|,$$

when $\delta = (z_1, z_2)$ is discrete, and by

$$|\delta| = \frac{|c|}{1 - |z|^2}$$

when $\delta = (z, c)$ is infinitesimal.

Thus $|\cdot|$ is the pseudohyperbolic distance on \mathbb{D} in the discrete case and the Poincaré metric on the tangent bundle of \mathbb{D} in the infinitesimal case.

For any domain U , the group of automorphisms of U will be denoted by $\text{Aut } U$. We shall make frequent use of the Blaschke factor B_α , defined for $\alpha \in \mathbb{D}$ by

$$B_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z} \quad \text{for } z \in \mathbb{D}.$$

3.1. The Carathéodory and Kobayashi extremal problems

If U is a domain (an open set in \mathbb{C}^n for some n) and δ is a datum in U , then we may consider the Carathéodory Optimization Problem, that is, to compute the quantity $|\delta|_{\text{car}}$ defined by

$$|\delta|_{\text{car}} = \sup_{F \in \mathbb{D}(U)} |F(\delta)|.$$

In the sequel we shall refer to this problem as $\text{Car}(\delta)$. The notation suppresses the dependence of both $|\delta|_{\text{car}}$ and $\text{Car}(\delta)$ on U . However, this will not be a cause for confusion, as the domain U will always be clear from the context. We say that C solves $\text{Car}(\delta)$ if $C \in \mathbb{D}(U)$ and

$$|\delta|_{\text{car}} = |C(\delta)|.$$

In [32, Section 2.1], the function $|\cdot|_{\text{car}}$, when applied to discrete datums, is called the Möbius pseudodistance for U , while, when applied to infinitesimal datums, $|\cdot|_{\text{car}}$ is called the Carathéodory-Reiffen pseudometric for U . For any nondegenerate datum δ in U , there exists $C \in \mathbb{D}(U)$ that solve $\text{Car}(\delta)$ [34].

If U is a domain and δ is a datum in U , then we may also consider the Kobayashi Extremal Problem for δ , that is, to compute the quantity $|\delta|_{\text{kob}}$ defined by

$$(3.2) \quad |\delta|_{\text{kob}} = \inf\{|\zeta| : \zeta \text{ is a datum in } \mathbb{D} \text{ and there exists } f \in U(\mathbb{D}) \text{ such that } f(\zeta) = \delta\}.$$

In the sequel we shall refer to this problem as $\text{Kob}(\delta)$. We say that k solves $\text{Kob}(\delta)$ if $k \in U(\mathbb{D})$ and there is a datum ζ in \mathbb{D} such that

$$k(\zeta) = \delta \quad \text{and} \quad |\delta|_{\text{kob}} = |\zeta|.$$

In [32, Section 3.1], for discrete datums, the function $\tanh^{-1}|\cdot|_{\text{kob}}$ is called the Lempert function for U . In [32, Section 3.5], for infinitesimal datums, the function $|\cdot|_{\text{kob}}$ is called the Kobayashi-Royden pseudometric for U .

A domain U is said to be *taut* if the space $U(\mathbb{D})$ is normal (for example, [32, Section 3.2]). If U is a taut domain then, for any nondegenerate datum δ in U , there exists $k \in U(\mathbb{D})$ that solve $\text{Kob}(\delta)$ [32, Section 3.2].

G is a hyperconvex domain [32, Remark 7.1.6] and therefore taut [32, Remark 3.2.3]. Hence $|\delta|_{\text{kob}}$ is attained for any datum δ in G .

It is immediate from the Schwarz-Pick Lemma that, for any nondegenerate datum δ in a domain U , $|\delta|_{\text{car}} \leq |\delta|_{\text{kob}}$.

PROPOSITION 3.1. *Let $h \in U(\mathbb{D})$, $C \in \mathbb{D}(U)$ be such that $C \circ h = \text{id}_{\mathbb{D}}$. If ζ is a datum in \mathbb{D} and $\delta = h(\zeta)$ then h solves $\text{Kob}(\delta)$.*

For then

$$|\delta|_{\text{car}} \geq |C(\delta)| = |C \circ h(\zeta)| = |\zeta| \geq |\delta|_{\text{kob}} \geq |\delta|_{\text{car}}$$

and so $|\zeta| = |\delta|_{\text{kob}}$.

3.2. The Carathéodory extremal problem $\text{Car}(\delta)$ for G

We say that a set $\mathcal{C} \subseteq \mathbb{D}(U)$ is a *universal set for the Carathéodory problem on U* if, whenever δ is a nondegenerate datum in U , there exists a function $C \in \mathcal{C}$ such that C solves $\text{Car}(\delta)$.

On the bidisc, the two co-ordinate functions constitute a universal set for the Carathéodory problem. On G , there is a one-parameter family that constitutes a universal set.

DEFINITION 3.2. The function Φ is defined for $(z, s^1, s^2) \in \mathbb{C}^3$ such that $zs^1 \neq 2$ by

$$(3.3) \quad \Phi(z, s^1, s^2) = \frac{2zs^2 - s^1}{2 - zs^1}.$$

We shall write $\Phi_z(s)$ as a synonym for $\Phi(z, s^1, s^2)$ where $s = (s^1, s^2)$.

Φ_ω does belong to $\mathbb{D}(G)$ for every $\omega \in \mathbb{T}$: see Proposition A.1 in Appendix A. The following statement is in Theorem 1.1 (for discrete datums) and Corollary 4.3 (for infinitesimal datums) in [5].

THEOREM 3.3. *The set $\mathcal{C} = \{\Phi_\omega : \omega \in \mathbb{T}\}$ is a universal set for the Carathéodory problem on G .*

Theorem 3.3 provides a concrete way of analyzing the Carathéodory problem on G . Indeed, if, for a nondegenerate datum δ , we define a function ρ_δ on the circle by the formula

$$(3.4) \quad \rho_\delta(\omega) = |\Phi_\omega(\delta)|^2, \quad \omega \in \mathbb{T},$$

then Theorem 3.3 implies that

$$(3.5) \quad |\delta|_{\text{car}}^2 = \max_{\omega \in \mathbb{T}} \rho_\delta(\omega).$$

3.3. Five types of datum δ in G

The following definition formalizes five qualitative cases for the behavior of the Carathéodory extremal problem for G .

DEFINITION 3.4. Let δ be a nondegenerate datum in G . We say that δ is

(1) *purely unbalanced* if ρ_δ has a unique maximizer $\omega_0 = e^{it_0}$ and

$$\frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{t=t_0} < 0;$$

(2) *exceptional* if ρ_δ has a unique maximizer $\omega_0 = e^{it_0}$ and

$$\frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{t=t_0} = 0;$$

(3) *purely balanced* if ρ_δ has exactly two maximizers in \mathbb{T} ;

(4) *royal* if ρ_δ is constant and $C(s) = \frac{1}{2}s^1$ solves $\text{Car}(\delta)$;

(5) *flat* if ρ_δ is constant and $C(s) = s^2$ solves $\text{Car}(\delta)$.

Finally, we say that δ is *balanced* if δ is either exceptional, purely balanced, or royal and we say that δ is *unbalanced* if δ is either purely unbalanced or flat.

It may be helpful to look at the cartoons in Appendix B at this point; they provide visual images of geodesics of the five types.

PENTACHOTOMY THEOREM 3.6. *If δ is a nondegenerate datum in G , then exactly one of the cases (1) to (5) in Definition 3.4 obtains. In particular, either δ is balanced or δ is unbalanced and not both.*

PROOF. Case 1: δ is discrete.

(i) Suppose that ρ_δ is constant on \mathbb{T} . By Theorem A.14 either (a) $(s_j^1)^2 = 4s_j^2$ for $j = 1, 2$ or (b) there exists $\beta \in \mathbb{D}$ such that $s_j^1 = \beta + \bar{\beta}s_j^2$ for $j = 1, 2$. In case (a) we may write $s_j = (2z_j, z_j^2)$ where $z_j = \frac{1}{2}s_j^1$. We then find that

$$\Phi_\omega(s_j) = -z_j$$

independently of ω , and so, by equations (3.4) and (3.5),

$$|\delta|_{\text{car}}^2 = \rho_\delta(\omega) = |\Phi_\omega(\delta)|^2 = |(\Phi_\omega(s_1), \Phi_\omega(s_2))|^2 = |(-z_1, -z_2)|^2 = |(z_1, z_2)|^2 = |(\frac{1}{2}s_1^1, \frac{1}{2}s_2^1)|^2.$$

Since the function $C(s) = \frac{1}{2}s^1$ belongs to $\mathbb{D}(G)$, it follows that C solves $\text{Car}(\delta)$, and so, according to Definition 3.4, δ is a royal datum.

In case (b) we have $s_j = (\beta + \bar{\beta}z_j, z_j)$, where $z_j = s_j^2$. A simple calculation shows that, for any $\omega \in \mathbb{T}$,

$$\Phi_\omega(s_j) = m_\omega(z_j) \quad \text{for } j = 1, 2,$$

for some $m_\omega \in \text{Aut } \mathbb{D}$. Again by equations (3.4) and (3.5),

$$|\delta|_{\text{car}}^2 = \sup_\omega |\Phi_\omega(\delta)|^2 = |(\Phi_\omega(s_1), \Phi_\omega(s_2))|^2 = |(m_\omega(z_1), m_\omega(z_2))|^2 = |(z_1, z_2)|^2 = |(s_1^2, s_2^2)|^2.$$

Hence $C(s) = s^2$ solves $\text{Car}(\delta)$, and so δ is a flat datum.

(ii) Now suppose that ρ_δ is not constant on \mathbb{T} . It is clear from the definition of ρ_δ ,

$$\rho_\delta(e^{it}) = |\Phi_{e^{it}}(s_1, s_2)|^2 = \left| \frac{\Phi_{e^{it}}(s_1) - \Phi_{e^{it}}(s_2)}{1 - \overline{\Phi_{e^{it}}(s_2)}\Phi_{e^{it}}(s_1)} \right|^2,$$

that the function $t \mapsto \rho_\delta(e^{it})$ is smooth and 2π -periodic, and so attains its maximum over $[0, 2\pi)$ at some point t_0 , which is a critical point satisfying

$$\frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{t=t_0} \leq 0.$$

In fact, by Theorem A.13, ρ_δ attains its supremum over \mathbb{T} at either one or two points. In the former case δ is either purely unbalanced or exceptional, and in the latter case δ is purely balanced. We have shown that, for any nondegenerate discrete datum δ in G , at least one of (1) to (5) is true.

Conditions (1), (2) and (3) are clearly mutually exclusive. Moreover, since they together correspond to ρ_δ being nonconstant on \mathbb{T} , they are inconsistent with (4) and (5). It remains to show that δ cannot be both flat and royal. If it is, then by the calculations above there exists $\beta \in \mathbb{D}$ such that both s_1 and s_2 lie in the intersection of the sets

$$\{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\} \quad \text{and} \quad \{s \in G : (s^1)^2 = 4s^2\}.$$

Thus the equation

$$(\beta + \bar{\beta}z)^2 = 4z$$

has two distinct roots for $z \in \mathbb{D}$; this is easily seen to be false. Hence a nondegenerate discrete datum cannot be both royal and flat. We have shown that exactly one of the cases (1) to (5) obtains for any discrete datum δ .

Case 2: $\delta = (s_1, v)$ is a nondegenerate infinitesimal datum in G .

(i) Suppose that ρ_δ is constant on \mathbb{T} . By Theorem A.15, ρ_δ is constant on \mathbb{T} if and only if one of the following conditions holds:

- (a) there exists $z \in \mathbb{D}$ such that $s_1 = (2z, z^2)$ and v is collinear with $(1, z)$,
- (b) there exist $\beta, z \in \mathbb{D}$ such that $s_1 = (\beta + \bar{\beta}z, z)$ and v is collinear with $(\bar{\beta}, 1)$.

Thus, according to Definition 3.4, δ is a royal datum in case (a) and a flat datum in case (b).

(ii) Now suppose that ρ_δ is not constant on \mathbb{T} . Let $\delta = (s_1, v)$. By the definition of ρ_δ , for $\omega \in \mathbb{T}$,

$$\rho_\delta(e^{it}) = |\Phi_{e^{it}}(s_1, v)|^2 = |(\Phi_{e^{it}}(s_1), D_v \Phi_{e^{it}}(s_1))|^2 = \frac{|D_v \Phi_{e^{it}}(s_1)|^2}{(1 - |\Phi_{e^{it}}(s_1)|^2)^2}.$$

It is clear that the function $t \mapsto \rho_\delta(e^{it})$ is smooth and 2π -periodic, and so attains its maximum over $[0, 2\pi)$ at some point t_0 , which is a critical point satisfying

$$\frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{t=t_0} \leq 0.$$

By Theorem A.13, ρ_δ attains its supremum over \mathbb{T} at either one or two points. In the former case δ is either purely unbalanced or exceptional, and in the latter case δ is purely balanced. We have shown that, for any nondegenerate infinitesimal datum δ in G , at least one of (1) to (5) is true.

Conditions (1), (2) and (3) are clearly mutually exclusive. Moreover, since they together correspond to ρ_δ being nonconstant on \mathbb{T} , they are inconsistent with (4) and (5). It remains to show that δ cannot be both flat and royal. If it is, then there exists $z \in \mathbb{D}$ such that

$$s_1 = (2z, z^2) \text{ and } v \text{ is collinear with } (1, z),$$

and there exists $\beta \in \mathbb{D}$ such that

$$s_1 = (\beta + \bar{\beta}z^2, z^2) \text{ and } v \text{ is collinear with } (\bar{\beta}, 1).$$

Thus $(1, z)$ is collinear with $(\bar{\beta}, 1)$, and so $1 = \bar{\beta}z$, which is impossible for $z, \beta \in \mathbb{D}$. Hence a nondegenerate infinitesimal datum cannot be both royal and flat. We have shown that exactly one of the cases (1) to (5) obtains for δ . \square

We shall have more to say about the above classification scheme of datums using the relation (3.5). Later, in Theorem 7.1 we shall give necessary and sufficient geometrical conditions for a datum to be one of the five types in Definition 3.4. Additionally, the balanced/unbalanced dichotomy is characterized in geometric terms in Section 8 (Theorem 7.1 below).

Let δ be a datum in G . Observe that if C solves $\text{Car}(\delta)$, then, as automorphisms of \mathbb{D} act as isometries in the pseudohyperbolic metric, $m \circ C$ also solves $\text{Car}(\delta)$. Thus the solution to $\text{Car}(\delta)$ is never unique. This motivates the following definition.

DEFINITION 3.5. If δ is a nondegenerate datum in G , we say that *the solution to $\text{Car}(\delta)$ is essentially unique*, if whenever C_1 and C_2 solve $\text{Car}(\delta)$ there exists $m \in \text{Aut } \mathbb{D}$ such that $C_2 = m \circ C_1$.

3.4. The Kobayashi extremal problem $\text{Kob}(\delta)$ for G

Let δ be a datum in G . Just as solutions to $\text{Car}(\delta)$ are never unique, nor are solutions to $\text{Kob}(\delta)$: if $m \in \text{Aut } \mathbb{D}$ and f solves $\text{Kob}(\delta)$, then $f \circ m$ solves $\text{Kob}(\delta)$. This suggests the following analog of Definition 3.5.

DEFINITION 3.6. If δ is a nondegenerate datum in a domain U , we say that *the solution to $\text{Kob}(\delta)$ is essentially unique*, if, whenever f_1 and f_2 solve $\text{Kob}(\delta)$, there exists $m \in \text{Aut } \mathbb{D}$ of \mathbb{D} such that $f_2 = f_1 \circ m$.

It is easy to see that if U is strictly convex and the boundary of U is smooth then the solution of any Kobayashi extremal problem in U is essentially unique [32, Proposition 11.3.3]. On the other hand, if U is merely assumed to be convex, it does not follow that Kobayashi extremals are essentially unique – witness the example $U = \mathbb{D}^2$. It is noteworthy that for G , despite its non-convexity, solutions of Kobayashi problems are nevertheless essentially unique. For discrete datums, the following result is contained in [6, Theorem 0.3]. For infinitesimal datums it is proved in Appendix A – see Theorem A.10.

THEOREM 3.7. *If δ is a nondegenerate datum in G then the solution to $\text{Kob}(\delta)$ is essentially unique.*

REMARK 3.8. This theorem implies that if, for some nondegenerate datum δ in G , two solutions to $\text{Kob}(\delta)$ agree at some nondegenerate datum in \mathbb{D} then they coincide.

Complex geodesics in G

In Section 4.1 we establish some facts required for the proof of Theorem 7.1. We begin with some generalities. In the present context it is convenient to use the term ‘geodesic’ to denote a *set*, rather than a holomorphic map on \mathbb{D} (as for example in [34]). For the latter notion we reserve the term ‘complex C-geodesic’.

4.1. Complex geodesics and datums in G

DEFINITION 4.1. Let U be a domain and let $\mathcal{D} \subset U$. We say that \mathcal{D} is a *complex geodesic* in U if there exists a function $k \in U(\mathbb{D})$ and a function $C \in \mathbb{D}(U)$ such that $C \circ k = \text{id}_{\mathbb{D}}$ and $\mathcal{D} = k(\mathbb{D})$.

Kobayashi [34, Chapter 4, Section 6], following [46], defines a *complex C-geodesic* in a domain U to be a map $k \in U(\mathbb{D})$ such that, for every nondegenerate datum ζ in \mathbb{D} ,

$$|k(\zeta)|_{\text{car}} = |\zeta|.$$

It follows easily from the Schwarz-Pick Lemma that $k \in U(\mathbb{D})$ is a complex C-geodesic if and only if k has a holomorphic left inverse. Thus a map k is a complex C-geodesic if and only if $k(\mathbb{D})$ is a complex geodesic in the sense of Definition 4.1¹.

A notable advance in the hyperbolic geometry of domains in \mathbb{C}^n was the seminal theorem of Lempert [36] to the effect that the Carathéodory and Kobayashi metrics agree on any bounded convex domain U : $|\cdot|_{\text{kob}} = |\cdot|_{\text{car}}$ in U . It implies that if δ is a nondegenerate datum in a bounded convex domain $U \subseteq \mathbb{C}^n$, then there exists a solution C to $\text{Car}(\delta)$ and a solution k to $\text{Kob}(\delta)$ such that

$$(4.1) \quad C \circ k = \text{id}_{\mathbb{D}}.$$

This result guarantees a plentiful supply of complex geodesics in bounded convex domains.

The relationship between datums in U and complex geodesics in U can be cleanly described using the notion of contact.

DEFINITION 4.2. Let U be a domain, let V be a subset of U and let δ be a datum in U . Then δ *contacts* V if either

- (1) $\delta = (s_1, s_2)$ is discrete and $s_1 \in V$ and $s_2 \in V$, or
- (2) $\delta = (s, v)$ is infinitesimal and either $v = 0$ and $s \in V$, or there exist two sequences of points $\{s_n\}$ and $\{t_n\}$ in V such that $s_n \neq t_n$ for all n , $s_n \rightarrow s$, $t_n \rightarrow s$ and

$$\frac{t_n - s_n}{\|t_n - s_n\|} \rightarrow v_0,$$

for some unit vector v_0 collinear with v .

¹Kobayashi also uses the term ‘complex geodesic’, in a sense which differs slightly from ours, though the difference is not significant for the domains studied in this paper.

REMARK 4.3. If a datum δ in U contacts a set $V \subseteq U$, then, for any holomorphic map F on U , the value of $F(\delta)$ only depends on $F|V$. It is obvious for discrete δ and for the infinitesimal datum $\delta = (s, 0)$. Consider a nondegenerate infinitesimal datum $\delta = (s, v)$ that contacts a subset V of U . Then there exist two sequences of points $\{s_n\}$ and $\{t_n\}$ in V such that $s_n \neq t_n$ for all n , $s_n \rightarrow s$, $t_n \rightarrow s$ and

$$\frac{t_n - s_n}{\|t_n - s_n\|} \rightarrow v_0,$$

for some unit vector v_0 collinear with v . Note that

$$t_n = s_n + \frac{t_n - s_n}{\|t_n - s_n\|} \|t_n - s_n\|,$$

and so, for any domain Ω and $F \in \Omega(U)$,

$$F(s) = \lim_{n \rightarrow \infty} F(s_n) \quad \text{and} \quad D_v F(s) = \lim_{n \rightarrow \infty} \frac{F(t_n) - F(s_n)}{\|t_n - s_n\|}.$$

Thus $F(\delta)$ is determined by $F|V$.

The notion of contact permits Lempert's discovery to be expressed in the following geometric fashion:

For every nondegenerate datum δ in a bounded convex domain U there is a complex geodesic in U that δ contacts.

The following proposition relates the concept of contact to holomorphic maps on \mathbb{D} .

PROPOSITION 4.4. *Let U be a domain in \mathbb{C}^n .*

- (1) *If $f \in U(\mathbb{D})$ and ζ is a datum in \mathbb{D} then $f(\zeta)$ contacts $f(\mathbb{D})$;*
- (2) *if k is a complex C -geodesic in U and δ is a datum in U that contacts $k(\mathbb{D})$, then there exists a datum ζ in \mathbb{D} such that $\delta = k(\zeta)$.*

PROOF. (1) The statement is trivial for discrete datums. Let $\zeta = (z, v)$ be an infinitesimal datum in \mathbb{D} . If $f'(z)v = 0$ then $f(\zeta) = (f(z), 0)$, which contacts $f(\mathbb{D})$. Consider the case that $f'(z)v \neq 0$. Let

$$u_n = z + \frac{1}{n}v, \quad v_n = z + \frac{1}{2n}v.$$

For n a sufficiently large integer, u_n and v_n are in \mathbb{D} , and we may define $s_n = f(u_n)$, $t_n = f(v_n)$ in U . Clearly $s_n \rightarrow f(z)$, $t_n \rightarrow f(z)$. Since

$$s_n - t_n = \frac{1}{2n}f'(z)v + o(1/n)$$

and $f'(z)v$ is nonzero, $s_n \neq t_n$ for sufficiently large n , and

$$\begin{aligned} \frac{s_n - t_n}{\|s_n - t_n\|} &= \frac{(2n)^{-1}f'(z)v + o(\frac{1}{n})}{\|(2n)^{-1}f'(z)v + o(\frac{1}{n})\|} \\ &\rightarrow \frac{f'(z)v}{\|f'(z)v\|}, \end{aligned}$$

which is collinear with $f'(z)v$. Thus $f(\zeta)$ contacts $f(\mathbb{D})$.

(2) Let k be a complex C -geodesic. Then k has a left inverse $C \in \mathbb{D}(U)$. Let $V = k(\mathbb{D})$. We have

$$k \circ C|V = \text{id}_U|V$$

Since δ contacts V , by Remark 4.3,

$$(k \circ C)(\delta) = \text{id}_U(\delta) = \delta.$$

Let $\zeta = C(\delta)$. Then ζ is a datum in \mathbb{D} and, by equation (3.1), $k(\zeta) = \delta$. \square

4.2. Uniqueness of complex geodesics for each datum in G

An important fact in the function theory of G is that Lempert's conclusion ($|\cdot|_{\text{kob}} = |\cdot|_{\text{car}}$) holds for G , even though G does not satisfy any of the various convexity hypotheses of versions of Lempert's theorem: G is not convex (nor even biholomorphic to a convex domain [23]).

THEOREM 4.5. *For any datum δ in G , $|\delta|_{\text{kob}} = |\delta|_{\text{car}}$.*

For discrete datums this is [5, Corollary 5.7]. In fact, since G is taut, it follows that equality also holds for infinitesimal datums [32, Proposition 11.1.7]. See also [32, Theorem 7.1.16].

The following property of G plays a fundamental role in this paper.

THEOREM 4.6. *For every nondegenerate datum δ in G there exists a unique complex geodesic \mathcal{D} in G such that δ contacts \mathcal{D} . Moreover $\mathcal{D} = k(\mathbb{D})$ where k solves $\text{Kob}(\delta)$.*

PROOF. Let δ be a nondegenerate datum in G . Let k solve $\text{Kob}(\delta)$ and C solve $\text{Car}(\delta)$. Then there exists a datum ζ in \mathbb{D} such that $k(\zeta) = \delta$ and

$$|\zeta| = |\delta|_{\text{kob}} = |\delta|_{\text{car}} = |C(\delta)| > 0.$$

Thus $C \circ k \in \mathbb{D}(\mathbb{D})$ and $|C \circ k(\zeta)| = |\zeta|$. By the Schwarz-Pick Lemma, $C \circ k \in \text{Aut } \mathbb{D}$. Let $\mathcal{D} = k(\mathbb{D})$. Then \mathcal{D} is a complex geodesic in G and $\delta = k(\zeta)$ contacts \mathcal{D} as desired.

To prove uniqueness, let \mathcal{D}' be any complex geodesic of G that is contacted by δ . By Definition 4.1 there exists $h \in G(\mathbb{D})$ and $F \in \mathbb{D}(G)$ such that $F \circ h = \text{id}_{\mathbb{D}}$ and $\mathcal{D}' = h(\mathbb{D})$. Since δ contacts $h(\mathbb{D})$, by Proposition 4.4, there exists a datum ζ' in \mathbb{D} such that $h(\zeta') = \delta$. Since $F \circ h = \text{id}_{\mathbb{D}}$ we have $|\zeta'| = |\delta| = |\zeta|$, and so h, k are both solutions of $\text{Kob}(\delta)$. By Theorem 3.7, $h = k \circ m$ for some $m \in \text{Aut } \mathbb{D}$. Thus $\mathcal{D}' = h(\mathbb{D}) = k(\mathbb{D}) = \mathcal{D}$. \square

As a consequence of Theorem 4.6 we may unambiguously attach to each datum in G a unique complex geodesic.

DEFINITION 4.7. For any nondegenerate datum δ in G the unique complex geodesic in G that is contacted by δ is denoted by \mathcal{D}_δ .

4.3. Flat C-geodesics

The symmetrized bidisc has the remarkable property that it is foliated by the ranges of its complex C-geodesics of degree one. For any $\beta \in \mathbb{D}$ we introduce the function $f_\beta : \mathbb{D} \rightarrow \mathbb{C}^2$ given by

$$(4.2) \quad f_\beta(z) = (\beta + \bar{\beta}z, z)$$

and the set

$$(4.3) \quad \mathcal{F}_\beta = f_\beta(\mathbb{D}) = \{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\}.$$

PROPOSITION 4.8. *For any $\beta \in \mathbb{D}$ the map f_β is a complex C-geodesic of G .*

PROOF. By Proposition A.1, f_β maps \mathbb{D} into G . Thus $f_\beta \in G(\mathbb{D})$, and clearly f_β has a holomorphic left inverse, to wit the co-ordinate function s^2 . Hence f_β is a complex C-geodesic of G . \square

It follows of course that \mathcal{F}_β is a complex geodesic of G .

The following proposition implies that G is foliated by the flat geodesics \mathcal{F}_β , $\beta \in \mathbb{D}$, in G . The proof is a simple calculation.

PROPOSITION 4.9. *If $s \in G$ then there is a unique $\beta \in \mathbb{D}$ such that $s \in \mathcal{F}_\beta$. Moreover β is given by*

$$\beta = \frac{s^1 - \bar{s}^1 s^2}{1 - |s^2|^2}.$$

The fact that β so defined lies in \mathbb{D} is contained in Proposition A.1.

PROPOSITION 4.10. *The following statements are equivalent for a complex C-geodesic k of G .*

- (1) $\deg(k) = 1$;
- (2) $k = f_\beta \circ m$ for some $\beta \in \mathbb{D}$ and $m \in \text{Aut } \mathbb{D}$;
- (3) $k(\mathbb{D})$ is the intersection of G with a complex line;
- (4) $k(\mathbb{D})$ is contained in the intersection of G with a complex line.

PROOF. (1) \Leftrightarrow (2) The complex C-geodesics of G are explicitly described in [6, 41]: they are Γ -inner functions of degree one or two, and those of degree one are precisely of the form in (2).

(2) \Rightarrow (3) \Rightarrow (4) is trivial. Conversely, suppose that $k(\mathbb{D})$ is contained in the intersection of a complex line with G . Let $\omega \in \mathbb{T}$ be such that Φ_ω is a left inverse of k . Then $m \stackrel{\text{def}}{=} \Phi_\omega \circ k \in \text{Aut } \mathbb{D}$. By Theorem A.7 and Proposition A.8, k^2 is a non-constant Blaschke product, φ say, of degree 1 or 2, and

$$(4.4) \quad k^1 = \bar{k}^1 k^2 \quad \text{on } \mathbb{T}.$$

Since $k(\mathbb{D})$ lies in a complex line we can write $k = (a + b\varphi, \varphi)$ for some $a, b \in \mathbb{C}$. Moreover the relation (4.4) implies that $b = \bar{a}$. Hence

$$(4.5) \quad k = (a + \bar{a}\varphi, \varphi).$$

Since φ is nonconstant it has a zero z_0 in \mathbb{D} . Since $k(z_0) = (a, 0)$ and $k(z_0) \in G$, we have $a \in \mathbb{D}$. Differentiate the identity

$$(2\omega\varphi - a - \bar{a}\varphi)(z) = m(z)(2 - \omega a - \omega\bar{a}\varphi(z))$$

and substitute $z = z_0$ to show that $\varphi'(z_0) \neq 0$ and thus that z_0 is a simple zero of φ . Since k has a holomorphic left inverse, k is injective, and therefore z_0 is the only zero of φ in \mathbb{D} . Hence φ has degree 1 and so $\varphi \in \text{Aut } \mathbb{D}$. Equation (4.5) shows that statement (2) holds. Thus (4) implies (2). \square

In the light of Proposition 4.10 we shall call any complex C-geodesic of degree one a *flat C-geodesic* of G . There is no conflict with the notion of flat datum, as introduced in Definition 3.4, for the following reason.

PROPOSITION 4.11. *A nondegenerate datum δ in G is a flat datum if and only if there is a flat C-geodesic k that solves $\text{Kob}(\delta)$.*

PROOF. Suppose that δ is a flat datum. Since ρ_δ is constant on \mathbb{T} , Theorems A.14 and A.15 imply that δ contacts either \mathcal{R} or \mathcal{F}_β for some $\beta \in \mathbb{D}$. In the former case the function $C(s) = \frac{1}{2}s^1$ solves $\text{Car}(\delta)$, which implies that the datum δ is both royal and flat, contrary to the Pentachotomy Theorem. Hence δ contacts \mathcal{F}_β , and the flat C-geodesic $k = f_\beta$ solves $\text{Kob}(\delta)$.

Conversely, suppose that $f_\beta \circ m$ solves $\text{Kob}(\delta)$ for some $\beta \in \mathbb{D}$, $m \in \text{Aut } \mathbb{D}$; then so does f_β . Since $C \circ f_\beta = \text{id}_\mathbb{D}$, where $C(s) = s^2$, it follows that C solves $\text{Car}(\delta)$. A simple calculation shows that, for any $\omega \in \mathbb{T}$,

$$\Phi_\omega \circ f_\beta = m_\omega \quad \text{for some } m_\omega \in \text{Aut } \mathbb{D}.$$

Since δ contacts $f_\beta(\mathbb{D})$, by Proposition 4.4, $\delta = f_\beta(\zeta)$ for some nondegenerate datum ζ in \mathbb{D} . We have, in view of the definition (3.4),

$$\rho_\delta(\omega) = |\Phi_\omega(\delta)|^2 = |\Phi_\omega \circ f_\beta(\zeta)|^2 = |m_\omega(\zeta)|^2 = |\zeta|^2.$$

Hence ρ_δ is constant on \mathbb{T} . Thus δ is a flat datum. \square

4.4. Rational Γ -inner functions

Let us recall some notions and results from [10]. The closure of G in \mathbb{C}^2 will be denoted by Γ ; thus

$$\Gamma = \{(z + w, zw) : z, w \in \mathbb{D}^-\}.$$

The distinguished boundary of Γ is denoted by $b\Gamma$.

DEFINITION 4.12. A *rational Γ -inner function* is a rational analytic map $h : \mathbb{D} \rightarrow \Gamma$ with the property that h maps \mathbb{T} into the distinguished boundary $b\Gamma$ of Γ .

All complex C-geodesics in G are rational Γ -inner functions (Theorem A.7).

For any rational Γ -inner function h , the second component h^2 is a finite Blaschke product. The degree $\deg(h)$ is equal to the degree of h^2 (in the usual sense). The following notion will play an important role.

DEFINITION 4.13. Let h be a rational Γ -inner function. A *royal node* of h is a point $\lambda \in \mathbb{D}^-$ such that $h(\lambda) \in \mathcal{R}^-$. A *royal point* of h is a point of $h(\mathbb{D}^-) \cap \mathcal{R}^-$.

The following result is [10, Theorem 1.1]. Observe that $\lambda \in \mathbb{D}^-$ is a royal node of h if and only if $h^1(\lambda)^2 = 4h^2(\lambda)$.

THEOREM 4.14. *If h is a nonconstant rational Γ -inner function then either $h(\mathbb{D}^-) = \mathcal{R}^-$ or h has exactly $\deg(h)$ royal nodes, counted with multiplicity.*

Here multiplicity is defined as follows.

DEFINITION 4.15. Let h be a rational Γ inner function. If σ is a zero of $(h^1)^2 - 4h^2$ of order ℓ , we define the *multiplicity* $\#\sigma$ of σ (as a royal node of h) by

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D} \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

LEMMA 4.16. *Let δ be a nondegenerate datum in G , let k solve $\text{Kob}(\delta)$ and suppose that k^2 is a Blaschke product of degree 2. Let $\omega \in \mathbb{T}$. The following conditions are equivalent:*

- (i) ω is a maximiser of ρ_δ on \mathbb{T} ;
- (ii) $(2\bar{\omega}, \bar{\omega}^2)$ is in $k(\mathbb{D})^- \cap \partial\mathcal{R}$;
- (iii) Φ_ω solves $\text{Car}(\delta)$.

PROOF. By the definition of ρ_δ , (i) \Leftrightarrow (iii).

(ii) \Rightarrow (iii). Suppose (ii), and let $\tau \in \mathbb{D}^-$ be such that $k(\tau) = (2\bar{\omega}, \bar{\omega}^2)$. By the Maximum Modulus Principle, $\tau \in \mathbb{T}$. Consider the rational function

$$(4.6) \quad \Phi_\omega \circ k(\lambda) = \frac{2\omega k^2(\lambda) - k^1(\lambda)}{2 - \omega s(\lambda)}$$

which is inner, by Proposition A.3. By [8, Corollary 6.10] k^1, k^2 are rational functions having the same denominator and therefore $\Phi_\omega \circ k$ has apparent degree 2, the degree of k . However, as $\lambda \rightarrow \tau$, both the numerator and the denominator of the right hand side of equation (4.6) tend to zero. Hence both numerator and denominator are divisible by $\lambda - \tau$, and so $\Phi_\omega \circ k$ is a rational inner function of degree at most one.

Suppose that $\Phi_\omega \circ k$ is constant. Then $k(\mathbb{D})$ is contained in a complex line, and so, by Proposition 4.10, k has degree one, contrary to hypothesis. Thus $\Phi_\omega \circ k$ is a rational inner function of degree exactly one, which is to say that $\Phi_\omega \circ k \in \text{Aut } \mathbb{D}$.

Since k solves $\text{Kob}(\delta)$ there exists a datum $\zeta \in \mathbb{D}$ such that $k(\zeta) = \delta$ and $|\zeta| = |\delta|_{\text{kob}}$. By the invariance of $|\cdot|$ under automorphisms,

$$|\Phi_\omega \circ k(\zeta)| = |\zeta|.$$

Therefore

$$|\Phi_\omega(\delta)| = |\Phi_\omega \circ k(\zeta)| = |\zeta| = |\delta|_{\text{kob}} = |\delta|_{\text{car}}.$$

Hence Φ_ω solves $\text{Car}(\delta)$. Thus (ii) implies (iii).

(iii) \Rightarrow (ii). Let Φ_ω solve $\text{Car}(\delta)$. Then $\Phi_\omega \circ k \in \text{Aut } \mathbb{D}$ has degree one. Thus

$$\Phi_\omega \circ k = \frac{2\omega k^2 - k^1}{2 - \omega k^1}$$

has a cancellation at some point $\tau \in \mathbb{D}^-$ (that is, $\lambda - \tau$ divides both numerator and denominator). By Theorem A.6, $\tau \in \mathbb{T}$, τ is a royal node for k and $k(\tau) = (2\bar{\omega}, \bar{\omega}^2)$. Therefore $(2\bar{\omega}, \bar{\omega}^2)$ is in $k(\mathbb{D})^- \cap \partial\mathcal{R}$. \square

PROPOSITION 4.17. *Let k be a rational Γ -inner function of degree 2. Then k is a complex C -geodesic of G if and only if k has a royal node in \mathbb{T} .*

PROOF. Let k be a complex C -geodesic, so that k has a holomorphic left inverse C . Pick a nondegenerate datum ζ in \mathbb{D} . By Proposition 3.1, k solves $\text{Kob}(\delta)$, where $\delta = k(\zeta)$. Let ω be a maximizer of ρ_δ in \mathbb{T} . By Lemma 4.16, there exists $z \in \mathbb{T}$ such that $k(z) = (2\bar{\omega}, \bar{\omega}^2)$. Then z is a royal node of k in \mathbb{T} .

