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Underdetermined Wideband DOA Estimation of Off-Grid Sources Employing the Difference Co-Array Concept

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Abstract

A wideband off-grid model is proposed to represent dictionary mismatch under the compressive sensing framework exploiting difference co-arrays. A group sparsity based off-grid method is proposed for underdetermined wideband direction of arrival (DOA) estimation which provides improved performance over the existing group sparsity based method with a same search grid. A two-step approach is then proposed which achieves an even better performance with significantly reduced computational complexity.

Keywords: Off-Grid, difference co-array, group sparsity, DOA estimation, compressive sensing.

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1. Introduction

Sparse array geometries exploiting nonuniform sensor positions can provide a large number of virtual sensors in the difference co-array context, enabling direction of arrival (DOA) estimation of far more sources than the number of

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physical sensors. Two important array geometries, i.e., nested array [1, 2] and co-prime array [3, 4, 17], have been proposed for systematic sparse array design. Several approaches, including subspace-based methods employing spatial smoothing [1, 2, 3, 4, 5, 6] and a reshaping process to form a Toeplitz matrix [7] were proposed to exploit the increased number of degrees of freedom (DOFs) offered by co-arrays. In [8], a hybrid approach is presented to use a low-rank matrix denoising algorithm followed by a MUSIC-like subspace based method for DOA estimation. Note that all these subspace-based methods only utilize the consecutive difference co-array lags corresponding to a virtual uniform linear array (ULA) for DOA estimation.

On the other hand, by using compressive sensing (CS) [9, 10] based signal reconstruction methods for underdetermined DOA estimation, a higher number of DOFs is achieved by effectively using all consecutive and non-consecutive lags of the resulting difference co-arrays [11, 12, 13, 14, 15, 16, 17]. A performance analysis of the CS-based methods is provided in [18].

One of the major issues involved with the CS-based approach is the off-grid problem. That is, the true signal DOAs may not necessarily fall on the exact discrete grid which is defined over a finite number of spatial angles. Off-grid sources cause so-called dictionary mismatch problem which violates the sparsity conditions and compromises the performance as well as the identifiability of the CS-based methods [19, 20, 21, 22, 23]. One solution to this problem is to use a denser search grid with an increased number of angles, thus leading to higher computational complexity. An iterative procedure reduces complexity by adaptively refining the grid only around the regions where the sources are located [24]. A sparse Bayesian learning solution to such iterative algorithm is proposed in [25]. The complexity of these iterative methods, however, still remains high. In [26], the off-grid DOA estimation problem is considered as a nonconvex positive perturbed basis pursuit denoising problem, which is then solved using a simpler alternating algorithm based on a convex optimization approach. In addition, a joint sparse recovery method is developed for underdetermined off-grid DOA estimation of narrowband signals [27, 28].

For wideband signals, by enforcing the same spatial support across the entire frequency band of interest, group sparsity (GS) based method has proven effective for DOA estimation [29]. However, except for using a denser grid, the off-grid problem for wideband DOA estimations has not yet been considered. In this paper, for the first time, the wideband dictionary mismatch problem is dealt with by employing the difference co-array equivalence concept. We first extend the narrowband off-grid model to the wideband case, and derive a GS-based off-grid (GS-OG) DOA estimation approach for joint recovery of the sparse signal power entries and the associated off-grid calibration terms. The wideband extension of the narrow off-grid algorithm is not a simple average of the results obtained from different frequencies and a new formulation based on the group sparsity concept has to be introduced to effectively exploit the information carried across the frequency band of a wideband signal. In addition, although such an extension is effective in theory, it has two additional challenges: 1) the number of parameters to be estimated is very large (including both the on-grid angles and the associated offsets), which renders the estimation problem difficult to solve and leads to inaccurate DOA estimation result; 2) the method's complexity is significantly high.

To tackle these two additional challenges, a two-step off-grid (TS-OG) approach is proposed for complexity reduction, where the signal powers and the off-grid terms are estimated separately. In the first step, the GS-based DOA estimation is utilized to yield a coarser grid estimation, whereas in the second step, off-grid optimization is performed to estimate the off-grid bias vector which is constrained to be identical across all frequency bins. Both off-grid wideband methods outperform the GS-based one, and the two-step approach offers a significant complexity reduction while achieving a better performance compared with the GS-OG method.

