This is a repository copy of *Pricing and hedging game options in currency models with proportional transaction costs*.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/id/eprint/104030/

Version: Accepted Version

Article:

PRICING AND HEDGING GAME OPTIONS IN CURRENCY MODELS WITH PROPORTIONAL TRANSACTION COSTS*

ALET ROUX

Department of Mathematics, University of York, Heslington, YO10 5DD, United Kingdom
alet.roux@york.ac.uk

The pricing, hedging, optimal exercise and optimal cancellation of game or Israeli options are considered in a multi-currency model with proportional transaction costs. Efficient constructions for optimal hedging, cancellation and exercise strategies are presented, together with numerical examples, as well as probabilistic dual representations for the bid and ask price of a game option.

Keywords: game options; game contingent claims; Israeli options; proportional transaction costs; currency model; superhedging; optimal exercise.

1. Introduction

The study of game options (also called Israeli options) date back to the seminal work of Kifer (2000); the recent survey paper by Kifer (2013a) provides a complete chronology and literature review. In addition to being of interest as derivative securities in their own right, game options have also played an important role in the study of other derivatives, for example callable options (e.g. Kühn & Kyprianou 2007) and convertible bonds (e.g. Kallsen & Kühn 2005, Bielecki *et al.* 2008, Wang & Jin 2009).

A game option is a contract between a writer (the *seller*) and the holder (the *buyer*) whereby a pre-specified payoff is delivered by the seller to the buyer at the earliest of the exercise time (chosen by the buyer) and the cancellation time (chosen by the seller). If the game option is cancelled before or at the same time as being exercised, then the seller also pays a cancellation penalty to the buyer. A game option is thus essentially an American option with the additional provision that the seller can cancel the option at any time before expiry, thus forcing early exercise at a price (the penalty). In practice, this feature tends to reduce costs for both

---

the seller and the buyer, which makes game options an attractive alternative to American options.

It has been well observed that arbitrage pricing of European and American options in incomplete friction-free models and models with proportional transaction costs result in a range of arbitrage-free prices, bounded from below by the *bid price* and from above by the *ask price* (see e.g. Föllmer & Schied 2002, Bensaid *et al.* 1992, Chalasani & Jha 2001, Roux & Zastawniak 2015). The same holds true for game options (Kallsen & Kühn 2005, Kifer 2013b).

The pricing and hedging of game options in the presence of proportional transaction costs also share a number of other important properties with their European and American counterparts. (The properties for European and American options mentioned below were all established by Roux & Zastawniak (2015) in a similar technical setting to the present paper.) Firstly, similar to European options, the hedging of game options is symmetric in the sense that the hedging problem for the buyer is exactly the same as the hedging problem for the seller (of a different game option with related payoff). Kifer (2013b) observed this property in a two-asset model.

Kifer (2013b) also showed that the probabilistic dual representations of the bid and ask prices of game options contain so-called randomised stopping times, a feature shared with the ask price of an American option (for which it was first observed by Chalasani & Jha 2001). Randomised (or mixed) stopping times have been studied by Baxter & Chacon (1977) and many others, primarily as an aid to show the existence and properties of optimal ordinary stopping times. Randomised stopping times can be thought of as convex combinations of ordinary stopping times in a well-defined sense. The reason for the appearance of randomised stopping times in the probabilistic dual representations of the bid and ask prices is that, in the presence of transaction costs, the most expensive exercise (cancellation) strategy for the seller (buyer) of a game option to hedge against is not necessarily the same as the exercise (cancellation) strategy that is most attractive to the buyer (seller). As a result, it generally costs the seller (buyer) more to hedge against all exercise (cancellation) strategies than against the best exercise (cancellation) strategy for the buyer (seller). It turns out that the seller (buyer) must in effect be protected against a certain randomised exercise (cancellation) time.

Furthermore, similar to a long American option (i.e. the buyer's case), the pricing and hedging problems for both the buyer and seller of game option are inherently non-convex. Thus ideas beyond convex duality are needed to study these problems. Nevertheless, the link between game options and short American options (i.e. the seller's case, a convex problem) means that convex duality methods still have an important role to play in establishing the probabilistic dual representations.

In this paper we consider the pricing and hedging of game options in the numéraire-free discrete-time model of foreign exchange markets introduced by Kabanov (1999), where proportional transaction costs are modelled as bid-ask spreads between currencies. This model has been well studied by Kabanov & Stricker (2001),

Kabanov *et al.* (2002), Schachermayer (2004) and others (see also Kabanov & Safarian 2009).

The main aims of our work are twofold. Firstly, we present constructive algorithms for computing optimal exercise and cancellation times together with optimal hedging strategies for both the buyer and seller of a game option in this model. The algorithmic constructions in this paper are closely related to previously developed algorithms for the pricing and hedging of European and American options under proportional transaction costs (see e.g. Löhne & Rudloff 2014, Roux & Zastawniak 2009, 2015). These existing constructions yield efficient numerical algorithms; in particular they are known to price path-independent options in polynomial time in recombinant models (which typically have exponentially-sized state spaces). Numerical examples that illustrate the constructions are provided. Secondly, we establish probabilistic dual representations for the bid and ask prices of game options. In both these contributions we extend the recent results of Kifer (2013b) for game options from two-asset to multi-asset models. Our proofs are rigorous, thus closing two gaps in the arguments of Kifer (2013b); see Remark 3.3, the comments below Proposition 3.2 and Example 5.2 for further details.

The methods used in this paper come from convex analysis and dynamic programming, and in particular we will use recent results from Roux & Zastawniak (2015) for an American option with random expiration date. The restriction to finite state space is motivated by the desire to produce computationally efficient algorithms for pricing and hedging. The restriction to discrete time is justified by a recent negative result by Dolinsky (2013) that the super-replication price of a game option in continuous time under proportional transaction costs is the initial value of a trivial buy-and-hold superhedging strategy.

The structure of this paper is as follows. Section 2 specifies the currency model with proportional transaction costs, and reviews various notions concerning randomised stopping times and approximate martingales. The main algorithms for pricing and hedging together with theoretical results for the seller's and buyer's position are presented in Section 3, with the proofs of all results deferred to Section 4. Section 5 concludes the paper with three numerical examples.

2. Preliminaries

2.1. *Proportional transaction costs*

The numéraire free currency model of Kabanov (1999) has discrete trading dates $t = 0, \ldots, T$ and is based on a finite probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t=0}^T$. The model contains d currencies (or *assets*), and at any time t, one unit of currency $j = 1, \ldots, d$ may be obtained by exchanging $\pi_t^{ij} > 0$ units of currency $i = 1, \ldots, d$. We assume that $\pi_t^{ii} = 1$ for $i = 1, \ldots, d$, i.e. every currency may be freely exchanged for itself.

Assume that the filtration $(\mathcal{F}_t)_{t=0}^T$ is generated by $(\pi_t^{ij})_{t=0}^T$ for $i, j = 1, \ldots, d$, and assume for simplicity that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, that $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ and that $P(\omega) > 0$ for

all $\omega \in \Omega$. Write $\mathcal{L}_\tau$ for the family of $\mathcal{F}_\tau$-measurable $\mathbb{R}^d$-valued random variables for every stopping time τ, and write $\mathcal{L}_\tau^+$ for the family of non-negative random variables in $\mathcal{L}_\tau$.

Let Ω_t be the set of atoms of $\mathcal{F}_t$ for $t = 0, \ldots, T$. The elements of Ω_t are called the *nodes* of the model at time t. A node $\nu \in \Omega_{t+1}$ is called a *successor* to a node $\mu \in \Omega_t$ if $\nu \subset \mu$. The collection of successors of μ is denoted succ μ. We shall implicitly and uniquely identify random variables f in $\mathcal{L}_t$ with functions $\mu \mapsto f^\mu \in \mathbb{R}^d$ on Ω_t, and likewise every set $A \in \mathcal{L}_t$ that we will consider will be implicitly and uniquely defined by a set-valued mapping $\mu \mapsto A^\mu \subseteq \mathbb{R}^d$ on Ω_t such that

$$A = \{f \in \mathcal{L}_t : f^\mu \in A^\mu \text{ for all } \mu \in \Omega_t\}.$$

A portfolio $x = (x^1, \ldots, x^d) \in \mathcal{L}_\tau$ is called *solvent* at a stopping time τ if it can be exchanged into a portfolio in $\mathcal{L}_\tau^+$ without any additional investment, i.e. if there exist non-negative $\mathcal{F}_\tau$-measurable random variables β^{ij} for $i, j = 1, \ldots, d$ such that

$$x^i + \sum_{j=1}^d \beta^{ji} - \sum_{j=1}^d \beta^{ij} \pi_\tau^{ij} \geq 0 \text{ for all } i = 1, \ldots, d.$$

Write $\mathcal{K}_\tau$ for the family of solvent portfolios at time τ; then the *solvency cone* $\mathcal{K}_\tau$ is the convex cone generated by the canonical basis $e^1, \ldots, e^d$ of $\mathbb{R}^d$ and the vectors $\pi_\tau^{ij} e^i - e^j$ for $i, j = 1, \ldots, d$. Observe that $\mathcal{K}_\tau$ is a polyhedral cone, hence closed.

A *self-financing trading strategy* $y = (y_t)_{t=0}^T$ is an $\mathbb{R}^d$-valued predictable process with initial endowment $y_0 \in \mathcal{L}_0$ satisfying $y_t - y_{t+1} \in \mathcal{K}_t$ for all $t = 0, \ldots, T-1$. Denote the family of self-financing trading strategies by Φ.

A self-financing trading strategy $y = (y_t) \in \Phi$ is called an *arbitrage opportunity* if $y_0 = 0$ and there exists some $x \in \mathcal{L}_T^+ \setminus \{0\}$ such that $y_T - x \in \mathcal{K}_T$. This definition of arbitrage is consistent with (though formally different to) that of Schachermayer (2004) and Kabanov & Stricker (2001) (who called it weak arbitrage).

For any non-empty convex cone $A \subseteq \mathbb{R}^d$, write A^* for the *positive polar* of A, i.e.

$$A^* := \{x \in \mathbb{R}^d : x \cdot y \geq 0 \text{ for all } y \in A\}.$$

Theorem 2.1 (Kabanov & Stricker (2001)). *The model is free of arbitrage if and only if there exists a probability measure $\mathbb{P}$ equivalent to P and an $\mathbb{R}^d$-valued $\mathbb{P}$-martingale $S = (S_t)_{t=0}^T$ such that*

$$S_t \in \mathcal{K}_t^* \setminus \{0\} \text{ for } t = 0, \ldots, T.$$

Any pair $(\mathbb{P}, S)$ satisfying the conditions of Theorem 2.1 is called an *equivalent martingale pair*. Denote the family of equivalent martingale pairs by $\mathcal{P}$; then $\mathcal{P} \neq \emptyset$ in the absence of arbitrage.

Remark 2.1. Theorem 2.1 and the other results in this paper can equivalently be formulated in terms of consistent pricing processes $(Z_t)_{t=0}^T$ where

$$Z_t = S_t \mathbb{E}_P \left(\frac{d\mathbb{P}}{dP} \middle| \mathcal{F}_t \right) \text{ for all } t = 0, \ldots, T.$$

Schachermayer (2004 pp. 24–25) provides further details on this equivalence.