Conversely, suppose that k has a royal node $z \in \mathbb{T}$, so that $k(z) = (2\bar{\omega}, \bar{\omega}^2)$ for some $\omega \in \mathbb{T}$. By Lemma 4.16, Φ_ω solves $\text{Car}(\delta)$. Since $|\cdot|_{\text{kob}} = |\cdot|_{\text{car}}$, it follows that Φ_ω is a holomorphic left inverse of k modulo $\text{Aut } \mathbb{D}$. Hence k is a complex C -geodesic of G . \square

The retracts of G and the bidisc \mathbb{D}^2

As was observed in the introduction, there are inclusions between the collections of complex geodesics, holomorphic retracts and sets with the norm-preserving extension property in a general domain. In this chapter we prove Theorem 1.3, to the effect that the nontrivial retracts in G are the complex geodesics. By way of a digression, we show that a similar geometric argument can be used to establish the (known) description of the retracts in \mathbb{D}^2 . Furthermore, we show by an explicit construction that complex geodesics in G are zero sets of polynomials of degree at most 2.

5.1. Retracts and geodesics of G

Let U be a domain. We say that ρ is a *retraction of U* if $\rho \in U(U)$ and $\rho \circ \rho = \rho$. A set $R \subseteq U$ is a *retract in U* if there exists a retraction ρ of U such that $R = \text{ran } \rho$. We say that a retraction ρ and the corresponding retract R are *trivial* if either ρ is constant (and R is a singleton set) or $\rho = \text{id}_U$ (and $R = U$).

A particularly simple way that a nontrivial retraction can arise is if $C \in \mathbb{D}(U)$, $k \in U(\mathbb{D})$ and $C \circ k = \text{id}_{\mathbb{D}}$. In that event, if we set $\rho = k \circ C$, then $\rho \in U(U)$ and

$$(5.1) \quad \rho \circ \rho = (k \circ C) \circ (k \circ C) = k \circ (C \circ k) \circ C = k \circ \text{id}_{\mathbb{D}} \circ C = k \circ C = \rho.$$

Therefore, if U has dimension greater than one, ρ is a nontrivial retraction and the complex geodesic $\text{ran } k = \text{ran } \rho$ is a nontrivial retract. Thus, on general domains in dimension greater than one, complex geodesics are nontrivial retracts. The question arises: for which domains is the converse of this statement true? See for example [28, 33]. It *is* true for the symmetrized bidisc.

THEOREM 5.1. *A map $\rho \in G(G)$ is a nontrivial retraction of G if and only if $\rho = k \circ C$ for some $C \in \mathbb{D}(G)$ and $k \in G(\mathbb{D})$ such that $C \circ k = \text{id}_{\mathbb{D}}$. Thus R is a nontrivial retract in G if and only if R is a complex geodesic in G .*

PROOF. Let ρ be a retraction of G that is neither constant nor id_G and let $R = \text{ran } \rho$. As ρ is not constant, there exists a pair s_1, s_2 of distinct points in $\rho(G)$. Let δ be the nondegenerate discrete datum (s_1, s_2) in G .

Choose a datum ζ in \mathbb{D} and $k \in G(\mathbb{D})$ such that k solves $\text{Kob}(\delta)$ and $k(\zeta) = \delta$. Since $\rho|_R$ is the identity, $\rho(\delta) = \delta$, and so both k and $\rho \circ k$ solve $\text{Kob}(\delta)$. Moreover $k(\zeta) = \delta = \rho \circ k(\zeta)$. By Remark 3.8, $\rho \circ k = k$. Therefore

$$\mathcal{D}_\delta = k(\mathbb{D}) = \rho \circ k(\mathbb{D}) \subseteq R.$$

If $\mathcal{D}_\delta \neq R$ then there exists $t \in R \setminus \mathcal{D}_\delta$. By the argument in the foregoing paragraph, for any $s \in \mathcal{D}_\delta$, the complex geodesic through t and s is contained in

R , that is, $\mathcal{D}_{(t,s)} \subseteq R$. In particular, as $s_1 \neq s_2$, the geodesics $\mathcal{D}_1 \stackrel{\text{def}}{=} \mathcal{D}_{(t,s_1)}$ and $\mathcal{D}_2 \stackrel{\text{def}}{=} \mathcal{D}_{(t,s_2)}$ are contained in R . Furthermore, Theorem 4.6 implies that $\mathcal{D}_1 \neq \mathcal{D}_2$.

Choose k_j to solve $\text{Kob}(t, s_j)$ and a vector v_j tangent to \mathcal{D}_j , $j = 1, 2$ at t . Since $\mathcal{D}_1 \neq \mathcal{D}_2$, it follows from Theorem 4.6 that v_1, v_2 are not collinear. Since ρ acts as the identity map on R ,

$$D\rho(t)v_j = v_j, \quad j = 1, 2.$$

Hence $D\rho(t)$ is an invertible linear transformation. By the Inverse Function Theorem $\rho(G)$ contains a neighborhood of t in \mathbb{C}^2 . Since ρ is the identity on its range, ρ agrees with id_G on a nonempty open set. Thus $\rho = \text{id}_G$, contrary to our assumption. Hence $R = \mathcal{D}_\delta$. \square

5.2. Retracts of \mathbb{D}^2

The argument used in the preceding section to characterize retractions in G applies in any domain U in \mathbb{C}^2 with the property that, for every nondegenerate datum δ in U , the solution to $\text{Kob}(\delta)$ is essentially unique. The bidisc does not have this property, but nevertheless, a variant of the argument is effective. We shall exploit the fact that, for datums λ in \mathbb{D}^2 , the solution of $\text{Kob}(\lambda)$ is not essentially unique precisely when the solution of $\text{Car}(\lambda)$ is essentially unique. Thereby we shall obtain known results on retractions of \mathbb{D}^2 (see [28, 27, 33]) by simple geometric methods.

Incidentally, the implications of uniqueness and nonuniqueness of solutions of Carathéodory and Kobayashi problems in many domains (including G) are explored in [35].

DEFINITION 5.2. Let $\lambda = (\lambda_1, \lambda_2)$ be a nondegenerate discrete datum in \mathbb{D}^2 . Then λ is *balanced*, of *type 1* or of *type 2* according as

$$|(\lambda_1^1, \lambda_2^1)| = |(\lambda_1^2, \lambda_2^2)|, \quad |(\lambda_1^1, \lambda_2^1)| > |(\lambda_1^2, \lambda_2^2)| \quad \text{or} \quad |(\lambda_1^1, \lambda_2^1)| < |(\lambda_1^2, \lambda_2^2)|$$

respectively. λ is *unbalanced* if λ is of type 1 or type 2.

If λ is a balanced datum in \mathbb{D}^2 then the unique geodesic in \mathbb{D}^2 that contacts λ is a balanced disc in the sense of Definition 8.3. It is not hard to see that, for the nondegenerate datum $\lambda = (\lambda_1, \lambda_2)$ in \mathbb{D}^2 , the Kobayashi extremal problem $\text{Kob}(\lambda)$ has an essentially unique solution if and only if λ is balanced, whereas the Carathéodory extremal problem $\text{Car}(\lambda)$ has an essentially unique solution if and only if λ is unbalanced.

THEOREM 5.3. *A subset R of \mathbb{D}^2 is a nontrivial retract in \mathbb{D}^2 if and only if R is a complex geodesic in \mathbb{D}^2 .*

PROOF. Since R is assumed to be nontrivial, in particular R is not a singleton. Choose points $\lambda_1, \lambda_2 \in R$ with $\lambda_1 \neq \lambda_2$.

If $\lambda = (\lambda_1, \lambda_2)$ is balanced, then by [2, page 293] the solution to $\text{Kob}(\lambda)$ is essentially unique. It then follows by the argument in the preceding section that $R = k(\mathbb{D})$ where k solves $\text{Kob}(\lambda)$.

Now assume that λ is not balanced, say

$$(5.2) \quad |(\lambda_1^1, \lambda_2^1)| > |(\lambda_1^2, \lambda_2^2)|.$$

Let ρ be a retraction of \mathbb{D}^2 with $\rho(\mathbb{D}^2) = R$. Let $f_1, f_2 : \mathbb{D}^2 \rightarrow \mathbb{D}$ be the first and second co-ordinate functions. Since $\{f_1, f_2\}$ constitutes a universal set for the

Carathéodory problem on \mathbb{D}^2 , the inequality (5.2) shows that f_1 solves $\text{Car}(\lambda)$. Since λ contacts R and $\rho|_R$ is the identity map, it is clear that $f_1 \circ \rho$ also solves $\text{Car}(\lambda)$. By the essential uniqueness of solutions of $\text{Car}(\lambda)$, there exists $m \in \text{Aut } \mathbb{D}$ such that $f_1 \circ \rho = m \circ f_1$. Apply this equation to the points $\lambda_1, \lambda_2 \in R$ to obtain the equations

$$\lambda_1^1 = m(\lambda_1^1) \quad \text{and} \quad \lambda_2^1 = m(\lambda_2^1).$$

The relation (5.2) shows that $\lambda_1^1 \neq \lambda_2^1$, and thus m fixes a pair of distinct points in \mathbb{D} . Hence $m = \text{id}_{\mathbb{D}}$, and so

$$(5.3) \quad f_1 \circ \rho = f_1$$

on \mathbb{D}^2 . Thus ρ has the form

$$\rho(\lambda) = (\lambda^1, \varphi(\lambda))$$

for some $\varphi \in \mathbb{D}(\mathbb{D}^2)$. Consequently, if for each fixed $\alpha \in \mathbb{D}$ we define $\varphi_\alpha \in \mathbb{D}(\mathbb{D})$ by the formula,

$$\varphi_\alpha(w) = \varphi(\alpha, w), \quad w \in \mathbb{D}.$$

For any $z \in \mathbb{D}^2$, the equation $\rho \circ \rho = \rho$ yields the relation

$$\rho(z^1, \varphi(z)) = (z^1, \varphi(z)).$$

Hence

$$(5.4) \quad \varphi(z) = \varphi(z^1, \varphi(z)), \quad z \in \mathbb{D}^2.$$

We claim that φ_α is a retraction of \mathbb{D} . Indeed, for $w \in \mathbb{D}$,

$$\begin{aligned} \varphi_\alpha(\varphi_\alpha(w)) &= \varphi(\alpha, \varphi(\alpha, w)) \\ &= \varphi(\alpha, w) \quad (\text{by equation (5.4)}) \\ &= \varphi_\alpha(w). \end{aligned}$$

This proves that φ_α is a retraction of \mathbb{D} for all $\alpha \in \mathbb{D}$.

Now retractions of \mathbb{D} are either constant maps or $\text{id}_{\mathbb{D}}$. If there exists $\alpha_0 \in \mathbb{D}$ such that $\varphi_{\alpha_0} = \text{id}_{\mathbb{D}}$, then by continuity, $\varphi_\alpha = \text{id}_{\mathbb{D}}$ for all α in a neighborhood of α_0 . But then $\rho = \text{id}_{\mathbb{D}^2}$, which contradicts the assumption that R is nontrivial. Therefore, φ_α is constant for all $\alpha \in \mathbb{D}$.

For each $\alpha \in \mathbb{D}$, choose $g(\alpha) \in \mathbb{D}$ where $\varphi_\alpha(w) = g(\alpha)$ for all $w \in \mathbb{D}$. Then

$$\rho(\lambda) = (\lambda_1, \varphi(\lambda)) = (\lambda_1, \varphi_{\lambda_1}(\lambda_2)) = (\lambda_1, g(\lambda_1))$$

for all $\lambda_1 \in \mathbb{D}$. This proves that if we define $k \in \mathbb{D}^2(\mathbb{D})$ by the formula $k(w) = (w, g(w))$, $w \in \mathbb{D}$, then

$$R = \text{ran } \rho = \text{ran } k.$$

Since in addition, $f_1 \circ k = \text{id}_{\mathbb{D}}$, it follows that R is a geodesic.

If λ is not balanced and $|(\lambda_1^2, \lambda_2^2)| > |(\lambda_1^1, \lambda_2^1)|$, then the reasoning as above, with the roles of the coordinates reversed, again yields the conclusion that R is a geodesic. \square

5.3. Geodesics in G are varieties

The relation in equation (5.1) has another valuable application, namely to the calculation of a quadratic polynomial that vanishes on a prescribed geodesic of G .

THEOREM 5.4. *For every complex geodesic \mathcal{D} in G there exists a polynomial P of total degree at most 2 such that $\mathcal{D} = \{s \in G : P(s) = 0\}$.*

If \mathcal{D} is flat then we may choose $P(s) = s^1 - \beta - \bar{\beta}s^2$ for some $\beta \in \mathbb{D}$. The result therefore follows from the following more detailed statement.

Warning: in the next proposition superscript 2 is used for the square of a number and for the second component in the same setting. For example, $(s^2)^2$ is the square of the the second component of $s = (s^1, s^2)$.

PROPOSITION 5.5. *Let δ be a nonflat nondegenerate datum in G , let $\omega \in \mathbb{T}$ be such that Φ_ω solves $\text{Car}(\delta)$ and let $k \in G(\mathbb{D})$ satisfy $\Phi_\omega \circ k = \text{id}_{\mathbb{D}}$. Then k^2 is expressible in the form*

$$(5.5) \quad k^2 = \frac{q^\sim}{q}$$

for some polynomial q of degree 2 that does not vanish on \mathbb{D}^- , where $q^\sim(z) = z^2 q(1/\bar{z})$.

Let

$$(5.6) \quad P(s) = (2 - \omega s^1)^2 \frac{\tilde{q} \circ \Phi_\omega(s) - s^2 q \circ \Phi_\omega(s)}{\omega^2 s^2 - 1}.$$

Then

- (1) P is a polynomial of total degree 2 that vanishes on \mathcal{D}_δ ;
- (2) P is irreducible;
- (3) $P(s) = (s^2)^2 \overline{P(\bar{s}^1/\bar{s}^2, 1/\bar{s}^2)}$ and
- (4) $\mathcal{D}_\delta = \{s \in G : P(s) = 0\}$.

PROOF. Proposition A.8 in the Appendix contains the statement that k^2 is expressible in the form (5.5) for some polynomial q of degree at most 2 that does not vanish on \mathbb{D}^- . Since δ is not flat, $\deg k^2 = 2$, which is to say that $\deg(q^\sim) = 2$.

By equation (5.1), $k \circ \Phi_\omega$ is a retraction onto $\text{ran } k = \mathcal{D}_\delta$. Therefore, for $s \in \mathcal{D}_\delta$,

$$(k \circ \Phi_\omega)(s) = s.$$

Writing $k = (k^1, k^2)$ and $s = (s^1, s^2)$, we obtain the equations

$$(5.7) \quad k^1 \circ \Phi_\omega(s) = s^1 \quad \text{and} \quad k^2 \circ \Phi_\omega(s) = s^2$$

for all $s \in \mathcal{D}_\delta$.

By Proposition A.8, $k^2(\bar{\omega}) = \bar{\omega}^2$, and therefore

$$(5.8) \quad q^\sim(\bar{\omega}) = \bar{\omega}^2 q(\bar{\omega}).$$

Substitute $z = \Phi_\omega(s)$ into equation (5.5) and use the second equation in (5.7) to obtain

$$(5.9) \quad q^\sim \circ \Phi_\omega(s) = s^2 q \circ \Phi_\omega(s)$$

for all $s \in \mathcal{D}_\delta$. By virtue of equation (3.3), if Q^\sim and Q are defined by the formulas

$$(5.10) \quad Q^\sim(s) = (2 - \omega s^1)^2 q^\sim \circ \Phi_\omega(s) \quad \text{and} \quad Q(s) = (2 - \omega s^1)^2 q \circ \Phi_\omega(s),$$

then Q^\sim and Q are quadratic polynomials in two variables, and equation (5.9) becomes

$$(5.11) \quad Q^\sim(s) = s^2 Q(s) \quad \text{for all } s \in \mathcal{D}_\delta.$$

Also, note that if $s = (s^1, \bar{\omega}^2)$ and $s^1 \neq 2\bar{\omega}$, then

$$\Phi_\omega(s) = \frac{2\omega\bar{\omega}^2 - s^1}{2 - \omega s^1} = \bar{\omega},$$

so that equations (5.10) and (5.8) imply that

$$(5.12) \quad Q^\sim(s) = (2 - \omega s^1)^2 q^\sim(\bar{\omega}) = (2 - \omega s^1)^2 \bar{\omega}^2 q(\bar{\omega}) = \bar{\omega}^2 Q(s).$$

Equations (5.11) and (5.12) together imply that

$$(5.13) \quad Q^\sim(s) - s^2 Q(s)$$

is a polynomial that vanishes on both \mathcal{D}_δ and $\{s : s^2 = \bar{\omega}^2\}$. It follows that

$$(5.14) \quad P(s) = \frac{Q^\sim(s) - s^2 Q(s)}{\omega^2 s^2 - 1}$$

is a well-defined polynomial that vanishes on \mathcal{D}_δ . From equation (5.14) one can derive an explicit formula for P in terms of q :

$$(5.15) \quad \begin{aligned} P(s) = & -q(\bar{\omega})(s^1)^2 + 2q'(\bar{\omega})s^1 s^2 - 2q''(0)(s^2)^2 \\ & + 2\overline{q'(\bar{\omega})}s^1 + 4(2\operatorname{Re} q(0) - q(\bar{\omega}))s^2 - 2\overline{q''(0)}. \end{aligned}$$

Note that, by equation (5.8), $q(\bar{\omega})$ is real.

P has total degree 2, and so (1) holds.

(2) Suppose P is not irreducible. Since P has degree 2, the zero set of P is the union of two complex lines. If \mathcal{D}_δ is contained in only one of the complex lines then, by Proposition 4.10, δ is a flat datum, contrary to hypothesis. Hence both lines meet \mathcal{D}_δ . Since \mathcal{D}_δ is connected, \mathcal{D}_δ contains the common point, $k(z_0)$ say, of the two lines. Thus both lines have the complex tangent $k'(z_0)$ at $k(z_0)$, and so the two lines coincide. Again \mathcal{D}_δ is contained in a complex line, and so Proposition 4.10 applies again to show that δ is a flat datum, contrary to hypothesis. Thus P is irreducible.

(3) Note that, for any $s \in G$,

$$\Phi_\omega \left(\frac{\bar{s}^1}{s^2}, \frac{1}{s^2} \right) = 1/\overline{\Phi_\omega(s)}.$$

Thus

$$\begin{aligned} P(\bar{s}^1/\bar{s}^2, 1/\bar{s}^2) &= (2 - \omega \bar{s}^1/\bar{s}^2)^2 \frac{\tilde{q} \circ \Phi_\omega \left(\frac{\bar{s}^1}{s^2}, \frac{1}{s^2} \right) - 1/\bar{s}^2 q \circ \Phi_\omega \left(\frac{\bar{s}^1}{s^2}, \frac{1}{s^2} \right)}{\omega^2/\bar{s}^2 - 1} \\ &= (2 - \omega \bar{s}^1/\bar{s}^2)^2 \frac{\bar{s}^2 \tilde{q}(1/\overline{\Phi_\omega(s)}) - q(1/\overline{\Phi_\omega(s)})}{\omega^2 - \bar{s}^2}. \end{aligned}$$

By definition, $q^\sim(z) = z^2 \overline{q(1/\bar{z})}$, hence

$$\overline{\tilde{q}(1/\overline{\Phi_\omega(s)})} = \frac{q \circ \Phi_\omega(s)}{\Phi_\omega(s)^2}$$

and

$$\overline{q(1/\overline{\Phi_\omega(s)})} = \frac{\tilde{q} \circ \Phi_\omega(s)}{\Phi_\omega(s)^2}.$$

Therefore

$$\begin{aligned} (s^2)^2 \overline{P(\overline{s^1/s^2}, 1/\overline{s^2})} &= \frac{(2\omega s^2 - s^1)^2}{1 - \omega^2 s^2} \left(s^2 \overline{\tilde{q}(1/\overline{\Phi_\omega(s)})} - \overline{q(1/\overline{\Phi_\omega(s)})} \right) \\ &= \frac{(2\omega s^2 - s^1)^2}{1 - \omega^2 s^2} \left(s^2 \frac{q \circ \Phi_\omega(s)}{\Phi_\omega(s)^2} - \frac{\tilde{q} \circ \Phi_\omega(s)}{\Phi_\omega(s)^2} \right) \\ &= P(s). \end{aligned}$$

Statement (3) holds.

(4) From statement (1), $\mathcal{D}_\delta \subset P^{-1}(\{0\})$. It is easy to see from the definition (5.6) of P that, for every $s \in G$,

$$\begin{aligned} P(s) &= (2 - \omega s^1)^2 \frac{\tilde{q} \circ \Phi_\omega(s) - s^2 q \circ \Phi_\omega(s)}{\omega^2 s^2 - 1} \\ &= \frac{(2 - \omega s^1)^2}{\omega^2 s^2 - 1} \left[\frac{\tilde{q} \circ \Phi_\omega(s)}{q \circ \Phi_\omega(s)} - s^2 \right] q \circ \Phi_\omega(s) \\ &= \frac{(2 - \omega s^1)^2}{\omega^2 s^2 - 1} q \circ \Phi_\omega(s) [k^2 \circ \Phi_\omega(s) - s^2]. \end{aligned}$$

Recall that the polynomial q does not vanish on \mathbb{D}^- , and $\frac{(2 - \omega s^1)^2}{\omega^2 s^2 - 1} \neq 0$ for all $s \in G$.

By assumption, $k \in G(\mathbb{D})$ satisfies $\Phi_\omega \circ k = \text{id}_{\mathbb{D}}$. We claim that, for each $s \in G$, the equation $k^2 \circ \Phi_\omega(s) = s^2$ implies that $k^1 \circ \Phi_\omega(s) = s^1$. For every $z \in \mathbb{D}$,

$$\frac{2\omega k^2(z) - k^1(z)}{2 - \omega k^2(z)} = z.$$

Hence

$$\frac{2\omega k^2(\Phi_\omega(s)) - k^1(\Phi_\omega(s))}{2 - \omega k^2(\Phi_\omega(s))} = \Phi_\omega(s)$$

and, since $k^2 \circ \Phi_\omega(s) = s^2$, we have

$$\frac{2\omega s^2 - k^1(\Phi_\omega(s))}{2 - \omega s^2} = \Phi_\omega(s).$$

This implies that

$$2\omega s^2 - k^1(\Phi_\omega(s)) = 2\omega s^2 - s^1, \text{ and so } k^1 \circ \Phi_\omega(s) = s^1.$$

Therefore

$$\begin{aligned} \{s \in G : P(s) = 0\} &= \{s \in G : k^2 \circ \Phi_\omega(s) = s^2\} \\ &= \{s \in G : k^2 \circ \Phi_\omega(s) = s^2 \text{ and } k^1 \circ \Phi_\omega(s) = s^1\}. \end{aligned}$$

Thus, by the equations (5.7), $\mathcal{D}_\delta = \{s \in G : P(s) = 0\}$. \square

LEMMA 5.6. *Let δ be a balanced datum in G , let Φ_{ω_1} and Φ_{ω_2} solve $\text{Car}(\delta)$ with $\omega_1 \neq \omega_2$, and let k solve $\text{Kob}(\delta)$. Choose $m_1, m_2 \in \text{Aut } \mathbb{D}$ such that*

$$m_1 \circ \Phi_{\omega_1} \circ k = \text{id}_{\mathbb{D}} \quad \text{and} \quad m_2 \circ \Phi_{\omega_2} \circ k = \text{id}_{\mathbb{D}},$$

and then let C_1 and C_2 be defined by

$$C_1 = m_1 \circ \Phi_{\omega_1} \quad \text{and} \quad C_2 = m_2 \circ \Phi_{\omega_2}.$$

There exists a nonzero constant c such that

$$C_1(s) - C_2(s) = \frac{cP_\delta(s)}{(2 - \omega_1 s^1)(2 - \omega_2 s^1)}$$

for all $s \in G$, where P_δ is defined by equation (5.6).

PROOF. Simply observe that

$$(2 - \omega_1 s^1)(2 - \omega_2 s^1)(C_1(s) - C_2(s))$$

is a quadratic polynomial that vanishes on $\text{ran } k$ and that

$$\text{ran } k = \mathcal{D}_\delta = \{s \in G : P_\delta(s) = 0\}.$$

□

Purely unbalanced and exceptional datums in G

The most delicate aspect of the geometric characterization of types of datum in Theorem 7.1 below is the distinction between purely unbalanced and exceptional datums, depending as it does on whether or not a certain second derivative is zero. In this chapter we calculate the relevant second derivatives.

Let us recall the following elegant parametrization of solutions to $\text{Kob}(\delta)$ due to Pflug and Zwonek [41, Theorem 2 part (ii)].

PROPOSITION 6.1. *Let δ be a nondegenerate datum in G and assume that $\mathcal{D}_\delta \cap \mathcal{R} = \emptyset$. If k solves $\text{Kob}(\delta)$ then there exist $m_1, m_2 \in \text{Aut } \mathbb{D}$ such that*

$$(6.1) \quad k = (m_1 + m_2, m_1 m_2).$$

PROPOSITION 6.2. *Let δ be a nondegenerate datum in G and suppose that $\mathcal{D}_\delta^- \cap \mathcal{R}^-$ is the singleton set $\{(2\bar{\omega}_0, \bar{\omega}_0^2)\}$ for some $\omega_0 \in \mathbb{T}$. Then*

$$(6.2) \quad \frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{e^{it}=\omega_0} = 0.$$

PROOF. Let k solve $\text{Kob}(\delta)$, so that there is a datum ζ in \mathbb{D} such that $k(\zeta) = \delta$ and $|\zeta| = |\delta|_{\text{kob}}$. Then $k(\mathbb{D}) = \mathcal{D}_\delta$, $k(\mathbb{D}^-) = \mathcal{D}_\delta^-$ and Φ_{ω_0} solves $\text{Car}(\delta)$ by Lemma 4.16.

Since k has no royal nodes in \mathbb{D} , by Proposition 6.1, $k = (m_1 + m_2, m_1 m_2)$ for some $m_1, m_2 \in \text{Aut } \mathbb{D}$. Replace k by $k \circ m_1^{-1}$: then we may write $k(z) = (z + m(z), zm(z))$, where $m = cB_\alpha$ for some $c \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. Consider any royal node $\tau \in \mathbb{D}^-$ of k . The royal nodes of k are the fixed points of m in \mathbb{D}^- , and so, since $k(\tau) \in E$,

$$k(\tau) = (2\tau, \tau^2) = (2\bar{\omega}_0, \bar{\omega}_0^2).$$

Thus $\tau = \bar{\omega}_0$. Hence k has the unique royal node $\bar{\omega}_0$, and so m has the unique fixed point $\bar{\omega}_0$, and this fixed point lies in \mathbb{T} .

It is elementary that, for the function $m = cB_\alpha$,

- (1) m has a unique fixed point which lies in \mathbb{T} if and only if $c \neq 1$ and $|\alpha| = \frac{1}{2}|1 - c|$, and
- (2) m has two fixed points which lie in \mathbb{T} if and only if $|\alpha| > \frac{1}{2}|1 - c|$.

For our chosen m , alternative (1) holds, and so $|\alpha| = \frac{1}{2}|1 - c| > 0$.

For $1 \leq r < |\alpha|^{-1}$ let $m_r = cB_{r\alpha}$. Then $m_r \in \text{Aut } \mathbb{D}$ and $m_1 = m$. For $r > 1$, since $|r\alpha| > \frac{1}{2}|1 - c|$, m_r has two fixed points lying in \mathbb{T} . These fixed points are easily seen to be

$$\tau_\pm(r) = \frac{1 - c \pm \sqrt{(1 - c)^2 + 4r^2|\alpha|^2 c}}{2r\bar{\alpha}}.$$

When $r = 1$ the discriminant in this formula is zero, since m_1 has a unique fixed point, and so $(1 - c)^2 = -4|\alpha|^2c$. Hence

$$(6.3) \quad \tau_{\pm}(r) = \frac{1 - c}{2r\bar{\alpha}}(1 \pm i\sqrt{r^2 - 1}) \in \mathbb{T}.$$

These two fixed points are distinct when $r > 1$ and coincide with τ when $r = 1$.

Let

$$(6.4) \quad k_r(z) = (z + m_r(z), zm_r(z))$$

for $z \in \mathbb{D}$, $1 \leq r < |\alpha|^{-1}$. Then k_r is a rational Γ -inner function of degree 2 having royal nodes $\tau_{\pm}(r) \in \mathbb{T}$. Since k_r^1 and k_r^2 have the same denominator, $\Phi_{\omega} \circ k_r$ is also rational of degree at most 2, for any ω . Moreover, $\Phi_{\omega} \circ k_r$ is inner, by Proposition A.3 in the Appendix. When $\omega = \overline{\tau_{\pm}(r)}$, the numerator and denominator of $\Phi_{\omega} \circ k_r(z)$ both vanish at $z = \tau_{\pm}(r)$. Thus cancellation occurs, and so $\Phi_{\overline{\tau_{\pm}(r)}} \circ k_r$ has degree 1. Thus k_r has the left inverse $\Phi_{\overline{\tau_{\pm}(r)}}$ modulo $\text{Aut } \mathbb{D}$, which is to say that k_r is a complex C-geodesic of G .

Let $\delta_r = k_r(\zeta)$, $1 \leq r < |\alpha|^{-1}$. Then k_r solves $\text{Kob}(\delta_r)$ and $\Phi_{\overline{\tau_{\pm}(r)}}$ solves $\text{Car}(\delta_r)$. Hence the point $\overline{\tau_{\pm}(r)}$ is a maximizer for ρ_{δ_r} , and consequently

$$(6.5) \quad \frac{d}{dt} \rho_{\delta_r}(e^{it}) \Big|_{e^{it} = \overline{\tau_{\pm}(r)}} = 0.$$

For $1 \leq r < |\alpha|^{-1}$ define $F_r : \mathbb{T} \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_r(\omega) &= \frac{d}{dt} \rho_{\delta_r}(e^{it}) \Big|_{e^{it} = \omega} \\ &= \frac{d}{dt} |\Phi_{e^{it}}(\delta_r)|^2 \Big|_{e^{it} = \omega}. \end{aligned}$$

By equation (6.5),

$$(6.6) \quad F_r(\overline{\tau_{\pm}(r)}) = 0.$$

From the formula (3.2) for Φ_{ω} it is easy to see that

$$|\Phi_{\omega}(\delta_r)| = |g_r(\omega)| \quad \text{for } \omega \in \mathbb{T}$$

for some fractional quadratic function g_r in which the coefficients depend continuously (indeed, polynomially) on r . It follows that $g_r \rightarrow g_1$ in $C^{\infty}(\mathbb{T})$ as $r \rightarrow 1$. Now

$$\begin{aligned} F_r(\omega) &= \frac{d}{dt} |g_r(e^{it})|^2 \Big|_{e^{it} = \omega} \\ &= -2 \text{Im} \left(\omega g_r'(\omega) \overline{g_r(\omega)} \right). \end{aligned}$$

Consequently $F_r \rightarrow F_1$ in $C^{\infty}(\mathbb{T})$ as $r \rightarrow 1$.

Let

$$G_r(t) = F_r(e^{it})$$

for $t \in \mathbb{R}$ and write

$$\tau = e^{it_0}, \quad \tau_{\pm}(r) = e^{it_{\pm}(r)},$$

where $t_{\pm}(r)$ is close to t_0 for r close to 1. Our goal (6.2) is to show that $G_1'(-t_0) = 0$. We have, for any $t \in \mathbb{R}$,

$$G_r(t) = G_r(-t_0) + (t + t_0)G_r'(-t_0) + \frac{1}{2}(t + t_0)^2 G_r''(\theta)$$

for some $\theta = \theta(r, t) \in \mathbb{R}$. By equation (6.6), $G_r(-t_{\pm}(r)) = 0$. Therefore

$$\begin{aligned} 0 &= G_r(-t_+(r)) - G_r(-t_-(r)) \\ &= (t_-(r) - t_+(r))G'_r(-t_0) + A_+(r)(t_0 - t_+(r))^2 + A_-(r)(t_0 - t_-(r))^2 \end{aligned}$$

for some real numbers $A_{\pm}(r)$ which are uniformly bounded for all r, t , say by $K > 0$. Thus, for $r > 1$,

$$|G'_r(-t_0)| \leq K \frac{(t_0 - t_+(r))^2 + (t_0 - t_-(r))^2}{|t_-(r) - t_+(r)|}$$

Now $|t_0 - t_{\pm}(r)|$ and $|t_+(r) - t_-(r)|$ are asymptotic to $|\tau - \tau_{\pm}(r)|$ and $|\tau_+(r) - \tau_-(r)|$ respectively as $r \rightarrow 1$, and from equation (6.3) it is easy to see that all these quantities are asymptotic to $\sqrt{r-1}$. It follows that $G'_r(-t_0) \rightarrow 0$ as $r \rightarrow 1$. Since $G_r \rightarrow G_1$ with respect to the C^1 norm, it follows that $G'_1(-t_0) = 0$, as required. \square

We shall need the following calculation.

LEMMA 6.3. *If a datum δ in G contacts \mathcal{R} then $\Phi_{\omega}(\delta)$ is independent of $\omega \in \mathbb{T}$.*

PROOF. If δ is discrete it has the form (s_1, s_2) where $s_j = (2z_j, z_j^2)$ for some $z_j \in \mathbb{D}$, $j = 1, 2$. One then finds that $\Phi_{\omega}(\delta) = (-z_1, -z_2)$ for all $\omega \in \mathbb{T}$. If δ is infinitesimal then $\delta = (s_1, v)$ where $s_1 = (2z_1, z_1^2)$ for some $z_1 \in \mathbb{D}$ and $v = \lambda(1, z_1)$ for some $\lambda \neq 0$. A straightforward calculation (more details are in the proof of Theorem A.15) yields

$$\Phi_{\omega}(\delta) = (-z_1, -\frac{1}{2}\lambda).$$

In either case, $\Phi_{\omega}(\delta)$ does not depend on ω . \square

For every $m \in \text{Aut } \mathbb{D}$ we define a map $\tilde{m} \in G(G)$ by

$$(6.7) \quad \tilde{m}(z + w, zw) = (m(z) + m(w), m(z)m(w)) \quad \text{for all } z, w \in \mathbb{D}.$$

Every automorphism of G is of the form \tilde{m} for some $m \in \text{Aut } \mathbb{D}$, by Theorem A.11.

We use the notation $f^{\vee}(z) = \overline{f(\bar{z})}$ for any function f on \mathbb{D} .

LEMMA 6.4. *Let $m = cB_{\alpha} \in \text{Aut } \mathbb{D}$ for some $c \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. For any $\omega \in \mathbb{T}$*

$$(6.8) \quad \Phi_{\omega} \circ \tilde{m} = m_1 \circ \Phi_{(m^{\vee})^{-1}(\omega)}$$

where $m_1 = cB_{-\alpha}$. Consequently, for any nondegenerate datum δ in G and any $m \in \text{Aut } \mathbb{D}$,

$$(6.9) \quad \rho_{\tilde{m}(\delta)} = \rho_{\delta} \circ (m^{\vee})^{-1}.$$

If $m(z) = cB_{\alpha}(z)$ then $(m^{\vee})^{-1}(\omega) = B_{-\bar{\alpha}}(c\omega)$.

PROOF. The identity (6.8) is a matter of verification. If $\omega_1 = (m^{\vee})^{-1}(\omega)$ then $\rho_{\tilde{m}(\delta)}(\omega) = |\Phi_{\omega} \circ \tilde{m}(\delta)|^2 = |m_1 \circ \Phi_{\omega_1}(\delta)|^2 = |\Phi_{\omega_1}(\delta)|^2 = \rho_{\delta}(\omega_1) = \rho_{\delta} \circ (m^{\vee})^{-1}(\omega)$. \square

LEMMA 6.5. *Let δ be a nondegenerate datum in G and let $m = cB_{\alpha} \in \text{Aut } \mathbb{D}$ for some $c \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. If ρ_{δ} attains its maximum over \mathbb{T} at ω_0 then $\rho_{\tilde{m}(\delta)}$ attains its maximum at $m^{\vee}(\omega_0)$ and*

$$(6.10) \quad \frac{d^2}{dt^2} \rho_{\tilde{m}(\delta)}(e^{it}) \Big|_{e^{it}=m^{\vee}(\omega_0)} = \frac{|1 - \alpha\omega_0|^4}{(1 - |\alpha|^2)^2} \frac{d^2}{dt^2} \rho_{\delta}(e^{it}) \Big|_{e^{it}=\omega_0}.$$

PROOF. It is immediate from equation (6.9) that $\rho_{\tilde{m}(\delta)}$ attains its maximum at $m^\vee(\omega_0)$.