This paper is organized as follows. A wideband signal model based on the difference co-array concept is presented in Sec. 2. A wideband off-grid model and two off-grid estimation methods are proposed in Sec. 3. Simulation results are provided in Sec. 4, and conclusions are drawn in Sec. 5.

2. Wideband Signal Model Exploiting the Difference Co-Array

Consider an N-sensor linear array and denote $\alpha_n d$ as the n-th sensor position and d as the unit inter-element spacing. The set of sensor positions S is expressed as

$$S = \{\alpha_n d, 0 \leq n \leq N-1\} \ . \tag{1}$$

As presented in [1, 3], the nested array and co-prime array are designed to optimize the virtual sensor positions corresponding to the difference co-array concept, defined as

$$\mathbb{C} = \{(\alpha_{n_1} - \alpha_{n_2})d, 0 \leq n_1, n_2 \leq N-1\} \ . \tag{2}$$

Assume that there are K mutually uncorrelated far-field wideband signals impinging from incident angles $\theta_k, \ k = 1, \ldots, K$. Then, the signal observed at the n-th sensor can be expressed as

$$x_n(t) = \sum_{k=1}^{K} s_k \left[t - \tau_n(\theta_k)\right] + \overline{n}_n(t) \ , \tag{3}$$

where $0 \leq n \leq N-1$, $s_k(t)$ is the k-th impinging signal, and $\overline{n}_n(t)$ represents the white noise at the corresponding sensor. Taking the zeroth sensor as the reference, $\tau_n(\theta_k)$ denotes the time delay of the k-th impinging signal with the incident angle θ_k arriving at the n-th sensor of the linear array.

After sampling with a frequency f_s, the discrete version of the observed signal vector in the time domain can be expressed as

$$\mathbf{x}[i] = \left[x_0[i], x_1[i], \ldots, x_{N-1}[i]\right]^T \ , \tag{4}$$

where i represents the discrete-time variable, $x_n[i]$ is the signal observed at the n-th sensor, and $\{\cdot\}^T$ denotes the transpose operation.

Then, an L-point discrete Fourier transform (DFT) is applied, where each received sensor signal is divided into P non-overlapping groups with length L for DFT application, and $p = 1, \ldots, P$ is the group index. We can obtain the observed signal vector $\mathbf{X}[l,p]$ at the p-th DFT group and the l-th frequency bin, given by

$$\mathbf{X}[l,p] = \left[X_0[l,p], X_1[l,p], \ldots, X_{N-1}[l,p]\right]^T , \tag{5}$$

where $l = 0, 1, \ldots, L-1$ and

$$X_n[l,p] = \sum_{i=0}^{L-1} x_n[L \cdot (p-1) + i] \cdot e^{-j\frac{2\pi}{L}il} . \tag{6}$$

Denote $\mathbf{S}[l,p]$ and $\overline{\mathbf{N}}[l,p]$ as the impinging source signal vector and the noise vector at the p-th DFT group and the l-th frequency bin, respectively. Then, the array output model in the frequency domain is given by

$$\mathbf{X}[l,p] = \mathbf{A}(l,\boldsymbol{\theta})\mathbf{S}[l,p] + \overline{\mathbf{N}}[l,p] , \tag{7}$$

where $\mathbf{A}(l,\boldsymbol{\theta})$ is the steering matrix whose column vector $\mathbf{a}(l,\theta_k)$ represents the steering vector at frequency f_l for the l-th frequency bin and angle θ_k

$$\mathbf{a}(l,\theta_k) = [e^{-j\frac{2\pi\alpha_0 d}{\lambda_l}\sin(\theta_k)}, \ldots, e^{-j\frac{2\pi\alpha_{N-1} d}{\lambda_l}\sin(\theta_k)}]^T , \tag{8}$$

where $\lambda_l = c/f_l$, and c is the signal propagation speed.

