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Thue-Morse constant is not badly approximable

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We prove that Thue-Morse constant $\tau_{TM} = 0.01101001\dots_2$ is not a badly approximable number. Moreover, we prove that $\tau_{TM}(a) = 0.01101001\dots_a$ is not badly approximable for every integer base $a \geq 2$ such that a is not divisible by 15. At the same time we provide a precise formula for convergents of the Laurent series $\tilde{f}_{TM}(z) = z^{-1} \prod_{n=1}^{\infty} (1 - z^{-2^n})$, thus developing further the research initiated by Alf van der Poorten and others.

1 Introduction

Let $\mathbf{t} = (t_0, t_1, \dots) = (0, 1, 1, 0, 1, 0, 0, \dots)$ be the Thue-Morse sequence, that is the sequence $(t_n)_{n \in \mathbb{N}_0}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, defined by recurrence relations $t_0 = 0$ and for all $n \in \mathbb{N}_0$

$$\begin{aligned}t_{2n} &= t_n, \\t_{2n+1} &= 1 - t_n.\end{aligned}$$

Thue-Morse sequence appears naturally in the description of many recurrent processes [2]. It is often considered as the simplest non-trivial example of so called automatic sequences [2], i.e., sequences generated by finite automata (which are, in simple words, Turing Machines without memory tape).

Allouche and Shallit asked the following question (see [2], Open Problem 9, p. 403).

Problem 1.1. Determine whether the partial quotients of the Thue-Morse constant τ_{TM} , defined by

$$\tau_{TM} := \sum_{k=0}^{\infty} \frac{t_k}{2^{k+1}}, \quad (1)$$

are bounded from above by a universal constant. \square

Remark 1.2. The name of the constant τ_{TM} defined by (1) varies slightly from one reference to another. In some sources it is called Prouhet-Thue-Morse constant, and in some others it is referred to as Thue-Morse-Mahler constant [4]. In this article we choose the name Thue-Morse as the shortest commonly used name for it. \square

Problem 1.1 can be easily translated into the language of Diophantine approximations. It is a well known fact that the number x has bounded partial quotients if and only if it is badly approximable, i.e., there is a constant $c > 0$ such that for any $p/q \in \mathbb{Q}$ we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2} \quad (2)$$

(for instance, see [5], Chapter 1, §2). So Problem 1.1 is equivalent to the question whether the Thue-Morse constant $x = \tau_{TM}$ is badly approximable or in other words whether it satisfies (2).

This problem attracted interest in the last years. In particular, Bugeaud [3] showed that the transcendence exponent of τ_{TM} is 2, that is for any $\varepsilon > 0$ there exists a constant c_ε such that

$$\left| \tau_{TM} - \frac{p}{q} \right| > \frac{c_\varepsilon}{q^{2+\varepsilon}}. \quad (3)$$

Later Bugeaud and Queffélec [4] proved that the sequence of partial quotients of τ_{TM} contains infinitely many values equal to 4 or 5 and at the same time it contains infinitely many values bigger than 50. Note that in terms of the inequality (2) this result implies an absolute upper bound on the constant c : if such positive c exists then $c < 1/50$.

In this paper we solve Problem 1.1 by showing that τ_{TM} is not a badly approximable number. This result is proved in Theorem 4.9 below. To establish it, we provide a sequence $(p_n/q_n)_{n \in \mathbb{N}} \in \mathbb{Q}$ with $\lim_{n \rightarrow \infty} q_n \rightarrow \infty$, of good rational approximations to the Thue-Morse constant. These approximations satisfy

$$q_n |q_n \tau_{TM} - p_n| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4)$$

It straightforwardly implies that the inequality (2) is not satisfied for any positive constant c and approximations p_n/q_n where n is large enough.

We construct a sequence of approximations $(p_n/q_n)_{n \in \mathbb{N}}$ by a specialization of good *functional* approximations to a so called Thue-Morse generating function:

$$f_{TM}(z) := \sum_{i=0}^{\infty} (-1)^{t_i} z^i. \quad (5)$$

It is easy to see that the Thue-Morse constant can be represented as

$$\tau_{TM} = \frac{1}{2} \left(1 - \frac{1}{2} f_{TM}(1/2) \right).$$

In our article we focus on a slightly modified version of this function:

$$\tilde{f}_{TM}(z) := \frac{1}{z} f_{TM}(1/z).$$

It is a Laurent series and one can easily check that τ_{TM} is a badly approximable number if and only if $\tilde{f}_{TM}(2)$ is badly approximable too.