Let $\omega_0 = e^{it_0}$ and define

$$r_\delta(t) = \rho_\delta(e^{it})$$

for $t \in \mathbb{R}$. Since r_δ attains its maximum over \mathbb{R} at t_0 , we have

$$(6.11) \quad r'_\delta(t_0) = 0.$$

Let $\tau(\cdot)$ be defined as a smooth function of t in a neighborhood of t_0 by

$$e^{i\tau(t)} = (m^\vee)^{-1}(e^{it}) = \frac{ce^{it} + \bar{\alpha}}{1 + c\alpha e^{it}}.$$

Then

$$(6.12) \quad \tau'(t) = \frac{1 - |\alpha|^2}{|1 + c\alpha e^{it}|^2}.$$

From equation (6.9) we have

$$r_{\tilde{m}(\delta)}(t) = \rho_{\tilde{m}(\delta)}(e^{it}) = \rho_\delta \circ (m^\vee)^{-1}(e^{it}) = \rho_\delta(e^{i\tau(t)}) = r_\delta \circ \tau(t).$$

Hence, by equation (6.12), for all $t \in \mathbb{R}$,

$$r'_{\tilde{m}(\delta)}(t) = r'_\delta \circ \tau(t) \tau'(t) = r'_\delta \circ \tau(t) \frac{1 - |\alpha|^2}{|1 + c\alpha e^{it}|^2}.$$

Differentiate again to obtain

$$(6.13) \quad r''_{\tilde{m}(\delta)}(t) = \frac{1 - |\alpha|^2}{|1 + c\alpha e^{it}|^4} \left(|1 + c\alpha e^{it}|^2 \frac{d}{dt} r'_\delta \circ \tau(t) - r'_\delta \circ \tau(t) \frac{d}{dt} |1 + c\alpha e^{it}|^2 \right).$$

Note that

$$e^{it} = m^\vee(\omega_0) \Leftrightarrow (m^\vee)^{-1}(e^{it}) = \omega_0 \Leftrightarrow e^{i\tau(t)} = e^{it_0}.$$

Let t_1 be such that $\tau(t_1) = t_0$. Then $e^{it_1} = m^\vee(\omega_0)$ and, by equation (6.11),

$$r'_\delta \circ \tau(t_1) = r'_\delta(t_0) = 0.$$

Hence the final term on the right hand side of equation (6.13) vanishes when $t = t_1$, and we have

$$(6.14) \quad \begin{aligned} r''_{\tilde{m}(\delta)}(t_1) &= \frac{1 - |\alpha|^2}{|1 + c\alpha e^{it_1}|^2} \frac{d}{dt} r'_\delta \circ \tau(t) \Big|_{t=t_1} \\ &= \frac{1 - |\alpha|^2}{|1 + c\alpha m^\vee(\omega_0)|^2} r''_\delta \circ \tau(t_1) \tau'(t_1) \\ &= \frac{(1 - |\alpha|^2)^2}{|1 + c\alpha m^\vee(\omega_0)|^4} r''_\delta(t_0). \end{aligned}$$

Now

$$1 + c\alpha m^\vee(\omega_0) = \frac{1 - |\alpha|^2}{1 - \alpha\omega_0},$$

and so equation (6.14) becomes

$$r''_{\tilde{m}(\delta)}(t_1) = \frac{|1 - \alpha\omega_0|^4}{(1 - |\alpha|^2)^2} r''_\delta(t_0).$$

The last equation is equivalent to equation (6.10). \square

LEMMA 6.6. *If \mathcal{D} is a complex geodesic in G such that*

$$(6.15) \quad \mathcal{D}^- \cap \mathcal{R}^- = \{(0, 0), (2, 1)\}$$

then $\mathcal{D} = k_r(\mathbb{D})$ for some $r \in (0, 1)$, where

$$(6.16) \quad k_r(z) = \frac{1}{1-rz} (2(1-r)z, z(z-r)) \quad \text{for } z \in \mathbb{D}.$$

PROOF. Let δ be a nondegenerate datum that contacts \mathcal{D} and let k solve $\text{Kob}(\delta)$. Then $\mathcal{D} = k(\mathbb{D})$. Since $(0, 0) \in k(\mathbb{D})$ and $(2, 1) \in k(\mathbb{D}^-)$ we may assume (composing k with an automorphism of \mathbb{D} if necessary) that $k(0) = (0, 0)$ and $k(1) = (2, 1)$. By [41, Remark 2], k can be written in the form

$$k(\lambda) = \frac{1}{1-\bar{\alpha}\lambda} (2\tau(1-|\alpha|)\lambda, \tau^2\lambda(\lambda-\alpha))$$

for some $\tau \in \mathbb{T}$ and $\alpha \in \mathbb{D}$. Since $k(1) = (2, 1)$ we have

$$\frac{\tau(1-|\alpha|)}{1-\bar{\alpha}} = 1 = \frac{\tau^2(1-\alpha)}{1-\bar{\alpha}}.$$

It follows that $1-|\alpha| = |1-\alpha|$, and so $0 \leq \alpha < 1$ and $\tau = 1$. Thus

$$k(\lambda) = \frac{1}{1-\alpha\lambda} (2(1-\alpha)\lambda, \lambda(\lambda-\alpha)).$$

If $\alpha = 0$ then $k(\mathbb{D}) = \mathcal{R}$, contrary to the hypothesis (6.15). Thus $0 < \alpha < 1$. \square

PROPOSITION 6.7. *If δ is a nondegenerate datum in G such that $\mathcal{D}_\delta^- \cap \mathcal{R}^-$ consists of two points, one lying in \mathcal{R} and one lying in the boundary $\partial\mathcal{R}$ of R , then*

$$\frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{e^{it}=1} < 0.$$

PROOF. First consider the case that

$$(6.17) \quad \mathcal{D}_\delta^- \cap \mathcal{R}^- = \{(0, 0), (2, 1)\}.$$

Let k solve $\text{Kob}(\delta)$. By Lemma 6.6, $k = k_r$ as in equation (6.16) for some $r \in (0, 1)$. Suppose that δ is a discrete datum. Then $\delta = k(\zeta)$ for some discrete datum ζ in \mathbb{D} , say $\zeta = (z_1, z_2)$ for some distinct points $z_1, z_2 \in \mathbb{D}$. Let $\omega = e^{it}$ throughout the proof. Observe that

$$\Phi_\omega \circ k(z) = \omega z \frac{z - \overline{\gamma(\omega)}}{1 - \gamma(\omega)z}$$

where

$$(6.18) \quad \gamma(\omega) = r1 + (1-r)\omega.$$

Here $\gamma \in \mathbb{D}$ if $\omega \in \mathbb{T} \setminus \{1\}$, while $\gamma(1) = 1$ and $\Phi_1 \circ k(z) = -\omega z$. Moreover

$$(6.19) \quad |\gamma(\omega)|^2 = 1 - 2r(1-r)(1 - \cos t).$$

We wish to differentiate with respect to t the function

$$\begin{aligned}
(6.20) \quad 1 - \rho_\delta(e^{it}) &= 1 - |\Phi_\omega(\delta)|^2 \\
&= 1 - |\Phi_\omega \circ k(\zeta)|^2 \\
&= 1 - \left| \frac{\Phi_\omega \circ k(z_1) - \Phi_\omega \circ k(z_2)}{1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)} \right|^2 \\
&= \frac{(1 - |\Phi_\omega \circ k(z_1)|^2)(1 - |\Phi_\omega \circ k(z_2)|^2)}{|1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)|^2}.
\end{aligned}$$

We have

$$(6.21) \quad 1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1) = (1 - \bar{z}_2 z_1) \left(1 + \frac{\bar{z}_2 z_1 (1 - |\gamma|^2)}{(1 - \bar{\gamma} \bar{z}_2)(1 - \gamma z_1)} \right)$$

$$(6.22) \quad = \frac{(1 - \bar{z}_2 z_1)(1 - \bar{\gamma} \bar{z}_2 - \gamma z_1 + \bar{z}_2 z_1)}{(1 - \bar{\gamma} \bar{z}_2)(1 - \gamma z_1)}.$$

By equation (6.20),

$$(6.23) \quad \begin{aligned} \log(1 - \rho_\delta(\omega)) &= \frac{1}{2} \log(1 - |\Phi_\omega \circ k(z_1)|^2)^2 + \frac{1}{2} \log(1 - |\Phi_\omega \circ k(z_2)|^2)^2 \\ &\quad - \log |1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)|^2. \end{aligned}$$

Consequently

$$(6.24) \quad \frac{d}{dt} \frac{(1 - \rho_\delta(\omega))}{1 - \rho_\delta(\omega)} = \frac{1}{2} \text{I} + \frac{1}{2} \text{II} - \text{III}$$

where

$$\begin{aligned}
(6.25) \quad \text{I} &= \frac{\frac{d}{dt} (1 - |\Phi_\omega \circ k(z_1)|^2)^2}{(1 - |\Phi_\omega \circ k(z_1)|^2)^2}, \\
\text{II} &= \frac{\frac{d}{dt} (1 - |\Phi_\omega \circ k(z_2)|^2)^2}{(1 - |\Phi_\omega \circ k(z_2)|^2)^2}, \\
\text{III} &= \frac{\frac{d}{dt} |1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)|^2}{|1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)|^2}.
\end{aligned}$$

Let us calculate III. From equations (6.18) and (6.19) we have

$$(6.26) \quad \frac{d\gamma}{dt} = i(1 - r)\omega, \quad \frac{d|\gamma|^2}{dt} = -2r(1 - r) \sin t.$$

Thus, on differentiating equation (6.21) we obtain

$$\begin{aligned}
(6.27) \quad \frac{d}{dt} (1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)) &= \frac{2r(1 - r) \bar{z}_2 z_1 (1 - \bar{z}_2 z_1)}{(1 - \bar{\gamma} \bar{z}_2)^2 (1 - \gamma z_1)^2} \times \\
&\quad ((1 - \bar{z}_2)(1 - z_1) \sin t + i(1 - r)(\bar{z}_2 - z_1)(1 - \cos t)).
\end{aligned}$$

On combining equations (6.27) and (6.22) we find that

$$(6.28) \quad \frac{d}{dt} |1 - \overline{\Phi_\omega \circ k(z_2)} \Phi_\omega \circ k(z_1)|^2 = \frac{4r(1 - r) |1 - \bar{z}_2 z_1|^2}{|1 - \gamma z_1|^2 |1 - \gamma z_2|^2} \operatorname{Re} A(t, z_1, z_2)$$

where

$$(6.29) \quad A(t, z_1, z_2) = \frac{\bar{z}_2 z_1 (1 - \bar{\gamma} \bar{z}_1 - \gamma z_2 + \bar{z}_1 z_2)}{(1 - \bar{\gamma} \bar{z}_2)(1 - \gamma z_1)} \times \\ ((1 - \bar{z}_2)(1 - z_1) \sin t + i(1 - r)(\bar{z}_2 - z_1)(1 - \cos t)).$$

Now use equations (6.22) and (6.29) to deduce that

$$(6.30) \quad \text{III} = 4r(1 - r) \operatorname{Re} \left\{ \frac{\bar{z}_2 z_1 ((1 - \bar{z}_2)(1 - z_1) \sin t + i(1 - r)(\bar{z}_2 - z_1)(1 - \cos t))}{(1 - \bar{\gamma} \bar{z}_2 - \gamma z_1 + \bar{z}_2 z_1)(1 - \bar{\gamma} \bar{z}_2)(1 - \gamma z_1)} \right\}.$$

By equation (6.24),

$$\frac{d}{dt} \rho_\delta(\omega) = -(1 - \rho_\delta(\omega)) \left(\frac{1}{2} \text{I} + \frac{1}{2} \text{II} - \text{III} \right).$$

We shall differentiate this equation and put $t = 0$, $\omega = 1$. Since $\omega = 1$ is a maximizer for ρ_δ , the point $t = 0$ is a critical point of $\rho_\delta(e^{it})$, and $\rho_\delta(1) = |\delta|_{\text{car}}^2$. Thus

$$(6.31) \quad \frac{d^2}{dt^2} \rho_\delta(\omega) \Big|_{t=0} = -(1 - |\delta|_{\text{car}}^2) \frac{d}{dt} \left(\frac{1}{2} \text{I} + \frac{1}{2} \text{II} - \text{III} \right) \Big|_{t=0}.$$

In view of equations (6.18) and (6.26), at the point $t = 0$,

$$\gamma = 1, \quad \frac{d\gamma}{dt} = i(1 - r) \quad \text{and} \quad \frac{d|\gamma|^2}{dt} = 0.$$

Hence

$$\begin{aligned} \frac{d}{dt} \text{III} \Big|_{t=0} &= 4r(1 - r) \operatorname{Re} \left\{ \bar{z}_2 z_1 \frac{d}{dt} \frac{(1 - \bar{z}_2)(1 - z_1) \sin t + i(1 - r)(\bar{z}_2 - z_1)(1 - \cos t)}{(1 - \bar{\gamma} \bar{z}_2 - \gamma z_1 + \bar{z}_2 z_1)(1 - \bar{\gamma} \bar{z}_2)(1 - \gamma z_1)} \right\} \Big|_{t=0} \\ &= 4r(1 - r) \operatorname{Re} \frac{\bar{z}_2 z_1}{(1 - \bar{z}_2)(1 - z_1)}. \end{aligned}$$

Since I, II are the special cases of III obtained when $z_2 = z_1$ and $z_1 = z_2$ respectively, it follows that

$$\begin{aligned} \frac{d}{dt} \text{I} \Big|_{t=0} &= \frac{4r(1 - r)|z_1|^2}{|1 - z_1|^2}, \\ \frac{d}{dt} \text{II} \Big|_{t=0} &= \frac{4r(1 - r)|z_2|^2}{|1 - z_2|^2}. \end{aligned}$$

By equation (6.31),

$$\begin{aligned} \frac{d^2}{dt^2} \rho_\delta(\omega) \Big|_{t=0} &= -(1 - |\delta|_{\text{car}}^2) 2r(1 - r) \left(\frac{|z_1|^2}{|1 - z_1|^2} + \frac{|z_2|^2}{|1 - z_2|^2} - 2 \operatorname{Re} \frac{\bar{z}_2 z_1}{(1 - \bar{z}_2)(1 - z_1)} \right) \\ &= -(1 - |\delta|_{\text{car}}^2) 2r(1 - r) \left| \frac{z_1}{1 - z_1} - \frac{z_2}{1 - z_2} \right|^2 \\ &< 0. \end{aligned}$$

We have proved Proposition 6.7 for discrete datums δ in the case that

$$\mathcal{D}_\delta^- \cap \mathcal{R}^- = \{(0, 0), (2, 1)\}.$$

Now consider the case that δ is an infinitesimal datum. Then $\delta = k(\zeta)$ for some infinitesimal datum $\zeta \in T\mathbb{D}$, say $\zeta = (z, v)$ where $v \neq 0$. We find that

$$\Phi_\omega \circ k(\zeta) = \left(\omega z B_{\bar{\gamma}}(z), -\omega v \frac{\bar{\gamma} - 2z + \gamma z^2}{(1 - \gamma z)^2} \right)$$

where γ is given by equation (6.18) and $B_{\bar{\gamma}}$ is a Blaschke factor. Therefore

$$\begin{aligned} |\Phi_{\omega} \circ k(\zeta)| &= \frac{|v| |\bar{\gamma} - 2z + \gamma z^2|}{|1 - \gamma z|^2 (1 - |z B_{\bar{\gamma}}(z)|^2)} \\ &= \frac{|v| |\bar{\gamma} - 2z + \gamma z^2|}{(1 - |z|^2)(1 - 2 \operatorname{Re}(\gamma z) + |z|^2)}. \end{aligned}$$

Hence

$$\begin{aligned} \log \rho_{\delta}(\omega) &= \log |\Phi_{\omega} \circ k(\zeta)|^2 \\ &= 2 \log \frac{|v|}{1 - |z|^2} + \log |\bar{\gamma} - 2z + \gamma z^2|^2 - 2 \log(1 - 2 \operatorname{Re}(\gamma z) + |z|^2), \end{aligned}$$

and so

$$(6.32) \quad \frac{d}{dt} \rho_{\delta}(\omega) = \rho_{\delta}(\omega)(\text{I} - 2\text{II})$$

where

$$\begin{aligned} \text{I} &= \frac{\frac{d}{dt} |\bar{\gamma} - 2z + \gamma z^2|^2}{|\bar{\gamma} - 2z + \gamma z^2|^2}, \\ \text{II} &= \frac{\frac{d}{dt} (1 - 2 \operatorname{Re}(\gamma z) + |z|^2)}{1 - 2 \operatorname{Re}(\gamma z) + |z|^2}. \end{aligned}$$

Now, by equations (6.26),

$$(6.33) \quad \frac{d}{dt} (\bar{\gamma} - 2z + \gamma z^2) = -i(1 - r)(\bar{\omega} - \omega z^2),$$

and therefore

$$(6.34) \quad \text{I} = 2(1 - r) \operatorname{Im} \frac{\bar{\omega} - \omega z^2}{\bar{\gamma} - 2z + \gamma z^2}.$$

Likewise

$$(6.35) \quad \text{II} = \frac{2(1 - r) \operatorname{Im}(\omega z)}{1 - 2 \operatorname{Re}(\gamma z) + |z|^2}.$$

Differentiate equation (6.32) and set $t = 0$. Since 0 is a critical point of $\rho_{\delta}(e^{it})$ and $\rho_{\delta}(1) = |\delta|_{\text{car}}^2$,

$$(6.36) \quad \frac{d^2}{dt^2} \rho_{\delta}(\omega) \Big|_{t=0} = |\delta|_{\text{car}}^2 \left(\frac{d}{dt} \text{I} - 2 \frac{d}{dt} \text{II} \right) \Big|_{t=0}.$$

From equations (6.34), (6.33) and (6.26) one has

$$\frac{d}{dt} \text{I} \Big|_{t=0} = -2(1 - r) \operatorname{Re} \frac{-2z + r(1 + z)^2}{(1 - z)^2},$$

while

$$\begin{aligned} \frac{d}{dt} \text{II} \Big|_{t=0} &= 2(1 - r) \operatorname{Im} \frac{d}{dt} \frac{\omega z}{1 - 2 \operatorname{Re}(\gamma z) + |z|^2} \Big|_{t=0} \\ &= \frac{2(1 - r)}{|1 - z|^4} \{ |1 - z|^2 \operatorname{Re} z - 2(1 - r)(\operatorname{Im} z)^2 \}. \end{aligned}$$

Hence, by equation (6.36),

$$\frac{d^2}{dt^2} \rho_{\delta}(\omega) \Big|_{t=0} = -2(1 - r) |\delta|_{\text{car}}^2 (\text{III} + r\text{IV})$$

where

$$\begin{aligned} \text{III} &= \operatorname{Re} \left\{ \frac{2z}{|1-z|^2} - \frac{2z}{(1-z)^2} \right\} - \frac{4(\operatorname{Im} z)^2}{|1-z|^4} \\ &= 0, \\ \text{IV} &= \operatorname{Re} \frac{(1+z)^2}{(1-z)^2} + \frac{4(\operatorname{Im} z)^2}{|1-z|^4} \\ &= \frac{(1-|z|^2)^2}{|1-z|^4}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{d^2}{dt^2} \rho_\delta(\omega) \Big|_{t=0} &= -2r(1-r) |\delta|_{\text{car}}^2 \frac{(1-|z|^2)^2}{|1-z|^4} \\ &< 0. \end{aligned}$$

We have proved the conclusion of the Proposition in the case that $\mathcal{D}_\delta^- \cap \mathcal{R}^- = \{(0, 0), (2, 1)\}$. Now consider the general case: suppose that

$$(6.37) \quad \mathcal{D}_\delta^- \cap \mathcal{R}^- = \{(2\alpha, \alpha^2), (2\bar{\omega}_0, \bar{\omega}_0^2)\}$$

for some $\alpha \in \mathbb{D}$, $\omega_0 \in \mathbb{T}$. Let

$$m = \overline{B_\alpha(\bar{\omega}_0)} B_\alpha \in \operatorname{Aut} \mathbb{D}.$$

Since \tilde{m} is an automorphism of G and δ contacts \mathcal{D}_δ , the datum \tilde{m} contacts $\tilde{m}(\mathcal{D}_\delta)$, and therefore

$$\tilde{m}(\mathcal{D}_\delta^-) = \mathcal{D}_{\tilde{m}(\delta)}^-, \quad \tilde{m}(\mathcal{R}^-) = \mathcal{R}^-$$

and so

$$\tilde{m}(\mathcal{D}_\delta^- \cap \mathcal{R}^-) = \mathcal{D}_{\tilde{m}(\delta)}^- \cap \mathcal{R}^-.$$

Hence, in view of the equation (6.37),

$$\mathcal{D}_{\tilde{m}(\delta)}^- \cap \mathcal{R}^- = \tilde{m}(\{(2\alpha, \alpha^2), (2\bar{\omega}_0, \bar{\omega}_0^2)\}) = \{(0, 0), (2, 1)\}.$$

By the above special case,

$$\frac{d^2}{dt^2} \rho_{\tilde{m}(\delta)}(e^{it}) \Big|_{e^{it}=1} < 0.$$

It follows from Lemma 6.5 that

$$\frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{e^{it}=\omega_0} < 0.$$

□

A geometric classification of geodesics in G

All geodesics in a domain U are coequal for Euclidean geometry, in the sense that they are all properly embedded analytic discs, and so conformally equivalent. However they can differ from the point of view of the intrinsic geometry of U . Say two geodesics \mathcal{D}_1 and \mathcal{D}_2 are *equivalent* in U (written $\mathcal{D}_1 \sim \mathcal{D}_2$) if there exists an automorphism τ of U such that $\tau(\mathcal{D}_1) = \mathcal{D}_2$. In the case when $U = \mathbb{B}$, the unit ball in \mathbb{C}^n , the group of automorphisms acts transitively on the set of complex geodesics, and so there is a single equivalence class. In G , on the other hand, we identify five distinct species of geodesic, and three of these five species consist of one-parameter families of pairwise inequivalent geodesics (Theorem 7.8), while the remaining two species comprise a single equivalence class.

In this chapter we prove a geometric characterization of the type of a datum δ (in the sense of Definition 3.4) in terms of the intersection of the complex geodesic \mathcal{D}_δ with the royal variety (more precisely, with the intersection of the corresponding closures). The result is Theorem 7.1. It has two important consequences: firstly, the type of a datum is preserved by automorphisms of G (Corollary 7.2), and secondly, if two datums contact the same geodesic then they have the same type (Corollary 7.3). The correspondence between nondegenerate datums and geodesics therefore permits a classification of geodesics too into five types. Recall the Pentachotomy Theorem 3.6, which states that, for a nondegenerate datum δ in G , exactly one of the cases (1) to (5) in Definition 3.4 holds.

The definitions of the five types of datum are not, at a first glance, geometric in character: it is not immediately apparent that types are preserved by automorphisms of G . However, the following theorem characterizes types of datums purely in terms of the geometry of G .

THEOREM 7.1. *Let δ be a nondegenerate datum in G and let $E = \mathcal{D}_\delta^- \cap \mathcal{R}^-$. Then*

- (1) *E consists of two points, one lying in \mathcal{R} and one lying in $\partial\mathcal{R} \iff \delta$ is purely unbalanced;*
- (2) *E consists of a single point that lies in $\partial\mathcal{R} \iff \delta$ is exceptional;*
- (3) *E consists of two points, both lying in $\partial\mathcal{R} \iff \delta$ is purely balanced;*
- (4) *E consists of a single point that lies in $\mathcal{R} \iff \delta$ is flat;*
- (5) *$E = \mathcal{R}^- \iff \delta$ is royal.*

PROOF. By Theorem 4.6, there exists a solution k of $\text{Kob}(\delta)$ and $\mathcal{D}_\delta = k(\mathbb{D})$. By Theorem A.7, this k is a rational Γ -inner function and k^2 is a Blaschke product of degree 1 or 2. Since k is Γ -inner, $k(\mathbb{T}) \subseteq b\Gamma$.

If $k(\mathbb{D}^-) \neq \mathcal{R}^- \cap \Gamma$, by Theorem 4.14, $k(\mathbb{D}^-)$ meets \mathcal{R}^- exactly $\deg(k)$ times, that is, once or twice, counted with multiplicity.

(4) Suppose that E consists of a single point $s_1 = k(z_1)$ that lies in \mathcal{R} (thus in G rather than ∂G). Now z_1 is a simple zero of the rational function $R = (k^1)^2 - 4k^2$ in \mathbb{D} . For suppose it has multiplicity greater than 1. By [10, Proposition 3.5], the zeros of R are symmetric with respect to \mathbb{T} , and it follows that R has at least 4 zeros lying off \mathbb{T} . However, since $\deg(k) \leq 2$, R has degree at most 4, and so $\deg(R) = 4$, $\deg(k) = 2$ and k has no royal nodes in \mathbb{T} . This contradicts the fact (see Proposition 4.17) that any complex C-geodesic of degree 2 has a royal node in \mathbb{T} . Hence z_1 is the unique royal node of k and is simple. By Theorem 4.14, the complex C-geodesic k has degree 1. Thus there is a flat complex C-geodesic that solves $\text{Kob}(\delta)$. By Proposition 4.11, δ is a flat datum.

Conversely, suppose δ is a flat datum. By Proposition 4.11, there is a flat complex C-geodesic, which has degree 1, that solves $\text{Kob}(\delta)$. By Theorem 4.14, k has exactly one royal node, counted according to multiplicity. However, by Proposition 4.10, any flat complex C-geodesic has the form $f_\beta \circ m$ for some $\beta \in \mathbb{D}$, $m \in \text{Aut } \mathbb{D}$ and so has a royal node in \mathbb{D} , by Proposition 7.7. Hence k has a single royal node, which lies in \mathbb{D} , and so E consists of a single point that lies in \mathcal{R} . The equivalence (4) is proved.

(5) Suppose $E = \mathcal{R}^-$. Then $\mathcal{R}^- = \mathcal{D}_\delta^- \cap \mathcal{R}^-$ and so $\mathcal{R}^- \subseteq \mathcal{D}_\delta^- = k(\mathbb{D}^-)$. Thus the holomorphic function $(k^1)^2 - 4k^2$ has uncountably many zeros in \mathbb{D} , and so vanishes identically. Thus $\mathcal{D}_\delta = \mathcal{R}$, and so δ contacts \mathcal{R} . By Lemma 6.3, $\Phi_\omega(\delta)$ is independent of ω . Thus ρ_δ is constant on \mathbb{T} , and so δ is either flat or royal. However, by statement (4), δ is not flat, and so δ is royal.

Conversely, suppose that δ is royal: ρ_δ is constant on \mathbb{T} and $C(s) = \frac{1}{2}s^1$ solves $\text{Car}(\delta)$. Thus Φ_ω solves $\text{Car}(\delta)$ for all $\omega \in \mathbb{T}$.

Define $h \in G(\mathbb{D})$ by $h(z) = (2z, z^2)$. Then $C \circ h = \text{id}_\mathbb{D}$. We claim that there exists a datum ζ in \mathbb{D} such that $h(\zeta) = \delta$. In the case that δ is discrete, say $\delta = (s_1, s_2)$, Theorem A.14 asserts that s_1 and s_2 either both lie in \mathcal{R} or both lie in $\{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\}$ for some $\beta \in \mathbb{D}$. If the second alternative holds then $C_1(s) = s^2$ is easily seen to solve $\text{Car}(\delta)$, and so δ is both flat and royal, contrary to Theorem 3.6. Thus s_1, s_2 both lie in \mathcal{R} , so that $s_1 = h(z_1)$, $s_2 = h(z_2)$ for some $z_1, z_2 \in \mathbb{D}$. Hence $\delta = h(\zeta)$ where $\zeta = (z_1, z_2)$.

In the case that δ is infinitesimal, say $\delta = (s_1, v)$, Theorem A.15 asserts that either (a) there exists $z_1 \in \mathbb{D}$ such that $s_1 = (2z_1, z_1^2)$ and v is collinear with $(1, z_1)$, or (b) there exist $\beta, z_1 \in \mathbb{D}$ such that $s_1 = (\beta + \bar{\beta}z_1, z_1)$ and v is collinear with $(\bar{\beta}, 1)$, say $v = c(\bar{\beta}, 1)$, $c \neq 0$.

We claim that (b) does not hold. For suppose it does. Let η be the datum (z_1, c) in \mathbb{D} and let $C_1(s) = s^2$. Then $h(\eta) = \delta$ and $C_1 \circ h = \text{id}_\mathbb{D}$. It follows that C_1 solves $\text{Car}(\delta)$, and therefore δ is both royal and flat, again contrary to Theorem 3.6. Hence (a) holds. Now, if ζ is the datum $(z_1, \frac{1}{2}c)$ in \mathbb{D} then $h(\zeta) = \delta$.

Thus, in either case, there is a datum ζ in \mathbb{D} such that $h(\zeta) = \delta$. By Lemma 3.1, since $C \circ h = \text{id}_\mathbb{D}$, it follows that h solves $\text{Kob}(\delta)$. Hence $\mathcal{D}_\delta = h(\mathbb{D}) = \mathcal{R}$, and so $E = \mathcal{R}^-$. Thus the equivalence (5) holds.

(3) Suppose E consists of two points, both in $\partial\mathcal{R}$. If $\deg(k) = 1$ then, as is shown in the proof of (4) above, δ is flat and hence $E \cap \partial\mathcal{R} = \emptyset$, contrary to hypothesis. Thus $\deg(k) = 2$. By Lemma 4.16, ρ_δ has exactly two maximizers in \mathbb{T} , which is to say that δ is purely balanced.

Conversely, suppose that δ is purely balanced, so that ρ_δ has exactly two maximizers, say $\omega_1 \neq \omega_2$ in \mathbb{T} . Then once again δ is not flat and so $\deg(k) = 2$. By Lemma 4.16,

$$\mathcal{D}_\delta^- \cap \partial\mathcal{R} = k(\mathbb{D})^- \cap \partial\mathcal{R} = \{(2\bar{\omega}_1, \bar{\omega}_1^2), (2\bar{\omega}_2, \bar{\omega}_2^2)\}.$$

Hence E contains two distinct points in $\partial\mathcal{R}$. Since k has exactly 2 royal nodes, by Theorem 4.14, it follows that k has no royal points in G , and so $E = \{(2\bar{\omega}_1, \bar{\omega}_1^2), (2\bar{\omega}_2, \bar{\omega}_2^2)\}$. Thus the equivalence (3) holds.

(1) Suppose that E consists of two points, one in \mathcal{R} and one, say $(2\bar{\omega}_0, \bar{\omega}_0^2)$, in $\partial\mathcal{R}$. By statement (4) above, δ is not flat and hence, by Propositions 4.11 and 4.10, $\deg(k) = 2$. By Lemma 4.16, ρ_δ has the unique maximizer ω_0 in \mathbb{T} , and by Proposition 6.7,

$$(7.1) \quad \frac{d^2}{dt^2} \rho_\delta(e^{it}) \Big|_{e^{it}=\omega_0} < 0.$$

By definition, δ is purely unbalanced.

Conversely, suppose that δ is purely unbalanced, so that ρ_δ has a unique maximizer, say ω_0 in \mathbb{T} , and the inequality (7.1) holds. By Proposition 6.2, $E \neq \{(2\bar{\omega}_0, \bar{\omega}_0^2)\}$. Thus k has a second royal node $z_0 \in \mathbb{D}^-$ such that $k(z_0) \neq (2\bar{\omega}_0, \bar{\omega}_0^2)$. If $z_0 \in \mathbb{T}$ then $k(z_0) \in \partial\mathcal{R}$ and, by Lemma 4.16, ρ_δ has a second maximizer in \mathbb{T} , contrary to the assumption that δ is purely unbalanced. Hence $z_0 \in \mathbb{D}$, so that $k(z_0) \in \mathcal{R}$. Thus E contains two points, one in \mathcal{R} and one in $\partial\mathcal{R}$. Since k has at most two royal nodes, by Theorem 4.14, E consists of these two points. The equivalence (1) is proved.

(2) Since $\deg(k)$ is one or two, by Theorem 4.14, either $k(\mathbb{D}) = \mathcal{R}$ or k has one or two royal nodes. There are thus five possibilities for E , as described in the statements (1)-(5). Thus E consists of a single point lying in $\partial\mathcal{R}$ if and only if E is not of the forms described in statements (1), (3), (4) and (5). Consequently E consists of a single point lying in $\partial\mathcal{R}$ if and only if δ is not purely unbalanced, purely balanced, flat or royal, which in turn is so if and only if δ is exceptional, by Theorem 3.6. \square

COROLLARY 7.2. *The type of a datum in G is holomorphically invariant: if δ is a nondegenerate datum in G and $\tilde{m} \in \text{Aut } G$ then δ and $\tilde{m}(\delta)$ are of the same type.*

The statement is immediate from Theorem 7.1 and the fact that every automorphism \tilde{m} of G extends continuously to a bijective self-map of Γ (Corollary A.12).

COROLLARY 7.3. *The type of a complex geodesic is unambiguously defined: if δ_1, δ_2 are nondegenerate datums in G and $\mathcal{D}_{\delta_1} = \mathcal{D}_{\delta_2}$ then δ_1 and δ_2 have the same type.*

This statement too is clear from Theorem 7.1.

The following geometric characterization of balanced geodesics in G is an immediate corollary of Theorem 7.1 and Definition 3.4.

PROPOSITION 7.4. *If δ is a nondegenerate datum in G , then δ is unbalanced if and only if \mathcal{D}_δ meets \mathcal{R} in a single point. δ is balanced if and only if either $\mathcal{D}_\delta \cap \mathcal{R} = \emptyset$ or $\mathcal{D}_\delta = \mathcal{R}$.*

In the light of Theorem 7.1, the terms *flat*, *exceptional*, *purely unbalanced*, *purely balanced*, and *royal* (as introduced in Definition 3.4) are all terms that refer not only to qualitative properties of the Carathéodory extremal problem associated with a given datum δ but also to geometric properties of the unique geodesic associated with δ .

DEFINITION 7.5. A geodesic \mathcal{D} in G is said to be *flat*, or of *flat type*, if there is a nondegenerate flat datum δ in G such that $\mathcal{D} = \mathcal{D}_\delta$. Analogous definitions apply for exceptional, purely unbalanced, purely balanced and royal geodesics.

PROPOSITION 7.6. *The flat geodesics in G are the sets \mathcal{F}_β (defined in equation (4.3)) for $\beta \in \mathbb{D}$.*

PROOF. Let \mathcal{D} be a flat geodesic in G and let δ be a flat datum such that $\mathcal{D} = \mathcal{D}_\delta$. By Proposition 4.11 there is a flat C-geodesic k that solves $\text{Kob}(\delta)$, and by Proposition 4.10, $k = f_\beta \circ m$ for some $\beta \in \mathbb{D}$ and $m \in \text{Aut } \mathbb{D}$. Hence

$$\mathcal{D} = \mathcal{D}_\delta = k(\mathbb{D}) = f_\beta(\mathbb{D}) = \mathcal{F}_\beta.$$

□

PROPOSITION 7.7. *The closure of a flat geodesic meets the closure of the royal variety \mathcal{R}^- exactly once, and the common point lies in \mathcal{R} .*

PROOF. Consider the flat geodesic $\mathcal{F}_\beta = f_\beta(\mathbb{D})$ where $\beta \in \mathbb{D}$. The royal nodes of f_β are the points $z \in \mathbb{D}^-$ such that $(\beta + \bar{\beta}z, z) \in \mathcal{R}^-$, that is, such that

$$(7.2) \quad (\beta + \bar{\beta}z)^2 = 4z.$$

If $\beta = 0$ then the unique royal node of f_β is 0 and $\mathcal{F}_0^- \cap \mathcal{R}^- = \{(0, 0)\}$. If $\beta \neq 0$ then the equation (7.2) has two distinct roots in \mathbb{C} which are symmetric with respect to the unit circle. Hence there is no root in \mathbb{T} and exactly one root in \mathbb{D} . Thus $\mathcal{F}_\beta^- \cap \mathcal{R}^-$ consists of a single point, which belongs to \mathcal{R} . □

The type of a complex geodesic is defined in terms of solutions of the Carathéodory extremal problem. It follows, though, that every solution k of a Kobayashi problem has a type, and one can ask for a characterization of types directly in terms of the complex C-geodesic $k \in G(\mathbb{D})$. Recall that two geodesics $\mathcal{D}_1, \mathcal{D}_2$ are equivalent (written $\mathcal{D}_1 \sim \mathcal{D}_2$) if there exists an automorphism \tilde{m} of G such that $\tilde{m}(\mathcal{D}_1) = \mathcal{D}_2$.

THEOREM 7.8. *Let \mathcal{D} be a complex geodesic of G .*

(1) *\mathcal{D} is purely unbalanced if and only if $\mathcal{D} \sim k_r(\mathbb{D})$ for some $r \in (0, 1)$ where*

$$k_r(z) = \frac{1}{1-rz} (2(1-r)z, z(z-r)) \quad \text{for all } z \in \mathbb{D};$$

(2) *\mathcal{D} is exceptional if and only if $\mathcal{D} \sim h_r(\mathbb{D})$ for some real number $r > 0$, where*

$$(7.3) \quad h_r(z) = (z + m_r(z), zm_r(z))$$

and

$$(7.4) \quad m_r(z) = \frac{(r-i)z+i}{r+i-iz} \quad \text{for } z \in \mathbb{D};$$

(3) \mathcal{D} is purely balanced if $\mathcal{D} \sim g_r(\mathbb{D})$ for some $r \in (0, 1)$, where

$$g_r(z) = (z + B_r(z), zB_r(z)) \quad \text{for all } z \in \mathbb{D};$$

(4) \mathcal{D} is flat if and only if $\mathcal{D} \sim k(\mathbb{D})$ where $k(z) = (0, z)$;

(5) \mathcal{D} is royal if and only if $\mathcal{D} \sim k(\mathbb{D})$ where $k(z) = (2z, z^2)$.

Moreover, in statements (1), (2) and (3), the corresponding geodesics are pairwise inequivalent for distinct values of r in the given range.

PROOF. (1) The royal nodes of k_r are 0 and 1 and the corresponding royal points are $(0, 0)$ and $(2, 1)$. Thus $k(\mathbb{D})^- \cap \mathcal{R}^- = \{(0, 0), (2, 1)\}$. Hence, if $\mathcal{D} \sim k(\mathbb{D})$ then $\mathcal{D}^- \cap \mathcal{R}^-$ consists of a point in \mathcal{R} and another in $\partial\mathcal{R}$. By Theorem 7.1, \mathcal{D} is purely unbalanced.

Conversely, let \mathcal{D} be purely unbalanced. By Theorem 7.1,

$$\mathcal{D}^- \cap \mathcal{R}^- = \{(2\alpha, \alpha^2), (2\tau, \tau^2)\}$$

for some $\alpha \in \mathbb{D}$ and $\tau \in \mathbb{T}$. Choose $m \in \text{Aut } \mathbb{D}$ such that $m(\alpha) = 0$ and $m(\tau) = 1$. Then $\tilde{m}(\mathcal{D}^-) \cap \mathcal{R}^- = \{(0, 0), (2, 1)\}$. By Lemma 6.6, $\tilde{m}(\mathcal{D}) = k_r(\mathbb{D})$ for some $r \in (0, 1)$, and then $\mathcal{D} = \tilde{m}^{-1}(k_r(\mathbb{D})) \sim k_r(\mathbb{D})$.