To exploit the increased DOFs provided by the virtual array based on the difference co-array, we first calculate the correlation matrix $\mathbf{R_{xx}}[l]$ as follows,

$$\begin{aligned}
\mathbf{R_{xx}}[l] &= \mathrm{E}\left\{\mathbf{X}[l,p] \cdot \mathbf{X}^H[l,p]\right\} \\
&= \sum_{k=1}^{K} \sigma_k^2[l]\mathbf{a}(l,\theta_k)\mathbf{a}^H(l,\theta_k) + \sigma_{\bar{n}}^2[l]\mathbf{I}_N \\
&\approx \frac{1}{P}\sum_{p=0}^{P-1} \mathbf{X}[l,p] \cdot \mathbf{X}^H[l,p] ,
\end{aligned} \tag{9}$$

where $\mathrm{E}\{\cdot\}$ is the expectation operator and $\{\cdot\}^H$ the Hermitian transpose operator. $\sigma_k^2[l]$ is the power of the k-th impinging signal at the l-th frequency bin, while $\sigma_{\bar{n}}^2[l]$ is the corresponding noise power. $\mathbf{I}_N$ is the $N \times N$ identity matrix. As shown in (9), P samples at the l-th frequency bin are used to estimate the correlation matrix.

By vectorizing $\mathbf{R_{xx}}[l]$, we obtain the following virtual array model

$$\mathbf{z}[l] = \text{vec}\{\mathbf{R_{xx}}[l]\} = \mathbf{B}[l]\mathbf{u}[l] + \sigma_{\bar{n}}^2[l]\widetilde{\mathbf{I}}_{N^2} \ , \tag{10}$$

where $\mathbf{B}[l]$ is the equivalent steering matrix of the difference co-array described by $\mathbb{C}$, with its k-th column vector $\mathbf{b}(l,\theta_k) = \mathbf{a}^*(l,\theta_k) \otimes \mathbf{a}(l,\theta_k)$ representing the corresponding steering vector ($\otimes$ denotes the Kronecker product). $\mathbf{u}[l] = \left[\sigma_1^2[l], \ldots, \sigma_K^2[l]\right]^T$ is the equivalent source signal vector holding all signal powers and $\widetilde{\mathbf{I}}_{N^2}$ is an $N^2 \times 1$ column vector obtained by vectorizing $\mathbf{I}_N$.

3. Group Sparsity Based Underdetermined Wideband DOA Estimation for Off-Grid Sources

3.1. Off-grid virtual model generation for a single frequency

Under the CS framework, we first generate a predefined search grid of K_g uniformly distributed potential incident angles $\theta_{g,0}, \ldots, \theta_{g,K_g-1}$, with $\{\cdot\}_g$ representing entries, vectors or matrices related to the predefined grid. Then, we construct an overcomplete representation of the equivalent steering matrix $\mathbf{B}_g[l] = \left[\mathbf{b}(l,\theta_{g,0}), \ldots, \mathbf{b}(l,\theta_{g,K_g-1})\right]$, and the corresponding unknown $K_g \times 1$ vector $\mathbf{u}_g[l]$ for possible source powers at directions θ_{k_g}, $0 \le k_g \le K_g - 1$, with its k_g-th entry denoted by $u_{g,k_g}[l]$. Under the perfect condition that the actual incident angles fall exactly on this predefined search grid, the virtual array model (10) can be transformed into

$$\mathbf{z}[l] = \mathbf{B}_g[l]\mathbf{u}_g[l] + \sigma_{\bar{n}}^2[l]\widetilde{\mathbf{I}}_{N^2} \ , \tag{11}$$

where for $1 \le k \le K$, we have

$$u_{g,k_g}[l] = \begin{cases} \sigma_k^2[l], & \theta_{g,k_g} = \theta_k \ , \\ 0, & \text{others} \ . \end{cases} \tag{12}$$

However, it is difficult to accurately represent the actual virtual structure model with a finite number of incident angles. Clearly, a more effective model approximation can be obtained by predefining a denser search grid with a much larger number of angles, but with a significantly increased computational complexity.

An off-grid model was investigated recently to overcome the dictionary mismatch problem [26, 27, 28]. For a single frequency, the equivalent steering vector at the actual incident angle θ_k can be approximated by applying the Taylor expansion to its nearest angle θ_{g,m_k} in the finite grid by

$$\mathbf{b}(l,\theta_k) \approx \sum_{\mu=0}^{\infty} \frac{\partial^{(\mu)}\mathbf{b}(l,\theta_{g,m_k})}{\mu! \cdot \partial\theta_{g,m_k}^{(\mu)}}(\theta_k - \theta_{g,m_k})^{\mu} \ , \tag{13}$$

where $-\frac{r}{2} \leq \theta_k - \theta_{g,m_k} \leq \frac{r}{2}$ with $r = \theta_{g,k_g+1} - \theta_{g,k_g}$ as the step size of the predefined grid, $\mu!$ denotes the factorial of μ, and $\frac{\partial^{(\mu)}f(\theta)}{\partial\theta^{(\mu)}}$ is the μ-th derivative of $f(\theta)$.