Assume for the remainder of this paper that the model contains no arbitrage.

2.2. *Randomised stopping times*

Definition 2.1 (Randomised stopping time). A *randomised (or mixed) stopping time* $\chi = (\chi_t)_{t=0}^T$ is an adapted nonnegative process satisfying

$$\sum_{t=0}^T \chi_t = 1.$$

Denote the set of randomised stopping times by $\mathcal{X}$, and the set of (ordinary) stopping times with values in $0, \ldots, T$ by $\mathcal{T}$. Every stopping time $\tau \in \mathcal{T}$ corresponds to a randomised stopping time $\chi^\tau = (\chi_t^\tau)_{t=0}^T$ defined as

$$\chi_t^\tau := \mathbf{1}_{\{\tau = t\}} \text{ for } t = 0, \ldots, T,$$

where $\mathbf{1}$ is the indicator function on Ω. The set $\mathcal{X}$ is the convex hull of $\{\chi^\tau : \tau \in \mathcal{T}\}$ and so $\mathcal{X}$ can be thought of as the linear relaxation of the set of ordinary stopping times in this sense.

Fix any process $A = (A_t)_{t=0}^T$ and randomised stopping time $\chi \in \mathcal{X}$. Define the processes $\chi^* = (\chi_t^*)_{t=0}^T$ and $A^{\chi^*} = (A_t^{\chi^*})_{t=0}^T$ by

$$\chi_t^* := \sum_{s=t}^T \chi_s, \qquad\qquad A_t^{\chi^*} := \sum_{s=t}^T \chi_s A_s$$

for $t = 0, \ldots, T$. For convenience also define $\chi_{T+1}^* := 0$ and $A_{T+1}^{\chi^*} := 0$. Observe that χ^* is a predictable process since

$$\chi_t^* = 1 - \sum_{s=0}^{t-1} \chi_s \text{ for } t = 1, \ldots, T.$$

The *value of A at* χ is defined as

$$A_\chi := A_0^{\chi^*} = \sum_{t=0}^T \chi_t A_t.$$

Observe that if $\chi = \chi^\tau$ for some $\tau \in \mathcal{T}$, then

$$\chi_t^{\tau^*} = \mathbf{1}_{\{\tau \geq t\}}, \qquad\qquad A_t^{\chi^{\tau^*}} = \sum_{s=t}^T \mathbf{1}_{\{\tau=s\}} A_s = A_\tau \mathbf{1}_{\{\tau \geq t\}}$$

for $t = 0, \ldots, T$, and in particular $A_{\chi^\tau} = A_\tau$.

Definition 2.2 (Approximate martingale pair). Fix any $\chi \in \mathcal{X}$. A pair $(\mathbb{P}, S)$ consisting of a probability measure $\mathbb{P}$ and an adapted $\mathbb{R}^d$-valued process S is called a χ-*approximate martingale pair* if

$$S_t \in \mathcal{K}_t^* \setminus \{0\}, \qquad\qquad \mathbb{E}_{\mathbb{P}}(S_{t+1}^{\chi_*} | \mathcal{F}_t) \in \mathcal{K}_t^*$$

for all $t = 0, \dots, T$. If $\mathbb{P}$ is in addition equivalent to P, then $(\mathbb{P}, S)$ is called a
χ-approximate equivalent martingale pair.

Denote the family of χ-approximate equivalent pairs $(\mathbb{P}, S)$ by $\mathcal{P}(\chi)$ and the set
of χ-approximate pairs by $\bar{\mathcal{P}}(\chi)$. For any $\chi \in \mathcal{X}$ and $i = 1, \dots, d$ define

$$\mathcal{P}^i(\chi) := \{(\mathbb{P}, S) \in \mathcal{P}(\chi) : S_t^i = 1 \text{ for } t = 0, \dots, T\},$$
$$\bar{\mathcal{P}}^i(\chi) := \{(\mathbb{P}, S) \in \bar{\mathcal{P}}(\chi) : S_t^i = 1 \text{ for } t = 0, \dots, T\}.$$

Since $\mathcal{P} \subseteq \mathcal{P}(\chi) \subseteq \bar{\mathcal{P}}(\chi)$ and $\mathcal{K}_t^*$ is a cone for all $t = 0, \dots, T$, the no-arbitrage
assumption implies that $\mathcal{P}^i(\chi)$ and $\bar{\mathcal{P}}^i(\chi)$ are non-empty.

Definition 2.3 (Truncated stopping time). Fix any $\chi \in \mathcal{X}$, $\sigma \in \mathcal{T}$. The *truncated randomised stopping time* $\chi \wedge \sigma = ((\chi \wedge \sigma)_t)_{t=0}^T$ is defined as

$$(\chi \wedge \sigma)_t := \chi_t \mathbf{1}_{\{t < \sigma\}} + \chi_t^* \mathbf{1}_{\{t = \sigma\}} \text{ for } t = 0, \dots, T.$$

The process $\chi \wedge \sigma$ is adapted and nonnegative, and moreover

$$\sum_{t=0}^T (\chi \wedge \sigma)_t = \sum_{t=0}^{\sigma-1} \chi_t + \chi_\sigma^* = 1,$$

so it is indeed a randomised stopping time. If $\chi = \chi^\tau$ for some $\tau \in \mathcal{T}$, then clearly
$(\chi \wedge \sigma)_t = \mathbf{1}_{\{\sigma \wedge \tau = t\}}$ for all $t = 0, \dots, T$, and so $\chi^\tau \wedge \sigma = \chi^{\tau \wedge \sigma}$. Denote the set of
randomised stopping times truncated at $\sigma \in \mathcal{T}$ by

$$\mathcal{X} \wedge \sigma := \{\chi \wedge \sigma : \chi \in \mathcal{X}\}.$$

3. Main Results and Discussion

In this section we formally define what we mean by a game option, and present the
constructions and main results.

Definition 3.1 (Game option). A *game option* is a derivative security that is
exercised at a stopping time $\tau \in \mathcal{T}$ chosen by the buyer and *cancelled* at a stopping
time $\sigma \in \mathcal{T}$ chosen by the seller. At time $\sigma \wedge \tau$ the buyer receives the payoff $Q_{\sigma\tau}$
from the seller, where

$$Q_{st} \equiv Q_{st}^{Y,X,X'} := Y_t \mathbf{1}_{\{s>t\}} + X_s \mathbf{1}_{\{s<t\}} + X_s' \mathbf{1}_{\{s=t\}} \tag{3.1}$$

for all $s, t = 0, \dots, T$, and $Y = (Y_t)_{t=0}^T$, $X = (X_t)_{t=0}^T$ and $X' = (X_t')_{t=0}^T$ are adapted
$\mathbb{R}^d$-valued processes such that

$$X_t - X_t' \in \mathcal{K}_t, \qquad\qquad X_t' - Y_t \in \mathcal{K}_t \tag{3.2}$$

for all $t = 0, \dots, T$.

In the event that the buyer exercises before the option is cancelled, i.e. on
$\{\tau < \sigma\}$, the buyer receives the payoff Y_τ from the seller at his exercise time τ.
If the seller cancels the option before it is exercised, i.e. on $\{\sigma < \tau\}$, the seller is

required to deliver the payoff X_σ to the buyer at the cancellation time σ, which consists of Y_σ and a penalty

$$X_\sigma - Y_\sigma = (X_\sigma - X'_\sigma) + (X'_\sigma - Y_\sigma) \in \mathcal{K}_\sigma.$$

In the event that the option is exercised and cancelled simultaneously, i.e. on $\{\sigma = \tau\}$, the seller pays X'_σ to the buyer, consisting of Y_σ and a penalty $X'_\sigma - Y_\sigma \in \mathcal{K}_\sigma$. The assumptions (3.2) mean that, at any time t, the portfolio X_t payable on cancellation is at least as attractive to the buyer as the portfolio X'_t payable in the event of simultaneous cancellation and exercise, which in turn is at least as attractive as the portfolio Y_t payable on exercise. It is therefore clear from Definition 3.1 that a game option is essentially an American option with payoff process Y with the additional feature that it may be cancelled by the seller at any time (upon payment of a cancellation penalty).

Remark 3.1. Definition 3.1 is slightly more general than the usual approach followed in the literature (see e.g. Kifer 2000, 2013b), where the standard assumption is that no penalty is paid if cancellation and exercise takes place simultaneously (i.e. $X'_t = Y_t$ for all t) and that no penalty is paid on maturity (i.e. $X_T = X'_T = Y_T$). The motivation for the generalization in the present paper is that it enables elegant exploitation of the symmetry between the seller's and buyer's hedging problems; see Proposition 3.2. Nevertheless, from a practical point of view, the pricing and hedging problems depend on X only through $(X_t)_{t<T}$ and on X' only through X'_T; see the key Constructions 3.1 and 3.3 as well as Lemma 4.1.

Remark 3.2. The property (3.2) imposes an ordering on the payoffs in the various scenarios for the seller and buyer, but there is no requirement in Definition 3.1 that any of the payoffs X_t, Y_t and X'_t should be solvent portfolios. The absence of such a solvency requirement makes it easy to adapt to the buyer's case, where in practice the payoffs tend to be "negative" in that they correspond to portfolios received rather than delivered. Typical cases are illustrated in Examples 5.1 and 5.2.

3.1. *Pricing and hedging for the seller*

A hedging strategy for the seller of a game option (Y, X, X') comprises a cancellation time σ and a self-financing trading strategy y that allows the seller to deliver the payoff without risk of loss.

Definition 3.2 (Hedging strategy for the seller). A *hedging strategy for the seller* is a pair $(\sigma, y) \in \mathcal{T} \times \Phi$ satisfying

$$y_{\sigma \wedge \tau} - Q_{\sigma\tau} \in \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}. \tag{3.3}$$

There exists at least one hedging strategy for the seller. Indeed, fixing $i = 1, \ldots, d$ and defining

$$m := \max \left\{ \sum_{j=1}^{d} \pi_t^{ij}(\omega) \max\{|Y_t^j(\omega)|, |X_t^j(\omega)|, |X_t'^j(\omega)|\} : t = 0, \ldots, T, \omega \in \Omega \right\},$$

8   *Alet Roux*

the (possibly expensive) buy-and-hold strategy $y = (y_t)_{t=0}^T$ with $y_t = me^i$ for $t = 0, \ldots, T$ hedges the game option for the seller with any choice of the cancellation time $\sigma \in \mathcal{T}$.

Consider now the following construction.