To prove pairwise inequivalence of the geodesics $k_r(\mathbb{D})$, $0 < r < 1$, suppose that $k_r(\mathbb{D}) \sim k_{r'}(\mathbb{D})$ for some pair r, r' . Then there exists $m_1 \in \text{Aut } \mathbb{D}$ such that $\tilde{m}_1(k_r(\mathbb{D})) = k_{r'}(\mathbb{D})$. According to [34, Corollary 4.6.4], in a general domain Ω , if f, g are complex C-geodesics in Ω having the same range then there is an automorphism α of Ω such that $g = f \circ \alpha$. Hence there exists $m_2 \in \text{Aut } \mathbb{D}$ such that

$$k_{r'} = \tilde{m}_1 \circ k_r \circ m_2.$$

Since \tilde{m}_1^{-1} preserves the royal variety, m_2 maps the royal nodes of $k_{r'}$ to the royal nodes of k_r . These royal nodes are 0 and 1 for both functions, and so m_2 leaves both 0 and 1 fixed. Hence $m_2 = \text{id}_{\mathbb{D}}$ and $k_{r'} = \tilde{m}_1 \circ k_r$. Similarly, \tilde{m}_1 maps the royal points of $k_{r'}$ to the royal points of k_r , which implies that m_1 also fixes the points 0 and 1, and so $m_1 = \text{id}_{\mathbb{D}}$. Thus $k_r = k_{r'}$, which easily implies that $r = r'$.

(2) We first show that $h_r(\mathbb{D})$ is an exceptional geodesic for any $r \in \mathbb{R} \setminus \{0\}$. Let

$$\tau = \frac{r-i}{r+i} \quad \text{and} \quad \alpha = \frac{1}{2}(1 - \bar{\tau}).$$

Then $\tau \in \mathbb{T} \setminus \{-1, 1\}$ and $\alpha \in \mathbb{D} \setminus \{0\}$, and we have $m_r = \tau B_\alpha \in \text{Aut } \mathbb{D}$.

Moreover, since $|1 - \tau| = 2|\alpha|$, it follows that the equation $m_r(z) = z$ has a unique root in \mathbb{T} , which is to say that h_r has a unique royal point in $\partial\mathcal{R}$, and no royal points in \mathcal{R} . By Theorem 3.7, $h_r(\mathbb{D})$ is an exceptional geodesic in G .

Conversely, suppose that \mathcal{D} is an exceptional geodesic in G . By Theorem 7.1, $\mathcal{D}^- \cap \mathcal{R}^-$ consists of a single point in $\partial\mathcal{R}$, which we may take (replacing \mathcal{D} by an equivalent geodesic) to be $(2, 1)$. Let k be a complex C-geodesic in G such that $\mathcal{D} = k(\mathbb{D})$. Since $\mathcal{D} \cap \mathcal{R} = \emptyset$, Proposition 6.1 implies that we can assume that $k(z) = (z + m(z), zm(z))$ for some $m \in \text{Aut } \mathbb{D}$, and since the royal points of this k are the points $(2z, z^2)$ where $z \in \mathbb{D}^-$ satisfies $m(z) = z$, we infer that m has the unique fixed point 1 in \mathbb{D}^- . Any such automorphism has the form $m = \tau B_\alpha$ for some $\alpha \in \mathbb{D} \setminus \{0\}$ and $\tau \in \mathbb{T} \setminus \{-1, 1\}$ such that $1 - \tau = 2\bar{\alpha}$.

Let

$$r = i \frac{1 + \tau}{1 - \tau}.$$

Then $r \in \mathbb{R} \setminus \{0\}$ and

$$\alpha = \frac{1}{2}(1 - \bar{\tau}) = -\frac{i}{r - i}.$$

One may then verify that $m = m_r$, and so $k = h_r$ for some $r \in \mathbb{R} \setminus \{0\}$. Now

$$m_r^{-1}(z) = \frac{(r + i)z - i}{r - i + iz} = m_{-r}(z)$$

and

$$h_r \circ m^{-1}(z) = (z + m_r^{-1}(z), zm_r^{-1}(z)) = h_{-r}(z).$$

Thus, for any $r \in \mathbb{R} \setminus \{0\}$, $h_r(\mathbb{D}) = h_{-r}(\mathbb{D})$. Hence $\mathcal{D} \sim h_r(\mathbb{D})$ for some $r > 0$.

Suppose that $h_r(\mathbb{D}) \sim h_{r'}(\mathbb{D})$ for some $r, r' > 0$. We shall show that $r = r'$. The equivalence implies that there exist $a_1, a_2 \in \text{Aut } \mathbb{D}$ such that

$$(7.5) \quad h_{r'} = \tilde{a}_2 \circ h_r \circ a_1.$$

Since a_1, \tilde{a}_2 fix the common royal nodes and points respectively of $h_r, h_{r'}$, necessarily $a_1(1) = 1 = a_2(1)$.

Equation (7.5) can be written

$$\begin{aligned} (z + m_{r'}(z), zm_{r'}(z)) &= \tilde{a}_2 \circ (a_1(z) + m_r \circ a_1(z), a_1(z)m_r \circ a_1(z)) \\ &= (a_2 \circ a_1(z) + a_2 \circ m_r \circ a_1(z), (a_2 \circ a_1(z))(a_2 \circ m_r \circ a_1(z))) \end{aligned}$$

for all $z \in \mathbb{D}$. It follows that, for every z , either

$$a_2 \circ a_1(z) = z \quad \text{and} \quad a_2 \circ m_r \circ a_1(z) = m_{r'}(z)$$

or

$$a_2 \circ a_1(z) = m_{r'}(z) \quad \text{and} \quad a_2 \circ m_r \circ a_1(z) = z.$$

Therefore either

$$(7.6) \quad a_2 \circ a_1 = \text{id}_{\mathbb{D}} \quad \text{and} \quad a_2 \circ m_r \circ a_1 = m_{r'}$$

or

$$(7.7) \quad a_2 \circ a_1 = m_{r'} \quad \text{and} \quad a_2 \circ m_r \circ a_1 = \text{id}_{\mathbb{D}}.$$

If the alternative (7.6) holds, then $a_2 = a_1^{-1}$ and

$$(7.8) \quad m_r \circ a_1 \circ m_{-r'} = a_1.$$

Let $a_1 = cB_\gamma$ for some $c \in \mathbb{T}$ and $\gamma \in \mathbb{D}$. Since $a_1(1) = 1$,

$$c = \frac{1 - \bar{\gamma}}{1 - \gamma}.$$

In view of equation (7.8) we have $m_r \circ a_1 \circ m_{-r'}(\gamma) = 0$, and therefore

$$a_1 \circ m_{-r'}(\gamma) = -\frac{i}{r - i}.$$

On writing out this equation in detail we obtain

$$\frac{1 - \bar{\gamma}}{1 - \gamma} \frac{(r' + i - i\gamma)\gamma - i - \gamma(r' - i)}{(-\bar{\gamma}(r' + i) + i)\gamma + \bar{\gamma}i + r' - i} = -\frac{i}{r - i}.$$

This equation simplifies to $r = r'$.

The remaining possibility is that the alternative (7.7) holds. In this case we may write

$$m_{-r'} \circ a_2 \circ a_1 = \text{id}_{\mathbb{D}}, \quad m_{-r'} \circ a_2 \circ m_r \circ a_1 = m_{-r'}.$$

Let $b = m_{-r'} \circ a_2$. Then the last equation becomes

$$b \circ a_1 = \text{id}_{\mathbb{D}}, \quad b \circ m_r \circ a_1 = m_{-r'}.$$

Since $m_{-r'}$ and a_2 both fix 1, so does the automorphism b , and we may therefore apply the first alternative, with a_2 replaced by b , to deduce that $r = -r'$. Since $r, r' > 0$, this cannot happen. Thus the first alternative (7.6) holds and $r = r'$.

(3) Suppose that $\mathcal{D} \sim g_r(\mathbb{D})$ for some $r \in (0, 1)$. Since B_r has the fixed points 1 and -1 , g_r has the royal nodes 1, -1 and $g(\mathbb{D})^-$ meets \mathcal{R}^- in the royal points $(\pm 2, 1)$. By Theorem 7.1, $g(\mathbb{D})$ and consequently \mathcal{D} are purely balanced geodesics.

Conversely, suppose that \mathcal{D} is purely balanced. By Theorem 7.1, $\mathcal{D}^- \cap \mathcal{R}^-$ consists of a pair of points in $\partial\mathcal{R}$, and we may take these points to be $(2, 1)$ and $(-2, 1)$. Choose a complex C-geodesic k such that $\mathcal{D} = k(\mathbb{D})$. Since $\mathcal{D} \cap \mathcal{R} = \emptyset$, by Proposition 6.1 we can assume that $k(z) = (z + m(z), zm(z))$ for some $m \in \text{Aut } \mathbb{D}$. The fixed points of m are the royal nodes of k , which are ± 1 . It follows that $m = B_r$ and hence $k = g_r$ for some nonzero $r \in (-1, 1)$. Since

$$g_r \circ B_{-r} = g_{-r},$$

we conclude that $\mathcal{D} \sim g_r(\mathbb{D})$ for some r in $(0, 1)$.

Suppose that $g_r(\mathbb{D}) \sim g_{r'}(\mathbb{D})$. Then there exist $a_1, a_2 \in \text{Aut } \mathbb{D}$ such that

$$(7.9) \quad g_{r'} = \tilde{a}_2 \circ g_r \circ a_1,$$

and moreover a_1, a_2 map the set $\{-1, 1\}$ of royal nodes of g_r and $g_{r'}$ to itself. Hence either a_1 or $-a_1$ fixes both 1 and -1 , and so $a_1 = \pm B_\gamma$ for some $\gamma \in (-1, 1)$. Then $a_1(\gamma) = 0$.

It follows from equation (7.9) that either

$$a_2 \circ a_1 = \text{id}_{\mathbb{D}} \quad \text{and} \quad a_2 \circ B_r \circ a_1 = B_{r'}$$

or

$$(7.10) \quad a_2 \circ a_1 = B_{r'} \quad \text{and} \quad a_2 \circ B_r \circ a_1 = \text{id}_{\mathbb{D}}.$$

Consider the first alternative. Then $a_2 = a_1^{-1}$ and $B_{-r} \circ a_1 = a_1 \circ B_{-r'}$. Since

$$B_{-r} \circ a_1(\gamma) = B_{-r}(0) = r,$$

we have

$$\pm B_\gamma \circ B_{-r'}(\gamma) = r.$$

Since r' and γ are real, $B_{-r'}(\gamma) = B_{-\gamma}(r')$, and so

$$r = \pm B_\gamma \circ B_{-\gamma}(r') = \pm r'.$$

Since $r, r' > 0$, we have $r = r'$.

If the second alternative (7.10) holds, then

$$(B_{-r'} \circ a_2) \circ a_1 = \text{id}_{\mathbb{D}} \quad \text{and} \quad (B_{-r'} \circ a_2) \circ B_r \circ a_1 = B_{-r'},$$

and so, by the first alternative, $r = -r'$. Again this contradicts $r, r' > 0$, and so we deduce that $r = r'$.

(4) and (5) follow from Theorem A.11. □

REMARK 7.9. By Definition 3.4, for a nondegenerate datum δ , there is a unique solution of $\text{Car}(\delta)$ among the Φ_ω if and only if δ is either purely unbalanced or exceptional. Hence the complex \mathbb{C} -geodesics k_r and h_r described in (1) and (2) of Theorem 7.8 have unique holomorphic left inverses (modulo $\text{Aut } \mathbb{D}$) of the form Φ_ω . In fact more is true: Kosiński and Zwonek [35, Theorem 5.3] show that the k_r and h_r have unique holomorphic left inverses (modulo $\text{Aut } \mathbb{D}$) of any form.

Balanced geodesics in G

In studying sets V with the norm-preserving extension property in G we shall exploit intersections of V with balanced and flat geodesics. Recall that a datum is defined to be balanced if it is exceptional, purely balanced or royal. We shall say that a complex geodesic is *balanced* if it is contacted by a nondegenerate balanced datum.

One characterization of balanced datums is given in Proposition 7.4 above; another is afforded by Proposition 8.2 below. The following is a simple observation.

LEMMA 8.1. *Let δ be a nondegenerate datum in G and let k be a solution to $\text{Kob}(\delta)$ such that*

$$k = (m_1 + m_2, m_1 m_2)$$

where $m_1, m_2 \in \text{Aut } \mathbb{D}$. Then the royal nodes of k are the solutions of $m_1 = m_2$ in \mathbb{D}^- and the roots of this equation in \mathbb{C} are symmetric with respect to \mathbb{T} .

PROPOSITION 8.2. *If δ is a nondegenerate datum in G and k is a solution to $\text{Kob}(\delta)$, then δ is balanced if and only if k factors through the bidisc, that is, there exists $f \in \mathbb{D}^2(\mathbb{D})$ such that $k = \pi \circ f$. Furthermore, if $f = (m_1, m_2)$ and $k = \pi \circ f$ then $m_1, m_2 \in \text{Aut } \mathbb{D}$ and exactly one of the following statements holds:*

- (1) $m_1 = m_2$ and δ is royal;
- (2) the equation $m_1(z) = m_2(z)$ has exactly two solutions in \mathbb{T} and δ is purely balanced;
- (3) the equation $m_1(z) = m_2(z)$ has exactly one solution in \mathbb{T} (which is of multiplicity two) and δ is exceptional.

PROOF. Fix a datum δ in G and a solution k to $\text{Kob}(\delta)$. Assume that δ is balanced. By Theorem 7.1 either $\mathcal{D}_\delta \cap \mathcal{R} = \emptyset$ or $\mathcal{D}_\delta = \mathcal{R}$. If $\mathcal{D}_\delta = \mathcal{R}$, then there exists $m \in \text{Aut } \mathbb{D}$ such that $k(z) = (2m(z), m(z)^2) = (\pi \circ f)(z)$ where $f : \mathbb{D} \rightarrow \mathbb{D}^2$ is defined by $f(z) = (m(z), m(z))$. If $\mathcal{D}_\delta \cap \mathcal{R} = \emptyset$ then, by Proposition 6.1, there exist $m_1, m_2 \in \text{Aut } \mathbb{D}$ such that $k = \pi \circ f$ where f is defined by $f(z) = (m_1(z), m_2(z))$. Thus, in either case, k factors through the bidisc.

To prove the converse, suppose that $f \in \mathbb{D}^2(\mathbb{D})$ and $k = \pi \circ f$ but that δ is not balanced, so that δ is either flat or purely unbalanced. Since k is Γ -inner, the components of f are inner.

If δ is flat, then since $C(s) = s^2$ solves $\text{Car}(\delta)$, $C \circ k \in \text{Aut } \mathbb{D}$. This implies that $f^1 f^2$ has degree 1. Since f^1 and f^2 are inner, it follows that either f^1 or f^2 is constant. But then $k = \pi \circ f$ maps \mathbb{D} into ∂G , contradicting the fact that k solves $\text{Kob}(\delta)$.

If δ is purely unbalanced then $\deg f^1 f^2 = \deg k^2 = 2$. Hence, since the argument in the previous paragraph implies that neither f^1 nor f^2 is constant, $\deg f^1 = \deg f^2 = 1$. Thus $m_1, m_2 \in \text{Aut } \mathbb{D}$. By Theorem 7.1, k has one royal

node in \mathbb{T} and one in \mathbb{D} . By Lemma 8.1, the quadratic equation $m_1 = m_2$ has a root in \mathbb{D} and another in \mathbb{T} , and its roots are symmetric with respect to \mathbb{T} . The equation thus has at least 3 distinct roots, which implies that m_1 and m_2 coincide, and so that δ is royal. This is a contradiction. Hence, if k factors through the bidisc then δ is balanced.

Suppose now that δ is balanced and $k = \pi \circ (m_1, m_2)$, where $m_1, m_2 \in \text{Aut}(\mathbb{D})$. Since k is Γ -inner, m_1 and m_2 are inner, and degree considerations show that $m_1, m_2 \in \text{Aut } \mathbb{D}$. Thus k is a rational Γ -inner function of degree 2 and is a complex C -geodesic of G . By Proposition 4.17, k has a royal node in \mathbb{T} .

By Lemma 8.1, the royal nodes of k are the solutions of $m_1 = m_2$ in \mathbb{D}^- and the roots of this equation in \mathbb{C} are symmetric with respect to \mathbb{T} . There are exactly three possible cases for the roots of the quadratic equation $m_1 = m_2$, given that there is a root in \mathbb{T} .

(1) There are infinitely many roots. Then m_1 coincides with m_2 and $k(\mathbb{D}) = \mathcal{R}$. By Theorem 7.1, δ is royal.

(2) The equation $m_1 = m_2$ has exactly two solutions in \mathbb{T} . Then k has exactly two royal nodes in \mathbb{T} . By Theorem 7.1, δ is purely balanced.

(3) The equation $m_1 = m_2$ has exactly one solution in \mathbb{T} , which is of multiplicity two. Then k has a unique royal node, which lies in $\partial\mathcal{R}$. By Theorem 7.1, δ is exceptional. \square

The term ‘balanced’ in Definition 3.4 was suggested by a similar notion for the bidisc.

DEFINITION 8.3. A *balanced disc* in \mathbb{D}^2 is a set of the form $\{(w, m(w)) : w \in \mathbb{D}\}$ for some $m \in \text{Aut } \mathbb{D}$.

According to [2, Definition 1.17], a nondegenerate discrete datum $\lambda = (\lambda_1, \lambda_2)$ in \mathbb{D}^2 is said to be *balanced* if the hyperbolic distance between the first co-ordinates of λ_1 and λ_2 is equal to the hyperbolic distance between the second co-ordinates of λ_1 and λ_2 ; otherwise λ is *unbalanced*. It was noted in [2] that, for a nondegenerate discrete datum λ in \mathbb{D}^2 , the solution to $\text{Kob}(\lambda)$ is essentially unique if and only if λ is balanced, and in that case the unique geodesic of \mathbb{D}^2 that contacts λ is a *balanced disc*. Any geodesic in \mathbb{D}^2 of the form $(m_1, m_2)(\mathbb{D})$, where $m_1, m_2 \in \text{Aut } \mathbb{D}$ is a balanced disc. The following lifting property of certain geodesics in G will be exploited in Section 9.

PROPOSITION 8.4. *Let δ be a nondegenerate datum in G that is either balanced or flat. There exists a balanced disc D in \mathbb{D}^2 such that $\mathcal{D}_\delta = \pi(D)$.*

PROOF. Suppose that δ is a balanced datum. By Proposition 8.2, \mathcal{D}_δ is of the form $\pi(D)$ where $D = (m_1, m_2)(\mathbb{D})$ for some $m_1, m_2 \in \text{Aut } \mathbb{D}$. This D is a balanced disc in \mathbb{D}^2 .

Consider the case of a flat datum δ . Note that

$$\{0\} \times \mathbb{D} = \mathcal{F}_0$$

is a flat geodesic. If $D = \{(z, -z) : z \in \mathbb{D}\}$ then D is a balanced disc in \mathbb{D}^2 , and

$$\pi(D) = \{(0, -z^2) : z \in \mathbb{D}\} = \mathcal{F}_0.$$

Now consider the general flat geodesic \mathcal{F}_β , where $\beta \in \mathbb{D}$. According to Theorem A.11, there exists $m \in \text{Aut } \mathbb{D}$ such that $\tilde{m}(\mathcal{F}_0) = \mathcal{F}_\beta$, where \tilde{m} is the automorphism

of G given by

$$\tilde{m}(z + w, zw) = (m(z) + m(w), m(z)m(w)).$$

Let

$$D_m = \{(m(z), m(-z)) : z \in \mathbb{D}\}.$$

Then D_m is a balanced disc in \mathbb{D}^2 , and

$$\pi(D_m) = \{(m(z) + m(-z), m(z)m(-z)) : z \in \mathbb{D}\} = \tilde{m}(\pi(D)) = \tilde{m}(\mathcal{F}_0) = \mathcal{F}_\beta.$$

□

Note that the image under π of a balanced datum in \mathbb{D}^2 need not be a balanced datum in G – it can be flat.

Geodesics and sets V with the norm-preserving extension property in G

We say that a set $A \subseteq \mathbb{C}^2$ is an *algebraic set* if there exists a set S of polynomials in two variables such that

$$A = \{\lambda \in \mathbb{C}^2 : p(\lambda) = 0 \text{ for all } p \in S\}.$$

We say that a set $V \subseteq \mathbb{C}^2$ is an *algebraic set in G* if there exists an algebraic set A in \mathbb{C}^2 such that $V = A \cap G$. If V is an algebraic set in G and f is a complex valued function defined on V , then we say that f is *holomorphic on V* if f can be *locally* extended to a holomorphic function on \mathbb{C}^2 , that is, if for each $\lambda \in V$ there exists a neighborhood U of λ in \mathbb{C}^2 and a holomorphic function F defined on U such that $F(\mu) = f(\mu)$ for all $\mu \in V \cap U$.

We recall Definition 1.2 from the introduction: a subset V of G has the norm-preserving extension property if every bounded holomorphic function f on V has a holomorphic extension to G of the same supremum norm. An immediate consequence of the definition is the following.

PROPOSITION 9.1. *Any subset of G that has the norm-preserving extension property is connected.*

PROOF. If not, choose two distinct components V_1 and V_2 of V . The norm-preserving extension property of V ensures the existence of a function $F \in \mathbb{C}(G)$ such that $\sup_G |F| = 1$, $F = 0$ on V_1 and $F = 1$ on V_2 , contrary to the Maximum Principle. \square

The following is a slight strengthening of an observation in the introduction.

PROPOSITION 9.2. *If R is a retract in G then R is an algebraic set in G having the norm-preserving extension property.*

PROOF. By Theorem 5.1 R is a geodesic. Hence, by Proposition 5.5, R is an algebraic set. If ρ is a retraction of G with range R and f is a bounded holomorphic function on V then $f \circ \rho$ is a holomorphic extension of f to G with the same supremum norm as f . \square

We initially hoped to prove the converse of Proposition 9.2 for G . After all, in the bidisc it *is* true that sets with the norm-preserving extension property are retracts [2]. However, it transpires that there are algebraic sets with the norm-preserving extension property in G which are not retracts; we call them *anomalous* sets. They are described in Lemma 9.10 below.

This chapter investigates some of the geometric consequences of the norm-preserving extension property. The argument will eventually culminate in Lemma 11.2, to the effect that if V is an algebraic set in G that has the norm-preserving

extension property and if V is not anomalous then V is a properly embedded finitely connected Riemann surface.

9.1. V and $\text{Car}(\delta)$

In this section we prove a technical result about solutions of the Carathéodory problem for datums that contact a set V having the norm-preserving extension property.

LEMMA 9.3. *Let $w_0 \in \mathbb{D}$ and let $D = \{z \in \mathbb{D} : |z - w_0| > r\}$ for some r such that $0 < r < 1 - |w_0|$. For any nondegenerate datum δ in D there exists a holomorphic map $\beta : D \rightarrow \mathbb{D}$ such that*

$$(9.1) \quad |\beta(\delta)|_{\mathbb{D}} > |\delta|.$$

We use the notation $|\cdot|_{\mathbb{D}}$ to emphasize that $\beta(\delta)$ is being regarded as a datum in \mathbb{D} .

PROOF. We can assume that $w_0 = 0$. Consider first the case of a nondegenerate discrete datum $\delta = (z_1, z_2)$. We shall choose $t > 0$ and $g \in \mathbb{C}(D)$ such that $g(z_1) = g(z_2) = 0$ and the function $z - tg(z)$ maps D into a disc of radius $R < 1$; then the function

$$\beta(z) = R^{-1}(z - tg(z))$$

will have the desired properties.

Let

$$\varepsilon = \frac{1}{2(1 + |z_1|)(1 + |z_2|)}.$$

Consider the Laurent expansion of the rational function $z/((z - z_1)(z - z_2))$, which is analytic in an annulus containing \mathbb{T} . A suitable partial sum of the expansion is a rational function $h \in \mathbb{C}(D)$ such that

$$(9.2) \quad \left| h(z) - \frac{z}{(z - z_1)(z - z_2)} \right| < \varepsilon$$

for all $z \in \mathbb{T}$, and the only poles of h are at 0 and ∞ . Let

$$(9.3) \quad g(z) = (z - z_1)(z - z_2)h(z) \quad \text{for } z \in D^-.$$

g is analytic in D and $g(z_1) = g(z_2) = 0$. Multiply the inequality (9.2) by $|(z - z_1)(z - z_2)|$ to deduce that, for $z \in \mathbb{T}$,

$$\begin{aligned} |\bar{z}g(z) - 1| &< \varepsilon \sup_{|z|=1} |(z - z_1)(z - z_2)| \\ &\leq \varepsilon(1 + |z_1|)(1 + |z_2|) \\ &= \frac{1}{2}. \end{aligned}$$

Hence, for $z \in \mathbb{T}$,

$$\text{Re}(\bar{z}g(z)) > \frac{1}{2}.$$

Consequently, for $z \in \mathbb{T}$,

$$(9.4) \quad \begin{aligned} |z - tg(z)|^2 &= 1 - 2t \text{Re}(\bar{z}g(z)) + t^2 |g(z)|^2 \\ &< 1 - t + t^2 \sup_{\mathbb{T}} |g|^2. \end{aligned}$$

Let $M = \sup_{|z|=r} |g(z)|$. For $|z| = r$ we have

$$|z - tg(z)| \leq r + tM.$$

Hence we can choose $R \in (0, 1)$ and t so small that $|z - tg(z)| < R < 1$ for all z in both the inner and outer bounding circles of D , and so, by the Maximum Principle, for all z in D . Consideration of the points z_1, z_2 shows that $R > \max\{|z_1|, |z_2|\}$, and it follows that

$$|\beta(\delta)|_{\mathbb{D}} = |(\beta(z_1), \beta(z_2))| = \left| \left(\frac{z_1}{R}, \frac{z_2}{R} \right) \right| > |(z_1, z_2)| = |\delta|.$$

Now consider the nondegenerate infinitesimal datum $(z_1; 1)$ in D . The proof is similar: construct $t > 0$ and $g \in \mathbb{C}(\mathbb{D})$ such that $g(z_1) = g'(z_1) = 0$ and the function $z - tg(z)$ maps D into a disc of radius $R < 1$. The rest of the construction is as in the discrete case, with $z_2 = z_1$. \square

LEMMA 9.4. *Let $V \subseteq G$ have the norm-preserving extension property and let δ be a nondegenerate datum in G that contacts V . If C is a solution to $\text{Car}(\delta)$ then $C(V)$ is dense in \mathbb{D} .*

PROOF. Let C be a solution to $\text{Car}(\delta)$ and let $\zeta = C(\delta)$. If $C(V)$ is not dense in \mathbb{D} then there exist $w_0 \in \mathbb{D}$ and r satisfying $0 < r < 1 - |w_0|$ such that if D is defined to be $\{z \in \mathbb{D} : |z - w_0| > r\}$, then $C(V) \subseteq D$ and ζ is a nondegenerate datum in D . By Lemma 9.3 there exists a holomorphic map $\beta : D \rightarrow \mathbb{D}$ such that $|\beta(\zeta)| > |\zeta|$. The map $f_0 = \beta \circ (C|_V)$ is well defined and holomorphic from V to \mathbb{D} . Consequently, since V has the norm-preserving extension property, there exists a holomorphic map $F : G \rightarrow \mathbb{D}^-$ satisfying $F|_V = f_0$. By the Open Mapping Theorem, $F(G) \subset \mathbb{D}$, and so F is a candidate for $\text{Car}(\delta)$. But C is a solution to $\text{Car}(\delta)$ and

$$\begin{aligned} |F(\delta)| &= |f_0(\delta)| && \text{by Remark 4.3} \\ &= |(\beta \circ C)(\delta)| \\ &= |(\beta(C(\delta)))| && \text{by equation (3.1)} \\ &= |\beta(\zeta)| \\ &> |\zeta| \\ &= |C(\delta)| \\ &= |\delta|_{\text{car}}. \end{aligned}$$

This contradiction shows that $C(V)$ is dense in \mathbb{D} . \square

9.2. V and balanced datums

Here is another technical result concerning datums that contact a set with the norm-preserving extension property.

LEMMA 9.5. *Let V be an algebraic set in G that has the norm-preserving extension property. Let δ be a nondegenerate datum in G such that there exist distinct points $\omega_1, \omega_2 \in \mathbb{T}$ for which $\Phi_{\omega_1}, \Phi_{\omega_2}$ solve $\text{Car}(\delta)$ (so that δ is either flat, royal or purely balanced). If δ contacts V then $\mathcal{D}_\delta \subseteq V$.*

PROOF. Pick $\omega_1, \omega_2 \in \mathbb{T}$ where $\omega_1 \neq \omega_2$ and Φ_{ω_1} and Φ_{ω_2} both solve $\text{Car}(\delta)$. Fix any solution k to $\text{Kob}(\delta)$, choose $m_1, m_2 \in \text{Aut } \mathbb{D}$ such that

$$m_1 \circ \Phi_{\omega_1} \circ k = \text{id}_{\mathbb{D}} \quad \text{and} \quad m_2 \circ \Phi_{\omega_2} \circ k = \text{id}_{\mathbb{D}},$$

and then let C_1 and C_2 be defined by

$$C_1 = m_1 \circ \Phi_{\omega_1} \quad \text{and} \quad C_2 = m_2 \circ \Phi_{\omega_2};$$

then

$$C_1 \circ k = \text{id}_{\mathbb{D}} = C_2 \circ k.$$

Define C by

$$C = \frac{1}{2}C_1 + \frac{1}{2}C_2.$$

CLAIM 9.6. If $\tau \in \mathbb{T}$, $\tau \neq C_2(2\bar{\omega}_1, \bar{\omega}_1^2)$, and $\tau \neq C_1(2\bar{\omega}_2, \bar{\omega}_2^2)$, then there exists $s_0 \in \mathcal{D}_\delta^- \cap V^-$ such that $s_0 \neq (2\bar{\omega}_1, \bar{\omega}_1^2)$, $s_0 \neq (2\bar{\omega}_2, \bar{\omega}_2^2)$, and

$$C_1(s_0) = C_2(s_0) = \tau.$$

To prove this claim, fix $\tau \in \mathbb{T} \setminus \{C_2(2\bar{\omega}_1, \bar{\omega}_1^2), C_1(2\bar{\omega}_2, \bar{\omega}_2^2)\}$. By construction, C solves $\text{Car}(\delta)$. Therefore, by Lemma 9.4 there exists a sequence of points $\{s_n\}$ in V such that $C(s_n) \rightarrow \tau$. Passing to a subsequence if necessary we may further assume that $s_n \rightarrow s_0 \in \partial G$. Since $C(s_n) \rightarrow \tau$, it follows that both $C_1(s_n) \rightarrow \tau$ and $C_2(s_n) \rightarrow \tau$. But then $C_1(s_n) - C_2(s_n) \rightarrow 0$. Consequently, by Lemma 5.6, it follows that $P_\delta(s_n) \rightarrow 0$. Since P_δ is continuous, it follows that $P_\delta(s_0) = 0$, that is, that $s_0 \in \mathcal{D}_\delta^-$. Since it is also the case that $s_0 \in V^-$, it follows that $s_0 \in \mathcal{D}_\delta^- \cap V^-$.

To see that $s_0 \neq (2\bar{\omega}_1, \bar{\omega}_1^2)$, assume to the contrary that $s_0 = (2\bar{\omega}_1, \bar{\omega}_1^2)$. Since $C_2 = m_2 \circ \Phi_{\omega_2}$ is continuous on a neighborhood of $(2\bar{\omega}_1, \bar{\omega}_1^2)$, it follows that

$$C_2(2\bar{\omega}_1, \bar{\omega}_1^2) = C_2(s_0) = \lim_{n \rightarrow \infty} C_2(s_n) = \tau,$$

contradicting the assumption that $\tau \neq C_2(2\bar{\omega}_1, \bar{\omega}_1^2)$. That $s_0 \neq (2\bar{\omega}_2, \bar{\omega}_2^2)$ follows in similar fashion. This completes the proof of Claim 9.6.

Now fix $\tau \in \mathbb{T}$ with $\tau \neq C_2(2\bar{\omega}_1, \bar{\omega}_1^2)$ and $\tau \neq C_1(2\bar{\omega}_2, \bar{\omega}_2^2)$. Let s_0 be as in the claim. Since $s_0 \in \mathcal{D}_\delta^-$, there exists $\eta \in \mathbb{T}$ such that $s_0 = k(\eta)$. But then

$$\tau = C_1(s_0) = C_1(k(\eta)) = (C_1 \circ k)(\eta) = \eta,$$

so that

$$k(\tau) = k(\eta) = s_0 \in V^-.$$

This proves that

$$k(\mathbb{T} \setminus \{C_2(2\bar{\omega}_1, \bar{\omega}_1^2), C_1(2\bar{\omega}_2, \bar{\omega}_2^2)\}) \subseteq V^-.$$

By continuity, $k(\mathbb{T}) \subseteq V^-$. Since V is algebraic there is a set S of polynomials such that

$$V = G \cap \{s : p(s) = 0 \text{ for all } p \in S\}.$$

Hence $p \circ k = 0$ on \mathbb{T} for all $p \in S$, and by the Maximum Principle, $p \circ k = 0$ on \mathbb{D} for all $p \in S$. Thus $\mathcal{D}_\delta = k(\mathbb{D}) \subseteq V$, as was to be proved. \square

REMARK 9.7. The above proof remains valid if V , rather than being assumed to be an algebraic set, is only assumed to be relatively polynomially convex.

9.3. V and flat or royal datums

In this section we show that if a set V having the norm-preserving extension property contacts either a flat datum or a royal datum then V has one of three concrete forms. To do this we first prove a result about the closest point in G to a flat or royal geodesic.

LEMMA 9.8. *Let δ be a nondegenerate datum in G such that Φ_ω solves $\text{Car}(\delta)$ for all $\omega \in \mathbb{T}$ (so that δ is flat or royal).*

(i) *If $t \in G \setminus \mathcal{D}_\delta$ then*

$$\inf_{s \in \mathcal{D}_\delta} |(s, t)|_{\text{car}}$$

is attained at some point $s_0 \in \mathcal{D}_\delta$. Furthermore,

(ii) *there are at least two values of $\omega \in \mathbb{T}$ such that Φ_ω solves $\text{Car}((s_0, t))$ (so that the datum (s_0, t) is either flat, royal or purely balanced);*

(iii) *if \mathcal{D}_δ is flat and $t \in \mathcal{R}$ then $s_0 \in \mathcal{R}$;*

(iv) *if \mathcal{D}_δ is flat and $t \notin \mathcal{R}$ then the datum (s_0, t) is purely balanced.*

PROOF. (i) Let k solve $\text{Kob}(\delta)$ and let $\omega_1 \in \mathbb{T}$ be such that $\Phi_{\omega_1} \circ k = \text{id}_{\mathbb{D}}$. The function φ on \mathcal{D}_δ defined by

$$\varphi(s) = |(s, t)|_{\text{car}}$$

is continuous and $\text{ran } \varphi \subseteq (0, 1)$. Note that

$$\begin{aligned} \varphi(s) &= |(s, t)|_{\text{car}} = \sup_{\omega \in \mathbb{T}} |(\Phi_\omega(s), \Phi_\omega(t))| \\ &\geq |(\Phi_{\omega_1}(s), \Phi_{\omega_1}(t))|. \end{aligned} \tag{9.5}$$

Let $w_1 = \Phi_{\omega_1}(t)$ and let $s = k(z)$, $z \in \mathbb{D}$. We have

$$\begin{aligned} \varphi(s) &\geq |(\Phi_{\omega_1} \circ k(z), w_1)| \\ &= |(z, w_1)|. \end{aligned} \tag{9.6}$$

Therefore

$$\begin{aligned} 1 - \varphi(s)^2 &\leq 1 - |(z, w_1)|^2 \\ &= \frac{(1 - |z|^2)(1 - |w_1|^2)}{|1 - \bar{w}_1 z|^2} \\ &\leq (1 - |z|^2) \frac{1 + |w_1|}{1 - |w_1|}. \end{aligned} \tag{9.7}$$

It is clear that $\varphi(s) \rightarrow 1$ as $s \rightarrow \partial G$ in \mathcal{D}_δ . Hence the function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(s) = \begin{cases} \varphi(s) & \text{if } s \in \mathcal{D}_\delta \\ 1 & \text{if } s \in \partial G \cap \overline{\mathcal{D}_\delta} \end{cases}$$

is continuous on the closure $\overline{\mathcal{D}_\delta}$ of \mathcal{D}_δ in G , and $\text{ran } \tilde{\varphi} \subseteq (0, 1]$. Note that $k(\mathbb{D}^-) = \overline{\mathcal{D}_\delta}$, and so $\overline{\mathcal{D}_\delta}$ is compact. Hence $\tilde{\varphi}$ attains its infimum at some point $s_0 \in \overline{\mathcal{D}_\delta}$. It is clear that s_0 is not in $k(\mathbb{T})$, since $\tilde{\varphi}(\mathbb{D}) \subseteq (0, 1)$ and $|\tilde{\varphi}(s)| = 1$ for all $s \in k(\mathbb{T})$. Thus $s_0 \in \mathcal{D}_\delta$.

(ii) Suppose that there exists a unique $\omega_0 \in \mathbb{T}$ such that

$$|\Phi_{\omega_0}((s_0, t))| = \sup_{\omega \in \mathbb{T}} |\Phi_\omega((s_0, t))| = |(s_0, t)|_{\text{car}}. \tag{9.8}$$

Let k be a solution to $\text{Kob}(\delta)$ and let $s_0 = k(z_0)$.

