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Identification of Continuous Time Models for Nonlinear Dynamic Systems from Discrete Data

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Abstract -- A new iOFR-MF (iterative Orthogonal Forward Regression – Modulating Function) algorithm is proposed to identify continuous time models from noisy data by combining the modulating function method and the iterative orthogonal forward regression (iOFR) algorithm. In the new method, a set of candidate terms, which describe different dynamic relationships among the system states or between the input and output, are first constructed. These terms are then modulated using the modulating function method to generate the data matrix. The iOFR algorithm is next applied to build the relationships between these modulated terms which includes detecting the model structure and estimating the associated parameters. The relationships between the original variables are finally recovered from the model of the modulated terms. Both nonlinear state-space models and a class of higher order nonlinear input-output models are considered. The new direct method is compared with the traditional finite difference method and results show that the new method performs much better than the finite difference method. The new method works well even when the measurements are severely corrupted by noise. The selection of appropriate modulating functions is also discussed.

Key words: *Continuous time model, iOFR algorithm, modulating function method, nonlinear system identification, orthogonal forward regression*

1. Introduction

The system identification of discrete time models has been well developed for both linear and nonlinear systems. Many different identification methods based on different criterions have been developed and used (Ljung, 1987; Söderström, 1989; Söderström & Stoica, 2002). Among the existing nonlinear system identification methods, the NARMAX (Nonlinear AutoRegressive Moving Average with eXogenous input) model and the associated Orthogonal Forward Regression (OFR) algorithm have been widely applied in the modelling of many engineering, chemical, biological, medical, geographical, and economic systems (Billings, 2013). However, a continuous time model is often favourable in some applications because of the inherent connection with physical systems.

There are essentially two classes of identification methods for continuous time models: the direct methods and the indirect methods (Unbehauen & Rao, 1998). The direct methods identify continuous time models directly from discrete data, but this often involves the reconstruction of the derivatives of variables from the discrete data (Coca & Billings, 1999). A very commonly used method is the finite difference method, for example, the methods based on the delta operator models (Anderson & Kadirkamanathan, 2007; Soderstrom, Fan, Carlsson, & Mossberg, 1997; Zhang & Billings, 2015). However, the numerical differentiation that is inherent in this approach may amplify the effects of noise, which makes the application of the finite difference impractical in many applications. The performance of these kinds of direct methods crucially depends on the filters used in the data pre-processing. Other direct methods include algorithms based on signal decompositions where the signals are reconstructed using a system of basis functions and the derivatives of the signals are calculated as the combination of the derivatives of these basis functions (Brewer, Barenco, Callard, Hubank, & Stark, 2008; Coca & Billings, 1997). To avoid numerical differentiation, indirect methods have been developed in the identification of continuous time models. These approaches involve the identification of discrete time models as a first step and then the transfer of the discrete-time models to continuous time models (Li & Billings, 2001). The transfer is based on some invariant properties between the discrete and continuous models, for example, the impulse response, the frequency response functions and so on. Despite the fact that the indirect methods often give good representations for a physical system, the transformation from discrete model to continuous model can become complicated when the system is nonlinear.

The modulating function method which directly produces continuous time models avoiding numerical differentiation of noisy data has been widely used in many applications (Preisig & Rippin, 1993; Saha & Rao, 1983; Unbehauen & Rao, 1990, 1998; Young, 1981). However, the modulating function method is well known as a parameter identification method rather than a model structure identification method and there are few studies that investigate the model structure detection in the modulating function method. In many applications, both the model structure and the unknown parameters need to be determined.

In this paper, a new iOFR- modulating function method is introduced, which combines the strengths of both the iOFR algorithm and the modulating function method. It will be shown that the model structure can be efficiently detected by adopting the iOFR algorithm to select the modulated candidate terms. The iOFR-MF algorithm provides a new and efficient method for identifying nonlinear continuous time models.

System identification involves two coupled problems: the detection of the model structure and estimation of the parameters. The detection of the model structure depends on the values of the associated parameters and the estimation of the parameters depends on the model structure. Therefore system identification searches for a solution in the Cartesian product of the set of candidate terms and the set of possible parameters. An exhaustive search on the whole solution space is often time-consuming and impractical in many applications. Evolutionary algorithms are often used to reduce the computation of the global search, for example the symbolic regression algorithms (Koza, 1992; Schmidt & Lipson, 2009). Even the evolutionary algorithms can be computationally intensive when the solution space is large. The OFR (Orthogonal Forward Regression) algorithm, which has successively decoupled these two processes by stepwise orthogonalising the candidate terms and selecting the significant terms one at a time, has been proved to be efficient in the identification of nonlinear systems. A new iOFR (iterative Orthogonal Forward Regression) algorithm has recently been proposed to improve the performance of the classic OFR algorithm (Yuzhu Guo, L.Z. Guo, S. A. Billings, & H. L. Wei, 2015b). In the iOFR algorithm, the classic OFR algorithm is iteratively applied where the next search is based on the suboptimal term set obtained at the previous stage. By slightly revising the classic OFR algorithm, the iOFR algorithm searches an optimal model on a global solution space and produces the optimal solution. In this paper, the iOFR algorithm will be combined with the modulating function method to provide a new system identification method where both the system structure can be efficiently detected and the parameters can be estimated avoiding all the problems caused by numerical differentiation of noisy data. The new algorithm can efficiently determine a parsimonious model structure without any a priori knowledge of the nonlinear system.