Construction 3.1. Construct adapted set-valued mappings $(\mathcal{Y}_t^a)_{t=0}^T$, $(\mathcal{X}_t^a)_{t=0}^T$, $(\mathcal{U}_t^a)_{t=0}^T$, $(\mathcal{V}_t^a)_{t=0}^T$, $(\mathcal{W}_t^a)_{t=0}^T$, $(\mathcal{Z}_t^a)_{t=0}^T$ as follows. For all $t = 0, \ldots, T$ let

$$\mathcal{Y}_t^a := Y_t + \mathcal{K}_t, \qquad \mathcal{X}_t^a := \begin{cases} X_T' + \mathcal{K}_T & \text{if } t = T, \\ X_t + \mathcal{K}_t & \text{if } t < T. \end{cases} \tag{3.4}$$

Define

$$\mathcal{W}_T^a := \mathcal{V}_T^a := \mathcal{L}_T, \qquad \mathcal{Z}_T^a := \mathcal{X}_T^a.$$

For $t = T - 1, \ldots, 0$ define by backward iteration

$$\mathcal{W}_t^a := \mathcal{Z}_{t+1}^a \cap \mathcal{L}_t, \tag{3.5}$$
$$\mathcal{V}_t^a := \mathcal{W}_t^a + \mathcal{K}_t, \tag{3.6}$$
$$\mathcal{Z}_t^a := (\mathcal{V}_t^a \cap \mathcal{Y}_t^a) \cup \mathcal{X}_t^a. \tag{3.7}$$

For each $t = 0, \ldots, T$, the set $\mathcal{Y}_t^a$ is the collection of portfolios in $\mathcal{L}_t$ that allows the seller to settle the option in the event that the buyer exercises at time t and the seller does not cancel the option at time t. The set $\mathcal{X}_t^a$ is the collection of portfolios that allows the seller to settle the option upon cancellation at time t, irrespective of whether the buyer exercises at time t or not. The property (3.2) gives that

$$\mathcal{X}_t^a = (X_t + \mathcal{K}_t) \cap (X_t' + \mathcal{K}_t) \text{ for } t < T.$$

The relation $\mathcal{X}_T^a = X_T' + \mathcal{K}_T$ follows from the fact that any cancellation at the final time T must be matched by simultaneous exercise.

The following result shows that $\mathcal{Z}_0^a$ is the set of initial endowments that allow the seller to hedge the game option.

Proposition 3.1. *We have*

$$\mathcal{Z}_0^a = \{y_0 : (\sigma, y) \text{ hedges } (Y, X, X') \text{ for the seller}\}. \tag{3.8}$$

It is demonstrated in the proof of Proposition 3.1, which is deferred to Section 4, that for each $t < T$ the sets $\mathcal{V}_t^a$, $\mathcal{W}_t^a$ and $\mathcal{Z}_t^a$ have natural interpretations that are important to the seller of the option. The set $\mathcal{W}_t^a$ consists of those portfolios at time t that allow the seller to hedge the option in the future (at time $t + 1$ or later), and $\mathcal{V}_t^a$ consists of those portfolios that may be rebalanced at time t into a portfolio in $\mathcal{W}_t^a$. The set $\mathcal{Z}_t^a$ consists of all portfolios that allow the seller to settle the option at time t or any time in the future without risk of loss.

Construction 3.1 is essentially an iteration (backwards in time) over the nodes of the price tree generated by the exchange rates; note in particular that (3.5) could

equivalently be written as

$$\mathcal{W}_t^{a\mu} := \bigcap_{\nu \in \text{succ}\,\mu} \mathcal{Z}_{t+1}^{a\nu} \text{ for } \mu \in \Omega_t. \tag{3.5$'$}$$

This property makes the construction particularly efficient for recombinant models, for which the number of nodes grow only polynomially with the number of steps in the model, despite the state space growing exponentially.

The sets $\mathcal{X}_t^a$ and $\mathcal{Y}_t^a$ in Construction 3.1 are clearly polyhedral and non-empty for all $t = 0, \ldots, T$, as are $\mathcal{V}_T^a$, $\mathcal{W}_T^a$ and $\mathcal{Z}_T^a$. The operations in Construction 3.1 are direct addition of polyhedral cones in (3.4) and (3.6), intersection in (3.5) and (3.7), and union in (3.7). The appearance of the union in (3.7) means that the sets $\mathcal{V}_t^a$, $\mathcal{W}_t^a$ and $\mathcal{Z}_t^a$ may be non-convex for some $t < T$. However, it is clear that these sets can be written as the finite union of non-empty (closed) polyhedra, and are therefore closed. In particular the closedness of $\mathcal{Z}_0^a$ is essential to Theorem 3.2 below.

The *ask price* of the game option in terms of any currency is defined as the infimal initial endowment in that currency that would allow the seller to hedge the game option without risk.

Definition 3.3 (Ask price). The *ask price* or *seller's price* or *upper hedging price* of a game option (Y, X, X') at time 0 in terms of currency $i = 1, \ldots, d$ is

$$\pi_i^a(Y, X, X') := \inf\{z \in \mathbb{R} : (\sigma, y) \in \mathcal{T} \times \Phi \text{ with } y_0 = ze^i$$
$$\text{hedges } (Y, X, X') \text{ for the seller}\}.$$

The existence of the buy-and-hold strategy for the seller means that the ask price is well defined. We now present a dual representation for the ask price in terms of randomised stopping times and approximate martingale pairs.

Theorem 3.1. *The ask price of a game option* (Y, X, X') *in terms of currency* $i = 1, \ldots, d$ *is*

$$\pi_i^a(Y, X, X') = \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \sup_{(\mathbb{P}, S) \in \mathcal{P}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((Q_{\sigma \cdot} \cdot S_{\sigma \wedge \cdot})_\chi)$$

$$= \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P}, S) \in \mathcal{P}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((Q_{\sigma \cdot} \cdot S_{\sigma \wedge \cdot})_\chi),$$

where $Q_{\sigma \cdot} \cdot S_{\sigma \wedge \cdot}$ *denotes the process* $(Q_{\sigma t} \cdot S_{\sigma \wedge t})_{t=0}^T$, *in other words,*

$$(Q_{\sigma \cdot} \cdot S_{\sigma \wedge \cdot})_\chi = \sum_{t=0}^{\sigma - 1} \chi_t Y_t \cdot S_t + \chi_{\sigma+1}^* X_\sigma \cdot S_\sigma + \chi_\sigma X_\sigma' \cdot S_\sigma.$$

The proof of Theorem 3.1 appears in Section 4.

Remark 3.3. Kifer (2013b Theorem 3.1) obtained a similar dual representation for a game option (Y, X, Y) in a two-currency model, that does not feature the truncated stopping time $\chi \wedge \sigma$ and stopped process $S_{\sigma \wedge \cdot}$, but rather χ and S. Example 5.2 demonstrates that these dual representations are not equivalent in general and that the representation in Theorem 3.1 is indeed the correct one.

10   *Alet Roux*

The reason for the difference between the two representations can be explained intuitively in the following way. The proof of Theorem 3.1 hinges on the fact that a pair (σ, y) hedges the game option for the seller if and only if y hedges an American option with payoff process H_σ (defined in (4.1)) and random expiration date σ for the seller. By contrast, the proof of Theorem 3.1 of Kifer (2013b) claims that (σ, y) hedges the game option (Y, X, Y) for the seller if and only if y hedges an American option with payoff process $(Q_{\sigma t}^{Y,X,Y})_{t=0}^T$ and expiration date T (rather than σ) for the seller. This claim does not hold true in general, because hedging such an American option would require the seller to be in a position to deliver $Q_{\sigma t} = X_\sigma$ on $\{\sigma > t\}$ at any time t, in other words, after the option has already been cancelled. As evidenced in Example 5.2, the non-equivalence is most easily noticed when transaction costs are large at time σ and/or X_σ is a non-solvent portfolio.

Returning to the problem of computing the ask price of a game option, the following result is a direct consequence of Proposition 3.1 and the closedness of $\mathcal{Z}_0^a$.

Theorem 3.2. *We have*

$$\pi_i^a(Y, X, X') = \min\{x \in \mathbb{R} : xe^i \in \mathcal{Z}_0^a\}. \tag{3.9}$$

Moreover, there exists a hedging strategy $(\hat{\sigma}, \hat{y})$ for the seller such that $\hat{y}_0 = \pi_i^a(Y, X, X')e^i$.

A hedging strategy (σ, y) for the seller is called *optimal* if it satisfies the properties in Theorem 3.2. A procedure for constructing such a strategy can be extracted from the proof of Proposition 3.1.

Construction 3.2. Construct an optimal strategy $(\hat{\sigma}, \hat{y})$ for the seller as follows. Let

$$\hat{y}_0 := \pi_i^a(Y, X, X')e^i.$$

For each $t = 0, \ldots, T-1$ and $\mu \in \Omega_t$, if $\hat{y}_t^\mu \in \mathcal{Z}_t^{a\mu} \setminus \mathcal{X}_t^{a\mu}$, then choose any

$$\hat{y}_{t+1}^\mu \in \mathcal{W}_t^{a\mu} \cap [\hat{y}_t^\mu - \mathcal{K}_t^\mu], \tag{3.10}$$

otherwise put $\hat{y}_{t+1}^\mu := \hat{y}_t^\mu$. Also define

$$\hat{\sigma} := \min\left\{t : \hat{y}_t \in \mathcal{X}_t^a\right\}.$$

The optimal strategy for the seller is not unique in general; this is reflected in the choice (3.10). In practice the seller might use secondary considerations, such as a preference for holding certain currencies over others, or optimality of a secondary hedging criterion, to guide the construction of a suitable optimal hedging strategy.

Two toy examples illustrating Constructions 3.1 and 3.2 and Theorem 3.1, as well as a third example with a more realistic flavour can be found in Section 5.

3.2. *Pricing and hedging for the buyer*

Consider now the hedging, pricing and optimal exercise problem for the buyer of a game option (Y, X, X').

Definition 3.4 (Hedging strategy for the buyer). A *hedging strategy for the buyer* is a pair $(\tau, y) \in \mathcal{T} \times \Phi$ satisfying

$$y_{\sigma \wedge \tau} + Q_{\sigma \tau} \in \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \sigma \in \mathcal{T}, \tag{3.11}$$

where the payoff process Q is defined in (3.1).

Observe from (3.1) that

$$Q_{\sigma \tau} = Q_{\sigma \tau}^{Y, X, X'} = -Q_{\tau \sigma}^{-X, -Y, -X'} \text{for all } \sigma, \tau \in \mathcal{T}, \tag{3.12}$$

and moreover from (3.2) that for all $t = 0, \ldots, T$

$$-Y_t - (-X_t') = X_t' - Y_t \in \mathcal{K}_t,$$
$$-X_t' - (-X_t) = X_t - X_t' \in \mathcal{K}_t.$$

Thus if (Y, X, X') is the payoff of a game option, then so is $(-X, -Y, -X')$, and (3.11) is equivalent to

$$y_{\tau \wedge \sigma} - Q_{\tau \sigma}^{-X, -Y, -X'} \in \mathcal{K}_{\tau \wedge \sigma} \text{ for all } \sigma \in \mathcal{T}.$$

Thus we arrive at the following result.

Proposition 3.2. *A pair* $(\tau, y) \in \mathcal{T} \times \Phi$ *hedges the game option* (Y, X, X') *for the buyer if and only if* (τ, y) *hedges the game option* $(-X, -Y, -X')$ *for the seller.*

This symmetry means that the results and constructions developed in the previous section for the seller's case can also be applied to the hedging and pricing problem for the buyer, thus substantiating the claim by Kifer (2013b pp. 679–80). In particular, Construction 3.1 can be applied directly, provided that $\mathcal{Y}_t^a$ and $\mathcal{X}_t^a$ in (3.4) is redefined to take into account the fact that the option is now $(-X, -Y, -X')$ rather than (Y, X, X'). The resulting construction reads as follows.