Since δ is assumed to be flat or royal, for each $\omega \in \mathbb{T}$, $\Phi_\omega \circ k$ is in $\text{Aut } \mathbb{D}$ and therefore has no critical points. For any direction $w \in \mathbb{C}$,

$$\begin{aligned} D_w |\Phi_{\omega_0}((k(z), t))|^2 \Big|_{z=z_0} &= D_w |(\Phi_{\omega_0} \circ k(z), \Phi_{\omega_0}(t))|^2 \Big|_{z=z_0} \\ &= \frac{d}{dy} |B \circ \Phi_{\omega_0} \circ k(z_0 + yw)|^2 \Big|_{y=0} \\ &= 2 \text{Re}\{w(B \circ \Phi_{\omega_0} \circ k)'(z_0) \overline{B \circ \Phi_{\omega_0} \circ k(z_0)}\}, \end{aligned} \tag{9.9}$$

where

$$B(z) = \frac{z - \Phi_{\omega_0}(t)}{1 - \overline{\Phi_{\omega_0}(t)}z}.$$

Now

$$|B \circ \Phi_{\omega_0} \circ k(z_0)| = |B \circ \Phi_{\omega_0}(s_0)| = |(s_0, t)|_{\text{car}} \neq 0.$$

Since, further, B' and $(\Phi_{\omega_0} \circ k)'$ are never zero on \mathbb{D} ,

$$(B \circ \Phi_{\omega_0} \circ k)'(z_0) \overline{(B \circ \Phi_{\omega_0} \circ k)(z_0)} \neq 0.$$

By equation (9.9), there is some direction w_0 such that $D_{w_0} |\Phi_{\omega_0}((k(z), t))|^2|_{z=z_0}$ is nonzero. After scaling w_0 , we can assume that

$$(9.10) \quad D_{w_0} |\Phi_{\omega_0}((k(z), t))|^2|_{z=z_0} = 1.$$

For $x \in [-\pi, \pi]$ and y in some sufficiently small neighborhood I of 0, let

$$f(x, y) = |\Phi_{e^{ix}\omega_0}((k(z_0 + yw_0), t))|^2.$$

Choose a sequence (y_j) in I , with $y_j < 0$, such that $y_j \rightarrow 0$.

CLAIM 9.9. There is a sequence (x_j) in $[-\pi, \pi]$ such that, for each j , $f(x, y_j)$ attains its maximum at $x = x_j$, and $x_j \rightarrow 0$.

Indeed, suppose not. Choose, for each j , any point x_j at which $f(x, y_j)$ attains its maximum over $[-\pi, \pi]$, then pass to a subsequence such that x_j converges to some point x_0 . By supposition, $x_0 \neq 0$, and so $e^{ix_0}\omega_0 \neq \omega_0$. We have $k(z_0 + y_j w_0) \rightarrow s_0$, and therefore

$$(9.11) \quad \begin{aligned} f(x_j, y_j) &= \max_{-\pi \leq x \leq \pi} f(x, y_j) \\ &= \max_{-\pi \leq x \leq \pi} |\Phi_{e^{ix}\omega_0}((k(z_0 + y_j w_0), t))|^2 \\ &= |(k(z_0 + y_j w_0), t)|_{\text{car}}^2 \\ &\rightarrow |(s_0, t)|_{\text{car}}^2. \end{aligned}$$

Hence

$$f(x_0, 0) = |(s_0, t)|_{\text{car}}^2.$$

Thus $\omega_1 = e^{ix_0}\omega_0$ is a second value of ω for which Φ_ω solves $\text{Car}((s_0, t))$, contrary to hypothesis. This establishes the Claim.

The function f has the following properties.

- (1) f is real analytic on $[-\pi, \pi] \times I$;
- (2) $\frac{\partial f}{\partial y}(0, 0) = 1$;
- (3) $f(x, 0)$ has a unique global maximum over $[-\pi, \pi]$, which is at $x = 0$.

Consider the power series expansion of f about $(0, 0)$. In view of statement (3) there exists $n \geq 1$ and $\rho > 0$ such that

$$\left. \frac{d^{2n}}{dx^{2n}} f(x, 0) \right|_{x=0} = -\rho$$

is the first nonzero derivative of $f(x, 0)$ at $x = 0$. There also exist analytic functions $\varphi(x), \psi(x, y)$ such that $\psi(0, 0) = 0$ and

$$(9.12) \quad f(x, y) = |(s_0, t)|_{\text{car}}^2 + y - \rho x^{2n} + x^{2n+1}\varphi(x) + y\psi(x, y)$$

in a neighborhood of $(0, 0)$. Consequently

$$\frac{\partial f}{\partial x}(x, y) = -2n\rho x^{2n-1} + (2n+1)x^{2n}\varphi(x) + x^{2n+1}\varphi'(x) + y\frac{\partial\psi}{\partial x}(x, y)$$

near $(0, 0)$. As x_j is a maximiser for $f(x, y_j)$, x_j is a critical point of $f(x, y_j)$. That is,

$$0 = -2n\rho x_j^{2n-1} + (2n+1)x_j^{2n}\varphi(x_j) + x_j^{2n+1}\varphi'(x_j) + y_j\frac{\partial\psi}{\partial x}(x_j, y_j).$$

It follows that

$$x_j^{2n-1} = O(y_j).$$

Since $x_j \rightarrow 0$,

$$-\rho x_j^{2n} + x_j^{2n+1}\varphi(x_j) = x_j^{2n-1}(-\rho x_j + x_j^2\varphi(x_j)) = o(y_j),$$

and, since $\psi(0, 0) = 0$,

$$y_j\psi(x_j, y_j) = o(y_j).$$

Hence the series expansion (9.12) yields

$$f(x_j, y_j) = |(s_0, t)|_{\text{car}}^2 + y_j + o(y_j),$$

and therefore (since $y_j < 0$) $f(x_j, y_j) < |(s_0, t)|_{\text{car}}^2$ for some j .

However, for all j we have, by the calculation (9.11),

$$\begin{aligned} f(x_j, y_j) &= |(k(z_0 + y_j w_0), t)|_{\text{car}}^2 \\ &\geq |(s_0, t)|_{\text{car}}^2 \end{aligned}$$

by choice of s_0 as a closest point in \mathcal{D}_δ to t . This is a contradiction, and so part (i) of the lemma is proved.

(iii) Suppose $t \in \mathcal{R}$, say $t = (2w, w^2)$ for some $w \in \mathbb{D}$. Without loss of generality we may assume that \mathcal{D}_δ is the flat geodesic $\mathcal{F}_0 = \{(0, z) : z \in \mathbb{D}\}$. This is because the automorphisms of G fix \mathcal{R} , permute the flat geodesics (by Proposition A.11) and preserve the Carathéodory distance. We claim that $(0, 0)$ is the closest point in \mathcal{F}_0 to t .

We must show that, for every $z \in \mathbb{D}$,

$$|(t, (0, 0))|_{\text{car}} \leq |(t, (0, z))|_{\text{car}}.$$

Now

$$\begin{aligned} |(t, (0, 0))|_{\text{car}} &= \sup_{\omega \in \mathbb{T}} |(\Phi_\omega(2w, w^2), \Phi_\omega(0, 0))| \\ &= \sup_{\omega \in \mathbb{T}} |(-w, 0)| \\ &= |w|. \end{aligned}$$

Furthermore

$$\begin{aligned} |(t, (0, z))|_{\text{car}} &= \sup_{\omega \in \mathbb{T}} |(\Phi_\omega(2w, w^2), \Phi_\omega(0, z))| \\ &= \sup_{\omega \in \mathbb{T}} |(-w, \omega z)| \\ &= \sup_{\omega \in \mathbb{T}} \left| \frac{\omega z + w}{1 + \bar{w}\omega z} \right|. \end{aligned}$$

Suppose $z \neq 0$. Choose $\omega \in \mathbb{T}$ and $r > 0$, such that $\omega z = rw$. Then

$$\begin{aligned} |(t, (0, z))|_{\text{car}} &\geq \left| \frac{\omega z + w}{1 + \bar{w}\omega z} \right| \\ &= \left| \frac{(1+r)w}{1+r|w|^2} \right| \\ &> |w| \\ &= |(t, (0, 0))|_{\text{car}}. \end{aligned}$$

Hence $(0, 0)$ is a closest point to t in \mathcal{D}_δ (in fact it is the unique closest point). Thus $s_0 \in \mathcal{R}$.

(iv) Since $t \notin \mathcal{R}$, by (ii) the datum (s_0, t) is either flat or purely balanced. If the datum (s_0, t) is flat then the $\mathcal{D}_\delta = \mathcal{D}_{(s_0, t)}$ and so $t \in \mathcal{D}_\delta$. This is a contradiction to the assumption that $t \in G \setminus \mathcal{D}_\delta$. Therefore the datum (s_0, t) is purely balanced. \square

LEMMA 9.10. *Let V be an algebraic set in G having the norm-preserving extension property. If δ is a flat datum in G that contacts V then either $V = \mathcal{D}_\delta$, $V = \mathcal{R} \cup \mathcal{D}_\delta$ or $V = G$.*

PROOF. Since δ contacts V , it follows from Lemma 9.5 that $\mathcal{D}_\delta \subseteq V$.

Suppose that $V \neq \mathcal{D}_\delta$. Then there exists a point $v_0 \in V$ such that $v_0 \notin \mathcal{D}_\delta$. Consider two cases: (1) $v_0 \notin \mathcal{R}$, and (2) $v_0 \in \mathcal{R}$.

Case (1): $v_0 \notin \mathcal{R}$. By Lemma 9.8 there exists a point $s_0 \in \mathcal{D}_\delta$ which is the closest point of \mathcal{D}_δ to v_0 and a pair ω_1, ω_2 of distinct points in \mathbb{T} such that both Φ_{ω_1} and Φ_{ω_2} solve $\text{Car}(\gamma)$, where $\gamma = (s_0, v_0)$. According to Definition 3.4 the datum γ is either flat, purely balanced or royal. Now γ is not royal, since $v_0 \notin \mathcal{R}$. Nor is γ flat; indeed, if γ is flat, then \mathcal{D}_γ and \mathcal{D}_δ are flat geodesics through s_0 and hence coincide, contrary to the fact that $v_0 \in \mathcal{D}_\gamma \setminus \mathcal{D}_\delta$. Hence γ is purely balanced, which is to say that there are exactly two values of ω for which Φ_ω solves $\text{Car}(\gamma)$. Consequently, by Lemma 9.5, $\mathcal{D}_\gamma \subseteq V$. Corollary 8.4 now implies that there is a balanced disc D in the bidisc such that $\pi(D) = \mathcal{D}_\gamma$.

Consider the case that \mathcal{D}_δ is the flat geodesic given by equation (4.2) with $\beta = 0$, that is,

$$\mathcal{D}_\delta = \mathcal{F}_0 = \{(0, z) : z \in \mathbb{D}\}.$$

By inspection,

$$\mathcal{D}_\delta = \pi(B)$$

where B is the balanced disc in \mathbb{D}^2 given by

$$(9.13) \quad B = \{(z, -z) : z \in \mathbb{D}\}.$$

Let $\lambda_0 = (z_0, -z_0) \in B$ be such that $\pi(\lambda_0) = s_0$. We can assume that also $\lambda_0 \in D$, for certainly D contains one of the (at most two) points of $\pi^{-1}(s_0)$, and we may replace D by the balanced disc $\{(\lambda^2, \lambda^1) : (\lambda^1, \lambda^2) \in D\}$ if necessary. Thus $\lambda_0 \in D \cap B$.

Choose $\mu_0 \in D$ such that $\pi(\mu_0) = v_0$. Since $v_0 \notin \mathcal{R}$, μ_0 does not lie in the diagonal set Δ in \mathbb{D}^2 , defined by $\Delta = \{(z, z) : z \in \mathbb{D}\}$. Since λ_0, μ_0 belong to the balanced disc D in \mathbb{D}^2 ,

$$|(z_0, \mu_0^1)| = |(-z_0, \mu_0^2)| = |(z_0, -\mu_0^2)|.$$

The locus of points $z \in \mathbb{D}$ such that

$$|(z, \mu_0^1)| = |(z, -\mu_0^2)|$$

is a geodesic $(z_t)_{t \in \mathbb{R}}$ through z_0 in the Poincaré disc (this statement is obvious if $\mu_0^1 = r$, $\mu_0^2 = -r$ for some $r > 0$, and the general case follows upon application of a suitable automorphism of \mathbb{D}). Thus

$$|(z_t, \mu_0^1)| = |(z_t, -\mu_0^2)| = |(-z_t, \mu_0^2)|.$$

Let $\lambda_t = (z_t, -z_t)$. Then (λ_t) is a curve in B such that (λ_t, μ_0) is a balanced pair for all t . Let the unique geodesic of \mathbb{D}^2 passing through λ_t and μ_0 be D_t (see the comment preceding Corollary 8.4). D_t is the balanced disc having the form

$$(9.14) \quad D_t = \{(w, m_t(w)) : w \in \mathbb{D}\}$$

where m_t is the unique automorphism of \mathbb{D} such that

$$(9.15) \quad m_t(\mu_0^1) = \mu_0^2 \quad \text{and} \quad m_t(z_t) = -z_t.$$

Clearly m_t depends continuously on $t \in \mathbb{R}$.

We claim that if $t_1 \neq t_2$ then $D_{t_1} \cap D_{t_2} = \{\mu_0\}$. Suppose, to the contrary, that the intersection contains a point $\nu \neq \mu_0$. Then, for some $w_1, w_2 \in \mathbb{D}$,

$$(w_1, m_{t_1}(w_1)) = \nu = (w_2, m_{t_2}(w_2)) \neq \mu_0.$$

Thus $w_1 = w_2$ and so

$$m_{t_1}(w_1) = m_{t_2}(w_1)$$

and either $w_1 \neq \mu_0^1$ or $m_{t_1}(w_1) \neq \mu_0^2$.

If $w_1 \neq \mu_0^1$ then m_{t_1}, m_{t_2} are automorphisms of \mathbb{D} taking the same values at the distinct points w_1 and μ_0^1 . Hence $m_{t_1} = m_{t_2}$ and so $D_{t_1} = D_{t_2}$. Thus D_{t_1} is a balanced disc containing the two distinct points $\lambda_{t_1}, \lambda_{t_2}$. The balanced disc B also contains these two points, and so, by uniqueness, $D_{t_1} = B$. It follows that $\mu_0 \in B$, and so, by application of π , we have $v_0 \in \mathcal{D}_\delta$, a contradiction. We deduce that $w_1 = \mu_0^1$ and $m_{t_1}(w_1) \neq \mu_0^2$. That is, $m_{t_1}(\mu_0^1) \neq \mu_0^2$, contrary to equation (9.15). Hence $D_{t_1} \cap D_{t_2} = \{\mu_0\}$.

Since D is the balanced disc in \mathbb{D}^2 containing λ_0 and μ_0 , again by uniqueness, $D_0 = D$. Hence $\pi(D_0) = \mathcal{D}_\gamma$ and so

$$(9.16) \quad \pi(D_0) \text{ is a purely balanced geodesic in } G.$$

Now $D_0 = \{(w, m_0(w)) : w \in \mathbb{D}\}$. By Proposition 6.1, m_0 has two distinct fixed points in \mathbb{T} . It follows that there is a neighborhood I of 0 in \mathbb{R} such that m_t has two distinct fixed points in \mathbb{T} for all $t \in I$. Then, for $t \in I$, $\pi(D_t)$ is a purely balanced geodesic in G containing the distinct points $v_0 = \pi(\mu_0)$ and $\pi(\lambda_t)$, both of which belong to V . Hence

$$\pi(D_t) = \mathcal{D}_{\delta_t} \quad \text{where } \delta_t = (v_0, \pi(\lambda_t)).$$

The nondegenerate discrete datum δ_t contacts V . By Lemma 9.5, $\mathcal{D}_{\delta_t} \subseteq V$ for all $t \in I$.

Now, since $\lambda_t \in B$, we have $\pi(\lambda_t) \in \mathcal{F}_0 \cap \mathcal{D}_{\delta_t}$ for all $t \in \mathbb{R}$. Hence, for $t \in I$ and β in a neighborhood N of zero,

$$(9.17) \quad \mathcal{F}_\beta \text{ meets } \mathcal{D}_{\delta_t} \quad \text{and} \quad \mathcal{D}_{\delta_t} \subset V.$$

Since $v_0 \notin \mathcal{F}_0$, we may ensure (replacing N by a smaller neighborhood of 0 if necessary) that $v_0 \notin \mathcal{F}_\beta$ for all $\beta \in N$.

Since $\mathcal{D}_{\delta_t} = \pi(D_t)$, statement (9.17) implies that there exists $\lambda_{\beta t} \in D_t \setminus \{\mu_0\}$ such that $\pi(\lambda_{\beta t}) \in \mathcal{F}_\beta \cap \mathcal{D}_{\delta_t} \subseteq V$ for $\beta \in N, t \in I$. Since distinct D_t s meet only at μ_0 , for fixed $\beta \in N$, the points $\lambda_{\beta t}, t \in I$, are pairwise distinct. Since each point of G has at most two preimages in \mathbb{D}^2 under π , it follows that the set $\{\pi(\lambda_{\beta t}) : t \in I\}$ is an uncountable subset of $\mathcal{F}_\beta \cap V$.

We can now show that $V = G$.

Suppose that p is a polynomial that vanishes on V . Then, for $\beta \in N$, p vanishes at the uncountably many points $\pi(\lambda_{\beta t}), t \in I$, in the one-dimensional disc \mathcal{F}_β , and hence $p|_{\mathcal{F}_\beta} = 0$. Since the union of the discs \mathcal{F}_β , for $\beta \in N$, is a neighborhood of zero in \mathbb{C}^2 , we have $p = 0$. Thus every polynomial that vanishes on V is the zero polynomial. Therefore, since V is an algebraic set in G , $V = G$.

We have shown that $V = G$ in the case that \mathcal{D}_δ is the flat geodesic \mathcal{F}_0 . If \mathcal{D}_δ is the flat geodesic \mathcal{F}_β , for some $\beta \in \mathbb{D}$, then the same conclusion can be deduced from the case $\beta = 0$ by the application of an automorphism of G , with the aid of Theorem A.11 in the Appendix. This concludes the proof of the lemma in Case (1).

Case (2): $v_0 \in \mathcal{R}$. By Lemma 9.8(ii) the closest point s_0 to v_0 in \mathcal{D}_δ lies in \mathcal{R} . Hence the nondegenerate discrete datum $\gamma = (s_0, v_0)$ is royal and contacts V . By Lemma 9.5, $\mathcal{D}_\gamma \subseteq V$, that is, $\mathcal{R} \subseteq V$. Hence $\mathcal{R} \cup \mathcal{D}_\delta \subseteq V$.

If $V \neq \mathcal{R} \cup \mathcal{D}_\delta$ then there exists a point $v_1 \in V \setminus (\mathcal{R} \cup \mathcal{D}_\delta)$. By Case (1), $V = G$.

We have shown that, if $V \neq \mathcal{D}_\delta$, then in Case (1) $V = G$ and in Case (2) either $V = \mathcal{R} \cup \mathcal{D}_\delta$ or $V = G$. \square

LEMMA 9.11. *Let V be an algebraic set in G having the norm-preserving extension property. If a royal datum contacts V then either $V = \mathcal{R}$, $V = \mathcal{R} \cup \mathcal{D}_\delta$ for some flat datum δ or $V = G$.*

PROOF. By Lemma 9.5, $\mathcal{R} \subseteq V$. Suppose that $V \neq \mathcal{R}$. There exists a point $t_0 \in V \setminus \mathcal{R}$. By Lemma 9.8 there is a point $s_0 \in \mathcal{R}$ such that the datum $\delta = (s_0, t_0)$ is either flat, royal or purely balanced. Since $t_0 \notin \mathcal{R}$, δ is not royal. Neither is δ purely balanced, because \mathcal{D}_δ meets \mathcal{R} at the point $s_0 \in G$, and Theorem 7.1(4) applies. Hence δ is flat. Again by Lemma 9.5, $\mathcal{D}_\delta \subseteq V$. Hence, by Lemma 9.10, either $V = \mathcal{R} \cup \mathcal{D}_\delta$ or $V = G$. \square

Anomalous sets $\mathcal{R} \cup \mathcal{D}$ with the norm-preserving extension property in G

Lemmas 9.10 and 9.11 imply that, among sets that contact a flat or royal datum, the only possible anomalous algebraic sets having the norm-preserving extension property are sets of the form $\mathcal{R} \cup \mathcal{D}$ for some flat complex geodesic \mathcal{D} . This chapter will be devoted to the proof that such sets do indeed have the norm-preserving extension property.

THEOREM 10.1. *If*

$$V = \mathcal{R} \cup \mathcal{D}$$

for some flat geodesic \mathcal{D} of G then V has the norm-preserving extension property.

Section 10.1 contains a number of definitions and lemmas to be used in the proof of Theorem 10.1 and the proof of the theorem is executed in Section 10.2.

10.1. Definitions and lemmas

DEFINITION 10.2. Let U be a domain and assume that $V \subseteq U$. We say that a function $f : V \rightarrow \mathbb{C}$ is *Herglotz on V* if f is analytic on V and $\operatorname{Re} f(s) > 0$ for all $s \in V$. We say that V has the *Herglotz-preserving extension property* if every function f that is Herglotz on V has an extension to a function that is Herglotz on U .

LEMMA 10.3. *Let U be a domain and assume that $V \subseteq U$. Then V has the Herglotz-preserving extension property if and only if V has the norm-preserving extension property.*

PROOF. The formula

$$\varphi(z) = \frac{1+z}{1-z}$$

defines a conformal automorphism of the Riemann sphere which maps \mathbb{D} onto the right halfplane $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. Consequently, if $V \subseteq U$ and f is an analytic function on V , then

$$f \text{ is Herglotz on } V \iff |\varphi^{-1} \circ f(s)| < 1 \text{ for all } s \in V.$$

Since this statement holds when $V = U$ as well, the lemma follows immediately. \square

DEFINITION 10.4. If f is a complex valued function on $\mathcal{R} \cup \mathcal{F}_0$ we define functions on \mathbb{D} by the formulas

$$(10.1) \quad f_{\mathcal{F}}(z) = f(0, z) \quad \text{and} \quad f_{\mathcal{R}}(z) = f(2z, z^2).$$

LEMMA 10.5. *Let f be a complex valued function defined on $\mathcal{R} \cup \mathcal{F}_0$. Then f is analytic on $\mathcal{R} \cup \mathcal{F}_0$ if and only if $f_{\mathcal{F}}$ and $f_{\mathcal{R}}$ are analytic on \mathbb{D} and $f_{\mathcal{F}}(0) = f_{\mathcal{R}}(0)$.*

PROOF. Clearly, if f is analytic on $\mathcal{R} \cup \mathcal{F}_0$, then $f_{\mathcal{F}}$ and $f_{\mathcal{R}}$ are analytic on \mathbb{D} and $f_{\mathcal{F}}(0) = f_{\mathcal{R}}(0)$. Conversely, assume that $f_{\mathcal{F}}$ and $f_{\mathcal{R}}$ are analytic on \mathbb{D} and

$$f_{\mathcal{F}}(0) = \eta = f_{\mathcal{R}}(0).$$

It is immediate from the first formula in (10.1) that f is analytic at each point $s = (0, z) \in \mathcal{F}_0 \setminus \{(0, 0)\}$. Likewise, the second formula in (10.1) implies that f is analytic at each point $s = (2z, z^2) \in \mathcal{R} \setminus \{(0, 0)\}$. To see that f is analytic at $s = (0, 0)$, define F by the formula

$$F(s) = f_{\mathcal{R}}\left(\frac{1}{2}s^1\right) + f_{\mathcal{F}}\left(s^2 - \frac{1}{4}(s^1)^2\right) - \eta$$

and observe that F is a holomorphic extension of f to a neighborhood of $(0, 0)$. \square

DEFINITION 10.6. For $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$ we define $f_{\tau} : \mathcal{R} \cup \mathcal{F}_0 \rightarrow \mathbb{D}$ by the formula

$$f_{\tau}(s) = \begin{cases} \tau_1 z & \text{if } s = (0, z) \in \mathcal{F}_0, \\ \tau_2 z & \text{if } s = (2z, z^2) \in \mathcal{R}. \end{cases}$$

LEMMA 10.7. *The function f_{τ} is analytic on $\mathcal{R} \cup \mathcal{F}_0$ for any $\tau \in \mathbb{T}^2$. Furthermore, if a complex-valued function Ψ_{τ} is defined on G by the formula*

$$(10.2) \quad \Psi_{\tau}(s) = -\tau_2 \Phi_{\omega}(s), \quad s \in G,$$

where $\omega = -\bar{\tau}_2 \tau_1$, then Ψ_{τ} is a norm-preserving analytic extension of f_{τ} to G .

PROOF. By Lemma 10.5 f_{τ} is analytic on $\mathcal{R} \cup \mathcal{F}_0$ and, clearly,

$$\sup_{s \in \mathcal{R} \cup \mathcal{F}_0} |f(s)| = 1.$$

Also, as Φ_{ω} is a Carathéodory extremal function on G ,

$$(10.3) \quad \sup_{s \in G} |\Psi_{\tau}(s)| = \sup_{s \in G} |\Phi_{\omega}(s)| = 1.$$

It is a simple to show from Definition 3.2 that $f_{\tau} = \Psi_{\tau}|_{(\mathcal{R} \cup \mathcal{F}_0)}$. This fact also implies that f_{τ} is analytic on $\mathcal{R} \cup \mathcal{F}_0$. \square

LEMMA 10.8 (A Herglotz Representation Theorem for $\mathcal{R} \cup \mathcal{F}_0$). *A function $h : \mathcal{R} \cup \mathcal{F}_0 \rightarrow \mathbb{C}$ is a Herglotz function on $\mathcal{R} \cup \mathcal{F}_0$ satisfying $h(0, 0) = 1$ if and only if there exists a probability measure μ on \mathbb{T}^2 such that*

$$(10.4) \quad h(s) = \int_{\mathbb{T}^2} \frac{1 + f_{\tau}(s)}{1 - f_{\tau}(s)} d\mu(\tau)$$

for all $s \in \mathcal{R} \cup \mathcal{F}_0$.

PROOF. Lemma 10.7 asserts that f_{τ} is analytic on $\mathcal{R} \cup \mathcal{F}_0$ for each $\tau \in \mathbb{T}^2$. Consequently, as $\sup_{\mathcal{R} \cup \mathcal{F}_0} |f_{\tau}| \leq 1$ and $f_{\tau}(0, 0) = 0$, it follows that the function

$$s \mapsto \frac{1 + f_{\tau}(s)}{1 - f_{\tau}(s)}, \quad s \in \mathcal{R} \cup \mathcal{F}_0,$$

is a Herglotz function on $\mathcal{R} \cup \mathcal{F}_0$ that assumes the value 1 at $(0, 0)$ for each $\tau \in \mathbb{T}^2$. Hence, if h is defined by equation (10.4), then h is Herglotz function on $\mathcal{R} \cup \mathcal{F}_0$ satisfying $h(0, 0) = 1$.

Now assume that h is a Herglotz function on $\mathcal{R} \cup \mathcal{F}_0$ satisfying $h(0, 0) = 1$. By Lemma 10.5 it follows that the functions $h_{\mathcal{F}}$ and $h_{\mathcal{R}}$ are analytic on \mathbb{D} . In addition

$h_{\mathcal{F}}(0) = 1$, $h_{\mathcal{R}}(0) = 1$, and both $h_{\mathcal{F}}$ and $h_{\mathcal{R}}$ have positive real part on \mathbb{D} . Therefore, by the classical Herglotz Representation Theorem there exist probability measures (i.e. finite positive Borel measures with total variation 1) $\mu_{\mathcal{F}}$ and $\mu_{\mathcal{R}}$ on \mathbb{T} such that

$$(10.5) \quad h_{\mathcal{F}}(z) = \int_{\tau_1 \in \mathbb{T}} \frac{1 + \tau_1 z}{1 - \tau_1 z} d\mu_{\mathcal{F}}(\tau_1)$$

and

$$(10.6) \quad h_{\mathcal{R}}(z) = \int_{\tau_2 \in \mathbb{T}} \frac{1 + \tau_2 z}{1 - \tau_2 z} d\mu_{\mathcal{R}}(\tau_2)$$

for all $z \in \mathbb{D}$. We define a probability measure μ on \mathbb{T}^2 by $d\mu(\tau) = d\mu_{\mathcal{F}}(\tau_1)d\mu_{\mathcal{R}}(\tau_2)$.

Observe that if $s = (0, z) \in \mathcal{F}_0$, then Definition 10.4 and formula (10.5) imply that

$$h(s) = h_{\mathcal{F}}(z) = \int_{\tau_1 \in \mathbb{T}} \frac{1 + \tau_1 z}{1 - \tau_1 z} d\mu_{\mathcal{F}}(\tau_1).$$

Therefore, since $\mu_{\mathcal{R}}$ has total variation 1, Fubini's Theorem implies that

$$\begin{aligned} h(s) &= \int_{\tau_2 \in \mathbb{T}} h(s) d\mu_{\mathcal{R}}(\tau_2) \\ &= \int_{\tau_2 \in \mathbb{T}} \left(\int_{\tau_1 \in \mathbb{T}} \frac{1 + \tau_1 z}{1 - \tau_1 z} d\mu_{\mathcal{F}}(\tau_1) \right) d\mu_{\mathcal{R}}(\tau_2) \\ &= \int_{\tau \in \mathbb{T}^2} \frac{1 + \tau_1 z}{1 - \tau_1 z} d\mu(\tau). \end{aligned}$$

But since $s = (0, z) \in \mathcal{F}_0$,

$$\frac{1 + \tau_1 z}{1 - \tau_1 z} = \frac{1 + f_{\tau}(s)}{1 - f_{\tau}(s)},$$

where f_{τ} is as in Definition 10.6. Therefore, equation (10.4) holds when $s \in \mathcal{F}_0$.

If $s = (2z, z^2) \in \mathcal{R}$, then by repeating the argument in the previous paragraph using the representation (10.6) for $h_{\mathcal{R}}$ one shows that equation (10.4) holds when $s \in \mathcal{R}$ as well. \square

10.2. The proof of the norm-preserving extension property for $\mathcal{R} \cup \mathcal{D}$

We first observe that Theorem A.11 guarantees that the group of automorphisms of G acts transitively on the flat geodesics in G and fixes \mathcal{R} . Therefore, it suffices to prove Theorem 10.1 in the special case when $\mathcal{D} = \mathcal{F}_0$.

Observe next that Lemma 10.3 implies that it suffices to prove that $\mathcal{R} \cup \mathcal{F}_0$ has the Herglotz-preserving extension property. However, if U is a general domain, V is a subset of U , and f is a Herglotz function on V , then

$$h = \frac{f - i \operatorname{Im} f(0, 0)}{\operatorname{Re} f(0, 0)}$$

is a Herglotz function on V satisfying $h(0, 0) = 1$. Therefore, as

$$f = (\operatorname{Re} f(0, 0))h + i \operatorname{Im} f(0, 0),$$

to establish the Herglotz-preserving extension property for V , it suffices to show that every Herglotz function h on V satisfying $h(0, 0) = 1$ has a Herglotz extension

to U . Hence, the proof of Theorem 10.1 will be complete if we can show that every Herglotz function h on $\mathcal{R} \cup \mathcal{F}_0$ satisfying $h(0, 0) = 1$ has a Herglotz extension to G .

Fix a Herglotz function h on $\mathcal{R} \cup \mathcal{F}_0$ satisfying $h(0, 0) = 1$. By Lemma 10.8 there exists a probability measure on \mathbb{T}^2 such that equation (10.4) holds. Define a function g on G by the formula

$$(10.7) \quad g(s) = \int_{\tau \in \mathbb{T}^2} \frac{1 + \Psi_\tau(s)}{1 - \Psi_\tau(s)} d\mu(\tau)$$

where for each $\tau \in \mathbb{T}^2$, Ψ_τ is the function defined by the formula (10.2).

First, note that Ψ_τ is analytic on G for each $\tau \in \mathbb{T}^2$ and that $\tau \mapsto \Psi_\tau$ is a continuous map from \mathbb{T} into $\mathbb{C}(G)$ (endowed with the topology of locally uniform convergence). Therefore,

$$(10.8) \quad g \text{ is a well defined analytic function on } G.$$

Next, note that since equation (10.3) implies that, for each $\tau \in \mathbb{T}^2$,

$$\operatorname{Re} \frac{1 + \Psi_\tau(s)}{1 - \Psi_\tau(s)} > 0$$

for all $s \in G$, and μ is a probability measure, necessarily the function g defined in equation (10.7) satisfies

$$(10.9) \quad \operatorname{Re} g(s) > 0 \quad \text{for all } s \in G.$$

Finally, note that Lemma 10.7 asserts that $f_\tau = \Psi_\tau|_{(\mathcal{R} \cup \mathcal{F}_0)}$ for each $\tau \in \mathbb{T}^2$. Consequently,

$$\frac{1 + f_\tau(s)}{1 - f_\tau(s)} = \frac{1 + \Psi_\tau(s)}{1 - \Psi_\tau(s)}$$

for each $\tau \in \mathbb{T}^2$ and each $s \in \mathcal{R} \cup \mathcal{F}_0$. Therefore, using equations (10.4) and (10.7), for each $s \in \mathcal{R} \cup \mathcal{F}_0$, we have

$$\begin{aligned} h(s) &= \int_{\mathbb{T}^2} \frac{1 + f_\tau(s)}{1 - f_\tau(s)} d\mu(\tau) \\ &= \int_{\mathbb{T}^2} \frac{1 + \Psi_\tau(s)}{1 - \Psi_\tau(s)} d\mu(\tau) \\ &= g(s). \end{aligned}$$

This completes the proof of Theorem 10.1. \square

V and a circular region R in the plane

In the light of Lemma 9.10 the study of algebraic sets V in G that have the norm-preserving extension property can be reduced to the case where V does not have contact with any flat datum. Our goal in this section is to show, under these assumptions, that V is a properly embedded planar Riemann surface of finite type [meaning that the boundary of V has finitely many connected components].

LEMMA 11.1. *Let V be an algebraic set in G with the norm-preserving extension property. If s is an isolated point in V then $V = \{s\}$.*

This statement follows from Proposition 9.1.

Recall that the map $\beta : G \rightarrow \mathbb{D}$ was defined in Proposition 4.9 by the formula

$$(11.1) \quad \beta(s) = \frac{s^1 - \bar{s}^1 s^2}{1 - |s^2|^2} \quad \text{for } s \in G.$$

LEMMA 11.2. *Let V be an algebraic set in G having the norm-preserving extension property. If V is not a singleton and V does not have contact with any flat datum, then V is a Riemann surface properly embedded in G and $\beta|_V$ is a homeomorphism onto its range.*

PROOF. Clearly, as V is assumed not to have contact with any flat datum, $V \neq G$. Therefore, if V is not a singleton, it follows from Lemma 11.1 that V has no isolated points and so is a one-dimensional algebraic set in G . We first prove that V has no singular points, and as a consequence that V is a one-dimensional complex manifold properly embedded in G . We then show that $\beta|_V$ is an open mapping. As Proposition 4.9 implies that $\beta|_V$ is injective, it follows that $\beta|_V$ is a homeomorphism onto its range.

V has no singular points. Fix $s_0 \in V$ and choose a neighborhood U_0 of s_0 and a holomorphic function ζ on U_0 such that

$$(11.2) \quad V \cap U_0 = Z_\zeta,$$

where Z_ζ denotes the zero set of ζ in U_0 . Let $\beta_0 = \beta(s_0)$. Introduce local coordinates (z, w) at s_0 by setting

$$s = s_0 + z(\bar{\beta}_0, 1) + w(1, -\beta_0),$$

and define η by

$$\eta(z, w) = \zeta(s).$$

Now, since $(\bar{\beta}_0, 1)$ points in the flat direction at s_0 , there does not exist $\varepsilon > 0$ such that η is identically zero on the set $\{(z, 0) : |z| < \varepsilon\}$; else V contacts the flat infinitesimal datum $(s_0, (\bar{\beta}_0, 1))$, contrary to hypothesis. Therefore, $\eta(z, 0)$ has a zero of finite positive order, ℓ say, at $z = 0$. Consequently, by the Weierstrass Preparation Theorem, there exist analytic functions of one variable b_1, \dots, b_ℓ , defined on a

neighborhood $\varepsilon\mathbb{D}$ of zero, for some $\varepsilon > 0$, and vanishing at zero, and a holomorphic function $h(z, w)$ on $(\varepsilon\mathbb{D})^2$ such that if P_w is the pseudopolynomial

$$P_w(z) = z^\ell + b_1(w)z^{\ell-1} + \dots + b_\ell(w),$$

then

$$(11.3) \quad \eta(z, w) = P_w(z)h(z, w) \quad \text{and} \quad h(z, w) \neq 0$$

on $(\varepsilon\mathbb{D})^2$. Evidently, it will be the case that s_0 is not a singular point of V if P_w has a unique irreducible factor, which has degree 1.

To see that P_w has a unique irreducible factor, assume to the contrary that Q_w and R_w are distinct irreducible monic factors of P_w in the ring

$$R = \mathbb{C}(\varepsilon\mathbb{D})[z]$$

of pseudopolynomials. Since the zeros of a polynomial depend continuously on its coefficients, there exists $\delta > 0$ such that the zeros of the polynomial $P_w(\cdot)$ lie in \mathbb{D}^- whenever $|w| < \delta$. It follows that the zeros of $Q_w(\cdot)$ and $R_w(\cdot)$ also lie in \mathbb{D}^- when $|w| < \delta$.