Using the first-order Taylor expansion, the off-grid model over the predefined search grid can be expressed as

$$\mathbf{z}[l] \approx \big(\mathbf{B}_g[l] + \mathbf{B}_g^{(1)}[l]\boldsymbol{\Delta}_g[l]\big)\mathbf{u}_g[l] + \sigma_{\bar{n}}^2[l]\widetilde{\mathbf{I}}_{N^2} \ , \tag{14}$$

where $\mathbf{B}_g^{(1)}[l] = \big[\frac{\partial\mathbf{b}(l,\theta_{g,0})}{\partial\theta_{g,0}}, \ldots, \frac{\partial\mathbf{b}(l,\theta_{g,K_g-1})}{\partial\theta_{g,K_g-1}}\big]$, and the diagonal matrix is generated by $\boldsymbol{\Delta}_g[l] = \mathrm{diag}\{\boldsymbol{\alpha}_g[l]\}$ with the k_g-th entry in the column bias vector $\boldsymbol{\alpha}_g[l]$ defined as

$$\alpha_{k_g}[l] = \begin{cases} \theta_k - \theta_{g,k_g}, & k_g = m_k \ , \\ 0, & \text{others} \ , \end{cases}$$

where $0 \leq k_g \leq K_g - 1$.

3.2. Group sparsity based off-grid wideband DOA estimation

In this section, we extend the narrowband off-grid model to the wideband case by proposing a GS-based off-grid DOA estimation method through simultaneous estimation of both the grid angles and the corresponding off-grid biases.

Assume that the frequency band of interest covers $Q \leq L$ frequency bins indexed by l_q, $q = 0, \ldots, Q-1$, which may or may not occupy consecutive frequency bands. Stack the virtual array vectors corresponding to the Q frequency bins as $\tilde{\mathbf{z}} = \big\{\mathbf{z}^T[l_0], \ldots, \mathbf{z}^T[l_{Q-1}]\big\}^T$ and construct a block diagonal matrix $\widetilde{\mathbf{B}}$ as

$$\widetilde{\mathbf{B}} = \mathrm{blkdiag}\left\{\mathbf{B}[l_0], \mathbf{B}[l_1], \ldots, \mathbf{B}[l_{Q-1}]\right\} \ , \tag{15}$$

where $\text{blkdiag}\{\cdot\}$ denotes an operation to construct a block diagonal matrix from the argument matrices.

The wideband model can then be expressed as

$$\tilde{\mathbf{z}} = \widetilde{\mathbf{B}}\tilde{\mathbf{u}} + \mathbf{W}\mathbf{v} \ , \tag{16}$$

where $\tilde{\mathbf{u}} = \left[\mathbf{u}^T[l_0], \ldots, \mathbf{u}^T[l_{Q-1}]\right]^T$, $\mathbf{W} = \text{blkdiag}\left\{\widetilde{\mathbf{I}}_{N^2}, \ldots, \widetilde{\mathbf{I}}_{N^2}\right\}$ is a $QN^2 \times Q$ matrix, and $\mathbf{v} = \left[\sigma_{\bar{n}}^2[l_0], \ldots, \sigma_{\bar{n}}^2[l_{Q-1}]\right]^T$ is a column vector holding all noise powers across the frequency bins of interest.

With the same search grid for each frequency bin, the sparse wideband model under the perfect condition of on-grid sources is given by

$$\tilde{\mathbf{z}} = \widetilde{\mathbf{B}}_g\tilde{\mathbf{u}}_g + \mathbf{W}\mathbf{v} \ , \tag{17}$$

where $\widetilde{\mathbf{B}}_g = \text{blkdiag}\left\{\mathbf{B}_g[l_0], \mathbf{B}_g[l_1], \ldots, \mathbf{B}_g[l_{Q-1}]\right\}$, and $\tilde{\mathbf{u}}_g = \left[\mathbf{u}_g^T[l_0], \mathbf{u}_g^T[l_1], \ldots, \mathbf{u}_g^T[l_{Q-1}]\right]^T$.