Construction 3.3. For all t let

$$\mathcal{Y}_t^b := -X_t + \mathcal{K}_t, \qquad\qquad \mathcal{X}_t^b := \begin{cases} -X_T' + \mathcal{K}_T & \text{if } t = T, \\ -Y_t + \mathcal{K}_t & \text{if } t < T. \end{cases}$$

Define

$$\mathcal{W}_T^b := \mathcal{V}_T^b := \mathcal{L}_T, \qquad\qquad \mathcal{Z}_T^b := \mathcal{X}_T^b.$$

For $t = T - 1, \ldots, 0$ let

$$\mathcal{W}_t^b := \mathcal{Z}_{t+1}^b \cap \mathcal{L}_t, \qquad \mathcal{V}_t^b := \mathcal{W}_t^b + \mathcal{K}_t^b, \qquad \mathcal{Z}_t^b := (\mathcal{V}_t^b \cap \mathcal{Y}_t^b) \cup \mathcal{X}_t^b.$$

It follows directly from Theorem 3.1 that $\mathcal{Z}_0^b$ is the set of initial endowments that allow the buyer to hedge the option, i.e.

$$\mathcal{Z}_0^b = \{z : (\tau, y) \in \mathcal{T} \times \Phi \text{ with } y_0 = ze^i \text{ hedges } (Y, X, X') \text{ for the buyer}\}.$$

The bid price of a game option in any currency is the largest amount that the buyer can raise in that currency at time 0 by using the payoff of the option as a guarantee.

Definition 3.5 (Bid price). The *bid price* or *lower hedging price* or *buyer's price* of a game option (Y, X, X') in currency $i = 1, \ldots, d$ is defined as

$$\pi_i^b(Y, X, X') := \sup\{-z : (\tau, y) \in \mathcal{T} \times \Phi \text{ with } y_0 = ze^i$$
$$\text{superhedges } (Y, X, X') \text{ for the buyer}\}.$$

Proposition 3.2 and Construction 3.3 give that

$$\pi_i^b(Y, X, X') = -\pi_i^a(-X, -Y, -X') = -\inf\left\{z : ze^i \in \mathcal{Z}_0^b\right\}. \qquad (3.13)$$

A hedging strategy (τ, y) for the buyer is called *optimal* if $y_0 = -\pi_i^b(Y, X, X')e^i$. Optimal hedging strategies can be generated by rewriting Construction 3.2 as follows.

Construction 3.4. Construct an optimal strategy $(\check{\tau}, \check{y})$ for the buyer as follows. Let

$$\check{y}_0 := -\pi_i^b(Y, X, X')e^i.$$

For each $t = 0, \ldots, T - 1$ and $\mu \in \Omega_t$, if $\check{y}_t^\mu \in \mathcal{Z}_t^{b\mu} \setminus \mathcal{X}_t^{b\mu}$, then choose any

$$\check{y}_{t+1}^\mu \in \mathcal{W}_t^{b\mu} \cap [\check{y}_t^\mu - \mathcal{K}_t^\mu], \qquad (3.14)$$

otherwise put $\check{y}_{t+1}^\mu := \check{y}_t^\mu$. Also define

$$\check{\tau} := \min\left\{t : \check{y}_t \in \mathcal{X}_t^b\right\}.$$

A toy example illustrating Constructions 3.3 and 3.4 can be found in Section 5. It demonstrates that the optimal cancellation time $\hat{\sigma}$ for the seller and the optimal exercise time $\check{\tau}$ for the buyer are not the same in general, and these times may also be different from the stopping time $\hat{\sigma} \wedge \check{\tau}$ at which the option payoff is paid.

Finally, combining Theorem 3.1 with (3.12) and (3.13) immediately gives the following dual representation for the bid price.

Theorem 3.3. *We have*

$$\pi_i^b(Y, X, X') = \max_{\tau \in \mathcal{T}} \min_{\chi \in \mathcal{X}} \inf_{(\mathbb{P}, S) \in \mathcal{P}^i(\chi \wedge \tau)} \mathbb{E}_\mathbb{P}((Q_{\cdot\tau} \cdot S_{\cdot \wedge \tau})_\chi)$$
$$= \max_{\tau \in \mathcal{T}} \min_{\chi \in \mathcal{X}} \min_{(\mathbb{P}, S) \in \bar{\mathcal{P}}^i(\chi \wedge \tau)} \mathbb{E}_\mathbb{P}((Q_{\cdot\tau} \cdot S_{\cdot \wedge \tau})_\chi),$$

where $Q_{\cdot\tau} \cdot S_{\cdot \wedge \tau}$ denotes the process $(Q_{s\tau}^{Y,X,X'} \cdot S_{s \wedge \tau})_{s=0}^T$, i.e.

$$(Q_{\cdot\tau} \cdot S_{\cdot \wedge \tau})_\chi = \sum_{s=0}^{\tau-1} \chi_s X_s \cdot S_s + \chi_{\tau+1}^* Y_\tau \cdot S_\tau + \chi_\tau X_\tau' \cdot S_\tau.$$

The representation in Theorem 3.3 is different from the representation by Kifer (2013b Theorem 3.1) for a game option (Y, X, X') in a two-currency model, for reasons already discussed in the context of Theorem 3.1; see Remark 3.3.

4. Proofs

Proof. [Proposition 3.1] Fix any $z \in \mathcal{Z}_0^a$. We claim that there exists a hedging strategy (σ, y) for the seller with $y_0 = z$. To this end, we construct $y = (y_t)_{t=0}^T$ together with a non-decreasing sequence $(\sigma_t)_{t=0}^T$ of stopping times. Define

$$y_0 := z, \qquad \sigma_0 := \begin{cases} 0 & \text{if } z \in \mathcal{X}_0^a, \\ 1 & \text{if } z \in \mathcal{Z}_0^a \setminus \mathcal{X}_0^a. \end{cases}$$

If $\sigma_0 = 0$, let $y_1 := y_0$. If $\sigma_0 = 1$, then

$$y_0 \in \mathcal{Z}_0^a \setminus \mathcal{X}_0^a \subseteq \mathcal{V}_0^a \cap \mathcal{Y}_0^a \subseteq \mathcal{W}_0^a + \mathcal{K}_0,$$

so that there exists some $y_1 \in \mathcal{W}_0^a$ such that $y_0 - y_1 \in \mathcal{K}_0$ and $y_1 \in \mathcal{Z}_1^a$.

Suppose by induction that for some $t > 0$ we have constructed $y_0, \ldots, y_t$ and non-decreasing $\sigma_0, \ldots, \sigma_{t-1}$ such that for $s = 0, \ldots, t-1$ we have y_{s+1} being $\mathcal{F}_s$-measurable, $\sigma_s \leq s+1$, $y_s - y_{s+1} \in \mathcal{K}_s$, $y_{\sigma_s} \in \mathcal{X}_{\sigma_s}^a$ on $\{\sigma_s \leq s\}$ and $y_u \in \mathcal{Z}_u^a \setminus \mathcal{X}_u^a$ on $\{\sigma_s > u\}$ for all $u = 0, \ldots, s$. Define

$$\sigma_t := \sigma_{t-1}\mathbf{1}_{\{\sigma_{t-1} < t\}} + t\mathbf{1}_{\{\sigma_{t-1}=t\} \cap \{y_t \in \mathcal{X}_t^a\}} + (t+1)\mathbf{1}_{\{\sigma_{t-1}=t\} \cap \{y_t \in \mathcal{Z}_t^a \setminus \mathcal{X}_t^a\}}.$$

On the set

$$\{\sigma_t > t\} = \{\sigma_t = t+1\} = \{\sigma_{t-1} = t\} \cap \{y_t \in \mathcal{Z}_t^a \setminus \mathcal{X}_t^a\}$$

we have

$$y_t \in \mathcal{Z}_t^a \setminus \mathcal{X}_t^a \subseteq \mathcal{V}_t^a \cap \mathcal{Y}_t^a \subseteq \mathcal{V}_t^a = \mathcal{W}_t^a + \mathcal{K}_t,$$

so there exists an $\mathcal{F}_t$-measurable random variable $x \in \mathcal{Z}_{t+1}^a$ such that $y_t - x \in \mathcal{K}_t$ on this set. Now define the $\mathcal{F}_t$-measurable random variable

$$y_{t+1} := x\mathbf{1}_{\{\sigma_t = t+1\}} + y_t\mathbf{1}_{\{\sigma_t \leq t\}}.$$

Since $0 \in \mathcal{K}_t$ we have $y_t - y_{t+1} \in \mathcal{K}_t$. Moreover,

$$y_{\sigma_t} = y_{\sigma_t}\mathbf{1}_{\{\sigma_t < t\}} + y_t\mathbf{1}_{\{\sigma_t = t\}} + y_{t+1}\mathbf{1}_{\{\sigma_t = t+1\}}$$
$$= y_{\sigma_{t-1}}\mathbf{1}_{\{\sigma_t < t\}} + y_t\mathbf{1}_{\{\sigma_{t-1}=t\} \cap \{y_t \in \mathcal{X}_t^a\}} + y_{t+1}\mathbf{1}_{\{\sigma_t = t+1\}}$$

so $y_{\sigma_t} \in \mathcal{X}_{\sigma_t}^a$ on the set

$$\{\sigma_t \leq t\} = \{\sigma_{t-1} < t\} \cup [\{\sigma_{t-1} = t\} \cap \{y_t \in \mathcal{X}_t^a\}].$$

This concludes the inductive step.

Let $\sigma := \sigma_T$; then the pair $(\sigma, y) \in \mathcal{T} \times \Phi$ satisfies

$$y_\sigma \in \mathcal{X}_\sigma^a, \qquad y_t \in \mathcal{Z}_t^a \setminus \mathcal{X}_t^a \text{ on } \{t < \sigma\} \text{ for } t = 0, \ldots, T-1.$$

Fix any stopping time τ. On $\{\tau < \sigma\}$ we have

$$y_\tau \in \mathcal{Z}_\tau^a \setminus \mathcal{X}_\tau^a \subseteq \mathcal{V}_\tau^a \cap \mathcal{Y}_\tau \subseteq \mathcal{Y}_\tau = Y_\tau + \mathcal{K}_\tau.$$

14 *Alet Roux*

On the set $\{\tau = \sigma\}$ we have $y_\sigma \in \mathcal{X}_\sigma^a \subseteq X_\sigma' + \mathcal{K}_\sigma$. On the set $\{\tau > \sigma\}$ we have $\sigma < \tau \leq T$ and $y_\sigma \in \mathcal{X}_\sigma^a = X_\sigma + \mathcal{K}_\sigma$. Thus $y_{\sigma \wedge \tau} - Q_{\sigma\tau} \in \mathcal{K}_{\sigma \wedge \tau}$, from which it follows that (σ, y) hedges the game option for the seller.

Conversely, suppose that (σ, y) hedges the game option for the seller. We show by backward induction that $y_t \in \mathcal{Z}_t^a$ on $\{t \leq \sigma\}$ and $y_t \in \mathcal{V}_t^a \cap \mathcal{Y}_t^a$ on $\{t < \sigma\}$ for all $t = 0, \ldots, T$, from which it can be deduced that $z = y_0 \in \mathcal{Z}_0^a$, which completes the proof. At time T we have $\{T = \sigma\} = \{T \leq \sigma\}$, so clearly $y_T \in X_T' + \mathcal{K}_T = \mathcal{X}_T^a = \mathcal{Z}_T^a$ on $\{T \leq \sigma\}$.