The remainder of the paper is organised as follows. Section 2 introduces the continuous-time models and the modulated function representation. The new iOFR-modulating function method is introduced in Section 3. In Section 4, the Lorenz system and van der Pol oscillator are identified to demonstrate the efficiency of the new method. Conclusions are finally drawn in Section 5.

2. Continuous models and the modulated representation

The main idea of the modulating function method is to transfer the derivatives of measured, noisy signals to analytical modulation functions. This transfer is achieved by integrating by parts the product of the modulating function with the terms in the model equation. A similar idea has also been used in the finite element method and distribution theory where the modulating function is referred to as the test function and the modulated equation is named as the weak formulation. In order to apply the modulating function method, the modulating functions are required to posses some good properties. The modulating functions are introduced in Section 2.1 and Section 2.2 and 2.3 show how to transform differential equations into algebraic relations between the modulated terms using these modulating functions.

2.1 Modulating functions

The modulating function is chosen to possess the following properties:

i). The modulating function $\varphi(t)$ has finite support, that is, $\varphi(t)$ is identically zero outside the finite interval $[a,b]$.

ii). The modulating function is sufficiently smooth, that is, the up to *d*-th derivatives exist.

iii). The derivatives of the modulating function are zero at the ends of the finite interval.

Many functions have been used as modulating functions, including trigonometric functions, the Poisson moment functions, Hartley functions, and so on. In this paper the B-spline function is used as the modulating function because of the simple structure and excellent approximation properties of these functions.

A kth order B-spline function can be recursively defined as follows given a set of $k+1$ knots s_0, s_1, ..., s_{k+1}.

$$B_{i,k}(t)=\frac{t-s_i}{s_{i+k-1}-s_i}B_{i,k-1}(t)+\frac{s_{i+k}-t}{s_{i+k}-s_{i+1}}B_{i+1,k-1}(t) \tag{1}$$

with

$$B_{i,1}(t)=\begin{cases}1, & s_i \le t < s_{i+1}\\ 0, & otherwise\end{cases} \tag{2}$$

The first index indicates the position of the B-spline function and the second index denotes the order of the B-spline functions. Using k+1 knots one *k*th order B-spline function can be defined and two *(k-1)* th order B-spline functions can be defined, in turn, three *(k-2)*th order B-spline functions, and so on.

The derivative of a k-th order B-spline can be calculated as

$$\frac{dB_{i,k}(t)}{dt} = (k-1)\left(\frac{B_{i,k-1}(t)}{s_{i+k-1}-s_i} - \frac{B_{i+1,k-1}(t)}{s_{i+k}-s_{i+1}}\right) \tag{3}$$

The higher order derivative of a B-spline basis function can be calculated according to the recursive formula:

$$\frac{d^r B_{i,k}(t)}{dt^r} = (k-1)\left(\frac{1}{s_{i+k-1}-s_i} \quad \frac{-1}{s_{i+k}-s_{i+1}}\right)\begin{pmatrix} \dfrac{d^{r-1}B_{i,k-1}(t)}{dt^{r-1}} \\ \dfrac{d^{r-1}B_{i+1,k-1}(t)}{dt^{r-1}} \end{pmatrix} \tag{4}$$

Figure 1 shows the cubic (4th order) B-spline function and its first and second order derivatives. It is easy to observe that the B-spline function satisfies all the requirements of a modulating function. The advantages of the spline function over other modulating functions has been discussed in the references (Preisig & Rippin, 1993).

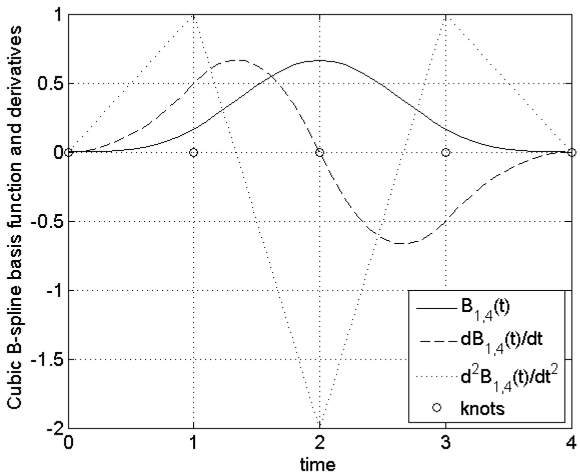


Fig 1 Cubic B-spline basis function and its derivatives

Using these modulating functions, some differential equations can then be transformed into algebraic relationships avoiding numerical differentiation. Commonly used models include state-space models and input-output models.