Note that the value $\tilde{f}_{TM}(2)$ also appears in the work of Dubickas [6]. It is shown there that for every irrational x one has

$$\inf_{n \in \mathbb{N}} \|2^n x\| \leq \tilde{f}_{TM}(2)$$

where $\|\cdot\|$ denotes the distance to the nearest integer. Moreover for $x = \tilde{f}_{TM}(2)$ the inequality becomes an equality.

To study functional approximations to $\tilde{f}_{TM}(z)$ we apply the theory of continued fractions for Laurent series, which is analogous to the classical theory of continued fractions of rational numbers (for instance, see [9]).

In our proof we take advantage of a functional equation for the function \tilde{f}_{TM} (see (10) below). This functional equation allows us, given one functional convergent to $\tilde{f}_{TM}(z)$, to produce an infinite sequence p_n/q_n of rational approximations to $\tilde{f}_{TM}(2)$ all satisfying

$$\left| \tilde{f}_{TM}(2) - \frac{p_n}{q_n} \right| \leq \frac{C}{q_n^2}, \quad (6)$$

where the constant C depends only on the initial functional approximation.

We find this construction interesting even on its own, as not only it allows to reproduce the results from [4] by choosing a good initial functional convergent to f_{TM} , but also it explains regularly situated large partial quotients in the continued fraction of τ_{TM} of the same value which can be observed numerically. For instance, in this way one can find an infinite sequence of partial quotients equal to 2569. With some computational efforts one can check that it is generated by the 15th convergent $P(z)/Q(z)$ of $\tilde{f}_{TM}(z)$.

Next, we manage to use arguments on congruences and primitive roots modulo 3^k , $k \in \mathbb{N}$, to justify that in a carefully chosen sequence of p_n/q_n satisfying (6) there will be arbitrarily large common factors r_n , hence after reducing by this common factor the couple of integers $(p_n/r_n, q_n/r_n)$ verifies (4) and so $\tilde{f}_{TM}(2)$ is not badly approximable.

In this paper we also provide the precise formulae for computing the convergents of Laurent power series $\tilde{f}_{TM}(z)$. Our interest in this subject is inspired by several papers by van der Poorten and others, where they study continued fractions for functions given by infinite products [1, 8, 9]. For instance, they numerically verified that the first partial quotients of $f_{TM}(1/z)$ have degree at most two. At the same time, the partial quotients of degree one have quickly growing coefficients, which is the generic behavior (see [9], section 2.1). The authors of [1] proved that all the partial quotients of Laurent power series

$$\sum_{k=0}^{\infty} (-1)^{t_k} z^{-3^k}$$

have degree one.

It appears that the continued fraction for $\tilde{f}_{TM}(z)$ is especially nice looking. Not only all its partial quotients have degree one, but, moreover, all the even ones are rational multiples of $z - 1$ and all the odd ones are rational multiples of $z + 1$ (see Proposition 3.2). We even provide simple recurrent formulae allowing to calculate rational factors of $z - 1$ and $z + 1$, thus giving the exact values of all the partial quotients of \tilde{f}_{TM} , see Proposition 3.3.

Generalizations. The natural question is whether it is possible to generalize our considerations to a broader framework. The launching site for our constructions in this article is the functional equation (10), so a natural extension is the following question.

Open Problem 1. Let $d \in \mathbb{N}$, $d \geq 2$ and let the function $f(z) \in \mathbb{Q}[[z]]$ satisfies the functional equation

$$f(z^d) = a(z)f(z) + b(z), \tag{7}$$

where $a(z), b(z) \in \mathbb{Q}(z)$. Moreover assume that $f(z)$ is a transcendental function. Let $a \in \mathbb{Q}$ be a non-zero rational within the radius of convergence of f . Determine the conditions for $f(a)$ to be a badly approximable number. \square

Our arguments in this article can be used to show that $\tilde{f}_{TM}(a)$ is not badly approximable for all $a \in \mathbb{N}$, $n \geq 2$, with possible exceptions when a is divisible by 15. This follows from Theorem 5.1 (see Corollary 5.2).

Note however that Van der Poorten and Shallit showed in [10] that the function $f_M(z) = \sum_{k=0}^{\infty} z^{2^k}$, which satisfies the functional equation

$$f_M(z^2) = f_M(z) - z,$$

has a badly approximable value at $z = 1/2$, moreover the continued fraction of $f_M(1/2)$ consists of just partial quotients 1 and 2 (actually in [10] this result is proved even for a much more general case of series $2 \sum_{k=0}^{\infty} \pm 2^{-2^k}$). So, the answer to Open Problem 1 definitively requires some additional conditions on the functional equation (7), separating the case of badly approximable values from not badly approximable ones.