For any $t < T$, suppose that $y_{t+1} \in \mathcal{Z}_{t+1}^a$ on $\{t + 1 \leq \sigma\} = \{t < \sigma\}$. This means that $y_{t+1} \in \mathcal{W}_t^a$ on $\{t < \sigma\}$ as $y_{t+1} \in \mathcal{L}_t$. Moreover $y_t - y_{t+1} \in \mathcal{K}_t$ implies that $y_t \in \mathcal{V}_t^a$ on $\{t < \sigma\}$, and therefore

$$y_t \in \mathcal{V}_t^a \cap [Q_{\sigma t} + \mathcal{K}_t] = \mathcal{V}_t^a \cap [Y_t + \mathcal{K}_t] = \mathcal{V}_t^a \cap \mathcal{Y}_t^a \subseteq \mathcal{Z}_t^a \text{ on } \{t < \sigma\}.$$

On the set $\{t = \sigma\} = \{t \leq \sigma\} \setminus \{t < \sigma\}$ we have

$$y_t \in Q_{tT} + \mathcal{K}_t = X_t + \mathcal{K}_t = \mathcal{X}_t^a \subseteq \mathcal{Z}_t^a.$$

This concludes the induction. $\qquad\square$

The next result will play an important role in the proof of Theorem 3.1. Define the auxiliary process

$$H_{st} \equiv H_{st}^{(Y,X,X')} := Y_t \mathbf{1}_{\{s>t\}} + X_s \mathbf{1}_{\{s=t<T\}} + X_s' \mathbf{1}_{\{s=t=T\}} \qquad (4.1)$$

for all $s, t = 0, \ldots, T$. Observe that the process $H_\sigma = (H_{\sigma t})_{t=0}^T$ is adapted for any $\sigma \in \mathcal{T}$, and that $H_{\sigma t} = Q_{\sigma t} = Y_t$ on $\{\sigma > t\}$ and $H_{\sigma t} = 0$ on $\{\sigma < t\}$ for all $t = 0, \ldots, T$.

The payoff $H_{\sigma t}$ can be interpreted as the payoff that a seller with pre-selected cancellation time σ needs to be prepared to deliver at time t if it is known that the buyer will not exercise at the same time as when the option is cancelled (except at time $t = T$); this is effectively the worst case scenario for such a seller because of (3.2).

Lemma 4.1. *A pair $(\sigma, y) \in \mathcal{T} \times \Phi$ hedges the game option for the seller if and only if*

$$y_{\sigma \wedge \tau} - H_{\sigma\tau} \in \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}. \qquad (4.2)$$

Proof. Throughout the proof we shall make frequent and implicit use of the fact that $\mathcal{K}_t$ is a pointed cone for all t, and in particular the property $0 \in \mathcal{K}_t$.

Fix any $(\sigma, y) \in \mathcal{T} \times \Phi$ satisfying (4.2). In view of (4.1), this implies that

$$(y_{\sigma \wedge \tau} - Y_{\sigma \wedge \tau}) \mathbf{1}_{\{\sigma > \tau\}} \in \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}, \qquad (4.3)$$

$$(y_\sigma - X_\sigma) \mathbf{1}_{\{\sigma = \tau < T\}} \in \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}, \qquad (4.4)$$

$$(y_\sigma - X_\sigma') \mathbf{1}_{\{\sigma = \tau = T\}} \in \mathcal{K}_{\sigma \wedge \tau}. \text{ for all } \tau \in \mathcal{T}, \qquad (4.5)$$

Substituting $\tau = 0, \ldots, T-1$ into (4.4) and $\tau = T$ into (4.5) gives

$$(y_\sigma - X_\sigma)\mathbf{1}_{\{\sigma < T\}} \in \mathcal{K}_\sigma, \tag{4.6}$$

$$(y_\sigma - X'_\sigma)\mathbf{1}_{\{\sigma = T\}} \in \mathcal{K}_\sigma. \tag{4.7}$$

Property (3.2) together with (4.6)–(4.7) then leads to $y_\sigma - X'_\sigma \in \mathcal{K}_\sigma$, whence

$$\begin{aligned}
(y_{\sigma \wedge \tau} - X'_{\sigma \wedge \tau})\mathbf{1}_{\{\sigma = \tau\}} &= (y_\sigma - X'_\sigma)\mathbf{1}_{\{\sigma = \tau\}} \\
&\in \mathbf{1}_{\{\sigma = \tau\}}\mathcal{K}_\sigma = \mathbf{1}_{\{\sigma = \tau\}}\mathcal{K}_{\sigma \wedge \tau} \subseteq \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}. \tag{4.8}
\end{aligned}$$

It also follows from (4.6) that

$$\begin{aligned}
(y_{\sigma \wedge \tau} - X_{\sigma \wedge \tau})\mathbf{1}_{\{\sigma < \tau\}} &= (y_\sigma - X_\sigma)\mathbf{1}_{\{\sigma < \tau\}} = (y_\sigma - X_\sigma)\mathbf{1}_{\{\sigma < T\}}\mathbf{1}_{\{\sigma < \tau\}} \\
&\in \mathbf{1}_{\{\sigma < \tau\}}\mathcal{K}_\sigma = \mathbf{1}_{\{\sigma < \tau\}}\mathcal{K}_{\sigma \wedge \tau} \subseteq \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}. \tag{4.9}
\end{aligned}$$

Properties (4.3), (4.8) and (4.9) lead to (3.3), and so (σ, y) hedges the game option for the seller.

Suppose conversely that $(\sigma, y) \in \mathcal{T} \times \Phi$ hedges the game option for the seller. Property (3.3) gives (4.3) and (upon choosing $\tau = T$)

$$y_\sigma - X_\sigma \mathbf{1}_{\{\sigma < T\}} - X'_\sigma \mathbf{1}_{\{\sigma = T\}} \in \mathcal{K}_\sigma,$$

from which it follows that

$$\begin{aligned}
(y_{\sigma \wedge \tau} - X_\sigma)\mathbf{1}_{\{\sigma = \tau < T\}} &+ (y_{\sigma \wedge \tau} - X'_\sigma)\mathbf{1}_{\{\sigma = \tau = T\}} \\
&= y_{\sigma \wedge \tau}\mathbf{1}_{\{\sigma = \tau\}} - X_\sigma \mathbf{1}_{\{\sigma = \tau < T\}} - X'_\sigma \mathbf{1}_{\{\sigma = \tau = T\}} \\
&= \left(y_{\sigma \wedge \tau} - X_\sigma \mathbf{1}_{\{\sigma < T\}} - X'_\sigma \mathbf{1}_{\{\sigma = T\}} \right)\mathbf{1}_{\{\sigma = \tau\}} \\
&= \left(y_\sigma - X_\sigma \mathbf{1}_{\{\sigma < T\}} - X'_\sigma \mathbf{1}_{\{\sigma = T\}} \right)\mathbf{1}_{\{\sigma = \tau\}} \\
&\in \mathbf{1}_{\{\sigma = \tau\}}\mathcal{K}_\sigma = \mathbf{1}_{\{\sigma = \tau\}}\mathcal{K}_{\sigma \wedge \tau} \subseteq \mathcal{K}_{\sigma \wedge \tau} \text{ for all } \tau \in \mathcal{T}. \tag{4.10}
\end{aligned}$$

Properties (4.3) and (4.10) together give (4.2), which completes the proof. □

Proof. [Theorem 3.1] Lemma 4.1 shows that, for $\sigma \in \mathcal{T}$ given, the pair $(\sigma, y) \in \mathcal{T} \times \Phi$ hedges the game option (Y, X, X') for the seller if and only if (4.2) holds, equivalently

$$y_\tau - H_{\sigma\tau} \in \mathcal{K}_\tau \text{ for all } \tau \in \mathcal{T} \text{ such that } \tau \leq \sigma.$$

Definition 3.3 and the finiteness of $\mathcal{T}$ then give that

$$\begin{aligned}
\pi_i^a&(Y, X, X') \\
&= \min_{\sigma \in \mathcal{T}} \inf\{z \in \mathbb{R} : (\sigma, y) \text{ hedges } (Y, X, X') \text{ for the seller and } y_0 = ze^i\} \\
&= \min_{\sigma \in \mathcal{T}} p_i^a(\sigma),
\end{aligned}$$

where

$$p_i^a(\sigma) := \inf\{z \in \mathbb{R} : y \in \Phi \text{ such that}$$
$$y_0 = ze^i, y_\tau - H_{\sigma\tau} \in \mathcal{K}_\tau \text{ for all } \tau \in \mathcal{T}, \tau \leq \sigma\}. \tag{4.11}$$

This means that any pair (σ, y) hedges the game option (Y, X, X') for the seller if and only if, in the terminology of Roux & Zastawniak (2015), the strategy y superhedges an option with payoff process $H_\sigma = (H_{\sigma t})_{t=0}^T$ that can be exercised by the buyer at any stopping time τ satisfying

$$\{\tau = t\} \subseteq \mathcal{E}_t := \{t \leq \sigma\} \text{ for all } t = 0, \ldots, T$$

for the seller. Intuitively, this is an American option with (random) expiration date σ. The quantity $p_i^a(\sigma)$ in (4.11) is the ask price of such an option in asset i, and Roux & Zastawniak (2015 Theorem 3) established that

$$p_i^a(\sigma) = \max_{\chi \in \mathcal{X}^{\mathcal{E}}} \sup_{(\mathbb{P},S) \in \mathcal{P}^i(\chi)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_\chi) = \max_{\chi \in \mathcal{X}^{\mathcal{E}}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^i(\chi)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_\chi)$$

where $H_\sigma \cdot S = (H_{\sigma t} \cdot S_t)_{t=0}^T$ and

$$\mathcal{X}^{\mathcal{E}} := \{\chi \in \mathcal{X} : \{\chi_t > 0\} \subseteq \mathcal{E}_t \text{ for all } t = 0, \ldots, T\} = \mathcal{X} \wedge \sigma.$$

It then follows that

$$p_i^a(\sigma) = \max_{\chi \in \mathcal{X}} \sup_{(\mathbb{P},S) \in \mathcal{P}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_{\chi \wedge \sigma}) = \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_{\chi \wedge \sigma}), \tag{4.12}$$

so that

$$\pi_i^a(Y, X, X') = \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \sup_{(\mathbb{P},S) \in \mathcal{P}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_{\chi \wedge \sigma})$$

$$= \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \mathcal{P}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_{\chi \wedge \sigma}). \tag{4.13}$$

Fix now any $\sigma \in \mathcal{T}$, $\chi \in \mathcal{X}$ and $(\mathbb{P}, S) \in \bar{\mathcal{P}}(\chi \wedge \sigma)$ and note that

$$(H_\sigma \cdot S)_{\chi \wedge \sigma}$$
$$= (Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_\chi + (\chi_\sigma^* \mathbf{1}_{\{\sigma < T\}} - \chi_{\sigma+1}^*) X_\sigma \cdot S_\sigma + (\chi_\sigma^* \mathbf{1}_{\{\sigma = T\}} - \chi_\sigma) X_\sigma' \cdot S_\sigma$$
$$= (Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_\chi + (\chi_\sigma \mathbf{1}_{\{\sigma < T\}} - \chi_{\sigma+1}^* \mathbf{1}_{\{\sigma = T\}}) X_\sigma \cdot S_\sigma - \chi_\sigma \mathbf{1}_{\{\sigma < T\}} X_\sigma' \cdot S_\sigma$$
$$= (Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_\chi + \chi_\sigma \mathbf{1}_{\{\sigma < T\}} (X_\sigma - X_\sigma') \cdot S_\sigma. \tag{4.14}$$