In the case of off-grid sources, we exploit the first-order Taylor expansion of the equivalent steering matrix $\widetilde{\mathbf{B}}_g$. Accordingly, the off-grid wideband model can be approximated by

$$\tilde{\mathbf{z}} \approx \left(\widetilde{\mathbf{B}}_g + \widetilde{\mathbf{B}}_g^{(1)}\widetilde{\boldsymbol{\Delta}}_g\right)\tilde{\mathbf{u}}_g + \mathbf{W}\mathbf{v} \ , \tag{18}$$

where $\widetilde{\mathbf{B}}_g^{(1)} = \text{blkdiag}\left\{\mathbf{B}_g^{(1)}[l_0], \ldots, \mathbf{B}_g^{(1)}[l_{Q-1}]\right\}$, and $\widetilde{\boldsymbol{\Delta}}_g = \text{diag}\left\{\widetilde{\boldsymbol{\alpha}}_g\right\}$ with $\widetilde{\boldsymbol{\alpha}}_g = \left[\boldsymbol{\alpha}_g^T[l_0], \ldots, \boldsymbol{\alpha}_g^T[l_{Q-1}]\right]^T$.

Construct a $K_g \times Q$ matrix $\widetilde{\mathbf{U}}_g = \left[\mathbf{u}_g[l_0], \ldots, \mathbf{u}_g[l_{Q-1}]\right]$, and use $\tilde{\mathbf{u}}_{g,k_g}$ to represent its k_g-th row. Then, we obtain the following column vector by an ℓ_2-norm operation

$$\hat{\mathbf{u}}_g = \left[\left\|\tilde{\mathbf{u}}_{g,0}\right\|_2, \left\|\tilde{\mathbf{u}}_{g,1}\right\|_2, \ldots, \left\|\tilde{\mathbf{u}}_{g,K_g-1}\right\|_2\right]^T \ , \tag{19}$$

where $\left\|\cdot\right\|_2$ denotes the ℓ_2 norm.

Joint recovery of $\tilde{\mathbf{u}}_g$ and $\widetilde{\boldsymbol{\Delta}}_g$ results in a non-convex optimization problem. To permit convexity, we define a column vector $\boldsymbol{\beta}_g[l] = \boldsymbol{\Delta}_g[l]\mathbf{u}_g[l]$ and a matrix $\mathbf{Y}_g = \left[\mathbf{u}_g[l_0], \ldots, \mathbf{u}_g[l_{Q-1}], \boldsymbol{\beta}[l_0], \ldots, \boldsymbol{\beta}[l_{Q-1}]\right]$, and enforce joint sparsity on $\mathbf{u}_g[l]$ and $\boldsymbol{\beta}_g[l]$. Then, a column vector $\hat{\mathbf{y}}_g^\circ$ is formed by $\hat{\mathbf{y}}_g^\circ = \left[\left\|\mathbf{y}_{g,0}\right\|_2, \ldots, \left\|\mathbf{y}_{g,K_g-1}\right\|_2, \left\|\mathbf{v}\right\|_2\right]^T$, where $\mathbf{y}_{g,k_g}$ is used to represent the k_g-th row of the matrix $\mathbf{Y}_g$. Finally, the proposed GS-based off-grid (GS-OG) method is formulated as the following convex optimization problem

$$\begin{aligned} \min_{\tilde{\mathbf{u}}_g,\widetilde{\boldsymbol{\beta}}_g,\mathbf{v}} \quad & \left\|\hat{\mathbf{y}}_g^{\circ}\right\|_1 \\ \text{subject to} \quad & \left\|\tilde{\mathbf{z}} - \widetilde{\mathbf{B}}_g\tilde{\mathbf{u}}_g - \widetilde{\mathbf{B}}_g^{(1)}\widetilde{\boldsymbol{\beta}}_g - \mathbf{W}\mathbf{v}\right\|_2 \leq \varepsilon \ , \\ & -\frac{r}{2}\tilde{\mathbf{u}}_g \preceq \widetilde{\boldsymbol{\beta}}_g \preceq \frac{r}{2}\tilde{\mathbf{u}}_g \ , \end{aligned} \tag{20}$$