2.2 State-space models

A state-space model is a canonical form model for both linear and nonlinear systems where the system can fully be described by a minimum number of state variables. Consider the state-space model in equation (5).

$$\begin{cases} \dot{\mathbf{x}}(t) = \boldsymbol{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ y(t) = x(t) + n(t) \end{cases} \tag{5}$$

where $x(t)$ is the state vector of the system; $y(t)$ is the measurements of the system states $x(t)$ and $n(t)$ is measurement noise. The function $f(x(t), u(t))$, which can be any linear or nonlinear function of the system states and inputs, defines a vector field which uniquely determines the transition of system states in the state space. The vector function $\boldsymbol{f}(x,u)$ can often be expanded as the superposition of a set of basis functions, that is.

$$\dot{x}(t) = \sum_{i=1}^{K_s} \theta_i \phi_i(x,u) \tag{6}$$

The model then becomes linear-in-the-parameters, and θ_i represents the system parameters. Multiplying both sides of equation (6) with the modulating function $\varphi(t)$ and integrating both sides over the time interval [a, b] yields

$$\int_a^b \varphi(\tau)\dot{x}(\tau)d\tau = \sum_{i=1}^{K_s} \theta_i \left(\int_a^b \varphi(\tau)\phi_i\left(x(\tau), u(\tau)\right)d\tau \right) \tag{7}$$

The left hand side of the equation can be written as (8) using the rule of integration by parts.

$$\int_a^b \varphi(\tau)\dot{x}(\tau)d\tau = \varphi(t)x(t)\Big|_a^b - \int_a^b \dot{\varphi}(\tau)x(\tau)d\tau \tag{8}$$

Because of the properties of the modulation function, the first term on the right hand side is always zero and the modulated system model becomes

$$-\int_a^b \dot{\varphi}(\tau)x(\tau)d\tau = \sum_{i=1}^{K_s} \theta_i \left(\int_a^b \varphi(\tau)\phi_i\left(x(\tau), u(\tau)\right)d\tau \right) \tag{9}$$

In system identification, modelling is usually performed based on the noisy observations $y(t)$ rather than the state variables $x(t)$. Using the modulating function method, the differentiation operation on the noisy observations $y(t)$ are successfully transferred onto the analytic modulating function $\varphi(t)$. At the same time, the structure of the model remains unchanged. Therefore, the identification of the state-space model (5) becomes the identification of the modulated model (9) from the observed input and output data. In practical applications, the form of the function $f(x,u)$ is often not known a priori, that is, what kind of terms $\phi_i(x,u)$ should be included in the model (6) needs to be determined before parameter estimation. This problem is crucial to the system identification and has not been well studied in the application of the modulating function method. In this paper, the iOFR algorithm will be adopted to solve the problem of model structure detection in the modulation function method. A large candidate term dictionary, in which all the possible terms are included, will initially be constructed. All these terms will then be modulated using the modulating function. The most significant terms will be selected from the modulated term dictionary using the new iOFR algorithm to constitute the system model. The new iOFR-modulating function method will be discussed in Section 3.

2.3 Higher - order continuous systems

When some of the inner-states in (5) cannot be directly measured the system can be described using the input-output model

$$y^{(n)} = f\left(y^{(n-1)}, y^{(n-2)}, \cdots, \dot{y}, y, u^{(n-1)}, u^{(n-2)}, \cdots, \dot{u}, u\right) \tag{10}$$

Here we assume the highest derivative $y^{(n)}$ is linear and with a unit coefficient in the model. Actually, the left hand side of the equation can be any linear combination of measurable variables and the associated derivatives. For example, a mechanical system can be written in the form $m\ddot{y} + c\dot{y} + ky = f\left(\dot{y}, y\right)$, where m, c, k represent the mass, damping coefficient, elastic coefficient, respectively, and $f\left(\dot{y}, y\right)$ is the nonlinear restoring force. However, in many applications the form and the nonlinear restoring force is not known and is usually approximated using a series of basis functions.

Expanding the function on the right hand side as the superposition of basis terms yields

$$y^{(n)} = \sum_{i=1}^{\kappa_1} \theta_i \phi_i \left(y^{(n-1)}, y^{(n-2)}, \cdots, \dot{y}, y, u^{(n-1)}, u^{(n-2)}, \cdots, \dot{u}, u\right) + \sum_{i=\kappa_1}^{\kappa_s} \theta_i \phi_i \left(y, u\right) \tag{11}$$

It is worth noticing that the modulating function cannot be used in the general input-output model in (11). The modulating function method requires the term $\phi_i\left(y^{(n-1)}, y^{(n-2)}, \cdots, \dot{y}, y, u^{(n-1)}, u^{(n-2)}, \cdots, \dot{u}, u\right)$ to be integrable, that is, ϕ_i (i = $1, 2, \ldots, \kappa_1$) is the derivative of another function $\psi_i\left(y,u\right)$ with respected to time, that is,

$$\frac{d^{n_i}}{dt^{n_i}} \psi_i\left(y,u\right) = \phi_i\left(y^{(n-1)}, y^{(n-2)}, \cdots, \dot{y}, y, u^{(n-1)}, u^{(n-2)}, \cdots, \dot{u}, u\right) \tag{12}$$

Term $\psi_i(y,u)$ can only be a function of the input and output not including their derivatives so that the time derivatives of the input and output can be fully transferred on to the analytic modulating function. When all the terms satisfy the requirements, the modulated model can be given as

$$\begin{aligned}&(-1)^n \int_a^b \varphi^{(n)}(\tau) y(\tau) d\tau = \sum_{i=1}^{\kappa_1} \theta_i (-1)^{n_i} \int_a^b \varphi^{(n_i)}(\tau) \psi(y(\tau) u(\tau)) d\tau \\ &+ \sum_{i=\kappa_1}^{\kappa_x} \theta_i \int_a^b \phi_i(y(\tau), u(\tau)) d\tau\end{aligned} \tag{13}$$

Again, the model structure needs to be detected before the parameters can be estimated based on the modulated model.