This follows from the properties of χ^*: in particular $\chi_{T+1}^* = 0$, $\chi_T^* = \chi_T$ and $\chi_\sigma^* = \chi_{\sigma+1}^* + \chi_\sigma$. Since $X_\sigma - X_\sigma' \in \mathcal{K}_\sigma$ by (3.2) and $S_\sigma \in \mathcal{K}_\sigma^*$, we immediately have

$$(H_\sigma \cdot S)_{\chi \wedge \sigma} \geq (Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_\chi. \tag{4.15}$$

Define the stopping time $\chi' = (\chi_t') \in \mathcal{X}$ by

$$\chi_t' := \chi_t \mathbf{1}_{\{t < \sigma\}} + \chi_\sigma^* \mathbf{1}_{\{t = T\}} \text{ for } t = 0, \ldots, T;$$

then $\chi \wedge \sigma = \chi' \wedge \sigma$ and so $(\mathbb{P}, S) \in \bar{\mathcal{P}}(\chi' \wedge \sigma)$. Moreover, since $\chi_\sigma' \mathbf{1}_{\{\sigma < T\}} = 0$ it follows from (4.14) that

$$(H_\sigma \cdot S)_{\chi \wedge \sigma} = (H_\sigma \cdot S)_{\chi' \wedge \sigma} = (Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_{\chi'}. \tag{4.16}$$

Combining (4.15) and (4.16) then gives

$$\max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((H_\sigma \cdot S)_{\chi \wedge \sigma}) = \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^i(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_\chi),$$

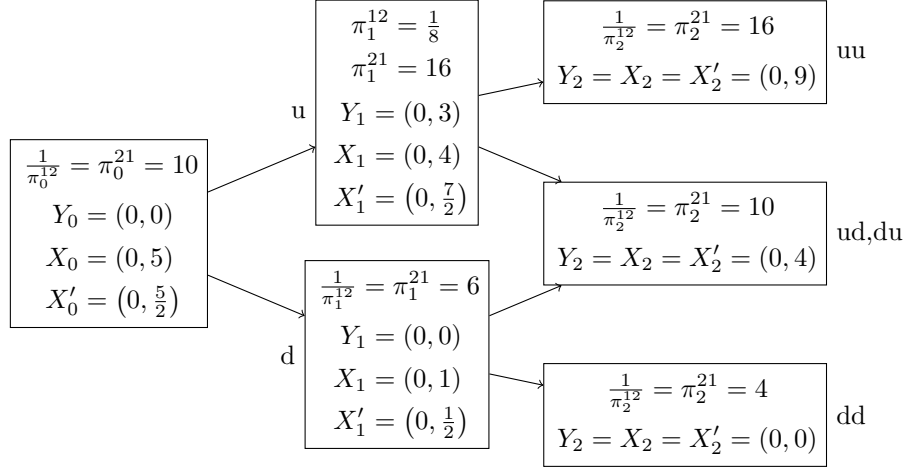and the result follows from (4.13). $\qquad \square$

Fig. 1. Game option in binary two-step two-currency model, Example 5.1

5. Numerical Examples

Three numerical examples are presented in this section. The first is a toy example to illustrate the constructions in Section 3. The second illustrates Theorem 3.1 and serves as a minimal counterexample to Theorem 3.1 of Kifer (2013b). The final example has a more realistic flavour.

Example 5.1. A game option (Y, X, X') in a binary two-step two-currency model is presented in Figure 1. The model is recombinant and has transaction costs only at node u at time 1, and the option is path-independent and has no cancellation penalties at time 2.

Let us use Construction 3.1 to find the set of hedging endowments for the seller. Clearly

$$\mathcal{Z}_2^{auu} = \left\{ \left(x^1, x^2 \right) \in \mathbb{R}^2 : 16x^1 + x^2 \geq 9 \right\},$$
$$\mathcal{Z}_2^{aud} = \left\{ \left(x^1, x^2 \right) \in \mathbb{R}^2 : 10x^1 + x^2 \geq 4 \right\},$$
$$\mathcal{Z}_2^{add} = \left\{ \left(x^1, x^2 \right) \in \mathbb{R}^2 : 4x^1 + x^2 \geq 0 \right\}$$

at time $t = 2$.

For time $t = 1$, consider the node u. We obtain $\mathcal{W}_1^{au}$ from (3.5') and $\mathcal{V}_1^{au}$ from (3.6); both procedures are shown graphically in Figure 2. Observe that the magnitude of the transaction costs at this node means that $\mathcal{W}_1^{au} + \mathcal{K}_1^u = \mathcal{W}_1^{au}$, whence $\mathcal{V}_1^{au} = \mathcal{W}_1^{au}$. The next step is to compute $\mathcal{Z}_1^{au}$ from (3.7). As shown in Figure 2, this can be done by finding the intersection of $\mathcal{V}_1^{au}$ and $\mathcal{Y}_1^{au}$ first and then taking the union with $\mathcal{X}_1^{au}$. The non-convexity of $\mathcal{Z}_1^{au}$ is due to the magnitude of the transaction costs and the shape of the payoff at this node. Similar considerations at the
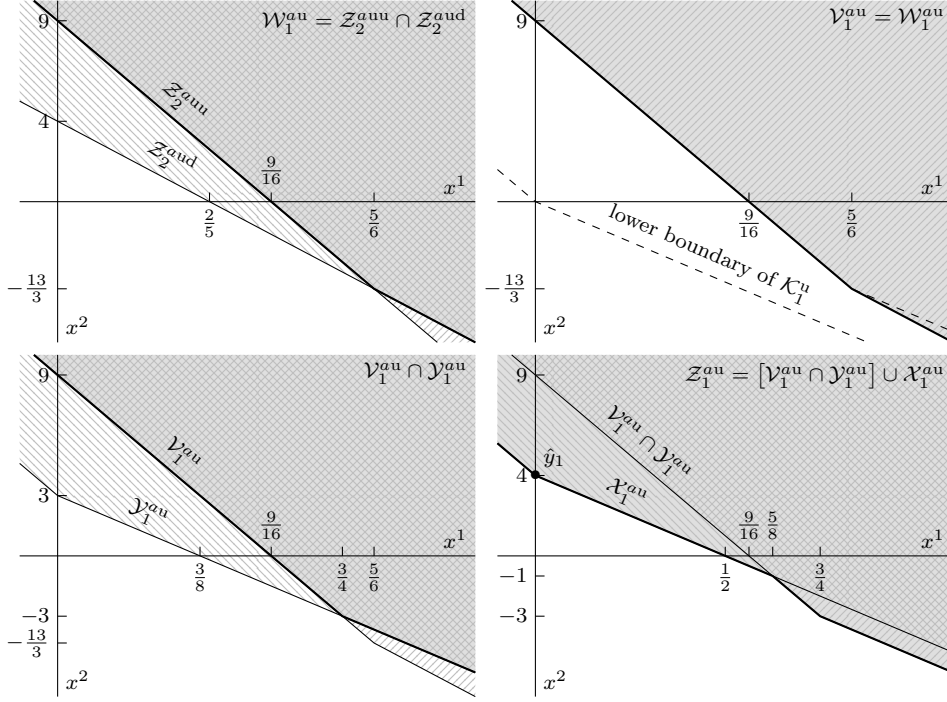
Fig. 2. $\mathcal{W}_1^{au}$, $\mathcal{V}_1^{au}$, $\mathcal{V}_1^{au} \cap \mathcal{Y}_1^{au}$, $\mathcal{Z}_1^{au}$ and $\hat{y}_1$, Example 5.1

node d give that

$$\mathcal{Z}_1^{ad} = \mathcal{X}_1^{ad} = \left\{ \left(x^1, x^2\right) : 6x^1 + x^2 \geq 1 \right\}. \tag{5.1}$$

Finally, following the same steps for time $t = 0$ results in the sets of portfolios $\mathcal{W}_0^a$, $\mathcal{V}_0^a$ and $\mathcal{Z}_0^a$ depicted in Figure 3. The ask prices of the game option in the two currencies are the intersections of the lower boundary of $\mathcal{Z}_0^a$ with the axes, and so can be read directly from the final graph in Figure 3, namely

$$\pi_1^a(Y, X, X') = \tfrac{2}{5}, \qquad\qquad \pi_2^a(Y, X, X') = 4.$$

Consider now the buyer's case. Observe in the context of Remark 3.2 that $(-X, -Y, -X')$ satisfies the conditions of Definition 3.1, even though some of the payoff portfolios are not solvent, for example, $-X_0 = (0, -5) \notin \mathcal{K}_0$. Construction 3.3 and similar calculations as in the seller's case yield

$$\mathcal{Z}_0^b = \left\{ \left(x^1, x^2\right) \in \mathbb{R}^2 : 6x^1 + x^2 \geq -\tfrac{11}{5} \right\}$$

and the bid prices

$$\pi_1^b(Y, X, X') = \tfrac{11}{50}, \qquad\qquad \pi_2^b(Y, X, X') = \tfrac{11}{5}.$$

Let us use Construction 3.4 to find an optimal hedging strategy $(\check{\tau}, \check{y})$ for the buyer in the scenario uu from the initial value $\check{y}_0 := \left(0, -\tfrac{11}{5}\right)$. Clearly $\check{y}_0 \notin \mathcal{X}_0^b = \mathcal{K}_0$
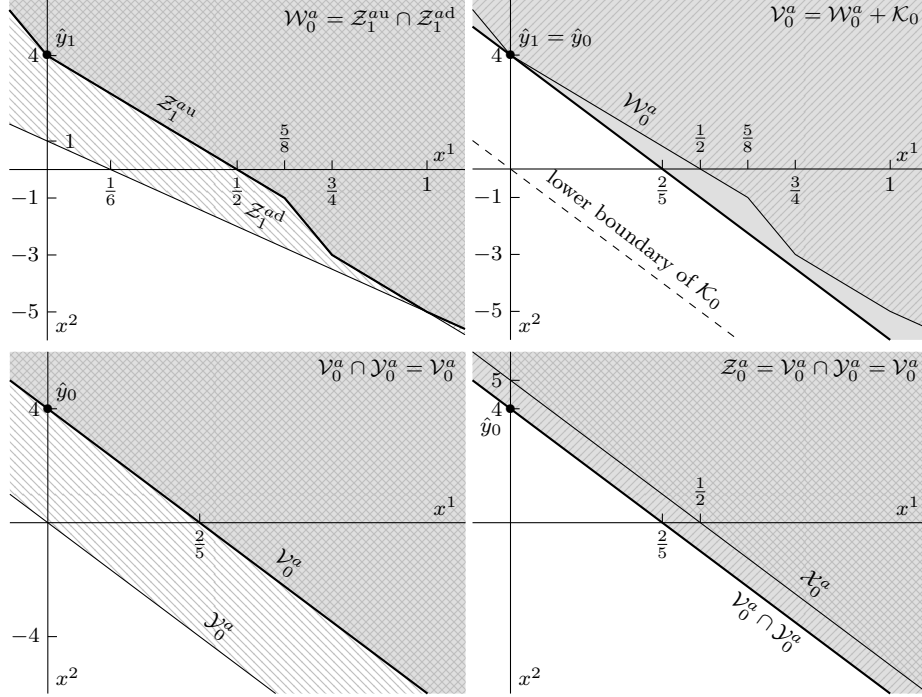
Fig. 3. $\mathcal{W}_0^a$, $\mathcal{V}_0^a$, $\mathcal{V}_0^a \cap \mathcal{Y}_0^a$, $\mathcal{Z}_0^a$ and $\hat{y}_0$, Example 5.1