where $\left\|\cdot\right\|_1$ is the ℓ_1 norm, ε is the allowable error bound, $\preceq$ represents $\leq$ elementwise, and $\widetilde{\boldsymbol{\beta}}_g = \left[\boldsymbol{\beta}_g^T[l_0], \ldots, \boldsymbol{\beta}_g^T[l_{Q-1}]\right]^T$. Note that $\hat{\mathbf{u}}_g$ in (19) is the initial DOA results over the search grid of K_g angles, and the associated off-grid bias vector is obtained by $\widehat{\boldsymbol{\alpha}}_g = \frac{1}{Q}\sum_{q=0}^{Q-1}\boldsymbol{\beta}[l_q] \oslash \mathbf{u}_g[l_q]$ with $\oslash$ representing the elementwise division of two vectors. Only the entries in $\widehat{\boldsymbol{\alpha}}_g$ corresponding to the non-zero entries in $\hat{\mathbf{u}}_g$ are considered for DOA calibration.

Note in (20) that, as a common practice, the problem is formulated as an ℓ_1-norm minimization [24, 30] because the ℓ_0-norm optimization is in general an NP-hard (Non-Deterministic Polynomial Hard) problem which is difficult to solve. On the other hand, the relaxed ℓ_1-norm problem is a convex optimization problem whose global optimum can be effectively determined by linear programming.

3.3. Two-step off-grid wideband DOA estimation

Estimation of $\tilde{\mathbf{u}}_g$ and $\widetilde{\boldsymbol{\beta}}_g$ simultaneously based on the GS concept in (20) is a time consuming process with extremely high complexity. Therefore, we propose a two-step method with simplified solution of the above two variables, leading to significant reductions in computational complexity.

The bias vectors $\boldsymbol{\alpha}_g[l_q]$, $q = 0, \ldots, Q-1$, share the same value across the entire frequency band. By enforcing $\boldsymbol{\alpha}_g[l_q] = \widehat{\boldsymbol{\alpha}}_g$, $\forall q = 0, \ldots, Q-1$, with $\widehat{\boldsymbol{\alpha}}_g$ representing the off-grid bias vector to be estimated, in lieu of $\boldsymbol{\beta}_g[l_q]$, a significant complexity reduction can be achieved. Toward this purpose, $\widetilde{\boldsymbol{\alpha}}_g$ can be expressed as

$$\widetilde{\boldsymbol{\alpha}}_g = \left[\widehat{\boldsymbol{\alpha}}_g^T, \widehat{\boldsymbol{\alpha}}_g^T, \ldots, \widehat{\boldsymbol{\alpha}}_g^T\right]^T = \mathbf{1}_{K_g} \otimes \widehat{\boldsymbol{\alpha}}_g \ , \tag{21}$$

where $\mathbf{1}_{K_g}$ is an all-one $K_g \times 1$ column vector.

A two-step (TS) approach to separately estimate $\tilde{\mathbf{u}}_g$ and $\widehat{\boldsymbol{\alpha}}_g$ can then be implemented. This approach is referred to as the TS-OG method, where two convex optimization problems are formulated as

$$\begin{aligned} \text{Step 1: } \min_{\tilde{\mathbf{u}}_g, \mathbf{v}} \quad & \left\|\hat{\mathbf{u}}_g^{\circ}\right\|_1 \\ \text{subject to} \quad & \left\|\tilde{\mathbf{z}} - \widetilde{\mathbf{B}}_g \tilde{\mathbf{u}}_g - \mathbf{W}\mathbf{v}\right\|_2 \leq \varepsilon\ , \\ \text{Step 2: } \min_{\widehat{\boldsymbol{\alpha}}_g} \quad & \left\|\tilde{\mathbf{z}} - \widetilde{\mathbf{B}}_g \tilde{\mathbf{u}}_g - \widetilde{\mathbf{B}}_g^{(1)} (\widetilde{\boldsymbol{\alpha}}_g \odot \tilde{\mathbf{u}}_g) - \mathbf{W}\mathbf{v}\right\|_2 \\ \text{subject to} \quad & -\frac{r}{2}\mathbf{1}_{K_g} \preceq \widehat{\boldsymbol{\alpha}}_g \preceq \frac{r}{2}\mathbf{1}_{K_g}, \end{aligned} \tag{22}$$