If the restrictions of Q_w, R_w to $(\varepsilon\mathbb{D}) \times n^{-1}\mathbb{D}$ are equal, for some $n > 1/\delta$, then in fact $Q_w = R_w$ on $(\varepsilon\mathbb{D})^2$. Since Q_w and R_w are distinct, for each $n > 1/\delta$, there exists $w_n \in n^{-1}\mathbb{D}$ such that $Q_{w_n}(\cdot) \neq R_{w_n}(\cdot)$. As $Q_{w_n}(\cdot), R_{w_n}(\cdot)$ are distinct monic polynomials, their zero sets are not identical, and so there exist complex numbers x_n and y_n such that $x_n \neq y_n$ and

$$Q_{w_n}(x_n) = 0 = R_{w_n}(y_n).$$

Since $x_n, y_n \in \mathbb{D}^-$, by passing to a subsequence if necessary, we may assume that the sequences $\{x_n\}$ and $\{y_n\}$ converge. Since, for all n , $Q_{w_n}(x_n) = 0$, we have $P_{w_n}(x_n) = 0$ and thus

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} P_{w_n}(x_n) \\ &= \lim_{n \rightarrow \infty} x_n^\ell + \lim_{n \rightarrow \infty} b_1(w_n)x_n^{\ell-1} + \dots + \lim_{n \rightarrow \infty} b_\ell(w_n) = \left(\lim_{n \rightarrow \infty} x_n \right)^\ell. \end{aligned}$$

Therefore $x_n \rightarrow 0$ and, similarly, $y_n \rightarrow 0$. Finally, define $s_n, t_n \in V$ by setting

$$s_n = s_0 + x_n(\bar{\beta}_0, 1) + w_n(1, -\beta_0)$$

and

$$t_n = s_0 + y_n(\bar{\beta}_0, 1) + w_n(1, -\beta_0).$$

With the above constructions we have $s_n \rightarrow s_0$ and

$$\frac{t_n - s_n}{\|t_n - s_n\|} = \frac{y_n - x_n}{\|y_n - x_n\|} \frac{(\bar{\beta}_0, 1)}{\|(\bar{\beta}_0, 1)\|}.$$

After passage to a further subsequence one infers that there exists $\chi \in \mathbb{T}$ such that

$$\frac{t_n - s_n}{\|t_n - s_n\|} \rightarrow \chi \frac{(\bar{\beta}_0, 1)}{\|(\bar{\beta}_0, 1)\|},$$

which is a flat direction at s_0 . Thus V is contacted by the flat infinitesimal datum $(s_0, (\bar{\beta}_0, 1))$, contrary to assumption. This completes the proof that P_w does not have a distinct pair of irreducible factors.

To see that each of the irreducible factors of P_w has degree 1, assume to the contrary that Q_w is an irreducible factor and $\deg Q_w = d \geq 2$. Since Q_w is irreducible, there exists $\varepsilon > 0$ such that for each w with $0 < |w| < \varepsilon$ the equation

$Q_w(z) = 0$ has exactly d distinct roots. Thus we may choose sequences $\{w_n\}$, $\{x_n\}$, and $\{y_n\}$ such that $w_n \rightarrow 0$ and

$$x_n \neq y_n \quad \text{and} \quad Q_{w_n}(x_n) = 0 = Q_{w_n}(y_n)$$

for all n . By the argument of the previous paragraph, these sequences lead to a contradiction to the assumption that V does not have contact with a flat datum.

We have shown that $\eta(z, w) = (z + b_1(w))^k h(z, w)$ for some positive integer k . Thus

$$(11.4) \quad V \cap U_0 = Z_\zeta = \{s \in U_0 : z + b_1(w) = 0\}$$

and $V \cap U_0 = f(N)$ for some neighborhood N of 0 in \mathbb{C} , where

$$(11.5) \quad f(w) = s_0 - b_1(w)(\bar{\beta}_0, 1) + w(1, -\beta_0).$$

Thus s_0 is not a singular point of V . Hence V has no singular points.

$\beta|V$ is an open mapping. As V is a one dimensional analytic set without singular points, it follows that V is a Riemann surface. Since algebraic sets are closed, the injection map of V into G is a proper embedding. There remains to show that the function $\beta|V$ is a homeomorphism onto its range. Clearly, $\beta|V$ is continuous. Furthermore, as V is assumed not to have contact with any flat geodesic, it is clear from Proposition 4.9 that $\beta|V$ is injective. We conclude the proof that β is a homeomorphism by showing that $\beta|V$ is an open mapping.

Firstly, for f as in equation (11.5),

$$df_0 = (-b'_1(0)\bar{\beta}_0 + 1, -b'_1(0) - \beta_0) dw_0.$$

Secondly, the total complex differential of β at a general point $s \in G$ is given by

$$\begin{aligned} d\beta &= \frac{\partial\beta}{\partial s^1} ds^1 + \frac{\partial\beta}{\partial s^{\bar{1}}} ds^{\bar{1}} + \frac{\partial\beta}{\partial s^2} ds^2 + \frac{\partial\beta}{\partial s^{\bar{2}}} ds^{\bar{2}} \\ &= \frac{1}{1 - |s^2|^2} ds^1 - \frac{s^2}{1 - |s^2|^2} ds^{\bar{1}} + \frac{-s^{\bar{1}} + s^1 s^{\bar{2}}}{(1 - |s^2|^2)^2} ds^2 + s^2 \frac{s^1 - s^{\bar{1}} s^2}{(1 - |s^2|^2)^2} ds^{\bar{2}} \\ &= \frac{1}{1 - |s^2|^2} \left(ds^1 - s^2 ds^{\bar{1}} - \bar{\beta} ds^2 + s^2 \beta ds^{\bar{2}} \right) \\ (11.6) \quad &= \frac{1}{1 - |s^2|^2} \left(ds^1 - \bar{\beta} ds^2 - s^2 (ds^{\bar{1}} - \beta ds^{\bar{2}}) \right). \end{aligned}$$

Thirdly, observe that if $f = (f^1, f^2)$ is given by equation (11.5), then

$$(df^1)_0 = (-b'_1(0)\bar{\beta}_0 + 1)dw_0 \quad \text{and} \quad (df^2)_0 = (-b'_1(0) - \beta_0)dw_0,$$

and so

$$\begin{aligned} (df^1 - \bar{\beta}_0 df^2)_0 &= (-b'_1(0)\bar{\beta}_0 + 1) - \bar{\beta}_0(-b'_1(0) - \beta_0)dw_0 \\ (11.7) \quad &= (1 + |\beta_0|^2)dw_0. \end{aligned}$$

Similarly

$$(11.8) \quad (d\bar{f}^1 - \beta_0 d\bar{f}^2)_0 = (1 + |\beta_0|^2)d\bar{w}_0.$$

Therefore, by virtue of equations (11.6), (11.7) and (11.8),

$$(11.9) \quad d(\beta \circ f)_0 = C(dw_0 - s_0^2 d\bar{w}_0)$$

where

$$C = \frac{1 + |\beta_0|^2}{1 - |s_0^2|^2} > 0.$$

Let $u(x, y), v(x, y)$ be the real and imaginary parts of $\beta \circ f(x + iy)$ for $x, y \in \mathbb{R}$.

It follows from equation (11.9) that the Jacobian determinant of $\beta \circ f$ at 0 is

$$\frac{\partial(u, v)}{\partial(x, y)}(0) = C^2 - |Cs_0^2|^2 = C^2(1 - |s_0^2|^2) > 0.$$

By the Inverse Function Theorem, $\beta \circ f$ is invertible in a neighborhood of s_0 in V . As the point $s_0 \in V$ is arbitrary, $\beta|_V$ is an open mapping, as was to be proved.

We have shown that V is a one-dimensional manifold. By Proposition 9.1, V is connected, and so V is a Riemann surface. \square

LEMMA 11.3. *If V is an algebraic set in G then $\beta(V)$ is a semialgebraic set and $\partial\beta(V)$ has a finite number of components.*

PROOF. G is the set of points $s \in \mathbb{C}^2$ such that

$$|s^1 - \overline{s^1}s^2|^2 < (1 - |s^2|^2)^2,$$

and therefore G is a semialgebraic set. Since $V = G \cap A$ for some algebraic set A , it follows that V is a semialgebraic set (see, for example, [20]). It is clear from the definition of β that β is a regular mapping on G (that is, the real and imaginary parts of β are rational functions whose denominators do not vanish on G). Hence β is a semialgebraic map, and so, by [20, Proposition 2.2.7], $\beta(V)$ is a semialgebraic set. It follows from [20, Proposition 2.2.2] that the boundary $\partial\beta(V)$ of $\beta(V)$ is a semialgebraic set. By [20, Theorem 2.4.5], $\partial\beta(V)$ has finitely many connected components. \square

A *circular region* is defined to be a region in \mathbb{C} whose boundary consists of a finite number of disjoint nondegenerate circles. These domains are canonical multiply connected domains, in the sense that every finitely connected planar domain is conformally equivalent to a circular region [22, Theorem 7.9]. A domain is *nondegenerate* if none of its boundary components is a singleton set.

LEMMA 11.4. *Let V be an algebraic set in G having the norm-preserving extension property. If V does not have contact with any flat datum, then either V is a singleton or there exists a circular region R in the plane and a bijective holomorphic mapping $\sigma : R \rightarrow V$ such that $\sigma'(z) \neq 0$ for all $z \in R$.*

PROOF. Suppose that V is not a singleton. By Lemma 11.2, V is a Riemann surface and V is homeomorphic to $\beta(V)$, an open subset of \mathbb{D} . Now any open planar Riemann surface is conformally equivalent to a plane region R [12, Chapter III, Paragraph 4]. By Lemma 11.3, $\partial\beta(V)$ has a finite number of components, and so $\beta(V)$ is finitely connected. Since R is homeomorphic to $\beta(V)$, R is a finitely connected plane region. We claim that R is nondegenerate. Suppose, to the contrary, that there is some point z_0 that is a singleton boundary component of R . Let $\sigma = (a, b) : R \rightarrow V$ be a conformal map onto the Riemann surface V . Since every bounded holomorphic function on $R \setminus \{z_0\}$ continues holomorphically to R ,

$$s_0 \stackrel{\text{def}}{=} \lim_{z \rightarrow z_0} (a(z), b(z))$$

exists. Furthermore, since $z_0 \in \partial R$, $s_0 \in \partial V \subseteq \partial G$. Hence there exists $\omega \in \mathbb{T}$ such that $|\Phi_\omega(s_0)| = 1$, for otherwise $|\Phi_\omega(s_0)| < 1$ for all $\omega \in \mathbb{T}$, which implies that $s_0 \in G$, by Proposition A.1. The holomorphic extension to $R \cup \{z_0\}$ of $\Phi_\omega \circ \sigma$ has modulus at most 1 and attains its maximum modulus at the interior point z_0 ,

contrary to the Maximum Modulus Principle. Hence the boundary of R has no singleton components, and so, by [22, Theorem 7.9], we can assume that R is a circular domain. Then $\sigma : R \rightarrow V$ is the required conformal map. \square

Proof of the main theorem

This chapter is devoted to the proof of Theorem 1.4. For convenience we repeat the statement.

THEOREM 12.1. *V is an algebraic subset of G having the norm-preserving extension property if and only if either V is a retract in G or V is the union of \mathcal{R} and a flat geodesic in G .*

By Proposition 9.2 and Theorem 10.1, if V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$, where \mathcal{D} is a flat geodesic in G , then V is an algebraic subset of G with the norm-preserving extension property. We shall prove the converse.

Accordingly, for the remainder of the chapter, fix a set $V \subseteq G$ such that

$$(12.1) \quad V \text{ is an algebraic set in } G, \text{ and}$$

$$(12.2) \quad V \text{ has the norm-preserving extension property.}$$

We wish to show that V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$, where \mathcal{D} is a flat geodesic in G . This conclusion certainly follows if V is a singleton set or if $V = G$. We may therefore assume that

$$(12.3) \quad V \text{ is not a singleton set, and}$$

$$(12.4) \quad V \neq G.$$

If V has contact with a flat datum δ , then by Lemma 9.10, either $V = G$ or $V = \mathcal{D}_\delta$, both of which are retracts in G , or $V = \mathcal{R} \cup \mathcal{D}_\delta$. If V has contact with a royal datum δ , then by Lemma 9.11, either $V = \mathcal{R}$, V is the union of \mathcal{R} and a flat geodesic in G or $V = G$. We may therefore assume further that

$$(12.5) \quad V \text{ does not have contact with any flat or royal datum.}$$

By Lemma 11.4 there exists a circular region R in the plane and a bijective holomorphic mapping $\sigma : R \rightarrow V$ such that $\sigma'(z) \neq 0$ for all $z \in R$.

LEMMA 12.2. *If V has contact with a purely balanced datum δ in G then (1) $V = \mathcal{D}_\delta$ and (2) V is a retract in G .*

PROOF. Suppose that V has contact with a purely balanced datum δ in G . By Lemma 9.5, $\mathcal{D}_\delta \subseteq V$. Choose a solution k of $\text{Kob}(\delta)$, so that $\mathcal{D}_\delta = k(\mathbb{D})$. Then k is a rational Γ -inner function and so *a fortiori* proper as a map from \mathbb{D} to V , and hence $f \stackrel{\text{def}}{=} \sigma^{-1} \circ k$ is a proper injective holomorphic map from \mathbb{D} to R . By the Open Mapping Theorem, f is a homeomorphism from \mathbb{D} to $f(\mathbb{D})$. We claim that $f(\mathbb{D}) = R$.

Suppose that $f(\mathbb{D}) \neq R$. Choose a point $w \in R \setminus f(\mathbb{D})$, a point $u \in f(\mathbb{D})$ and a continuous path γ from u to w in R . Let $z_0 = \gamma(t_0)$ where $t_0 = \sup\{t : \gamma(t) \in f(\mathbb{D})\}$. Then $z_0 \in (\partial f(\mathbb{D})) \cap R$.

Pick a sequence (λ_n) in \mathbb{D} such that $f(\lambda_n) \rightarrow z_0$. If (λ_n) has a subsequence which converges to a limit μ in \mathbb{D} then $f(\mu) = z_0$ and hence $z_0 \in f(\mathbb{D})$, a contradiction. Hence $|\lambda_n| \rightarrow 1$, and so, by the propriety of f , $f(\lambda_n)$ tends to ∂R , and so $z_0 \in \partial R$, contrary to the fact that $z_0 \in R$. Hence $f(\mathbb{D}) = R$, and so $k(\mathbb{D}) = \sigma \circ f(\mathbb{D}) = \sigma(R) = V$. Thus $\mathcal{D}_\delta = V$, and so V is a retract in G (see equation (5.1)). \square

In the light of Lemma 12.2 we may further assume that

(12.6) V does not have contact with any purely balanced datum.

Let

(12.7)
$$\partial R = \partial_0 \cup \partial_1 \cup \cdots \cup \partial_n$$

be a decomposition of ∂R into its connected components, where ∂_0 is the boundary of the unbounded component of the complement of R^- . We may take ∂_0 to be the unit circle.

LEMMA 12.3. *If $n = 0$ then V is a nontrivial retract in G and is a complex geodesic of G .*

PROOF. Since $n = 0$ we have $R = \mathbb{D}$. By assumption V is an algebraic subset of G having the norm-preserving extension property, and thus the map $\sigma^{-1} : V \rightarrow \mathbb{D}$ has a holomorphic extension $f : G \rightarrow \mathbb{D}$. By the Open Mapping Theorem, $f(G) \subset \mathbb{D}$, and so $\sigma \circ f$ is a holomorphic map from G to V which is the identity on V . That is, V is a retract in G . By statements (12.3) and (12.4), V is neither a singleton nor the whole of G , and so V is a nontrivial retract. By Theorem 5.1, V is a complex geodesic in G . \square

The rest of the proof consists of a demonstration that $n = 0$.

Suppose $n \neq 0$. Write $\sigma = (a, b)$, so that a and b are holomorphic functions on R and, for all $z \in R$, $a'(z)$ and $b'(z)$ are not both zero.

12.1. Preliminary lemmas

For $\omega \in \mathbb{T}$ let $f_\omega = \Phi_\omega \circ \sigma : R \rightarrow \mathbb{D}$. Define $\mathcal{A} \subseteq \mathbb{D}(R)$ by

$$\mathcal{A} = \{f_\omega : \omega \in \mathbb{T}\}.$$

LEMMA 12.4. *If $\omega_1 \neq \omega_2$ then $f_{\omega_1} \neq f_{\omega_2}$.*

PROOF. Assume, to the contrary, that ω_1, ω_2 are distinct point in \mathbb{T} but $f_{\omega_1} = f_{\omega_2}$. Straightforward manipulation of the equation

$$\frac{2\omega_1 b - a}{2 - \omega_1 a} = \frac{2\omega_2 b - a}{2 - \omega_2 a}$$

yields the relation $a^2 = 4b$, so that $V = \sigma(R) \subseteq \mathcal{R}$, contrary to the statement (12.5). \square

LEMMA 12.5. *\mathcal{A} is a universal set for the Carathéodory Problem on R . Furthermore, if ζ is a nondegenerate datum in R , then there exists a unique $f \in \mathcal{A}$ that solves $\text{Car}(\zeta)$.*

PROOF. If ζ is a datum in R then $\sigma(\zeta)$ is a datum in V . Furthermore, since σ is biholomorphic, the holomorphic invariance of the Carathéodory metric implies that

$$|\zeta|_R = |\sigma(\zeta)|_V.$$

Here we use a self-explanatory variant on the notation $|\cdot|_{\text{car}}$ which indicates the domain in question. Also, as V has the norm-preserving extension property,

$$|\sigma(\zeta)|_V = |\sigma(\zeta)|_G.$$

Therefore, if we view $\sigma(\zeta)$ as a datum in G and we choose $\omega \in \mathbb{T}$ such that Φ_ω solves $\text{Car}(\sigma(\zeta))$, then $f_\omega = \Phi_\omega \circ \sigma$ solves $\text{Car}(\zeta)$.

To see that f_ω is unique, assume to the contrary that f_{ω_1} and f_{ω_2} solve $\text{Car}(\zeta)$ and $\omega_1 \neq \omega_2$. Then Φ_{ω_1} and Φ_{ω_2} both solve $\text{Car}(\sigma(\zeta))$. This means that $\sigma(\zeta)$ is a flat, royal or purely balanced datum that contacts V , contrary to statements (12.5) and (12.6). \square

In the light of Lemma 12.3 it is clear that the proof of Theorem 1.4 will be complete if we establish that $n = 0$. Assume to the contrary that

$$(12.8) \quad n \geq 1.$$

While \mathcal{A} is a universal set for the Carathéodory Problem on R , by Lemma 12.5, it is not necessarily the case that each $f \in \mathcal{A}$ solves $\text{Car}(\zeta)$ for some nondegenerate datum ζ in R . Accordingly, we define a set $E \subseteq \mathbb{T}$ by

$$E = \{\omega \in \mathbb{T} : f_\omega \text{ solves } \text{Car}(\zeta) \text{ for some nondegenerate datum } \zeta \text{ in } R\}$$

It is well known that the Carathéodory extremal functions on R (the *Ahlfors functions*) are $(n+1)$ -valent inner functions that analytically continue to a neighborhood of R^- (for example, [25, Chapter 5, Theorem 1.6]). In particular the functions f_ω , $\omega \in E$ enjoy these properties. Of course f_ω is nonconstant for $\omega \in E$.

LEMMA 12.6. *There exist two datums ζ_1 and ζ_2 in R such that f_{ω_1} solves $\text{Car}(\zeta_1)$, f_{ω_2} solves $\text{Car}(\zeta_2)$, and $\omega_1 \neq \omega_2$.*

PROOF. Fix a datum ζ_1 in R and let f_{ω_1} solve $\text{Car}(\zeta_1)$. As f is $(n+1)$ -valent, there exist $z_1, z_2 \in R$ such that $z_1 \neq z_2$ and $f_{\omega_1}(z_1) = f_{\omega_2}(z_2)$. Let $\zeta_2 = (z_1, z_2)$. As $|f_{\omega_1}(\zeta_2)| = 0$, f_{ω_1} cannot possibly solve $\text{Car}(\zeta_2)$. Therefore, the lemma follows by choice of ω_2 such that f_{ω_2} solves $\text{Car}(\zeta_2)$. \square

LEMMA 12.7. *If $n \geq 1$ then E is an infinite set.*

PROOF. Assume to the contrary that E is finite.

Fix a nondegenerate discrete datum $\zeta_1 = (z_0, z_1)$ in R and let f_{ω_1} solve $\text{Car}(\zeta_1)$. Since f_{ω_1} is $(n+1)$ -valent and $n \geq 1$, there exists $z_2 \in R$ with $z_2 \neq z_1$ and $f_{\omega_1}(z_1) = f_{\omega_1}(z_2)$. Construct a continuous curve $\gamma : [1, 2] \rightarrow R$ with $\gamma(1) = z_1$ and $\gamma(2) = z_2$ and define a curve of discrete datums δ_t in R by the formula

$$\zeta_t = (z_0, \gamma(t)), \quad 1 \leq t \leq 2.$$

Since Lemma 12.5 guarantees that for each $t \in [1, 2]$ there exists a unique $\omega \in E$ such that f_ω solves $\text{Car}(\zeta_t)$, it follows that there exists a well defined function $\Omega : [1, 2] \rightarrow E$ such that

$$f_{\Omega(t)} \text{ solves } \text{Car}(\zeta_t) \text{ for all } t \in [1, 2].$$

Also, observe that by construction, $\Omega(1) = \omega_1$.

We claim that for each $\tau \in E$, $\Omega^{-1}(\{\tau\})$ is an open subset of $[1, 2]$. To prove this claim, fix $\tau_0 = \Omega(t_0) \in E$ and note that since the solution to $\text{Car}(\zeta_{t_0})$ is unique,

$$(12.9) \quad |f_\tau(\zeta_{t_0})| < |f_{\tau_0}(\zeta_{t_0})| \text{ for all } \tau \in E \setminus \{\tau_0\}.$$

It follows by continuity, that equation (12.9) holds, with t_0 replaced by t , for all t in a neighborhood I of t_0 . But then,

$$f_{\tau_0} \text{ solves } \text{Car}(\zeta_t) \text{ for all } t \in I,$$

that is, $I \subseteq \Omega^{-1}(\{\tau_0\})$. This proves that $\Omega^{-1}(\{\tau\})$ is an open subset of $[1, 2]$ for each $\tau \in E$.

Since Ω is continuous and $[1, 2]$ is connected, Ω is constant. Hence

$$\Omega(2) = \Omega(1) = \omega_1$$

and f_{ω_1} solves $\text{Car}(\zeta_2)$. But ζ_2 is nondegenerate while $|f_{\omega_1}(\zeta_2)| = 0$. \square

12.2. $\sigma : R \rightarrow V$ is analytic on R^-

LEMMA 12.8. *The map σ extends holomorphically to a neighborhood of R^- .*

PROOF. By Lemma 12.7 there exist distinct points $\omega_1, \omega_2 \in E$. Let

$$(12.10) \quad f_1 = f_{\omega_1} = \frac{2\omega_1 b - a}{2 - \omega_1 a} \quad \text{and} \quad f_2 = f_{\omega_2} = \frac{2\omega_2 b - a}{2 - \omega_2 a}.$$

Solve these equations for a and b in terms of f_1, f_2 :

$$a = 2 \frac{\omega_1 f_2 - \omega_2 f_1}{\omega_2 - \omega_1 + \omega_1 \omega_2 (f_2 - f_1)}$$

$$b = \frac{f_2 - f_1 + (\omega_2 - \omega_1) f_1 f_2}{\omega_2 - \omega_1 + \omega_1 \omega_2 (f_2 - f_1)}.$$

Since f_1 and f_2 are holomorphic on a neighborhood of R^- , a and b extend to meromorphic functions defined on a neighborhood of R^- . But a and b are bounded on R , and so extend to holomorphic functions on a neighborhood of R^- . \square

LEMMA 12.9. *$\sigma(\partial R)$ is contained in the distinguished boundary $b\Gamma$ of the closure Γ of G , and therefore*

$$|a| \leq 2, \quad |b| = 1 \quad \text{and} \quad a = \bar{a}b \quad \text{on } \partial R.$$

PROOF. We are using the characterization of $b\Gamma$ given in Proposition A.2. First notice that as $\sigma \in G(R)$, $|a(z)| < 2$ for all $z \in R$. By Lemma 12.8, σ extends continuously to ∂R , and therefore $|a(z)| \leq 2$ for all $z \in \partial R$. There remains to prove that $a = \bar{a}b$ and $|b| = 1$ on ∂R .

Let f_1 and f_2 be as in Lemma 12.8, let $E_1 = \{z \in \partial R : a(z) = 2\bar{\omega}_1\}$ and $E_2 = \{z \in \partial R : a(z) = 2\bar{\omega}_2\}$. Clearly, Lemma 12.8 guarantees that E_1 and E_2 are finite. As

$$f_1 = \frac{2\omega_1 b - a}{2 - \omega_1 a}$$

is inner, it follows that $|\Phi_{\omega_1}(a(z), b(z))| = 1$ for all $z \in \partial R \setminus E_1$. Since, for such z , $(a(z), b(z)) \in \Gamma \setminus \{(2\bar{\omega}_1, \bar{\omega}_1^2)\}$, Proposition A.5 shows that

$$(12.11) \quad \omega_1(a(z) - \overline{a(z)}b(z)) = 1 - |b(z)|^2$$

for all $z \in \partial R \setminus E_1$. But Lemma 12.8 implies that a and b are continuous on ∂R . Therefore, equation (12.11) holds for all $z \in \partial R$. In similar fashion we deduce that

$$(12.12) \quad \omega_2(a(z) - \overline{a(z)}b(z)) = 1 - |b(z)|^2 \quad \text{for all } z \in \partial R.$$

Since $\omega_1 \neq \omega_2$, it follows immediately from equations (12.11) and (12.12) that

$$(12.13) \quad a(z) - \overline{a(z)}b(z) = 0 \quad \text{and} \quad |b(z)| = 1$$

for all $z \in \partial R$. Thus $\sigma(z) \in b\Gamma$ for all $z \in \partial R$. \square

The following lemma gives an additional relationship between a and b that must hold whenever $\omega \in E$.

LEMMA 12.10. *If $\omega \in E$ and $a(\eta) = 2\bar{\omega}$ for some $\eta \in \partial R$, then $b(\eta) = \bar{\omega}^2$. In consequence*

$$a(\eta)^2 = 4b(\eta).$$

PROOF. Suppose that $b(\eta) \neq \bar{\omega}^2$. Then Φ_ω is continuous at $\sigma(\eta)$, and

$$f_\omega = \frac{2\omega b - a}{2 - \omega a}$$

is bounded on a neighborhood of $\sigma(\eta)$. Since $2 - \omega a = 0$ at η , necessarily $2\omega b - a = 0$ at η , so that $b(\eta) = \bar{\omega}^2$. \square

LEMMA 12.11. *The set*

$$F \stackrel{\text{def}}{=} \{\omega \in \mathbb{T} : 2\bar{\omega} \in a(\partial R)\}$$

is finite.

PROOF. If not then there is an infinite sequence (ω_i) of distinct points in \mathbb{T} and a sequence (η_i) of distinct points in ∂R such that $a(\eta_i) = 2\bar{\omega}_i$ for each i . Lemma 12.10 then implies that $a(\eta_i)^2 = 4b(\eta_i)$ for all i . Hence, by Lemma 12.8, $a(\eta)^2 = 4b(\eta)$ for all $\eta \in R$, so that $\sigma(R)$ is contained in the royal variety \mathcal{R} . Then each datum in G that contacts V also contacts \mathcal{R} , contrary to assumption (12.5). \square

LEMMA 12.12. *There exists $\omega \in E$ such that $2\bar{\omega} \notin a(\partial R)$.*

PROOF. Lemma 12.7 asserts that E is infinite, while Lemma 12.11 states that the subset F of E is finite. Therefore $E \setminus F$ is nonempty, that is, there exists $\omega \in E$ such that $2\bar{\omega} \notin a(\partial R)$. \square

12.3. Degree considerations

For $r = 0, \dots, n$ and φ a nonvanishing continuously differentiable complex-valued function on ∂_r we define

$$\#_r(\varphi) = \frac{1}{2\pi i} \int_{\partial_r} \frac{\varphi'(z)}{\varphi(z)} dz.$$

Here, in the decomposition (12.7) of the boundary of R , ∂_r is oriented counter-clockwise if $r = 0$ and clockwise if $r > 0$.

If φ is defined on ∂R we set

$$\#(\varphi) = \sum_{r=0}^n \#_r(\varphi).$$

We note that if φ is a nonconstant inner function on R that extends to be holomorphic on a neighborhood U of R^- , then $\#_r(\varphi) \geq 1$ for each r so that $\#(\varphi) \geq n + 1$. Furthermore, since the cycle $\partial_0 + \cdots + \partial_n$ is clearly homologous to 0 in U , we may apply the Argument Principle to deduce that in this case $\#(\varphi)$ is equal to the number of zeros of φ in R (for example, [11, Chapter 4, Subsection 5.2]).

LEMMA 12.13.

$$(12.14) \quad \#(b) = n + 1.$$

PROOF. By Lemma 12.12, there is an $\omega \in E$ such that $2\bar{\omega} \notin a(\partial R)$. Thus $\omega a(z) \neq 2$ for all $z \in \partial R$. It follows from Lemma 12.9 that $|a(z)| \leq 2$ for all $z \in \partial R$. Hence the curves

$$\frac{2 - \bar{\omega}a}{2 - \omega a}(z), \quad z \in \partial_r,$$

are well defined and do not meet the closed negative real axis for $r = 0, \dots, n$. Therefore

$$\# \left(\frac{2 - \bar{\omega}a}{2 - \omega a} \right) = 0.$$

Since

$$(12.15) \quad f_\omega = \frac{2\omega b - a}{2 - \omega a} = \frac{2\omega b - \bar{a}b}{2 - \omega a} = \omega b \frac{2 - \bar{\omega}a}{2 - \omega a},$$

we have $\#(f_\omega) = \#(b)$. Now f_ω is not a constant function, for otherwise $\Phi_\omega \circ \sigma$ is a constant of unit modulus on R , and hence $\sigma(R) \subseteq \partial G$, which is false. Thus the Ahlfors function f_ω for R has $n + 1$ zeros counting multiplicity, so that

$$(12.16) \quad \#(f_\omega) = n + 1.$$

Therefore $\#(b) = n + 1$. \square

Notice that the equation (12.15) is valid for all $\omega \in \mathbb{T} \setminus F$, and therefore the index formula (12.16) is also valid for such ω .

LEMMA 12.14. *The set F of Lemma 12.11 is empty.*

PROOF. Assume to the contrary that $\omega_0 \in F$, say $a(z_0) = 2\bar{\omega}_0$ where $z_0 \in \partial R$. Since F is finite, there exists a sequence $\{\omega_i\}_{i \geq 1}$ in $\mathbb{T} \setminus F$ such that $\omega_i \rightarrow \omega_0$. Consequently, $f_{\omega_i} \rightarrow f_{\omega_0}$ uniformly on compact subsets of R . Furthermore, the denominator of f_{ω_i} , $2 - \omega_i a$, converges uniformly on a neighborhood U of \bar{R} to the denominator of f_{ω_0} . Hence, by Hurwitz's Theorem, $2 - \omega_i a$ has a zero at a point $w_i \in U$, and since $\omega_i \notin F$, we have $w_i \notin R^-$. We can assume that $w_i \rightarrow z_0$. Thus f_{ω_i} has a pole at a point $w_i \in U \setminus \bar{R}$. By the Schwarz Reflection Principle, for $i \geq 0$, f_{ω_i} has a zero at a point $z_i \in R$, where $z_i \rightarrow z_0$. Now $\omega_i \notin F$ for each $i \geq 1$, and so the formula (12.16) obtains. Hence f_{ω_i} is an inner function on R of minimal degree $n + 1$. Since cancellation occurs in the limit f_{ω_0} of (f_{ω_i}) , it follows that $\#(f_{\omega_0}) < n + 1$, and therefore $\#(f_{\omega_0}) = 0$, which is to say that f_{ω_0} is constant. However, since $\omega_0 \in E$, there is a nondegenerate datum δ in R such that f_{ω_0} solves $\text{Car}(\delta)$, which implies that f_{ω_0} is nonconstant, a contradiction. Hence $F = \emptyset$. \square

Conclusion. It remains to show that $n = 0$. Suppose not.

By Lemma 12.14, the set F is empty. Therefore, on ∂R , $|a| < 2$ and

$$|4b - (4b - a^2)| = |a^2| < 4 = |4b|.$$

Hence, by Rouché's Theorem, $4b - a^2$ has the same number of zeros as b , which, in view of Lemma 12.13, is to say that

$$\#(4b - a^2) = \#(b) = n + 1.$$

Since $n \geq 1$, this implies that $4b - a^2$ has at least two zeros in R , and hence that $V = \sigma(R)$ meets \mathcal{R} at least twice. Thus there exists a royal datum in G which contacts V , contrary to statement (12.5).

We have established that $n = 0$. This completes the proof of Theorem 1.4.

Sets in \mathbb{D}^2 with the symmetric extension property

Motivation for the study of the norm-preserving extension property in a domain of holomorphy is described in the introduction of [2]. One motive is to understand the set of solutions of a Nevanlinna-Pick interpolation problem on \mathbb{D}^2 . Another is to prove refinements of the inequalities of von Neumann [37] and Andô [14]. In this section we shall apply the preceding results to obtain some statements relevant to interpolation problems on \mathbb{D}^2 having a symmetry property; in the next section we shall apply our results to the theory of spectral sets of commuting pairs of operators.

In [2, Theorem 1.20] Agler and McCarthy described all sets in the bidisc that have the norm-preserving extension property. The following statement is contained their result.

THEOREM 13.1. *An algebraic set V in \mathbb{D}^2 has the norm-preserving extension property if and only if V has one of the following forms.*

- (1) $V = \{\lambda\}$ for some $\lambda \in \mathbb{D}^2$;
- (2) $V = \mathbb{D}^2$;
- (3) $V = \{(z, f(z)) : z \in \mathbb{D}\}$ for some $f \in \mathbb{D}(\mathbb{D})$;
- (4) $V = \{(f(z), z) : z \in \mathbb{D}\}$ for some $f \in \mathbb{D}(\mathbb{D})$.

The authors also generalized the norm-preserving extension property in [2, Definition 1.2] as follows.

DEFINITION 13.2. Let Ω be a domain of holomorphy, V be a subset of Ω and A be a collection of bounded holomorphic functions on V . Then V is said to have the *A-extension property* (relative to Ω) if, for every $f \in A$, there is a bounded holomorphic function g on Ω such that $g|_V = f$ and

$$\sup_{\Omega} |g| = \sup_V |f|.$$

Let V be a symmetric algebraic set in \mathbb{D}^2 ('symmetric' meaning that $(\lambda^1, \lambda^2) \in V$ implies that $(\lambda^2, \lambda^1) \in V$). Let $H_{\text{sym}}^{\infty}(V)$ denote the algebra of bounded holomorphic functions g on V which are symmetric, in the sense that $g(\lambda^1, \lambda^2) = g(\lambda^2, \lambda^1)$ for all $(\lambda^1, \lambda^2) \in V$. We say that V has the *symmetric extension property* if V has the $H_{\text{sym}}^{\infty}(V)$ -extension property. In this section we shall describe all symmetric algebraic sets in \mathbb{D}^2 that have the symmetric extension property. It is striking that there are three species of set in \mathbb{D}^2 that have the symmetric extension property but do not resemble any of the types in Theorem 13.1.

The symmetric extension property in \mathbb{D}^2 is closely related to the norm-preserving extension property in G . We shall denote by t the transposition map $t(\lambda^1, \lambda^2) = (\lambda^2, \lambda^1)$.

LEMMA 13.3. *A symmetric subset V of \mathbb{D}^2 has the symmetric extension property if and only if $\pi(V)$ has the norm-preserving extension property in G .*

PROOF. Note that since V is symmetric, $V = \pi^{-1}(\pi(V))$.

Suppose that $V \subseteq \mathbb{D}^2$ has the symmetric extension property. Let f be a bounded holomorphic function on $\pi(V)$; then $f \circ \pi$ is a bounded symmetric function on V . Moreover, $f \circ \pi$ is holomorphic on V . For consider any $\lambda \in V$. Since f is holomorphic, there is a neighborhood U of $\pi(\lambda)$ in G and a holomorphic function g on U which agrees with f on $U \cap \pi(V)$. Then $g \circ \pi$ is a holomorphic function on the neighborhood $\pi^{-1}(U)$ of λ which agrees with $f \circ \pi$ on $\pi^{-1}(U) \cap V$. Thus $f \circ \pi$ is a symmetric bounded holomorphic function on V . Since V has the symmetric extension property, there exists a bounded holomorphic function f_1 on \mathbb{D}^2 which agrees with $f \circ \pi$ on V and satisfies

$$\sup_{\mathbb{D}^2} |f_1| = \sup_V |f \circ \pi|.$$

Let $\tilde{f} = \frac{1}{2}(f_1 + f_1 \circ t)$. Then \tilde{f} is a symmetric bounded holomorphic function on \mathbb{D}^2 which agrees with $f \circ \pi$ on V and satisfies

$$\sup_{\mathbb{D}^2} |\tilde{f}| = \sup_V |f \circ \pi|.$$

There exists a holomorphic function F on G such that $\tilde{f} = F \circ \pi$. Then F agrees with f on $\pi(V)$ and

$$\sup_G |F| = \sup_{\mathbb{D}^2} |\tilde{f}| = \sup_V |f \circ \pi| = \sup_{\pi(V)} |f|.$$

Hence $\pi(V)$ has the norm-preserving extension property in G .