and so $\check{\tau} > 0$. As can be seen from Figure 4,

$$\mathcal{W}_0^b \cap [\check{y}_0 - \mathcal{K}_0] = \left\{ \left(-\tfrac{3}{10}, \tfrac{4}{5}\right) \right\}$$

and so $\check{y}_1 = \left(-\tfrac{3}{10}, \tfrac{4}{5}\right)$ by (3.14). Observing that

$$\check{y}_1 \notin \mathcal{X}_1^{bu} = \left\{ \left(x^1, x^2\right) \in \mathbb{R}^2 : 16x^1 + x^2 \geq -3, 8x^1 + x^2 \geq -3 \right\},$$

in other words that $\check{\tau}(\mathrm{uu}) > 1$, we proceed to select $\check{y}_2^{\mathrm{u}}$. Figure 4 also shows that

$$\mathcal{W}_1^{bu} \cap [\check{y}_1 - \mathcal{K}_1^{\mathrm{u}}] = \mathrm{conv}\left\{ \check{y}_1, \left(-\tfrac{37}{40}, \tfrac{29}{5}\right), \left(-\tfrac{5}{6}, \tfrac{13}{3}\right), (0, -4) \right\},$$

so there is some freedom in the selection of $\check{y}_2^{\mathrm{u}}$. In fact

$$\mathcal{W}_1^{bu} \cap [\check{y}_1 - \mathcal{K}_1^{\mathrm{u}}] \subseteq \mathcal{X}_2^{buu} = \left\{ \left(x^1, x^2\right) \in \mathbb{R}^2 : 16x^1 + x^2 \geq -9 \right\},$$

which means that every choice of $\check{y}_2^{\mathrm{u}} \in \mathcal{W}_1^{bu} \cap [\check{y}_1 - \mathcal{K}_1^{\mathrm{u}}]$ would enable the buyer to exercise the option at time 2 at the node uu and remain in (or return to) a solvent position. It is also clear that $\check{\tau}(\mathrm{uu}) = 2$.

Returning to the seller, it is straightforward to use Construction 3.2 to create an optimal hedging strategy $(\hat{\sigma}, \hat{y})$ for the seller from the initial endowment $\hat{y}_0 := (0, 4)$. From Figure 3 it is clear that $\hat{y}_0 \notin \mathcal{X}_0^a$ (and so $\hat{\sigma} > 0$), and also that

$$\mathcal{W}_0^a \cap [\hat{y}_0 - \mathcal{K}_0] = \{(0, 4)\} = \{\hat{y}_0\},$$

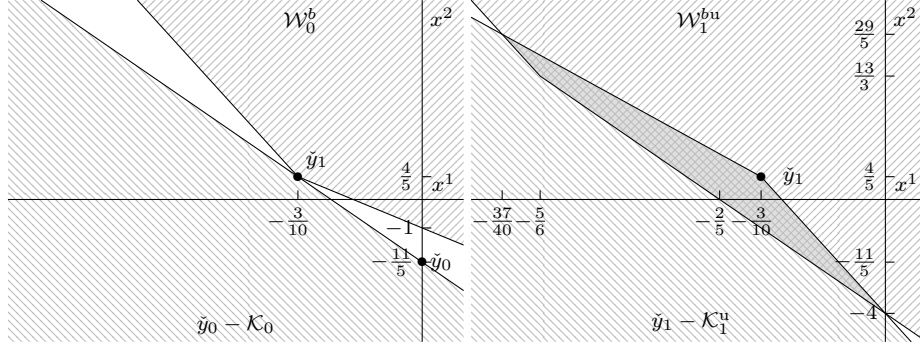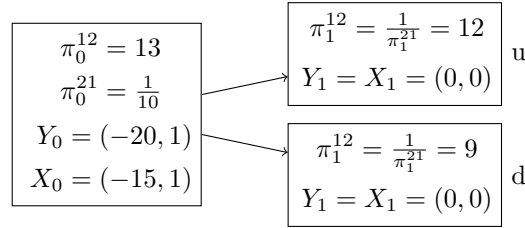Fig. 4. $\check{y}_0$ and $\check{y}_1$, Example 5.1

Fig. 5. Game option in binary single-step two-currency model, Example 5.2

and so by (3.10) we must choose $\hat{y}_1 := \hat{y}_0$. Figure 2 and (5.1) shows that $\hat{y}_1 \notin \mathcal{X}_1^a$
and so $\hat{\sigma} = 1$. This means that if the seller follows $(\hat{\sigma}, \hat{y})$ and the buyer $(\check{\tau}, \check{y})$, then
in scenario uu the option will be cancelled (but not exercised) at the node u, at
which time the seller delivers $(0, 4)$ to the buyer.

Example 5.2. Figure 5 presents a game option (Y, X, Y) in a binary single-step
two-currency model. The option satisfies Definition 3.1 despite the fact that neither
Y_0 nor X_0 are solvent at time 0 (cf. Remark 3.2).

It would be tempting for the seller to cancel the option at time 0, because
delivering X_0 to the buyer at time 0 is effectively the same as receiving the portfolio
$(15, -1)$, which the seller could immediately convert into

$$15 - \pi_0^{12} = 2 > 0$$

units of currency 1. If the buyer exercises at time 0 then the seller is in an even
better position, because $-Y_0$ could similarly be converted into 7 units of currency 1.
Applying Construction 3.1 to (Y, X, Y) gives the set $\mathcal{Z}_0^a$ presented in Figure 6.
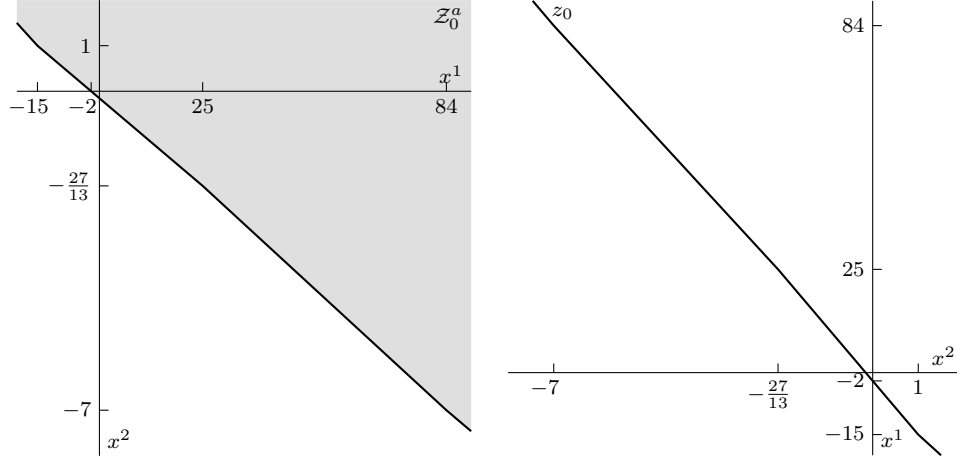Clearly

$$\pi_1^a(Y, X, Y) = -2,$$

Fig. 6. $\mathcal{Z}_0^a$ and z_0, Example 5.2

which means that cancellation at time 0 is indeed optimal for the seller. Theorem 3.2(i) of Kifer (2013b) describes the construction of a function z_0, also shown in Figure 6, and gives the ask price as

$$z_0(0) = -2 = \pi_1^a(Y, X, Y).$$

Note that $\mathcal{Z}_0^a$ is the epigraph of z_0, reflected around the line $x^1 = x^2$; in other words, $\mathcal{Z}_0^a$ would have been the epigraph of z_0, had the currencies been ordered differently.

We will compare the dual representations for $\pi_1^a(Y, X, Y)$ in Theorem 3.1 above and Theorem 3.1 of Kifer (2013b). The model has only two stopping times, 0 and 1, and so this can be done easily and directly. Observe also that

$$\mathcal{K}_0^* = \text{cone}\left\{(1, 10), (1, 13)\right\},$$
$$\mathcal{K}_1^{*\mathrm{u}} = \text{cone}\left\{(1, 12)\right\} \subset \mathcal{K}_0^*,$$
$$\mathcal{K}_1^{*\mathrm{d}} = \text{cone}\left\{(1, 9)\right\}.$$

This means (cf. Definition 2.2) that the property $(\mathbb{P}, S) = \left(\mathbb{P}, \left(S^1, S^2\right)\right) \in \bar{\mathcal{P}}^1(\chi)$ is equivalent to

$$\mathbb{P}(\mathrm{u}) \geq \tfrac{1}{3}, \qquad\qquad \mathbb{P}(\mathrm{d}) = 1 - \mathbb{P}(\mathrm{u}) \geq 0,$$

together with

$$S_0^1 = S_1^1 = 1, \qquad 10 \leq S_0^2 \leq 13, \qquad S_1^2(\mathrm{u}) = 12, \qquad S_1^2(\mathrm{d}) = 9.$$

For the representation for $\pi_1^a(Y, X, Y)$ in Theorem 3.1, take first $\sigma = 0$. For every $\chi \in \mathcal{X}$ and $(\mathbb{P}, S) \in \bar{\mathcal{P}}^1(\chi \wedge 0)$ we have

$$(Q_{0\cdot} \cdot S_{0\wedge\cdot})_\chi = \chi_1^* X_0 \cdot S_0 + \chi_0 Y_0 \cdot S_0 = -5\chi_0 - 15 + S_0^2,$$

22   *Alet Roux*

and therefore

$$\max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi \wedge 0)} \mathbb{E}_{\mathbb{P}}((Q_{0\cdot} \cdot S_{0 \wedge \cdot})_\chi)$$
$$= \max\{-5\chi_0 - 15 + S_0^2 : \chi_0 \in [0,1], S_0^2 \in [10,13]\} = -2.$$

Similarly, for $\sigma = 1$,

$$(Q_{1\cdot} \cdot S_{1 \wedge \cdot})_\chi = \chi_0 Y_0 \cdot S_0 + \chi_2^* X_1 \cdot S_1 + \chi_1 Y_1 \cdot S_1 = \chi_0(S_0^2 - 20)$$

and so

$$\max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi \wedge 1)} \mathbb{E}_{\mathbb{P}}((Q_{1\cdot} \cdot S_{1 \wedge \cdot})_\chi)$$
$$= \max\{\chi_0(S_0^2 - 20) : \chi_0 \in [0,1], S_0^2 \in [10,13]\} = 0. \quad (5.2)$$

Finally,

$$\min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi \wedge \sigma)} \mathbb{E}_{\mathbb{P}}((Q_{\sigma\cdot} \cdot S_{\sigma \wedge \cdot})_\chi) = \min\{-2, 0\} = -2 = \pi_1^a(Y, X, Y)$$

as expected.