where $\hat{\mathbf{u}}_g^{\circ} = \left[\hat{\mathbf{u}}_g^T, \left\|\mathbf{v}\right\|_2\right]^T$, and $\odot$ represents the elementwise multiplication of two vectors. The first step is the former GS-based formulation used to recover $\tilde{\mathbf{u}}_g$, followed by a minimization problem with a bounded constraint to obtain the off-grid bias vector $\widehat{\boldsymbol{\alpha}}_g$. By estimating $\tilde{\mathbf{u}}_g$ and $\widehat{\boldsymbol{\alpha}}_g$ separately, the increase of complexity associated with Step 2 becomes limited, while an improved performance can be achieved, as shown in our simulations.

4. Simulation Results

We consider an example of $K = 12$ wideband source signals with their off-grid incident angles uniformly distributed between $-59.25°$ and $58.75°$. An $L = 64$ point-DFT is applied. The frequency bins of interest cover the range from 17 to 31 with $Q = 15$ bins in total, corresponding to the normalized frequency range from 0.5π to π. Setting $N_1 = 3$ and $N_2 = 4$, a co-prime array of $2N_1 + N_2 - 1 = 9$ sensors is considered with sensor position set $S = \{0d, 3d, 4d, 6d, 8d, 9d, 12d, 16d, 20d\}$, where $d = \lambda_{\min}/2$ with $\lambda_{\min} = 2c/f_s$ being the minimum wavelength within the frequency band of interest. The number of samples used for the correlation matrix calculation at each frequency bin is set to be $P = 1000$. A search grid of $K_g = \frac{180}{r} + 1$ potential angles associated with the step size r is generated within the full angle range from $-90°$ to $90°$, and the allowable error bound ε is chosen to give the best result through trial-and-error in every experiment. In our simulations, a software package called CVX

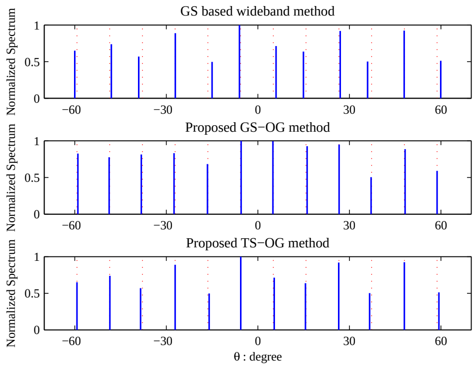


Figure 1: Results obtained by different wideband DOA estimation methods, where the dotted lines represent the actual incident angles of the impinging signals, while the solid lines represent the estimation results.

for specifying and solving convex programs [31, 32] is used to solve all these optimization problems.

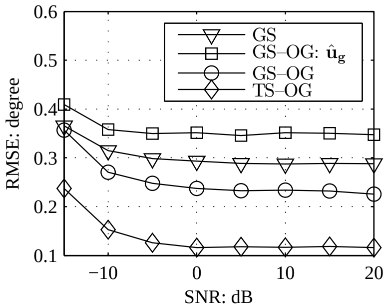
For the first set of simulations, we compare the DOA estimation performance of the existing GS-based method, the proposed GS-OG method and the TS-OG method. The input SNR is 0 dB, and a large step size of $r = 3^\circ$ is used for clear demonstration. As shown in Fig. 1, all the 12 sources (more than the number of physical sensors) have been distinguished successfully by the three methods. With calibration using the bias vector $\widehat{\boldsymbol{\alpha}}_g$, the results of the proposed two off-grid methods provide closer DOA estimates to the true values.

To further compare the estimation accuracy of different methods, we focus on the root mean square error (RMSE) results with respect to a varied input SNR through Monte Carlo simulations of 500 trials. The RMSE of the estimated DOAs is defined as

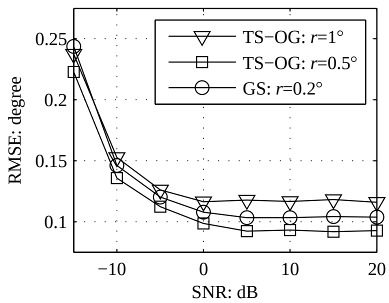
$$\text{RMSE} = \sqrt{\frac{1}{MK}\sum_{m=1}^{M}\sum_{k=1}^{K}\left(\hat{\theta}_k(m) - \theta_k\right)^2}\,, \tag{23}$$

where $M = 500$ is the number of independent simulation trials, and $\hat{\theta}_k(m)$ represents the estimate of θ_k at the m-th trial.