Conversely, suppose that $\pi(V)$ has the norm-preserving extension property in G . We must show that V has the symmetric extension property. To this end consider any bounded symmetric holomorphic function f on V . There exists a unique function F on $\pi(V)$ such that $f = F \circ \pi$. We claim that F is holomorphic on $\pi(V)$.

Consider any $s \in \pi(V)$ and let $\mu \in V$ be a preimage of s under π . Since f is holomorphic on V there exists a neighborhood U of μ in \mathbb{D}^2 and a holomorphic function g on U such that f and g agree on $U \cap V$. We claim that both U and g can be taken to be symmetric. Indeed, if $t(\mu) = \mu$ then define U_1 to be $U \cap t(U)$ and let $g_1 = \frac{1}{2}(g + g \circ t)$; then U_1 is a symmetric neighborhood of μ and g_1 is a symmetric holomorphic function on U_1 such that g_1 and f agree on $U_1 \cap V$. If $t(\mu) \neq \mu$, let U_0 be an open ball centred at μ and contained in U such that U_0 is disjoint from $t(U_0)$, let $U_1 = U_0 \cup t(U_0)$ and extend g to a symmetric holomorphic function on U_1 by $g_1(\lambda) = g \circ t(\lambda)$ for $\lambda \in t(U_0)$. In either case U_1 is a symmetric neighborhood of μ and g_1 agrees with f on $U_1 \cap V$.

The symmetric function g determines a holomorphic function H on the neighborhood $\pi(U)$ of s such that $g = H \circ \pi$ on U . Then H is a holomorphic function on the neighborhood $\pi(U)$ of s such that H and F agree on $\pi(U) \cap \pi(V)$. Thus F is bounded and holomorphic on $\pi(V)$.

Since $\pi(V)$ has the norm-preserving extension property, there exists a bounded holomorphic function \tilde{F} on G which extends F and satisfies

$$\sup_G |\tilde{F}| = \sup_{\pi(V)} |F| = \sup_V |f|.$$

Then $\tilde{F} \circ \pi$ is a bounded holomorphic function on \mathbb{D}^2 that extends f and has the same supremum norm as f . Hence V has the symmetric extension property in \mathbb{D}^2 . \square

LEMMA 13.4. *If V is a symmetric algebraic set in \mathbb{D}^2 then $\pi(V)$ is an algebraic set in G .*

PROOF. Suppose that

$$V = \{\lambda \in \mathbb{D}^2 : f(\lambda) = 0 \text{ for all } f \in S\}$$

for some set $S \subseteq \mathbb{C}[\lambda^1, \lambda^2]$. Since V is symmetric, for $\lambda \in \mathbb{D}^2$ we have $\lambda \in V$ if and only if $t(\lambda) \in V$. Hence $\lambda \in V$ if and only if both $(f + f \circ t)(\lambda) = 0$ and $(f - f \circ t)^2(\lambda) = 0$ for all $f \in S$. On replacing S by the set

$$\{f + f \circ t : f \in S\} \cup \{(f - f \circ t)^2 : f \in S\},$$

we may assume that S is a set of *symmetric* polynomials, and so can be expressed in the form

$$S = \{F \circ \pi : F \in S^b\}$$

for some set $S^b \subseteq \mathbb{C}[s^1, s^2]$. Then

$$\pi(V) = \{s \in G : F(s) = 0 \text{ for all } F \in S^b\}.$$

Hence $\pi(V)$ is an algebraic set. \square

Recall that the notion of a balanced disc in \mathbb{D}^2 was given in Definition 5.2. It is a subset D of \mathbb{D}^2 having the form $D = \{(z, m(z)) : z \in \mathbb{D}\}$ for some $m \in \text{Aut } \mathbb{D}$.

THEOREM 13.5. *A symmetric algebraic set V in \mathbb{D}^2 has the symmetric extension property if and only if one of the following six alternatives holds.*

- (1) $V = \{\lambda, t(\lambda)\}$ for some $\lambda \in \mathbb{D}^2$;
- (2) $V = \mathbb{D}^2$;
- (3) $V = D \cup t(D)$ for some balanced disc D in \mathbb{D}^2 such that D^- meets the set $\{(z, z) : z \in \mathbb{T}\}$;
- (4) $V = V_\beta$ for some $\beta \in \mathbb{D}$, where

$$(13.1) \quad V_\beta \stackrel{\text{def}}{=} \{(z, w) \in \mathbb{D}^2 : z + w = \beta + \bar{\beta}zw\};$$

- (5) $V = \Delta \cup V_\beta$ for some $\beta \in \mathbb{D}$, where $\Delta = \{(z, z) : z \in \mathbb{D}\}$;
- (6) $V = V_{m,r}$ for some $r \in (0, 1)$ and $m \in \text{Aut } \mathbb{D}$, where

$$(13.2) \quad V_{m,r} \stackrel{\text{def}}{=} \{(z, w) \in \mathbb{D}^2 : H_r(m(z), m(w)) = 0\}$$

and

$$(13.3) \quad H_r(z, w) \stackrel{\text{def}}{=} 2zw(r(z+w) + 2 - 2r) - (1+r)(z+w)^2 + 2r(z+w).$$

Moreover, the six types of sets V in (1) to (6) are mutually exclusive.

PROOF. Let V be a symmetric algebraic set in \mathbb{D}^2 . By Lemma 13.4, $\pi(V)$ is an algebraic set in G . By Lemma 13.3, V has the symmetric extension property if and only if $\pi(V)$ has the norm-preserving extension property in G . By Theorem 1.4, $\pi(V)$ has the norm-preserving extension property if and only if either $\pi(V)$ is a retract in G or $\pi(V) = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic \mathcal{D} of G .

Suppose that V has the symmetric extension property. Consider the two cases (i) $\pi(V)$ is a retract and (ii) $\pi(V) = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic \mathcal{D} .

In case (ii), by Proposition 7.6, $\mathcal{D} = \mathcal{F}_\beta$ for some $\beta \in \mathbb{D}$, and so $V = \pi^{-1}(\mathcal{R} \cup \mathcal{F}_\beta) = \Delta \cup V_\beta$, as in alternative (5) of the theorem.

In case (i), by Theorem 1.3, the retracts in G are the singleton sets, G itself and the complex geodesics in G . If $\pi(V)$ is either a singleton or G then alternative (1) or (2) respectively holds.

The remaining possibility in case (i) is that $\pi(V)$ is a complex geodesic in G , so that $\pi(V) = \mathcal{D}_\delta$ for some nondegenerate datum δ , of one of the five types described in Definition 3.4. If δ is flat then $\mathcal{D} = \mathcal{F}_\beta$ for some $\beta \in \mathbb{D}$, and $V = V_\beta$, so that alternative (4) holds. By Proposition 8.4, if δ is balanced then there exists a balanced disc D in \mathbb{D}^2 such that $\pi(D) = \pi(V)$, and hence $V = D \cup t(D)$. Moreover, if k solves $\text{Kob}(\delta)$, then $k(z) = (z + m(z), zm(z))$ for some $m \in \text{Aut } \mathbb{D}$, and the disc D is given by $\{(z, m(z)) : z \in \mathbb{D}\}$. Furthermore, since k has a royal node in \mathbb{T} , the disc D^- meets $\{(z, z) : z \in \mathbb{T}\}$. Thus alternative (3) holds.

There remains the case that $\pi(V)$ is a purely unbalanced geodesic. By Theorem 7.8, then $\pi(V) \sim k_r(\mathbb{D})$ for some $r \in (0, 1)$, where

$$(13.4) \quad k_r(z) = \frac{1}{1-rz} (2(1-r)z, z(z-r)) \quad \text{for all } z \in \mathbb{D}.$$

Let $h_r(z) = k_r(-z)$. Then $h_r(\mathbb{D}) = k_r(\mathbb{D})$ and one finds that $\Phi_1 \circ h_r = \text{id}_{\mathbb{D}}$. We may apply Proposition 5.5 to write $h_r(\mathbb{D})$ as the zero set of a polynomial. We have $h_r = \tilde{q}_r/q_r$, where

$$q_r(z) = 1 + rz, \quad \tilde{q}_r(z) = z(z+r).$$

As in equation (5.6), let

$$\begin{aligned} P_r(s) &= (2 - \omega s^1)^2 \frac{\tilde{q}_r \circ \Phi_\omega(s) - s^2 q_r \circ \Phi_\omega(s)}{s^2 - \bar{\omega}^2} \\ &= 2s^2(rs^1 + 2 - 2r) - (1+r)(s^1)^2 + 2rs^1. \end{aligned}$$

By Proposition 5.5, $k_r(\mathbb{D}) = G \cap P_r^{-1}(0)$. Hence $\pi(V) = G \cap \tilde{m}^{-1}(P_r^{-1}(0))$ for some $m \in \text{Aut } \mathbb{D}$. That is, there exist $r \in (0, 1)$ and $m \in \text{Aut } \mathbb{D}$ such that $(z, w) \in V$ if and only if

$$P_r(m(z) + m(w), m(z)m(w)) = 0.$$

Since the function H_r is defined in equation (13.3) so that $H_r(z, w) = P_r(z+w, zw)$, we have $V = V_{m,r}$, and alternative (6) of the theorem holds.

We have shown that if V has the symmetric extension property then V is of one of the forms (1) to (6). We prove the converse. If either of alternatives (1) or (2) holds, then $\pi(V)$ is a singleton or G respectively and so $\pi(V)$ has the norm-preserving extension property. If (5) holds then $\pi(V) = \mathcal{R} \cup \mathcal{F}_\beta$, and so, by Theorem 1.4, $\pi(V)$ has the norm-preserving extension property in G .

We claim that, in the remaining cases (3), (4) and (6), $\pi(V)$ is a complex geodesic in G .

If (3) holds then $D = \{(z, m(z)) : z \in \mathbb{D}\}$ for some $m \in \text{Aut } \mathbb{D}$ having a fixed point in \mathbb{T} . The map $k(z) = (z + m(z), zm(z))$ is a complex C-geodesic and so $\pi(V) = \pi(D) = k(\mathbb{D})$ is a geodesic of degree 2.

If (4) holds and $V = V_\beta$ for some $\beta \in \mathbb{D}$, then $\pi(V)$ is the flat geodesic \mathcal{F}_β .

If (6) holds and $V = V_{m,r}$ for some $m \in \text{Aut } \mathbb{D}$ and $r \in (0, 1)$, then $\pi(V) \sim k_r(\mathbb{D})$, where k_r is given by equation (13.4), and so, by Theorem 7.8, $\pi(V)$ is a (purely unbalanced) geodesic in G . Thus $\pi(V)$ has the norm-preserving extension property in G .

In each of the six cases $\pi(V)$ has the norm-preserving extension property in G , and therefore V has the symmetric extension property in \mathbb{D}^2 .

To see that the classes (1) to (6) of sets V in \mathbb{D}^2 with the symmetric extension property are mutually exclusive, one can simply consider the corresponding sets $\pi(V)$. Our reasoning shows that, in cases (1) to (6), the corresponding set $\pi(V)$ is respectively (1) a singleton (2) G (3) an exceptional, purely balanced or royal geodesic (4) a flat geodesic (5) $\mathcal{R} \cup \mathcal{F}_\beta$ for some β and (6) a purely unbalanced geodesic. \square

Applications to the theory of spectral sets

Two fundamental results in the harmonic analysis of operators on Hilbert space [44] are the inequalities of von Neumann and Andô [37, 14], which can be expressed by the statements

- (1) the closed unit disc is a spectral set for every contraction, and
- (2) the closed unit bidisc is a spectral set for every pair of commuting contractions.

In this chapter an *operator* means a bounded linear operator on a Hilbert space, and a *contraction* means an operator of norm at most 1. A *spectral set* for a commuting n -tuple T of operators is a set $V \subseteq \mathbb{C}^n$ such that $\sigma(T) \subseteq V$ and, for every holomorphic function f in a neighborhood of V ,

$$\|f(T)\| \leq \sup_V |f|.$$

The terminology *spektrale Menge* is due to von Neumann [37]. Spectral sets for commuting tuples of operators are commonly defined to be closed subsets of \mathbb{C}^n , but in [2] the notion was broadened considerably, with the goal of a refinement of Andô's and von Neumann's inequalities.

DEFINITION 14.1. Let V be a subset of \mathbb{C}^n and T be an n -tuple of commuting operators on a Hilbert space. T is *subordinate* to V if the spectrum $\sigma(T)$ is a subset of V and every holomorphic function on a neighborhood of V that vanishes on V annihilates T .

Clearly, if T is subordinate to V and g is the restriction to V of a holomorphic function f on a neighborhood of V then we may uniquely define $g(T)$ to be $f(T)$, where $f(T)$ is defined by the Taylor functional calculus. We denote by $H^\infty(V)$ the algebra of functions $f|_V$ where f is bounded and holomorphic in some neighborhood U_f of V . Thus, if T is subordinate to V then the map $g \mapsto g(T)$ is a functional calculus for $H^\infty(V)$.

DEFINITION 14.2. Let $V \subseteq \mathbb{C}^n$, let $A \subseteq H^\infty(V)$ and let T be an n -tuple of commuting operators. V is an *A-spectral set* for T if T is subordinate to V and, for every $f \in A$,

$$(14.1) \quad \|f(T)\| \leq \sup_V |f|.$$

Another formulation of Andô's inequality is that \mathbb{D}^2 is a spectral set for any commuting pair of contractions whose joint spectrum is contained in \mathbb{D}^2 . Isolating the role of \mathbb{D}^2 in this statement and generalizing it to arbitrary subsets of $H^\infty(V)$, in [2, Definition 1.12] the authors introduced the following notion.

DEFINITION 14.3. Let $V \subseteq \mathbb{C}^2$ and let $A \subseteq H^\infty(V)$. Then V is an A -von Neumann set if the inequality

$$\|f(T)\| \leq \sup_V |f|$$

holds for all $f \in A$ and all pairs T of commuting contractions which are subordinate to V , in the sense of Definition 14.1.

Thus V is an A -von Neumann set if V is an A -spectral set for every pair T of commuting contractions which is subordinate to V .

One of the main results of [2], Theorem 1.13, states that if $V \subseteq \mathbb{D}^2$ and $A \subseteq H^\infty(V)$ then V is an A -von Neumann set if and only if V has the A -extension property relative to \mathbb{D}^2 . In the case that V is a symmetric subset of \mathbb{D}^2 and A is the algebra $H_{\text{sym}}^\infty(V)$ (see page 77), Theorem 13.5 enables us to give an explicit description of the A -von Neumann sets in \mathbb{D}^2 .

THEOREM 14.4. *Let V be a symmetric algebraic set in \mathbb{D}^2 . Then V is an $H_{\text{sym}}^\infty(V)$ -von Neumann set if and only if V has one of the six forms (1) to (6) in Theorem 13.5.*

PROOF. By [2, Theorem 1.13], V is an $H_{\text{sym}}^\infty(V)$ -von Neumann set in \mathbb{D}^2 if and only if V has the $H_{\text{sym}}^\infty(V)$ -extension property relative to \mathbb{D}^2 . By Theorem 13.5, the latter is so if and only if V has one of the six forms (1) to (6) in the theorem. \square

The A -von Neumann sets of Definition 14.3 are very much tied to the bidisc. One can define a similar notion for other subsets of \mathbb{C}^2 . Let us illustrate with the symmetrized bidisc.

DEFINITION 14.5. A pair T of commuting bounded linear operators is a Γ -contraction if Γ is a spectral set for T .

Let $V \subseteq G$ and let $A \subseteq H^\infty(V)$. Then V is a (G, A) -von Neumann set if V is an A -spectral set for every Γ -contraction T subordinate to V .

THEOREM 14.6. *Let $V \subseteq G$ and let $A \subseteq H^\infty(V)$. Then V is a (G, A) -von Neumann set if and only if V has the A -extension property relative to G .*

PROOF. Suppose that V has the A -extension property relative to G . Consider any $f \in A$. There exists $g \in H^\infty(G)$ such that $g|_V = f$ and

$$(14.2) \quad \sup_G |g| = \sup_V |f|.$$

Let T be a Γ -contraction subordinate to V : then

$$\|g(T)\| \leq \sup_G |g|,$$

and so, by equation (14.2),

$$(14.3) \quad \|f(T)\| = \|g(T)\| \leq \sup_\Gamma |g| = \sup_V |f|.$$

This inequality holds for all $f \in A$, and so V is an A -spectral set for T . Since this is true for every Γ -contraction T subordinate to V , it follows that V is a (G, A) -von Neumann set.

Conversely, suppose that V is a (G, A) -von Neumann set. We claim that $\pi^{-1}(V)$ is an $A \circ \pi$ -von Neumann set contained in \mathbb{D}^2 . Consider a commuting pair of contractions T subordinate to $\pi^{-1}(V)$. Then $\pi(T)$ is a Γ -contraction and

$$\sigma(\pi(T)) = \pi(\sigma(T)) \subseteq \pi(\pi^{-1}(V)) = V.$$

Let us show that $\pi(T)$ is subordinate to V . Consider g holomorphic on a neighborhood of V and such that $g|_V = 0$. Then $g \circ \pi$ is holomorphic on a neighborhood of $\pi^{-1}(V)$ and is zero on $\pi^{-1}(V)$. Since T is subordinate to $\pi^{-1}(V)$, $g \circ \pi(T) = 0$ and hence $\pi(T)$ is subordinate to V .

By assumption, V is a (G, A) -von Neumann set, and so V is an A -spectral set for $\pi(T)$. Therefore

$$(14.4) \quad \|f \circ \pi(T)\| \leq \sup_V |f| = \sup_{\pi^{-1}(V)} |f \circ \pi|$$

for all $f \in A$. Thus, for all $F \in A \circ \pi$,

$$(14.5) \quad \|F(T)\| \leq \sup_{\pi^{-1}(V)} |F|$$

whenever T is a commuting pair of contractions subordinate to $\pi^{-1}(V)$. That is, $\pi^{-1}(V)$ is an $A \circ \pi$ -von Neumann set in \mathbb{D}^2 .

By [2, Theorem 1.13], a subset W of \mathbb{D}^2 is an $A \circ \pi$ -von Neumann set if and only if W has the $A \circ \pi$ -extension property relative to \mathbb{D}^2 . On applying this result to $W = \pi^{-1}(V)$ we deduce that $\pi^{-1}(V)$ has the $A \circ \pi$ -extension property relative to \mathbb{D}^2 . It follows that V has the A -extension property relative to G . Indeed, consider $f \in A \subseteq H^\infty(V)$. Since $\pi^{-1}(V)$ has the $A \circ \pi$ -extension property in \mathbb{D}^2 , there exists a bounded holomorphic function g on \mathbb{D}^2 which agrees with $f \circ \pi$ on $\pi^{-1}(V)$ and satisfies

$$\sup_{\mathbb{D}^2} |g| = \sup_{\pi^{-1}(V)} |f \circ \pi| = \sup_V |f|.$$

Let $\tilde{g} = \frac{1}{2}(g + g \circ t)$. Then \tilde{g} is a symmetric bounded holomorphic function on \mathbb{D}^2 which agrees with $f \circ \pi$ on $\pi^{-1}(V)$ and satisfies

$$\sup_{\mathbb{D}^2} |\tilde{g}| = \sup_{\pi^{-1}(V)} |f \circ \pi|.$$

There exists a holomorphic function F on G such that $\tilde{g} = F \circ \pi$. Then F agrees with f on V and

$$\sup_G |F| = \sup_{\mathbb{D}^2} |\tilde{g}| = \sup_{\pi^{-1}(V)} |f \circ \pi| = \sup_V |f|.$$

Hence V has the A -extension property relative to G . □

The same theorem is stated in [19].

In the event that $A = H^\infty(V)$ for a subset V of G , we can describe all (G, A) -von Neumann sets.

THEOREM 14.7. *Let V be an algebraic subset of G . Then V is a $(G, H^\infty(V))$ -von Neumann set in G if and only if either V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic \mathcal{D} in G , where \mathcal{R} is the royal variety.*

PROOF. By Theorem 14.6, V is a $(G, H^\infty(V))$ -von Neumann set if and only if V has the $H^\infty(V)$ -extension property relative to G . By Theorem 1.4, the latter holds if and only if either V is a retract in G or $V = \mathcal{R} \cup \mathcal{D}$ for some flat geodesic \mathcal{D} in G . □

Anomalous sets with the norm-preserving extension property in some other domains

At little extra cost we can deduce from Theorems 1.3 and 1.4 that there are anomalous sets having the norm-preserving extension property in some other domains besides G – to wit, domains containing G as a retract. In particular, the statement holds for the 2×2 spectral ball, the tetrablock and the pentablock, defined below.

LEMMA 15.1. *If Ω is a domain in \mathbb{C}^d such that G can be embedded in Ω as a retract then there exist sets which have the norm-preserving extension property but are not retracts in Ω .*

PROOF. G can be embedded in Ω as a retract if and only if there exist $\iota \in \Omega(G)$ and $\kappa \in G(\Omega)$ such that κ is a left inverse of ι .

Suppose such maps ι and κ exist. Let \mathcal{D} be a flat geodesic in G and let $V = \mathcal{R} \cup \mathcal{D}$. By Theorem 1.4, V is a subset of G that has the norm-preserving extension property. By Lemma 15.2 below, $\iota(V)$ has the norm-preserving extension property in Ω_2 .

However, $\iota(V)$ is not a retract of Ω . For suppose that ρ is a retraction in Ω with range $\iota(V)$. Let

$$\rho_1 = \kappa \circ \rho \circ \iota : G \rightarrow G;$$

then

$$\rho_1 \circ \rho_1 = \kappa \circ \rho \circ (\iota \circ \kappa) \circ \rho \circ \iota.$$

Now $\iota \circ \kappa$ acts as the identity mapping on $\iota(G)$, and so *a fortiori* on $\text{ran } \rho$. Hence

$$\begin{aligned} \rho_1 \circ \rho_1 &= \kappa \circ \rho \circ \rho \circ \iota \\ &= \kappa \circ \rho \circ \iota \\ &= \rho_1. \end{aligned}$$

Thus ρ_1 is a retraction on G with range $\kappa \circ \iota(V)$, which is V . Therefore V is a nontrivial retract in G . This contradicts Theorem 1.3, since V is clearly not a geodesic (it is not homeomorphic to a disc). Hence $\iota(V)$ is not a retract of Ω_2 . \square

LEMMA 15.2. *Let U_1, U_2 be domains and suppose that $\iota \in U_2(U_1)$, $\kappa \in U_1(U_2)$ satisfy $\kappa \circ \iota = \text{id}_{U_1}$. If a subset V of U_1 has the norm-preserving extension property in U_1 then $\iota(V)$ has the norm-preserving extension property in U_2 .*

PROOF. Consider a bounded holomorphic function f on $\iota(V)$. Then $f \circ \iota$ is bounded and holomorphic on V , and so has a norm-preserving extension g to U_1 . Thus $g|_V = f \circ \iota$ and

$$\sup_{U_1} |g| = \sup_V |f \circ \iota| = \sup_{\iota(V)} |f|.$$

Now $g \circ \kappa$ is bounded and holomorphic on U_2 , and

$$\sup_{U_2} |g \circ \kappa| = \sup_{U_1} |g| = \sup_{\iota(V)} |f|.$$

Moreover, for any point of $\iota(V)$, say $\iota(z)$ where $z \in V$,

$$g \circ \kappa(\iota(z)) = g \circ \kappa \circ \iota(z) = g(z) = f \circ \iota(z).$$

Hence $g \circ \kappa$ is a norm-preserving extension of f to U_2 . \square

The $n \times n$ *spectral ball* is the set

$$\Omega_n \stackrel{\text{def}}{=} \{A \in \mathbb{C}^{n \times n} : r(A) < 1\}$$

where $r(\cdot)$ denotes the spectral radius of a square matrix. The domain Ω_n is of interest principally in the theory of invariant distances and metrics [32, 38, 39, 15, 16]. Holomorphic interpolation from \mathbb{D} to Ω_n , the ‘spectral Nevanlinna-Pick problem’, is a test case of the μ -synthesis problem of H^∞ control (see for example [17, 47]).

The symmetrized bidisc was introduced as a tool for the study of Ω_2 , and so it is natural to consider the implications of our main theorems for the function theory of Ω_2 .

There are many ways in which G can be embedded as a retract in Ω_2 . Let $\kappa : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^2$ be defined by $\kappa(A) = (\text{tr } A, \det A)$. Clearly a matrix $A \in \mathbb{C}^{2 \times 2}$ belongs to Ω_2 if and only if $\kappa(A) \in G$. For any holomorphic function

$$(15.1) \quad F : G \rightarrow GL_2(\mathbb{C})$$

define a holomorphic map $\iota_F : G \rightarrow \Omega_2$ by

$$(15.2) \quad \iota_F(s) = F(s)^{-1} \begin{bmatrix} 0 & 1 \\ -s^2 & s^1 \end{bmatrix} F(s).$$

Then $\kappa \circ \iota_F = \text{id}_G$, and so ι_F is injective and $\iota_F \circ \kappa$ is a retraction in Ω_2 . Thus $\iota_F(G)$ is a retract in Ω_2 .

The *tetrablock* is the domain

$$\mathcal{E} \stackrel{\text{def}}{=} \{(a_{11}, a_{22}, \det A) : A = [a_{ij}]_{i,j=1}^2, \|A\| < 1\}$$

in \mathbb{C}^3 . There are many equivalent characterizations of this domain [1]. It is easy to show that the map $\iota : G \rightarrow \mathcal{E}$ given by

$$\iota(s) = \left(\frac{1}{2}s^1, \frac{1}{2}s^1, s^2\right)$$

is a holomorphic injection with left inverse

$$\kappa(z) = (z^1 + z^2, z^3)$$

that maps \mathcal{E} to G . Thus E contains G as a retract.

The *pentablock* is the domain

$$\mathcal{P} = \{(a_{21}, \text{tr } A, \det A) : A = [a_{ij}]_{i,j=1}^2, \|A\| < 1\}$$

in \mathbb{C}^3 . The pentablock also admits numerous characterizations [9]. The map $\iota : G \rightarrow \mathcal{P}$ given by

$$\iota(s) = (0, s^1, s^2)$$

has a left inverse $\kappa : \mathcal{P} \rightarrow G$ given by $\kappa(z^1, z^2, z^3) = (0, z^2, z^3)$. Thus G is embeddable as a retract in \mathcal{P} .

The tetrablock and the pentablock are both of interest in the theory of invariant distances [32] and in the analysis of special cases of the μ -synthesis problem [47, 9].

The following statement is an immediate consequence of Lemma 15.1.

THEOREM 15.3. *The 2×2 spectral ball Ω_2 , the tetrablock and the pentablock all contain sets having the norm-preserving extension property which are not retracts in the respective domains.*

REMARK 15.4. For a domain U in \mathbb{C}^N , $N \geq 3$, one cannot expect all nontrivial retracts to be complex geodesics. For example, let Ω be one of the following domains: the 2×2 spectral ball Ω_2 , the tetrablock and the pentablock. Then the image of G under ι in Ω is a retract which is not a complex geodesic (since it has complex dimension 2).

APPENDIX A

Some useful facts about the symmetrized bidisc

In this appendix we gather together some established properties of the symmetrized bidisc G relevant to the paper, with appropriate citations. We also prove analogs for infinitesimal datums of some statements in the literature about discrete datums. An alternative source for many of these results is [32, Chapter 7].

A.1. Basic properties of G and Γ

There are numerous criteria for a point of \mathbb{C}^2 to belong to G .

PROPOSITION A.1. *The following statements are equivalent for any $s = (s^1, s^2) \in \mathbb{C}^2$.*

- (1) $s \in G$ (that is, there exist $z, w \in \mathbb{D}$ such that $s = (z + w, zw)$);
- (2) $|s^1 - \overline{s^1}s^2| < 1 - |s^2|^2$;
- (3) $|\Phi_\omega(s)| < 1$ for all $\omega \in \mathbb{T}$;
- (4) there exists $\beta \in \mathbb{D}$ such that $s^1 = \beta + \overline{\beta}s^2$;
- (5) $2|s^1 - \overline{s^1}s^2| + |(s^1)^2 - 4s^2| < 4 - |s^1|^2$.

The proof of this proposition can be found in [5, Theorem 2.1]. The functions Φ_ω , $\omega \in \mathbb{T}$, in (3) are those introduced in Definition 3.2.

A simple calculation shows that, for any $s \in G$, the corresponding β in (4) is unique and is given by

$$(A.1) \quad \beta = \frac{s^1 - \overline{s^1}s^2}{1 - |s^2|^2}.$$

Recall from Section 4.2 that Γ denotes the closure of G and $b\Gamma$ the distinguished boundary of Γ . It means that $b\Gamma$ is the Shilov boundary of the uniform algebra of continuous complex-valued functions on Γ that are analytic in G . The following descriptions of $b\Gamma$ are to be found in [5, Theorem 2.4, equation (2.1)].

PROPOSITION A.2.

$$\begin{aligned} b\Gamma &= \{(z + w, zw) : |z| = 1 = |w|\} \\ &= \{s \in \mathbb{C}^2 : |s^1| \leq 2, |s^2| = 1 \text{ and } s^1 = \overline{s^1}s^2\}. \end{aligned}$$

A Γ -inner function is a holomorphic map $h : \mathbb{D} \rightarrow G$ such that, for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure, the radial limit $\lim_{r \rightarrow 1^-} f(r\lambda) \in b\Gamma$. Rational Γ -inner functions were defined in Definition 4.12.

PROPOSITION A.3. *If k is a nonconstant Γ -inner function and $\omega \in \mathbb{T}$ then $\Phi_\omega \circ k$ is an inner function.*

PROOF. For if k^* is the radial limit function of k then, for almost all $\tau \in \mathbb{T}$ there exist $z, w \in \mathbb{T}$, not both equal to $\bar{\omega}$, such that $k^*(\tau) = (z + w, zw)$. Then

$$\begin{aligned} |\Phi_\omega \circ k^*(\tau)| &= \left| \frac{2\omega zw - z - w}{2 - \omega(z + w)} \right| \\ &= \left| \omega zw \frac{2 - \bar{\omega}\bar{w} - \bar{\omega}\bar{z}}{2 - \omega z - \omega w} \right| \\ &= 1. \end{aligned}$$

□

The functions Φ_ω , $\omega \in \mathbb{T}$, are central to the theory of G . Their most significant property is the following.

THEOREM A.4. *The functions Φ_ω , $\omega \in \mathbb{T}$, constitute a universal set for the Carathéodory problem on G*

This statement is in [5, Theorem 1.1 and Corollary 4.3].

It is easy to see that, for any $\omega \in \mathbb{T}$, the function Φ_ω extends continuously to $\Gamma \setminus \{(2\bar{\omega}, \bar{\omega}^2)\}$ (but not to Γ). The range of the extended function is \mathbb{D}^- . It is useful to know which points are mapped into \mathbb{T} .

PROPOSITION A.5. *For any $\omega \in \mathbb{T}$ and $s \in \Gamma \setminus \{(2\bar{\omega}, \bar{\omega}^2)\}$,*

$$|\Phi_\omega(s)| = 1 \quad \text{if and only if} \quad \omega(s^1 - \bar{s}^1 s^2) = 1 - |s^2|^2.$$

The proof is a simple calculation and is given in [5, Theorem 2.5].

THEOREM A.6. *Let $h = (s^1, s^2)$ be a nonconstant rational Γ -inner function and let v be a finite Blaschke product. Then $\Phi_v \circ h$ has a cancellation at ζ if and only if the following conditions are satisfied: $\zeta \in \mathbb{T}$, ζ is a royal node for h and $v(\zeta) = \frac{1}{2}s^1(\bar{\zeta})$. Moreover $\Phi_v \circ h$ has at most one cancellation at any royal node ζ .*

The proof of this theorem is given in [8, Theorem 7.12].

A.2. Complex C-geodesics in G

Recall from Section 4.1 that a map $k \in G(\mathbb{D})$ is a complex C-geodesic if k has a holomorphic left inverse. Such maps can be concretely described. The following statement is [6, Lemma 1.1 and Theorem 1.2].

THEOREM A.7. *Let $k : \mathbb{D} \rightarrow G$ be a complex C-geodesic of G . Then k^2 is a Blaschke product of degree 1 or 2, k extends to a continuous function on \mathbb{D}^- which maps \mathbb{T} into $b\Gamma$ and there exist a Möbius function v and an $\omega_0 \in \mathbb{T}$ such that $\Phi_{\omega_0} \circ k = v$. Furthermore,*

$$k^2 \circ v^{-1}(\bar{\omega}_0) = \bar{\omega}_0^2$$

and, for all $\lambda \in \mathbb{D}$,

$$k^1(\lambda) = 2 \frac{\omega_0 k^2(\lambda) - v(\lambda)}{1 - \omega_0 v(\lambda)}.$$

A fortiori, k is a rational Γ -inner function.

The following statement is a simple corollary.

PROPOSITION A.8. *Let δ be a nondegenerate non-flat datum in G . If Φ_ω solves $\text{Car}(\delta)$, k solves $\text{Kob}(\delta)$ and k is normalised so that $\Phi_\omega \circ k = \text{id}_{\mathbb{D}}$, then k^2 is a Blaschke product of degree 2 and $k(\bar{\omega}) = (2\bar{\omega}, \bar{\omega}^2)$.*

COROLLARY A.9. *A map $k \in G(\mathbb{D})$ is a complex C -geodesic of G if and only if there is a nondegenerate datum δ in G such that k solves $\text{Kob}(\delta)$.*

The following uniqueness result for complex C -geodesics (stated in the introduction as Theorem 3.7) plays a vital role in the paper.

THEOREM A.10. *If δ is a nondegenerate datum in G then the solution to $\text{Kob}(\delta)$ is essentially unique.*

PROOF. For discrete datums this statement is contained in [6, Theorem 0.3]; we prove the infinitesimal case. Let $\delta = (s_1, v)$ be a nondegenerate infinitesimal datum in G . According to the definition (3.2),

$$|\delta|_{\text{kob}} = \inf\{|\zeta| : \zeta \text{ is a datum in } \mathbb{D} \text{ and there exists } h \in G(\mathbb{D}) \text{ such that } h(\zeta) = \delta\}.$$

Note that, for a datum $\zeta = (z_1, w)$ in \mathbb{D} and $h \in G(\mathbb{D})$,

$$h(\zeta) = (h(z_1), D_w h(z_1)) = (h(z_1), wh'(z_1)).$$

Thus

$$h(\zeta) = \delta \Leftrightarrow h(z_1) = s_1, \quad w(h^1)'(z_1) = v^1 \text{ and } w(h^2)'(z_1) = v^2.$$

Hence, for a datum $\delta = (s_1, v)$ in G ,

$$|\delta|_{\text{kob}} = \inf_{z_1 \in \mathbb{D}, w \in \mathbb{C}} \left\{ \frac{|w|}{1 - |z_1|^2} : \text{there exists } h \in G(\mathbb{D}) \text{ such that} \right. \\ \left. h(z_1) = s_1, w(h^1)'(z_1) = v^1 \text{ and } w(h^2)'(z_1) = v^2 \right\}.$$

Pick an infinitesimal datum $\zeta = (z_1, w)$ in \mathbb{D} such that $|\zeta| = |\delta|_{\text{kob}}$; then ζ is nondegenerate. Let $h = (s, p)$ solve $\text{Kob}(\delta)$ and satisfy $h(\zeta) = \delta$. Let Φ_ω solve $\text{Car}(\delta)$. Since $|\cdot|_{\text{kob}} = |\cdot|_{\text{car}}$ on G , $\Phi_\omega \circ h \in \text{Aut } \mathbb{D}$. Say $\Phi_\omega \circ h = q$. Then, by [6, Lemma 1.1], p is a Blaschke product of degree at most 2 and

$$(A.2) \quad s = 2 \frac{\omega p - q}{1 - \omega q}.$$

Hence, if $\tau = q^{-1}(\bar{\omega}) \in \mathbb{T}$, then $(\omega p - q)(\tau) = 0$, and so $p(\tau) = \bar{\omega}^2$. Thus p is a Blaschke product of degree at most 2 which satisfies

$$p(z_1) = s_1, \quad w(p)'(z_1) = v^2 \quad \text{and } p(\tau) = \bar{\omega}^2.$$

These properties determine p uniquely.

Since q is an automorphism of \mathbb{D} such that

$$q(\zeta) = \Phi_\omega \circ h(\zeta) = \Phi_\omega(\delta),$$

q is uniquely determined. Thus s is also uniquely determined by equation (A.2). Hence there is a unique solution h of $\text{Kob}(\delta)$ for which $h(\zeta) = \delta$. It follows that the solution of $\text{Kob}(\delta)$ is essentially unique. \square

A.3. Automorphisms of G

For every $m \in \text{Aut } \mathbb{D}$ we define a map $\tilde{m} \in G(G)$ by

$$(A.3) \quad \tilde{m}(z + w, zw) = (m(z) + m(w), m(z)m(w)) \quad \text{for all } z, w \in \mathbb{D}.$$

It is easy to see that \tilde{m} is an automorphism of G and that $m \mapsto \tilde{m}$ is a homomorphism from $\text{Aut } \mathbb{D}$ to $\text{Aut } G$. In fact it is an isomorphism.

THEOREM A.11. (1) *Every automorphism of G is of the form \tilde{m} for some m in $\text{Aut } \mathbb{D}$.*

- (2) *Every automorphism of G leaves the royal variety \mathcal{R} invariant.*
(3) *The automorphisms of G act transitively on the set of flat geodesics of G .*

Statement (1) is [7, Theorem 4.1], proved by an operator-theoretic method; a proof which uses Cartan's Classification Theorem is given in [31] and [32]. Given statement (1), statements (2) and (3) are easy to verify.

COROLLARY A.12. *Every automorphism of G extends continuously to a bijective self-map of Γ .*

It follows from Theorem A.11 and the fact that every automorphism of \mathbb{D} extends continuously to a bijective self-map of \mathbb{D}^- .