Now turn to the dual representation in Theorem 3.1 of Kifer (2013b). It can be written in our notation as

$$V^a := \min_{\sigma \in \mathcal{T}} \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi)} \mathbb{E}_{\mathbb{P}}((Q_{\sigma\cdot} \cdot S)_\chi),$$

where

$$(Q_{\sigma\cdot} \cdot S)_\chi = \sum_{t=0}^{\sigma} \chi_t Y_t \cdot S_t + \sum_{t=\sigma+1}^{T} \chi_t X_\sigma \cdot S_t$$

and $T = 1$. For $\sigma = 1$, the calculation (5.2) gives that

$$\max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi)} \mathbb{E}_{\mathbb{P}}((Q_{1\cdot} \cdot S)_\chi) = \max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi \wedge 1)} \mathbb{E}_{\mathbb{P}}((Q_{1\cdot} \cdot S_{1 \wedge \cdot})_\chi) = 0.$$

For $\sigma = 0$ and every $\chi \in \mathcal{X}$ and $(\mathbb{P}, S) \in \bar{\mathcal{P}}^1(\chi)$ we have

$$(Q_{\sigma\cdot} \cdot S)_\chi = \chi_0 Y_0 \cdot S_0 + \chi_1 X_0 \cdot S_1 = \chi_0(S_0^2 - 5) + (1 - \chi_0)S_1^2 - 15,$$

so that

$$\mathbb{E}_{\mathbb{P}}((Q_{0\cdot} \cdot S)_\chi) = \chi_0(S_0^2 - 5) + (1 - \chi_0)(3\mathbb{P}(\mathrm{u}) + 9) - 15.$$

This means that

$$\max_{\chi \in \mathcal{X}} \max_{(\mathbb{P},S) \in \bar{\mathcal{P}}^1(\chi)} \mathbb{E}_{\mathbb{P}}((Q_{0\cdot} \cdot S)_\chi)$$
$$= \max\{\chi_0(S_0^2 - 5) + (1 - \chi_0)(3\mathbb{P}(\mathrm{u}) + 9) - 15$$
$$: \chi_0 \in [0,1], S_0^2 \in [10,13], \mathbb{P}(\mathrm{u}) \in [\tfrac{1}{3}, 1]\} = -3.$$

Finally,

$$V^a = \min\{0, -3\} = -3 \neq \pi_1^a(Y, X, Y),$$

which demonstrates that V^a is not the correct dual representation for $\pi_1^a(Y, X, Y)$.

Example 5.3. Consider a three-currency model with T steps and time horizon 1 based on the two-asset recombinant Korn-Muller model (Korn & Müller 2009) with Cholesky decomposition, i.e. friction-free exchange rates in terms of currency 3 are modelled by the process $(S_t)_{t=0}^T$, where

$$S_t = \left(\varepsilon_t^1 S_{t-1}^1, \varepsilon_t^2 S_{t-1}^2, 1\right) \text{ for } t = 1, \ldots, T$$

and $(\varepsilon_t)_{t=1}^T = \left((\varepsilon_t^1, \varepsilon_t^2)\right)_{t=1}^T$ is a sequence of independent identically distributed random variables taking the values

$$\left(e^{-\frac{1}{2}\sigma_1^2\Delta - \sigma_1\sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2\Delta - (\rho+\sqrt{1-\rho^2})\sigma_2\sqrt{\Delta}}\right),$$

$$\left(e^{-\frac{1}{2}\sigma_1^2\Delta - \sigma_1\sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2\Delta - (\rho-\sqrt{1-\rho^2})\sigma_2\sqrt{\Delta}}\right),$$

$$\left(e^{-\frac{1}{2}\sigma_1^2\Delta + \sigma_1\sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2\Delta + (\rho-\sqrt{1-\rho^2})\sigma_2\sqrt{\Delta}}\right),$$

$$\left(e^{-\frac{1}{2}\sigma_1^2\Delta + \sigma_1\sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2\Delta + (\rho+\sqrt{1-\rho^2})\sigma_2\sqrt{\Delta}}\right),$$

each with positive probability, where $\Delta := \frac{1}{T}$ is the step size. The exchange rates with transaction costs are modelled as

$$\pi_t^{ij} := \begin{cases} \frac{S_t^j}{S_t^i}(1+k) & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

for $i, j = 1, \ldots, 3$ and $t \leq T$, where $k \in [0, 1)$. We take

$$\left(S_0^1, S_0^2\right) = (40, 50), \qquad \sigma_1 = 0.15, \qquad \sigma_2 = 0.1, \qquad \rho = 0.5.$$

Consider a game put option with physical delivery on a basket containing one unit each of currencies 1 and 2 and with strike K in currency 3, i.e.

$$Y_t = (-1, -1, K) \text{ for } t = 0, \ldots, T.$$

On cancellation the seller delivers the above payoff to the buyer, together with a cancellation penalty $p \geq 0$ in currency 3, so that

$$X_t = X_t' = (-1, -1, K+p) \text{ for } t = 0, \ldots, T.$$

We allow for the possibility that the seller may choose not to cancel the option, and the buyer may choose not to exercise, by adding an additional time step $T + 1$ and taking

$$Y_{T+1} = X_{T+1} = X_{T+1}' = (0, 0, 0).$$

Except for the union, the operations in Constructions 3.1 and 3.3, namely intersection and direct addition of a polyhedral cone, are standard geometric procedures when applied to polyhedra, and can be implemented using existing software libraries. Both these operations are union-preserving, and so the extension to unions

of polyhedra is straightforward. The numerical results below were produced using Maple with the *Convex* package (Franz 2009).

Table 1 contains bid and ask prices in currency 3 of the basket put with penalty $p = 5$ for a range of strike prices. Both bid and ask prices increase with the strike. Note the appearance of negative bid and ask prices for out-of-the-money options (i.e. $K < S_0^1 + S_0^2 = 90$). The reason for this is that the seller can cancel the option at any time, and cancellation tends to be particularly attractive to the seller (and very costly for the buyer) when the option is far out of the money; for example, if $K = 80$ then, from the point of view of the seller, cancelling the option at time 0 is equivalent to receiving the basket from the buyer and paying $K + p = 85$ in currency 3, which is less than the market price $S_0^1 + S_0^2 = 90$ (ignoring transaction costs) of the basket.

Table 1. Bid and ask prices of game basket put option with $N = 10$, $p = 5$ and different strikes K, Example 5.3

K	$k = 0$		$k = 0.005$	
	π_3^b	π_3^a	π_3^b	π_3^a
100	10.043290	11.033942	9.568590	11.687749
95	5.266479	6.817389	4.706543	7.480028
90	0.967824	2.774598	0.367975	3.423798
85	−2.934360	−1.091048	−3.587214	−0.444708
80	−6.910514	−5.131149	−7.584034	−4.614029

Bid and ask prices of the game basket put with strike $K = 100$ and a range of penalty values are reported in Table 2, together with bid and ask prices of the American basket put with the same strike (using the constructions of Roux & Zastawniak 2015). In practical terms, game options with large penalties resemble American options (because larger penalties make it less attractive for the seller to cancel the option early), and this explains the convergence of the bid and ask prices of the game option to that of the American option as the penalty increases.

Table 2. Bid and ask prices of game basket put option with $N = 10$, $K = 100$ and different penalties, Example 5.3

p	$k = 0$		$k = 0.005$	
	π_3^b	π_3^a	π_3^b	π_3^a
0	10.000000	10.000000	9.550000	10.447761
1	10.014709	10.278348	9.556726	10.790910
2	10.027095	10.497310	9.562075	11.051351
5	10.043290	11.033942	9.568590	11.687749
10	10.050958	11.571315	9.571850	12.297913
20	10.052026	11.796921	9.572414	12.575621
American	10.052027	11.812658	9.572414	12.589930

References

S. Baxter & R. Chacon (1977) Compactness of stopping times *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **40**, 169–181.

B. Bensaid, J.-P. Lesne, H. Pagès, H. & J. Scheinkman (1992) Derivative asset pricing with transaction costs *Mathematical Finance* **2**, 63–86.

T. R. Bielecki, S. Crépey, M. Jeanblanc, M. & M. Rutkowski (2008) Arbitrage pricing of defaultable game options with applications to convertible bonds *Quantitative Finance* **8**(8), 795–810.

P. Chalasani & S. Jha (2001) Randomized stopping times and American option pricing with transaction costs *Mathematical Finance* **11**(1), 33–77.

Y. Dolinsky (2013) Hedging of game options with the presence of transaction costs *Annals of Applied Probability* **23**(6), 2212–2237.

H. Föllmer & A. Schied (2002) *Stochastic Finance: An Introduction in Discrete Time.* Berlin: Walter de Gruyter.

M. Franz (2009) Convex—a Maple package for convex geometry.
   URL: *http://www.math.uwo.ca/∼mfranz/convex/*

Y. M. Kabanov (1999) Hedging and liquidation under transaction costs in currency markets *Finance and Stochastics* **3**, 237–248.

Y. M. Kabanov, M. Rásonyi & C. Stricker (2002) No-arbitrage criteria for financial markets with efficient friction *Finance and Stochastics* **6**, 371–382.

Y. M. Kabanov & C. Stricker (2001) The Harrison-Pliska arbitrage pricing theorem under transaction costs, *Journal of Mathematical Economics* **35**, 185–196.

Y. Kabanov & M. Safarian (2009) *Markets with Transaction Costs: Mathematical Theory.* New York: Springer-Verlag.

J. Kallsen & C. Kühn (2005) Convertible bonds: financial derivatives of game type. In: *Exotic Option Pricing and Advanced Lévy Models* (A. Kyprianou, W. Schoutens & P. Wilmott, eds.), 277–288. New York: Wiley.

Y. Kifer (2000) Game options *Finance and Stochastics* **4**, 443–463.

Y. Kifer (2013a) Dynkin's games and Israeli options *ISRN Probability and Statistics* **2013**, Article ID 856458.

Y. Kifer (2013b) Hedging of game options in discrete markets with transaction costs *Stochastics* **85**(4).

R. Korn & S. Müller (2009) The decoupling approach to binomial pricing of multi-asset options *Journal of Computational Finance* **12**(3), 1–30.

C. Kühn & A. E. Kyprianou (2007) Callable puts as composite exotic options *Mathematical Finance* **17**(4), 487–502.

A. Löhne & B. Rudloff (2014) An algorithm for calculating the set of superhedging portfolios in markets with transaction costs *International Journal of Theoretical and Applied Finance* **17**(2), Article ID 1450012.

A. Roux & T. Zastawniak (2009) American options under proportional transaction costs: Pricing, hedging and stopping algorithms for long and short positions *Acta Applicandae Mathematicae* **106**, 199–228.

A. Roux & T. Zastawniak (2015) American and Bermudan options in currency markets under proportional transaction costs *Acta Applicandae Mathematicae*. In press.

W. Schachermayer (2004) The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time *Mathematical Finance* **14**(1), 19–48.

L. Wang & Z. Jin (2009) Valuation of game options in jump-diffusion model and with applications to convertible bonds *Journal of Applied Mathematics and Decision Sciences*, Art. ID 945923.