2.4 Relationship with the finite difference method

In order to explain why the modulating function method works better than a finite difference method, the relationship between the finite difference method and the modulating function method will be discussed. It will be shown that the finite difference method is a specialisation of the modulating function method where the modulating function is the second order B-spline function.

Assume the data is uniformly sampled with the sampling interval Δt and the continuous signal can be constructed using a zero-order hold, that is,

$$x(t) = x(k-1), \quad k-1 \le t < k \tag{14}$$

The finite difference formulae

$$\dot{x}(k) \approx \frac{x(k) - x(k-1)}{\Delta t} \tag{15}$$

can then be thought as the modulated form of the following modulating function.

$$\varphi(t) = \begin{cases} \dfrac{t-(k-1)}{(\Delta t)^2} & k-1 \le t < k \\ \dfrac{(k+1)-t}{(\Delta t)^2} & k \le t < k+1 \end{cases} \tag{16}$$

The derivative of $\varphi(t)$ is

$$\dot{\varphi}(t) = \begin{cases} \dfrac{1}{(\Delta t)^2} & k-1 \le t < k \\ -\dfrac{1}{(\Delta t)^2} & k \le t < k+1 \end{cases} \tag{17}$$

Compare the modulating function in (16) and (17) with formulae (1) and (3). It is easy to see that modulating function (16) is similar to the second order B-spline function where the three knots used are $\{k-1, k, k+1\}$. Consequently, the finite difference method is a specialisation of the modulating function method where the modulating function is defined as (16) and the derivative is (17). However, the second order B-spline function with the finite support $[k\text{-}1, k+1]$ is not a good choice for the modulating function and the estimated parameters are often not satisfactory. How to choose an appropriate modulating function will be discussed in the next section.

3. Iterative orthogonal forward regression modulating function method

When the structure of the differential equation models are known, the differential equations can be transformed into algebraic equations with the structure of the equation unchanged using the modulating function. The associated parameters can then be estimated without any numerical differentiation. However, the structure of the model is often unknown. Hence a system identification method which can identify both the model structure and the associated parameters is needed. In this section a new iterative orthogonal forward regression modulating function method is introduced by combining the modulating function method with the iterative orthogonal forward regression algorithm.

3.1 Application of the iterative orthogonal forward regression algorithm in the modulating function method

In order to determine the model structure, a term dictionary $\mathcal{D} = \{\phi_1, \cdots, \phi_K\}$ is initially defined and the model to be identifed is assumed to be composed of a linear combination of a subset of the terms in the dictionary. To apply the iOFR algorithm, the dependent variable and these terms will be modulated first and these modulated candidate terms are then used to construct the modulated regression matrix.

In the previous section, only one set of data was generated through the integration over the support of the modulating function. In order to identify the modulated model, more data are needed. Additional data for the modulated model can be generated in two ways: shifting the position of the modulating function over the signals or alternatively, using different modulating functions. Either way, different modulating functions, which can be different in the form of function or different in the interval of support, are obtained. Define these modulating functions as $\varphi_k(t)$, k=1,2,…, N, and denote the associated nth order derivative as $\varphi_k^{(n)}(t)$.

Collecting *N* sets of the modulated terms yields the matrix form of equation

$$Y = \mathbf{\Phi}\Theta + \Xi \tag{18}$$

where $Y=[y(1),\ y(2),\ \ldots,\ y(N)]$, and $y(k)=(-1)^n \int_a^b \varphi_k^{(n)}(\tau) y(\tau) d\tau$. Matrix $\mathbf{\Phi}=[\phi_1 \quad \phi_2 \quad \cdots \quad \phi_K]$ is known as the modulated regression matrix. Each column $\phi_i=[\eta_i(1) \quad \cdots \quad \eta_i(N)]^T$ represents N realization of a modulated term.

When the original term does not contain the derivatives of the input or output, the corresponding modulated term of $\phi_i(y(\tau),u(\tau))$ is

$$\eta_i(k)=\int_a^b \varphi_k^{(n_i)}(\tau)\phi_i(y(\tau),u(\tau))d\tau \tag{19}$$

Otherwise, the modulated term will be

$$\eta_i(k)=(-1)^{n_i}\int_a^b \varphi_k^{(n_i)}(\tau)\psi_i(y(\tau)u(\tau))d\tau \tag{20}$$

when the original term also depends on the derivatives of the input or output, where $\psi_i(y(\tau)u(\tau))$ is defined in (12).