Fig. 2(a) gives the RMSE results obtained by different wideband DOA es-



(a) Comparison of different methods



(b) Effect of step size

Figure 2: RMSE results of different wideband DOA estimation methods with different step sizes.

timation methods with a fixed step size of $r = 1^\circ$, where "GS-OG: $\hat{\mathbf{u}}_g$" is for the initial estimation results $\hat{\mathbf{u}}_g$ before calibration of the GS-OG method, while "GS-OG" is for the final results after calibration using $\widehat{\boldsymbol{\alpha}}_g$. Clearly, with the same step size, the two proposed methods consistently outperform the existing GS one. Furthermore, the initial estimation results of the GS-OG before calibration is worse than the existing GS-based method, which is not surprising since there are more variables to estimate in the GS-OG method and therefore it represents a more difficult problem.

However, we can also observe from Fig. 2(a) that the TS-OG method has performed better than the GS-OG method. This may be explained as follows. Since $\boldsymbol{\beta}_g[l] = \boldsymbol{\Delta}_g[l]\mathbf{u}_g[l]$, the recovery of the bias vector $\boldsymbol{\alpha}_g[l]$ in $\boldsymbol{\Delta}_g[l]$ relies on the accuracy of both $\boldsymbol{\beta}_g[l]$ and $\mathbf{u}_g[l]$. However, as we discussed earlier, the recovered values of $\boldsymbol{\beta}_g[l]$ and $\mathbf{u}_g[l]$ may deviate from the true values due to the ℓ_1-norm approximation to the ℓ_0-norm. Therefore, the calculated bias vector as the element-wise ratio between the estimated $\boldsymbol{\beta}_g[l]$ and $\mathbf{u}_g[l]$ will not be as accurate as expected.

For the third set of simulations, we compare the estimation accuracy of our proposed TS-OG method with large step sizes of $r = 1^\circ$ and $r = 0.5^\circ$, with that of the GS-based method with a small step size of $r = 0.2^\circ$,and the result is shown in Fig. 2(b). As we can see, the GS-based method with $r = 0.2^\circ$ performs

Table 1: Running Time of Different Methods with Different Step Sizes

Step Size	K_g	GS	GS-OG	TS-OG
$r = 1^\circ$	181	9.158s	63.625s	11.419s
$r = 0.5^\circ$	361	54.503s	227.202s	62.771s
$r = 0.2^\circ$	901	170.622s	2503.378s	222.35s

a little better than the TS-OG method with $r = 1^\circ$, while the performance of the TS-OG method with $r = 0.5^\circ$ is the best. Therefore, by introducing the GS-based wideband off-grid model, a better performance can be achieved with a larger step size, leading to a reduced complexity.

For different step sizes, the computation time required by the MATLAB profiler under the environment of Intel CPU I5-3470 with a clock speed of 3.20 GHz and 12 GB RAM, is listed in Table 1 as an indication of their computational complexity. The proposed GS-OG method has the longest running time. For each method, a smaller the step size corresponds to a longer running time. It is noted that a better performance with a shorter running time is achieved by our proposed TS-OG method for a larger step size $r = 0.5^\circ$ compared with the GS-based method with a smaller step size of $r = 0.2^\circ$.

5. Conclusion

To overcome the dictionary mismatch problem in CS-based DOA estimation, a wideband off-grid signal model was developed, and two DOA estimation methods were proposed to achieve accurate results with a coarse search grid. The first one (GS-OG) is a direct extension of the narrowand case employing the group sparsity concept, while the other one is a two-step method (TS-OG) to reduce the computational complexity. Simulation results demonstrated that, for the same grid, the proposed methods provided improved performance as compared to the existing GS-based method. In particular, a significant reduction in computations has been achieved by the two-step method, which also provides a superior performance to the GS-OG method.

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