The automorphisms of G are linear fractional maps. If $m = \tau B_\alpha$ for some $\tau \in \mathbb{T}$, $\alpha \in \mathbb{D}$ then, for $s \in G$,

$$(A.4) \quad \tilde{m}(s) = \frac{1}{1 - \bar{\alpha}s^1 + \bar{\alpha}^2 s^2} (\tau(-2\alpha + (1 + |\alpha|^2)s^1 - 2\bar{\alpha}s^2), \tau^2(s^2 - \alpha s^1 + \alpha^2)).$$

A.4. A trichotomy theorem

Since the classification of datums in Definition 3.4 depends on the number of functions Φ_ω that solve the corresponding Carathéodory problem, it is important to know what the possibilities are for this number.

TRICHOTOMY THEOREM A.13. Let δ be a nondegenerate datum in G . Exactly one of the following assertions is true.

- (1) There is a unique $\omega \in \mathbb{T}$ such that Φ_ω solves $\text{Car}(\delta)$;
(2) there are exactly two points $\omega_1, \omega_2 \in \mathbb{T}$ such that $\Phi_{\omega_1}, \Phi_{\omega_2}$ solve $\text{Car}(\delta)$;
(3) for all $\omega \in \mathbb{T}$, Φ_ω solves $\text{Car}(\delta)$.

Moreover, all three possibilities do occur.

PROOF. In the case that δ is a discrete datum, the statement is [6, Theorem 1.6]. We sketch a proof for a infinitesimal datum $\delta = (s_1, v)$: it is a combination of the arguments in [5, Section 4] and [6, Theorem 1.6], where fuller versions of the reasoning can be found.

For $\kappa \geq 0$ let S_κ, P_κ be the operators on the Hilbert space \mathbb{C}^2 (with its standard inner product) given by

$$(A.5) \quad S_\kappa = \begin{bmatrix} s_1^1 & \kappa v^1 \\ 0 & s_1^1 \end{bmatrix}, \quad P_\kappa = \begin{bmatrix} s_1^2 & \kappa v^2 \\ 0 & s_1^2 \end{bmatrix}.$$

S_κ, P_κ are commuting operators and $\sigma(S_\kappa, P_\kappa) = \{s_1\} \subset G$. Moreover the spectral radius of S_κ is $|s_1^1|$, which is less than 2. For any $f \in \mathbb{D}(G)$,

$$f(S_\kappa, P_\kappa) = \begin{bmatrix} f(s_1) & \kappa D_v f(s_1) \\ 0 & f(s_1) \end{bmatrix}.$$

Hence

$$\begin{aligned} \|f(S_\kappa, P_\kappa)\| \leq 1 &\Leftrightarrow \frac{\kappa |D_v f(s_1)|}{1 - |f(s_1)|^2} \leq 1 \\ &\Leftrightarrow \kappa |f(\delta)| \leq 1. \end{aligned}$$

Recall that G is said to be a *spectral domain* for the commuting pair (S, P) if $\sigma(S, P) \subset G$ and

$$\|f(S, P)\| \leq \sup_G |f|$$

for every $f \in \mathbb{C}(G)$. It follows that G is a spectral domain for (S_κ, P_κ) if and only if $\kappa|\delta|_{\text{car}} \leq 1$.

On the other hand, by [5, Theorem 3.2], G is a spectral domain for (S_κ, P_κ) if and only if $\|\Phi_\omega(S_\kappa, P_\kappa)\| \leq 1$ for all $\omega \in \mathbb{T}$, or equivalently, if and only if

$$(A.6) \quad (2 - \omega S_\kappa)^*(2 - \omega S_\kappa) - (2\omega P_\kappa - S_\kappa)^*(2\omega P_\kappa - S_\kappa) \geq 0$$

for all $\omega \in \mathbb{T}$. The inequality (A.6) can be written

$$\operatorname{Re}\{\omega(S_\kappa - S_\kappa^* P_\kappa)\} \leq 1 - P_\kappa^* P_\kappa.$$

In the event that $\|P_\kappa\| < 1$ (that is, $\kappa|v^2| < 1 - |s_1^2|^2$) we define

$$T_\kappa = (1 - P_\kappa^* P_\kappa)^{-\frac{1}{2}}(S_\kappa - S_\kappa^* P_\kappa)(1 - P_\kappa^* P_\kappa)^{-\frac{1}{2}};$$

in the case that $\|P_\kappa\| = 1$, we take T_κ to be the unique operator on $\operatorname{ran}(1 - P_\kappa^* P_\kappa)$ such that

$$(1 - P_\kappa^* P_\kappa)^{\frac{1}{2}} T_\kappa (1 - P_\kappa^* P_\kappa)^{\frac{1}{2}} = S_\kappa - S_\kappa^* P_\kappa.$$

Now the condition (A.6) can be written

$$\operatorname{Re}\{\omega T_\kappa\} \leq 1 \quad \text{for all } \omega \in \mathbb{T},$$

which is equivalent to the statement that the numerical range $W(T_\kappa)$ is contained in \mathbb{D}^- . For the extremal choice $\kappa = 1/|\delta|_{\text{car}}$, $W(T_\kappa)$ touches the unit circle at one or more points ω , and these values of ω are precisely those for which Φ_ω solves $\operatorname{Car}(\delta)$. Since T_κ has rank one or two, $W(T_\kappa)$ is either an ellipse (with its interior), a point or a line segment. It follows that $W(T_\kappa) \cap \mathbb{T}$ consists of either a single point, a pair of points or the whole of \mathbb{T} . Thus the three possibilities for the solutions Φ_ω of $\operatorname{Car}(\delta)$ are as described in the theorem.

Examples to show that all three possibilities in Theorem A.13 do occur may be found in the proof of [6, Theorem 1.6]. \square

A.5. Datums for which all Φ_ω are extremal

For the proof of the Pentachotomy Theorem in Section 3.3 it is required to characterize the datums δ for which ρ_δ is constant on \mathbb{T} , or in other words, for which Φ_ω is extremal for $\operatorname{Car}(\delta)$ for all $\omega \in \mathbb{T}$. The next result is a re-statement of [5, Theorem 5.5].

THEOREM A.14. *Let $\delta = (s_1, s_2)$ be a nondegenerate discrete datum in G . The following statements are equivalent.*

- (1) Φ_ω solves $\operatorname{Car}(\delta)$ for every $\omega \in \mathbb{T}$;
- (2) s_1, s_2 either both lie in the royal variety \mathcal{R} or both lie in the set

$$\mathcal{F}_\beta = \{(\beta + \bar{\beta}z, z) : z \in \mathbb{D}\}$$

for some $\beta \in \mathbb{D}$.

We also need the infinitesimal version of Theorem A.14.

THEOREM A.15. *Let $\delta = (s_1, v)$ be a nondegenerate infinitesimal datum in G . The following statements are equivalent.*

- (1) Φ_ω solves $\operatorname{Car}(\delta)$ for every $\omega \in \mathbb{T}$;

- (2) either (a) there exists $z \in \mathbb{D}$ such that $s_1 = (2z, z^2)$ and v is collinear with $(1, z)$ or (b) there exist $\beta, z \in \mathbb{D}$ such that $s_1 = (\beta + \bar{\beta}z, z)$ and v is collinear with $(\bar{\beta}, 1)$.

PROOF. First show that (2) \Rightarrow (1). By the definition of ρ_δ , for $\omega \in \mathbb{T}$,

$$\rho_\delta(\omega) = |\Phi_\omega(s_1, v)|^2 = |(\Phi_\omega(s_1), D_v \Phi_\omega(s_1))|^2.$$

Suppose (a) holds, that is, $s_1 = (2z, z^2)$ and $v = \lambda(1, z)$ for some $z \in \mathbb{D}$ and $\lambda \neq 0$. One can see that $\Phi_\omega(s_1) = -z$ and

$$\begin{aligned} D_{(1,z)} \Phi_\omega(s_1) &= \left[\frac{\partial}{\partial s^1} \frac{2\omega s^2 - s^1}{2 - \omega s^1} + z \frac{\partial}{\partial s^2} \frac{2\omega s^2 - s^1}{2 - \omega s^1} \right]_{|s=(2z, z^2)} \\ &= \left[\frac{-2 + 2\omega^2 s^2 + 2z\omega(2 - \omega s^1)}{(2 - \omega s^1)^2} \right]_{|s=(2z, z^2)} \\ &= -\frac{1}{2}. \end{aligned}$$

Thus

$$\Phi_\omega(\delta) = (\Phi_\omega(s_1), D_{\lambda(1,z)} \Phi_\omega(s_1)) = (-z, -\frac{1}{2}\lambda).$$

Hence

$$\rho_\delta(\omega) = |(-z, -\frac{1}{2}\lambda)|^2.$$

Thus (a) implies that ρ_δ is constant on \mathbb{T} .

Suppose (b) holds, that is, $s_1 = (\beta + \bar{\beta}z, z)$ and v is collinear with $(\bar{\beta}, 1)$ for some $\beta, z \in \mathbb{D}$.

$$\begin{aligned} \Phi_\omega(s_1) &= \frac{2\omega z - (\beta + \bar{\beta}z)}{2 - \omega(\beta + \bar{\beta}z)} \\ &= m_\omega(z), \end{aligned}$$

where

$$m_\omega(z) = c \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad c = \omega \frac{2 - \bar{\omega}\bar{\beta}}{2 - \omega\beta} \in \mathbb{T} \quad \text{and} \quad \alpha = \frac{\beta}{2\omega - \bar{\beta}} \in \mathbb{D}.$$

Now

$$\begin{aligned} D_{(\bar{\beta}, 1)} \Phi_\omega(s_1) &= \left[\bar{\beta} \frac{\partial}{\partial s^1} \frac{2\omega s^2 - s^1}{2 - \omega s^1} + \frac{\partial}{\partial s^2} \frac{2\omega s^2 - s^1}{2 - \omega s^1} \right]_{|s=(\beta + \bar{\beta}z, z)} \\ &= \left[\bar{\beta} \frac{-2(1 - \omega^2 s^2)}{(2 - \omega s^1)^2} + \frac{2\omega}{2 - \omega s^1} \right]_{|s=(\beta + \bar{\beta}z, z)} \\ &= \frac{4\omega(1 - \operatorname{Re}(\omega\beta))}{(2 - \omega(\beta + \bar{\beta}z))^2}. \end{aligned}$$

It is routine to check that

$$m'_\omega(z) = \frac{4\omega(1 - \operatorname{Re}(\omega\beta))}{(2 - \omega(\beta + \bar{\beta}z))^2},$$

and so

$$\Phi_\omega(\delta) = (m_\omega(z), m'_\omega(z)).$$

Therefore, for all $\omega \in \mathbb{T}$,

$$\rho_\delta(\omega) = |\Phi_\omega(\delta)|^2 = |(m_\omega(z), m'_\omega(z))|^2 = |m_\omega((z, 1))|^2.$$

Since the Poincaré metric on \mathbb{D} is invariant under $\text{Aut } \mathbb{D}$,

$$\rho_\delta(\omega) = |(z, 1)|^2 = \frac{1}{(1 - |z|^2)^2},$$

which is independent of $\omega \in \mathbb{T}$. Thus (b) implies that ρ_δ is constant on \mathbb{T} . Hence, for each $\omega \in \mathbb{T}$, Φ_ω solves $\text{Car}(\delta)$ in both cases (a) and (b).

Now we show that (1) \Rightarrow (2). Let $\delta = (s_1, v)$ be a nondegenerate infinitesimal datum in G and suppose that ρ_δ is constant on \mathbb{T} . Write

$$s_1 = (\beta + \bar{\beta}p, p)$$

for some $\beta, p \in \mathbb{D}$. Calculation shows that

$$|\Phi_\omega(\delta)| = \frac{1}{1 - |p|^2} \left| \frac{-v^1 + 2v^2\omega + (pv^1 - s_1^1 v^2)\omega^2}{\bar{\beta} - 2\omega + \beta\omega^2} \right|.$$

Thus the rational function

$$(A.7) \quad F(w) = \frac{-v^1 + 2v^2w + (pv^1 - s_1^1 v^2)w^2}{\bar{\beta} - 2w + \beta w^2}$$

has constant modulus, say $C > 0$, for $w \in \mathbb{T}$. Hence

$$(A.8) \quad \begin{aligned} C^2 &= F(w)\overline{F(w)} \\ &= F(w)\overline{F(1/\bar{w})} \end{aligned}$$

for all $w \in \mathbb{T}$. Since the right hand side of equation (A.8) is a rational function of w , the equation remains true for all $w \in \mathbb{C}$ for which $F(w)$ and $F(1/\bar{w})$ are finite.

First we consider the case that $\beta = 0$, when $s_1 = (0, p)$. Since $|F|$ is constant on \mathbb{T} we have

$$|-v^1 + 2v^2w + pv^1w^2| = \text{const}$$

for $w \in \mathbb{T}$. It follows that either $v^1 = 0$ or $v^2 = 0$ and $p = 0$. In the former case

$$\delta = ((0, p), (0, v^2))$$

and in the latter case

$$\delta = ((0, 0), (v^1, 0)),$$

and the statement of the theorem holds.

Now consider the case that $\beta \neq 0$. We claim that cancellation occurs in the fractional quadratic F . The zeros of the denominator $\bar{\beta} - 2w + \beta w^2$ in the right hand side of equation (A.7) are

$$w_\pm = \frac{1 \pm \sqrt{1 - |\beta|^2}}{\beta}.$$

Observe that $w_+\bar{w}_- = 1$, that is, w_+ and w_- are symmetric with respect to \mathbb{T} . Moreover, $|w_+| > 1$ and $|w_-| < 1$.

Suppose there is no cancellation in equation (A.7). Then the points w_\pm are both poles of F . Equation (A.8) implies that $\frac{1}{\bar{w}_+}$ and $\frac{1}{\bar{w}_-}$ are zeros of F , that is, w_\pm are zeros of the numerator of F , contrary to assumption.

If there is only one cancellation in equation (A.7), say at w_+ and not at w_- , then $F(w_+)$ is finite and w_- is a pole of F . Since, by equation (A.8),

$$C^2 = F(w_+)\overline{F(w_-)},$$

we have $F(w_+) = 0$.

There are only two cases to consider:

Case 1: $\beta \neq 0$ and there exists exactly one cancellation in equation (A.7), which is at w_+ (when $F(w_+) = 0$) or at w_- (when $F(w_-) = 0$). We shall show that in this case the datum δ is such that $s_1 = (2z, z^2)$ and v is collinear with $(1, z)$ for some $z \in \mathbb{D}$.

Case 2: $\beta \neq 0$ and there exist 2 cancellations in equation (A.7), at both w_+ and w_- . We shall show that in this case the datum δ is such that $s_1 = (\beta + \bar{\beta}z, z)$ and v is collinear with $(\bar{\beta}, 1)$ for some $\beta, z \in \mathbb{D}$.

Case 1. Let w_0 be one of the roots of equation

$$(A.9) \quad \bar{\beta} - 2w + \beta w^2 = 0.$$

On taking the complex conjugate of this equation we obtain the system:

$$(A.10) \quad \begin{cases} w_0^2 \beta + \bar{\beta} & = 2w_0 \\ \beta + w_0^2 \bar{\beta} & = 2\bar{w}_0, \end{cases}$$

which can be solved to give $\beta = \frac{2\bar{w}_0}{1+|w_0|^2}$.

If there exists exactly one cancellation in equation (A.7), which is at w_0 , then, by L'Hôpital's Rule,

$$(A.11) \quad F(w_0) = \frac{2v^2 + 2(pv^1 - s_1^1 v^2)w_0}{-2 + 2\beta w_0}$$

Therefore, the facts that there exists exactly one cancellation in equation (A.7) at w_0 and that $F(w_0) = 0$ yield the system

$$(A.12) \quad \begin{cases} -v^1 + 2v^2 w_0 + (pv^1 - s_1^1 v^2)w_0^2 & = 0 \\ 2v^2 + 2(pv^1 - s_1^1 v^2)w_0 & = 0, \end{cases}$$

which is equivalent to

$$(A.13) \quad \begin{cases} v^1 & = v^2 w_0 \\ (pv^1 - s_1^1 v^2)w_0 & = -v^2. \end{cases}$$

Since δ is nondegenerate $v \neq 0$. Hence $v^2 \neq 0$ and

$$(A.14) \quad \begin{cases} v^1 & = v^2 w_0 \\ (pw_0 - s_1^1)w_0 & = -1. \end{cases}$$

Recall that

$$s_1^1 = \beta + \bar{\beta}p \quad \text{and} \quad \beta = \frac{2\bar{w}_0}{1+|w_0|^2}.$$

Thus the equation $(pw_0 - s_1^1)w_0 = -1$ implies that $p = \frac{1}{w_0}$, and so $s_1^1 = \frac{2}{w_0}$. Hence the datum $\delta = \left(\left(\frac{2}{w_0}, \frac{1}{w_0^2} \right), \left(1, \frac{1}{w_0} \right) \right)$ for $\frac{1}{w_0} \in \mathbb{D}$.

Case 2: there exist 2 cancellations in equation (A.7), at both w_+ and w_- . Equation (A.8) implies that $\frac{1}{\bar{w}_+}$ and $\frac{1}{\bar{w}_-}$ are zeros of F , that is, w_{\pm} are zeros of the numerator of F . Therefore, there exists $\lambda \neq 0$ such that the numerator and the denominator of F are connected by the following equation

$$(A.15) \quad -v^1 + 2v^2 w + (pv^1 - s_1^1 v^2)w^2 = -\lambda(\bar{\beta} - 2w + \beta w^2).$$

Since $s_1^1 = \beta + \bar{\beta}p$, we have

$$-v^1 + 2v^2 w + (pv^1 - (\beta + \bar{\beta}p)v^2)w^2 = -\lambda\bar{\beta} + 2\lambda w - \beta\lambda w^2, \quad w \in \mathbb{C}.$$

Therefore, $v^1 = \lambda\bar{\beta}$, $v^2 = \lambda$, $pv^1 - (\beta + \bar{\beta}p)v^2 = -\beta\lambda$. Thus, $v^1 = \bar{\beta}v^2$, $pv^1 - \bar{\beta}pv^2 = 0$. The datum $\delta = ((\beta + \bar{\beta}p, p), v^2(\bar{\beta}, 1))$ for some $\beta, p \in \mathbb{D}$. \square

APPENDIX B

Types of geodesic: a crib and some cartoons

The five types of datum δ and their associated complex geodesics \mathcal{D}_δ play a central role in the paper. The types are defined in Definition 3.4 on page 9, and the geometric characterization of types in terms of the intersection of \mathcal{D}_δ^- with the royal variety \mathcal{R}^- is contained in Theorem 7.1 on page 37. We are grateful to a referee for a very good suggestion: to include a ‘crib sheet’ containing the essentials of the terminology, and also cartoons to give the reader a visual representation of the five types.

B.1. Crib sheet

Type of datum δ and geodesic D_δ	For how many $\omega \in \mathbb{T}$ is Φ_ω extremal?	How many points are in $\mathcal{D}_\delta^- \cap \mathcal{R}^-$?	Balanced?
Purely unbalanced	Unique ω_0 , $\frac{d^2 \rho_\delta}{dt^2} _{\omega_0} < 0$	One in \mathcal{R} , one in $\partial\mathcal{R}$	No
Exceptional	Unique ω_0 , $\frac{d^2 \rho_\delta}{dt^2} _{\omega_0} = 0$	One (double) ¹ in $\partial\mathcal{R}$	Yes
Purely balanced	Two	Two, both in $\partial\mathcal{R}$	Yes
Flat	All $\omega \in \mathbb{T}$ $C(s) = s^2$ solves $\text{Car}(\delta)$	One, in \mathcal{R}	No
Royal	All $\omega \in \mathbb{T}$ $C(s) = \frac{1}{2}s^1$ solves $\text{Car}(\delta)$	All of \mathcal{R}^-	Yes

¹ In the sense that the multiplicity of the royal point is 2 – see Definition 4.15.

Balanced geodesics are those which are parallel to \mathcal{R} , in the sense that they either coincide with or are disjoint from \mathcal{R} in G .

B.2. Cartoons

We draw the intersection with \mathbb{R}^2 of a representative of each type of geodesic. The set $G \cap \mathbb{R}^2$ in the (s, p) plane is the interior of the isosceles triangle with vertices $(\pm 2, 1)$ and $(0, -1)$, as in Figure 1.

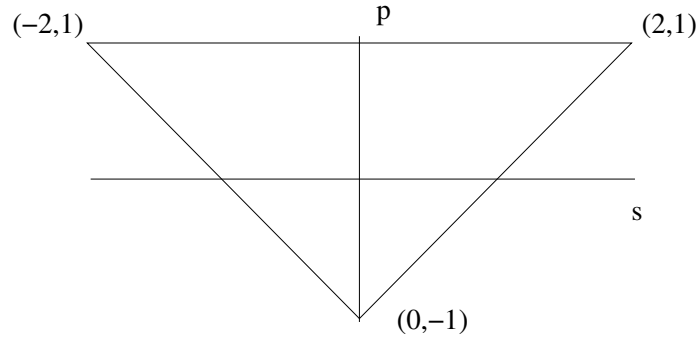


FIGURE 1. The real symmetrized bidisc $G \cap \mathbb{R}^2$

The **royal variety** is the locus $s^2 = 4p$, as in Figure 2.

Flat geodesics have the form $s = \beta + \bar{\beta}p$ for some $\beta \in \mathbb{D}$. This locus meets $G \cap \mathbb{R}^2$ if and only β is real, in which case its cartoon is as in Figure 3. A flat geodesic has a unique royal point, which is in \mathcal{R} .

Purely unbalanced geodesics, by Theorem 7.8, up to an automorphism of G , have the form $k_r(\mathbb{D})$ for some $r \in \mathbb{D}$, where

$$k_r(z) = \frac{1}{1-rz} (2(1-r)z, z(z-r))$$

for some $r \in (0, 1)$. The geodesic $k_r(\mathbb{D}^-)$ has royal points $(0, 0)$ in \mathcal{R} and $(2, 1)$ in $\partial\mathcal{R}$. Taking $r = \frac{1}{2}$ we obtain the curve

$$p = \frac{s(3s-2)}{2(s+2)}, \quad -\frac{2}{3} \leq s \leq 2,$$

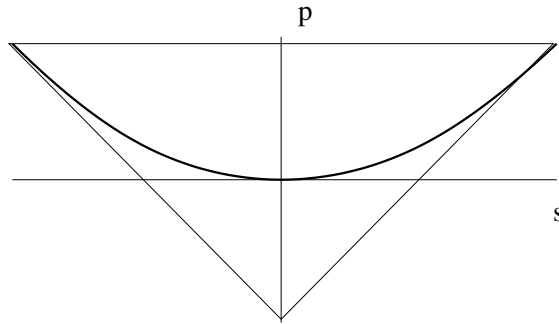


FIGURE 2. The royal variety

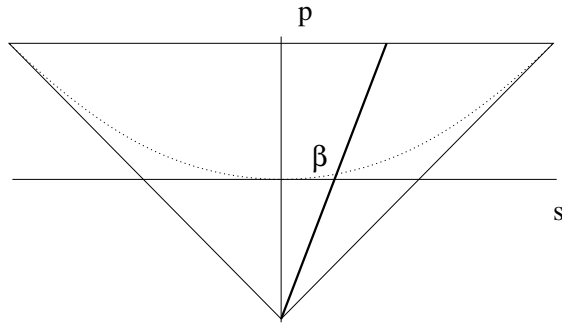


FIGURE 3. A flat geodesic

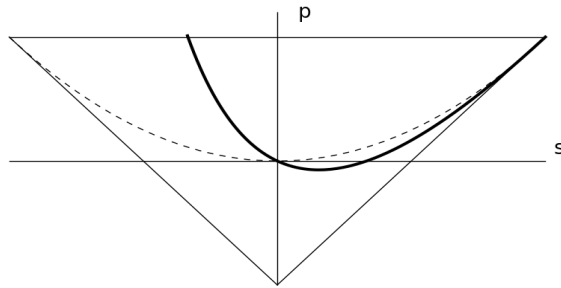


FIGURE 4. A purely unbalanced geodesic

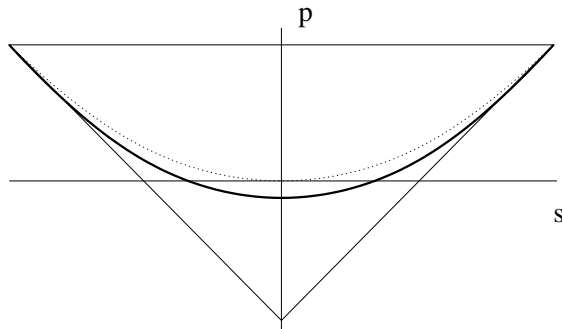


FIGURE 5. A purely balanced geodesic

which has graph as in Figure 4.

Purely balanced geodesics, up to automorphisms of G , are of the form $g_r(\mathbb{D})$, for some $r \in (0, 1)$, by Theorem 7.8, where

$$g_r(z) = (z + B_r(z), zB_r(z)).$$

The geodesic $g_r(\mathbb{D}^-)$ has the two royal points $(\pm 2, 1)$, both in $\partial\mathcal{R}$. In Figure 5 we have taken $r = \frac{1}{2}$.

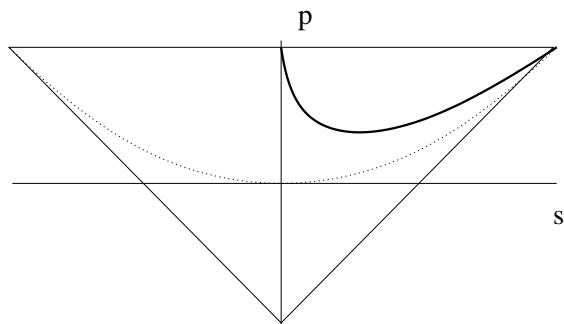


FIGURE 6. An exceptional geodesic

The remaining type of geodesic is the **exceptional type**. According to Theorem 7.8, up to automorphisms of G they have the form $h_r(\mathbb{D})$ for some $r > 0$, where

$$h_r(z) = (z + m_r(z), zm_r(z))$$

and

$$m_r(z) = \frac{(r-i)z+i}{r+i-iz}.$$

Any such geodesic $h_r(\mathbb{D}^-)$ has the unique royal point $(2, 1)$, and $h_r(z) \in \mathbb{R}^2$ for any z lying on the circle of centre $1 - ir$ and radius r . When r is chosen to be 1 the locus in Figure 6 results.

Bibliography

- [1] A. A. Abouhajar, M. C. White and N. J. Young, A Schwarz lemma for a domain related to mu-synthesis, *J. Geom. Anal.* **17** (2007) 717–750.
- [2] J. Agler and J. E. McCarthy. Norm preserving extensions of holomorphic functions from subvarieties of the bidisk. *Ann. of Math.*, 157(1):289–312, 2003.
- [3] J. Agler and J. E. McCarthy. Nevanlinna-Pick interpolation on the bidisk, *J. Reine Angew. Math.*, **506** (1999) 191-204.
- [4] J. Agler and N. J. Young, A model theory for Γ -contractions, *J. of Operator Theory*, **49** (2003) 45-60.
- [5] J. Agler and N. J. Young, The hyperbolic geometry of the symmetrised bidisc, *J. Geometric Analysis* **14** (2004) 375-403.
- [6] J. Agler and N. J. Young, The complex geodesics of the symmetrised bidisc, *Int. J. Math.* **17** (2006) 375-391.
- [7] J. Agler and N. J. Young, The magic functions and automorphisms of a domain, *Complex Analysis and Operator Theory* **2** (2008) 383-404.
- [8] J. Agler, Z. A. Lykova and N. J. Young, Extremal holomorphic maps and the symmetrised bidisc, *Proc. London Math. Soc.* **106**(4) (2013) 781-818.
- [9] J. Agler, Z. A. Lykova and N. J. Young, The complex geometry of a domain related to mu-synthesis, *J. Math. Analysis and its Applications* **422** (2015) 508-543.
- [10] J. Agler, Z. A. Lykova and N. J. Young, Algebraic and geometric aspects of rational Γ -inner functions, arXiv: 1502.04216, 22 pp.
- [11] L. V. Ahlfors, *Complex Analysis*, second edition, McGraw-Hill, New York, 1966.
- [12] L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, 1960.
- [13] H. Alexander, Extending bounded holomorphic functions from certain subvarieties of a polydisk, *Pacific J. Math.* **29** (1969) 458-490.
- [14] T. Andô, On a pair of commutative contractions, *Acta Sci. Math. (Szeged)* **24** (1963) 88-90.
- [15] R. B. Andrist, Lifting to the spectral ball with interpolation, *J. Math. Anal. Appl.* **435** (2016), no. 1, 315–320.
- [16] R. B. Andrist and F. Kutzschebauch, The fibred density property and the automorphism group of the spectral ball, arXiv:1501.07475v2 .
- [17] H. Bercovici, C. Foias, and A. Tannenbaum, Spectral variants of the Nevanlinna-Pick interpolation problem, commutant lifting theorem, *Signal processing, scattering and operator theory, and numerical methods*, Progr. Systems Control Theory, Vol. 5, Birkhäuser, Boston, 1990, pp. 23–45.
- [18] T. Bhattacharyya, S. Pal and S. Shyam Roy, Dilations of Γ -contractions by solving operator equations, *Advances in Mathematics* **230** (2012) 577-606.
- [19] T. Bhattacharyya, S. Pal and H. Sau, Holomorphic functions on the symmetrized bidisk - realization, interpolation and extension, *arXiv:1511.08962v4*, May 2016, 21 pp.
- [20] J. Bochnak, M. Coste and M. F. Roy, *Real Algebraic Geometry*, Springer Verlag, Berlin, 1998.
- [21] H. Cartan, *Séminaire Henri Cartan 1951/52*, W. A. Benjamin, New York, 1967.
- [22] J. B. Conway, *Functions of One Complex Variable II*, Springer-Verlag, New York, 1995.
- [23] C. Costara, The symmetrized bidisc and Lempert’s theorem, *Bull. Lond. Math. Soc.* **36** (2004) 656–662.
- [24] M. A. Dritschel, D. Estevez and D. Yakubovich, Traces of analytic uniform algebras on subvarieties and text conditions, arXiv:1505.01838 .
- [25] S. D. Fisher, *Function Theory on Planar Domains*, Wiley-Interscience, New York, 1983.

- [26] B. A. Fuks, *Special chapters in the theory of analytic functions of several complex variables*. Translated from the Russian by A. Jeffrey and N. Mugibayashi. Translations of Mathematical Monographs, **14** American Mathematical Society, Providence, R. I. 1965.
- [27] K. Guo, H. Huang and K. Wang, Retracts in polydisk and analytic varieties with the H^∞ -extension property, *J. Geom. Anal.* **18** (2008) 148-171.
- [28] L. F. Heath and T. J. Suffridge, Holomorphic retracts in complex n -space, *Illinois J. Math.* **25** (1981) 125-135.
- [29] G. M. Henkin, Continuation of bounded holomorphic functions from submanifolds in general position to strictly pseudoconvex domains, *Izv. Akad. Nauk SSSR Ser. Mat.*, **36** (1972) 540567.
- [30] L-K. Hua, Harmonic analysis of functions of several complex variables in the classical domains, Translated from the Russian, which was a translation of the Chinese original, by Leo Ebner and Adam Koranyi. Reprint of the 1963 edition. Translations of Mathematical Monographs **6**. American Mathematical Society, Providence, R.I., 1979.
- [31] M. Jarnicki and P. Pflug, On automorphisms of the symmetrised bidisc, *Arch. Math. (Basel)* **83** (2004) 264-266.
- [32] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, 2nd Extended Edition, De Gruyter, Berlin, 2013.
- [33] G. Knese, A refined Agler decomposition and geometric applications, *Indiana University Math. J.*, **60** (2011) 1831-1841.
- [34] S. Kobayashi, *Hyperbolic Complex Spaces*, Grundlehren der mathematischen Wissenschaften **318**, Springer Verlag 1998.
- [35] Ł. Kosiński and W. Zwonek, Nevanlinna-Pick problem and uniqueness of left inverses in convex domains, symmetrized bidisc and tetrablock. *J. Geom. Anal.* **26** (2016), no. 3, 1863–1890.
- [36] L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule, *Bull. Soc. Math. France*, 109:427–484, 1981.
- [37] J. von Neumann, Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, *Math Nachr.*, **4** (1951) 258-281.
- [38] N. Nikolov and P. Pflug, Kobayashi-Royden pseudometric vs. Lempert function, *Ann. Math. Pura Appl.* **190** (2008) 547-553.
- [39] N. Nikolov, P. J. Thomas and W. Zwonek, Discontinuity of the Lempert function and the Kobayashi metric of the spectral ball, *Integral Equations and Operator Theory* **61** (2008) 401-412.
- [40] P. L. Polyakov and G. M. Henkin, Integral formulas for solution of the $\bar{\partial}$ -equation and interpolation problems in analytic polyhedra, *Trudy. Mosk. Mat. Obs.* **53** (1990) 130170.
- [41] P. Pflug and W. Zwonek, Description of all complex geodesics in the symmetrized bidisc, *Bull. London Math. Soc.*, **37** (2005) 575–584.
- [42] W. Rudin, *Function theory in the unit ball of C^n* . Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
- [43] W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam 1969.
- [44] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*. Translated from the French and revised. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akademiai Kiad, Budapest 1970 .
- [45] J. L. Taylor, The analytic functional calculus for several commuting operators, *Acta Math.* **125** (1970) 1-38.
- [46] E. Vesentini, Complex geodesics, *Comp. Math.* **44** (1981) 375-394.
- [47] N. J. Young, Some analysable instances of μ -synthesis, *Mathematical methods in systems, optimization and control*, Editors: H. Dym, M. de Oliveira, M. Putinar, Operator Theory: Advances and Applications **222** 349–366, Birkhäuser, Basel, 2012.

Index

- B_α , 8
- G , 1
- $H^\infty(V)$, 83
- $H_{\text{sym}}^\infty(V)$, 77
- $\text{Car}(\delta)$, 8
- Γ , 17
- Γ -contraction, 84
- $\text{Kob}(\delta)$, 8
- $\Omega(U)$, 7
- Ω_n , 88
- Φ_ω , 4
- $\Phi_\omega(s)$, 9
- $\text{Aut } U$, 8
- \mathbb{C} , 1
- \mathcal{D}_δ , 15
- \mathcal{F}_β , 4, 15
- $|\delta|_{\text{car}}$, 8
- \mathbb{D} , 1
- id_X , 7
- $|\delta|_{\text{kob}}$, 8
- \mathcal{E} , 88
- \mathcal{P} , 88
- ρ_δ , 9
- \tilde{m} , 29, 94
- $b\Gamma$, 17, 91
- f^V , 29
- f_β , 15
- $f_{\mathcal{F}}$, 59
- $f_{\mathcal{R}}$, 59
- g_r , 41
- h_r , 41
- k_r , 41
- m_r , 41
- automorphisms of G , 94
- balanced disc, 46
- Blaschke factor, 8
- Carathéodory extremal problem, 8
 - universal set for, 9
- cartoons, 100
- circular region, 66
- complex C-geodesic, 13
 - type of, 4
- contact, 3, 14
- contraction, 83
- crib sheet, 99
- datum, 3, 7
 - balanced, 9
 - degenerate, 3
 - discrete, 7
 - exceptional, 9, 37, 99
 - flat, 9, 37, 99
 - infinitesimal, 7
 - modulus of, 8
 - nondegenerate, 7
 - purely balanced, 9, 37, 99
 - purely unbalanced, 9, 37, 99
 - royal, 9, 37, 99
 - type of, 9
 - unbalanced, 9
- distinguished boundary, 17
- domain, nondegenerate, 66
- extension property
 - A -, 77
 - Herglotz-preserving, 59
 - norm-preserving, 1
 - symmetric, 77
- flat C-geodesic, 16
- function
 - Γ -inner, 17
 - analytic on a subset of a domain, 1
 - rational Γ -inner, 17
 - symmetric, 77
- geodesics
 - equivalence of, 37
 - exceptional, 40, 41
 - flat, 40, 41
 - purely balanced, 40, 41
 - purely unbalanced, 40, 41
 - royal, 40, 41
 - types of, 40, 99

- Herglotz function, 59
- Kobayashi extremal problem, 8
- operator, 83
- pentablock, 88
- pentachotomy, 10
- retract, 1, 19
- retraction, 1, 19
- royal node, 17
 - multiplicity of, 17
- royal point, 17
- semialgebraic set, 66
- set
 - (G, A) -von Neumann, 84
 - A -spectral, 83
 - A -von Neumann, 84
 - algebraic, 49
 - anomalous, 49
 - spectral, 83
 - symmetric in \mathbb{D}^2 , 77
- spectral ball, 88
- spectral Nevanlinna-Pick problem, 88
- subordinate, 83
- tetrablock, 88
- theorem
 - (G, A) -von Neumann sets are sets with
 - A -extension property relative to G , 84
 - 1.3, 2
 - algebraic sets in G which are
 - $(G, H^\infty(V))$ -von Neumann sets, 85
 - characterization of geodesics in terms of
 - solutions of $\text{Car}(\delta)$, 41
 - complex geodesics are varieties, 22
 - geometric classification of geodesics, 37
 - Lempert's, 13
 - main, 2
 - retracts are complex geodesics in G , 19
 - symmetric algebraic sets in \mathbb{D}^2 which are
 - $H_{\text{sym}}^\infty(V)$ -von Neumann sets, 84
 - symmetric algebraic sets in \mathbb{D}^2 with the
 - symmetric extension property, 79
 - uniqueness of complex geodesics for each
 - datum, 15
- transposition map, 77
- trichotomy, 94