However, which terms should be included in a model is often unknown a priori. Hence, system identification involves the determination of the model structure which consists of a subset $\{\phi_j\}$ of the term in the dictionary $\mathcal{D}=\{\phi_1,\cdots,\phi_K\}$ and the estimation of the associated parameters. These two processes are firmly coupled with each other. Ranking of the significance of a term depends on the weight (coefficient) of the term in a model while the estimation of the parameters depends on what terms are included in a model. The OFR algorithm decouples the interactions between these two processes and provides an efficient method for the identification of nonlinear systems. In the OFR algorithm, the terms ϕ_j are orthogonalised stepwise into the orthogonal terms w_j and the associated coefficients can then be estimated as

$$g_j=\frac{\langle w_j, y\rangle}{\langle w_j, w_j\rangle} \tag{21}$$

The significance of the term can then be evaluated using the Error Reduction Ratio or ERR criterion defined as

$$ERR(w_j)=\frac{\langle g_j w_j, g_j w_j\rangle}{\langle y, y\rangle}=\frac{\langle w_j, y\rangle^2}{\langle w_j, w_j\rangle\langle y, y\rangle} \tag{22}$$

The terms can then be selected into the model according to the ERR criterion.

The regression will stop when all the significant terms have been detected. A commonly used stop condition can be set as

$$1-\sum_{j} ERR\left(w_{j}\right) \leq \rho \tag{23}$$

The sum of ERR (denoted as SERR) indicates that a proportion of $\sum_{j} ERR\left(w_{j}\right)$ information in the output has been explained by the terms $\{\phi_j\}$ which consists the model.

The standard orthogonal forward regression algorithm consists of the following steps:

(1) Sufficiently excite the system and measure the inputs and outputs of the system.

(2) Specify an initial full model set of κ candidate terms and the value of ρ .

(3) Compute the values of the ERR for each of the κ candidate terms and select the term which gives the largest value of ERR into the model as the first term.

(4) At the k th ($k \geq 2$) stages: compute the values of the error reduction ratio for each of the $(\kappa - k + 1)$ remaining candidate terms by assuming that each is the k th term in the selected model and perform the corresponding orthogonalisation; the term that gives the largest value of the error reduction ratio is then selected into the model as the k th term. If condition (23) is satisfied, finish the process and go to (5). Otherwise set $k = k + 1$ and repeat step (4).

(5) The final model contains κ_s terms and the parameter estimates can be calculated using a least squares formulae.

The classic OFR algorithm can occasionally produce sub-optimal models, for example, when the system is not persistently excited. An iterative OFR (iOFR) algorithm has recently been introduced to solve this problem. The iOFR algorithm comprises two steps, the first step is to obtain a suboptimal model set and the second step uses a subset of the terms which were obtained in the first step as the starting point of a global search. The new iterative OFR algorithm can be summarised in the following steps.

i) Preset a tolerance ρ and apply the standard OFR algorithm on the whole term dictionary $\mathbf{\Phi}$ to produce a suboptimal term set $\mathbf{\Phi}_s = \left\{\phi_{s_1} \quad \phi_{s_2} \quad \cdots \quad \phi_{s_{\kappa_s}}\right\}$;

ii) Select a small number $\Delta\rho$ as an amendment to the tolerance in the first step;

iii) Select a subset $\mathbf{\Phi}_{pre} \subset \mathbf{\Phi}_s$ of the terms ϕ_j, where $j = s_1, s_2, \cdots, s_{\kappa_s}$, in $\mathbf{\Phi}_s$ as preselected terms and search the other terms on the term set $\mathbf{\Phi} \backslash \mathbf{\Phi}_{pre}$ to construct a suboptimal solution satisfying $1 - \sum ERR_i < \rho + \Delta\rho$;

iv) Repeat iii) for different subset $\mathbf{\Phi}_{pre}$'s of $\mathbf{\Phi}_s$ and obtain some suboptimal models;

v) Compare the obtained suboptimal models and choose the best one as the final model $\mathbf{\Phi}_{op}$.

Remarks:

The subset $\mathbf{\Phi}_{pre}$ is often selected as a combination of p terms in $\mathbf{\Phi}_s$. There are a total number of $\binom{\kappa_s}{p}$ combinations. All the combinations are evaluated in step iv) and $\binom{\kappa_s}{p}$ candidate models are obtained. However, in most cases, the new iOFR can produce the optimal model using the setting p=1 (Yuzhu Guo, L. Z. Guo, S. A. Billings, & H. L. Wei, 2015a).

3.2 Selection of appropriate modulating functions

Using the B-spline form modulating functions three parameters need to be determined before the application of the OFR-modulating function method: the order of the B-spline function k, the length of the support interval $(b\text{-}a)$ of the modulating function, and the shift time with which the modulating functions are sliding over the signal.

A kth order B-spline function is $(k\text{-}1)$ times differentiable. Hence, in order to identify an mth order dynamic system, at least an $(m+1)$th B-spline function is needed. When the modulating function smoothly slides over the signal, the modulating process is actually a convolution of the modulating function and the signal. From the frequency domain viewpoint, the modulating function acts as a filter of which the impulse response function is the modulating function. The frequency response functions of the modulating function and the associated derivatives have been discussed in the reference where the length of the support interval $(b\text{-}a)$ is suggested to take a value of the dominating time constant of the signal(Preisig & Rippin, 1993). Actually a modulating function with a smaller support interval which means a wider pass band may keep more information of the original system in the transformed data but the elimination of the noise will be less significant. Fortunately, the frequency of noise is often much higher than the system frequencies. The selection of the sliding interval is relatively flexible, which depends on the number of the data available and the number of data needed for parameter estimation. A very small sliding interval time may cause the modulated data sets to be similar to each other and result in the singularity of the information matrix.

4. Test examples

In this section, the Lorenz system and the van der Pol oscillator will be used to show the efficiency of the iOFR-MF method. Because of the fact that both examples are autonomous systems, no inputs can be designed to guarantee the persistent excitation of the systems. Some of the system characteristics may therefore not be fully exhibited in the observed data. For this reason, the iterative OFR algorithm is used to avoid any potential problems caused by non-persistent excitation (Guo et al., 2015a). Notice that in both examples we make no assumptions about the form of the model, the model structure and the unknown parameters are all determined from the data with no a priori information to simulate the conditions that would exist in the identification of real systems.

4.1 Identification of a Lorenz system

The Lorenz system is widely used as the prototype for the study of chaos. The Lorenz system can be represented using the following nonlinear state-space equations.

$$\begin{cases} \dfrac{dx_1}{dt} = \sigma(x_2 - x_1) \\ \dfrac{dx_2}{dt} = x_1(\rho - x_3) - x_2 \\ \dfrac{dx_3}{dt} = x_1 x_2 - \beta x_3 \end{cases} \tag{24}$$

The parameters are set as σ =10, ρ =28, and β =8/3. The system exhibits chaotic behaviour for these parameters. The system was simulated with a sampling time of 0.002s. In this example, we assume all the system states are measurable. The simulated data were corrupted by 10% Gaussian white noise to mimic the effects of measurement noise, that is,

$$z_i(t) = x_i(t) + n_i(t) \quad i = 1, 2, 3 \tag{25}$$

where $n_i(t)$ is a Gaussian white noise process.

The noisy data are show in Fig 2. Figure 3(a) shows the phase portrait of the noisy data in the three-dimensional state space.

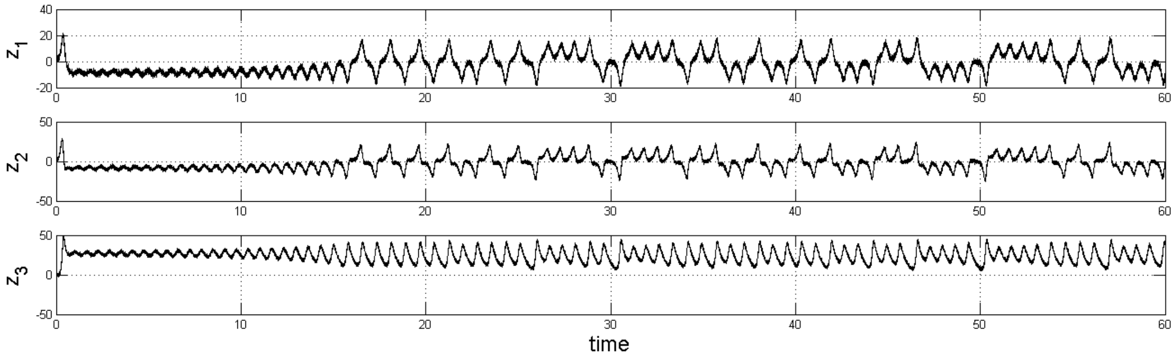
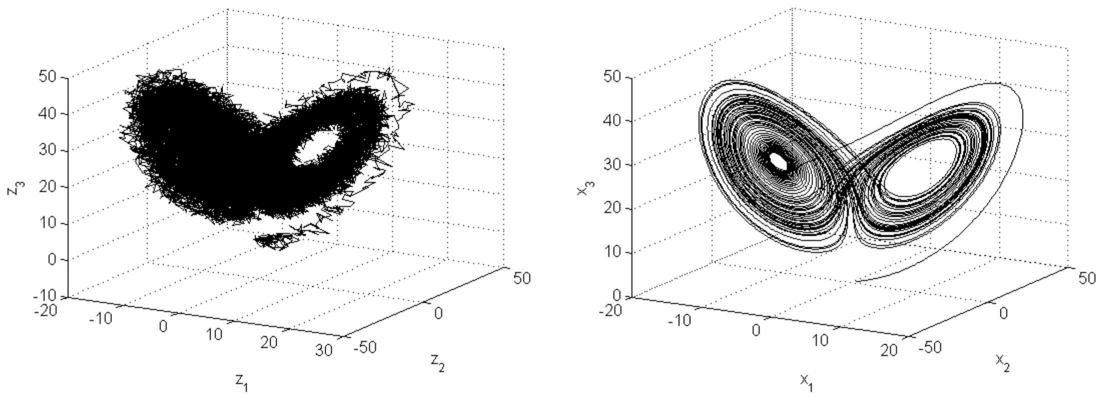


Fig 2 Data used for the system identification of the Lorenz system

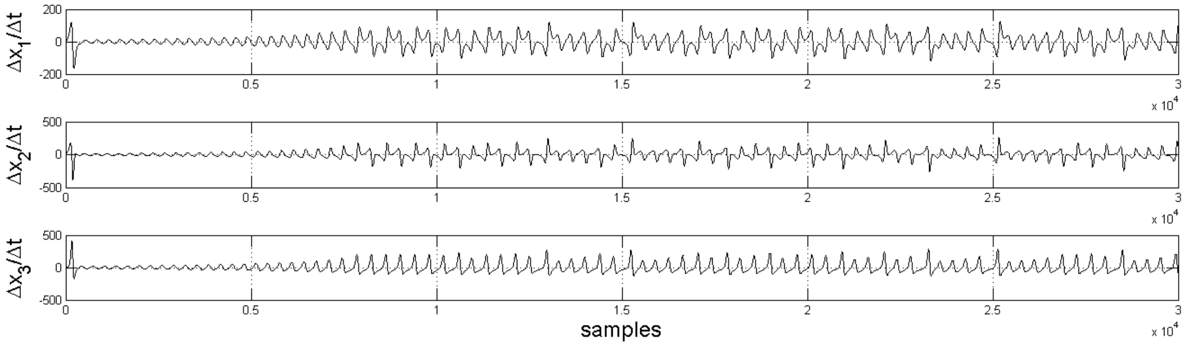


(a) (b)

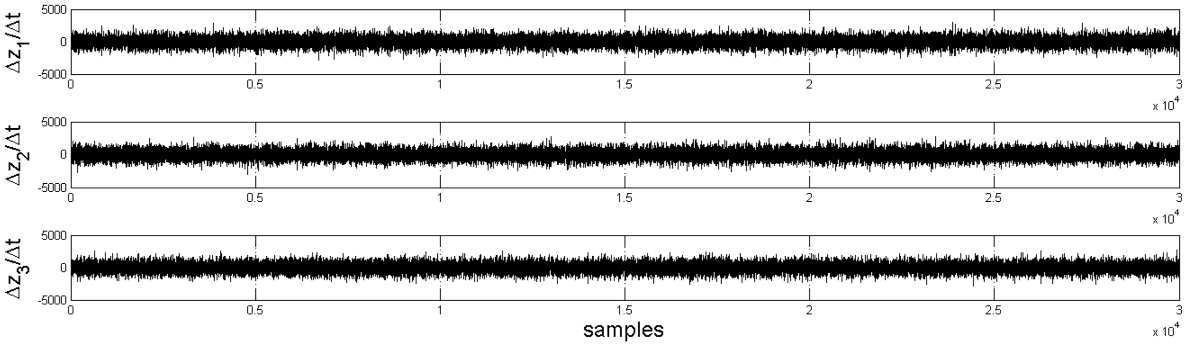
Fig 3 Phase portrait of the Lorenz system

(a) Measurement of data (b) Model prediction output of identified model in Table 3

In this example, we assume the relationships among the three variables are not known. Both the finite difference method and the new iOFR-MF method will be used discover both the nonlinear interactions among the system states and also the associated parameters. The Lorenz system was firstly identified using the finite difference method where the derivatives of the system states were approximated using the finite difference method. The finite differences of the measured states with 10% measurement noise are shown in Fig 4(b). Figure 4(a) shows the finite differences of the noise-free signals. It can be observed that the useful information in Figure 4(a) is overwhelmed by the magnified noise after the finite difference operation in Fig 4(b).



(a) finite differences of the noise-free state variables



(b) finite differences of the noisy state variables

Fig 4 The finite difference of the system state variables

The model structures were assumed unknown a priori and a term dictionary which consisted of all the up to third order polynomial terms of the system states was used, that is, $\mathcal{D} = \left\{ z_1, z_2, z_3, z_1^2, z_1 z_2, z_1 z_3, z_2^2, z_2 z_3, z_3^2, z_1^3, z_1^2 z_2, z_1^2 z_3, z_1 z_2^2, z_1 z_2 z_3, z_1 z_3^2, z_2^2 z_3, z_2 z_3^2, z_3^3 \right\}$. The significant terms were selected from the dictionary to form the models using the iOFR algorithm. The identified results are shown in Table 2. It can be observed that the model structures have been correctly detected by the iOFR algorithm even through the finite differences of the signals have been severely corrupted by the noise.However, the parameters are far from the real values.

Table 2 Model identified using the finite difference method for example 1

	No.	Terms	ERRs	Coefficients	Standard Deviation
subsystem1	1	z_2	0.058907	35.1577	0.9219
	2	z_1	5.319224	-42.2815	1.033
	SERR	--	5.38	--	--
subsystem2	1	z_2	0.615734	-47.6721	1.225
	2	z_1	0.773745	134.306	3.419

	3	z_1z_3	4.175286	-3.04449	0.08364
	SERR	--	5.56	--	--
subsystem3	1	z_3	0.074343	-4.05181	0.2148
	2	z_1z_2	1.645203	1.22792	0.05353
	SERR	--	1.72	--	--

The iOFR-modulating function method was then used to identify the system models where the cubic (4th order) B-spline function was used as the modulating functions with the length of support of 0.2s. The modulated derivatives $\dot{z}_i$ become $-\int_{t_0}^{t_0+0.2}\dot{B}_{1,4}(\tau-t_0)z_i(\tau)d\tau$, where t_0 represnets the time-shift of the cubical B-spline function. The dictionary of the modulated terms become

$$\mathcal{D}=\left\{\begin{array}{l}\int_{t_0}^{t_0+0.2}B_{1,4}(\tau-t_0)z_1(\tau)d\tau,\int_{t_0}^{t_0+0.2}B_{1,4}(\tau-t_0)z_2(\tau)d\tau,\int_{t_0}^{t_0+0.2}B_{1,4}(\tau-t_0)z_3(\tau)d\tau,\ldots,\\ \int_{t_0}^{t_0+0.2}B_{1,4}(\tau-t_0)z_2z_3^2(\tau)d\tau,\int_{t_0}^{t_0+0.2}B_{1,4}(\tau-t_0)z_3^3(\tau)d\tau\end{array}\right\}$$

Changing t_0 and collecting the data yields the data matrix $\mathbf{\Phi}$. The iOFR is then appliend and the identified models are shown in Table 3. The correct model structures have again been successfully identified and now the estimated parameters are very close to the correct values.

Table 3 Results produced by the iOFR-MF method for example 1

	No.	Terms	ERRs	Coefficients	Standard Deviation
subsystem1	1	z_2	19.6571	9.9502	0.00753
	2	z_1	78.65796	- 9.94434	0.008413
	SERR	--	98.32	--	--
subsystem2	1	z_1z_3	29.37048	- 0.997504	0.000732
	2	z_1	69.72317	27.9166	0.03067
	3	z_2	0.191828	- 0.973564	0.01086
	SERR	--	99.29	--	--
subsystem3	1	z_1z_2	53.07191	1.0024	0.000381
	2	z_3	46.49793	- 2.67029	0.001485
	SERR	--	99.57	--	--

4.2 Identification of a van der Pol oscillator

In this example, the van der Pol oscillator (26) will be identified to illustrate the identification of higher order input-output models. The van der Pol oscillator is widely used in the study of the limit cycles where a time-varying damping is used.

$$
\begin{cases}
\dfrac{d^2x}{dt^2} - \mu\left(1-x^2\right)\dfrac{dx}{dt} + x = 0 \\
z(t) = x(t) + n(t)
\end{cases}
\tag{26}
$$

The coefficient of the nonlinear damping was set as μ =3 and the period of the stable limit cycle is about 7.7s. The van der Pol oscillator was simulated using the 4th order Runge-Kutta algorithm at a sampling interval 0.01s and 10% Gaussian noise is added to the signal as the measurement noise.

This time, only the displacement $x(t)$ of the system is observed. The new iOFR-MF method is employed to identify the nonlinear relationship between the displacement, velocity and the acceleration. The structure of the system model is assumed unknown and assumed to have the form in (11). The left hand side of the equation was selected as the second order derivative $\ddot{x}$ of the system state with respect to time. The iOFR-modulating function method is used to detect the structure of the right hand side of the equation. The initial term dictionary was composed of the up to 5th order integrable polynomial terms $\left\{z, z^2, z^3, z^4, z^5, \dot{z}, z\dot{z}, z^2\dot{z}, z^3\dot{z}, z^4\dot{z}\right\}$. Obviously, more other intergrable terms can be included in the dictionary, for example $\sin(x)\dot{x}$. All the terms in the dictionary were modulated by a cubical B-spline function with a finite support 2s. The modulated terms are

$$
\left\{
\begin{array}{l}
\int_{t_0}^{t_0+0.2} B_{1,4}(\tau-t_0)z(\tau)d\tau, \int_{t_0}^{t_0+0.2} B_{1,4}(\tau-t_0)z^2(\tau)d\tau, \cdots, \int_{t_0}^{t_0+0.2} B_{1,4}(\tau-t_0)z^5(\tau)d\tau, \\
-\int_{t_0}^{t_0+0.2} \dot{B}_{1,4}(\tau-t_0)z(\tau)d\tau, -\dfrac{1}{2}\int_{t_0}^{t_0+0.2} \dot{B}_{1,4}(\tau-t_0)z^2(\tau)d\tau, \cdots, -\dfrac{1}{5}\int_{t_0}^{t_0+0.2} \dot{B}_{1,4}(\tau-t_0)z^5(\tau)d\tau
\end{array}
\right\}.
$$

Change the time-shift t_0 and construct the data matrix $\mathbf{\Phi}$. The system was then identified using the iOFR-modulating function method and the results are shown in Table 3. It can be observed that all the correct terms are selected and the associated coefficients are very close to the real values.

Table 3 Results produced by the iOFR-MF algorithm for example 2

No.	Terms	ERRs	Coefficients	Standard Deviation
1	x	39.53174	-0.9966	0.00112
2	$x^2\dot{x}$	13.23519	-2.00579	0.002159
3	$\dot{x}$	46.89805	2.0219	0.002464
SERR	--	99.66	--	--

5. Conclusions

Although the modulating method has been widely used for the identification of continuous models since it was introduced in 1954 (Shinbrot), the crucial problem about model structure detection has not been studied. In this paper, an efficient iOFR algorithm is used to extend the modulating function method to determine the correct model structure. A new iOFR-modulating function method is proposed to identify continuous time models direct from discrete data, involving both model structure detection and the associated parameter estimation. Simulations showed that the new algorithm performs significantly better than the finite difference method especially when the data is seriously polluted by noise.

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