While Open Problem 1 itself seems already enigmatic, we can consider even broader framework. The equation (7) is a classical example of so called Mahler's functional equation. In the most general framework, the following system of functional equations is known as *Mahler's system*:

$$a(z)\underline{f}(z^d) = A(z)\underline{f}(z) + B(z), \quad (8)$$

where $d \geq 2$ is an integer, $\underline{f}(z) = (f_1(z), \dots, f_n(z)) \in \mathbb{Q}[[z]]^n$, $a(z) \in \mathbb{Q}[z]$, A (resp. B) is an $n \times n$ (resp. $n \times 1$) matrix with coefficients in $\mathbb{Q}[z]$.

Open Problem 2. Let $d \in \mathbb{N}$, $d \geq 2$ and let the system of functions $\underline{f}(z) = (f_1(z), \dots, f_n(z)) \in (\mathbb{Q}[[z]])^n$ be a solution to the system (8). Moreover assume that $f_1(z)$ is a transcendental function. For a non-zero rational $a \in \mathbb{Q}$ within the radius of convergence of $\underline{f}(z)$ decide whether $f_1(a)$ is a badly approximable number or not. \square

2 General facts

It is well known ([2], §13.4) that the function $f_{TM}(z)$, defined by (5), admits the following presentation:

$$f_{TM}(z) = \prod_{k=0}^{\infty} (1 - z^{2^k}),$$

and the following functional equation holds:

$$f_{TM}(z^2) = \frac{f_{TM}(z)}{1 - z}. \quad (9)$$

As we have mentioned in the introduction, we will focus on the study of a slightly modified version of f_{TM} :

$$\tilde{f}_{TM}(z) := \frac{1}{z} f_{TM}(1/z).$$

Substituting $1/z$ in place of z into (9) we find that \tilde{f}_{TM} satisfies the following functional equation:

$$\tilde{f}_{TM}(z^2) = \frac{\tilde{f}_{TM}(z)}{z - 1}. \quad (10)$$

Easy verification shows that $f_{TM}(z)$ and $\tilde{f}_{TM}(z)$ are closely linked with Thue-Morse constant by

$$\tau_{TM} = \frac{1}{2} \left(1 - \frac{1}{2} f_{TM}(1/2) \right) = \frac{1}{2} \left(1 - \tilde{f}_{TM}(2) \right). \quad (11)$$

By rewriting (5) for $\tilde{f}_{TM}(z)$ one can easily check that $\tilde{f}_{TM}(z) \in \mathbb{Q}[[z^{-1}]]$. Moreover, all the coefficients of the resulting Laurent series are either 1 or -1 , therefore it converges for $|z| > 1$. Another consequence is that $\tilde{f}_{TM}(z)$ has the following continued fraction expansion

$$\tilde{f}_{TM}(z) = [0; a_1(z), a_2(z), \dots] = \frac{1}{a_1(z) + \frac{1}{a_2(z) + \dots}},$$

where $a_i(z) \in \mathbb{Q}[z]$, $i \in \mathbb{N}$ (the details can be found in [9]).

The important consequences of this fact are that the convergents $P_n(z)/Q_n(z)$ can be computed by the following recurrent formulae

$$\begin{aligned} P_{n+1}(z) &= a_{n+1}(z)P_n(z) + P_{n-1}(z), \\ Q_{n+1}(z) &= a_{n+1}(z)Q_n(z) + Q_{n-1}(z), \end{aligned} \quad (12)$$

for $n \geq 1$. Moreover Proposition 1 from [9] implies the following.

Proposition 2.1. Let $P(z), Q(z) \in \mathbb{Q}[z]$ be two polynomials. Then $P(z)/Q(z)$ is a convergent to $\tilde{f}_{TM}(z)$ if and only if

$$\deg(Q(z)\tilde{f}_{TM}(z) - P(z)) < -\deg Q(z), \quad (13)$$

where the degree of Laurent series $G(z) = \sum_{k=h}^{\infty} a_k z^{-k}$, $a_h \neq 0$ is minus the smallest index of a non-zero coefficient, that is in our notation we have $\deg G = -h$. \square

Note that unlike the classical setup of rational numbers, where the numerators and denominators p_n and q_n of convergents are defined uniquely, $P_n(z)$ and $Q_n(z)$ are only unique up to multiplication by a non-zero constant. At the same time, the polynomials computed by formulae (12) in general are not monic. So, we can add a condition that the numerator P_n has to be monic, producing a unique representative for each functional convergent to \tilde{f}_{TM} .

These canonical representatives $\hat{P}_n(z)/\hat{Q}_n(z)$, with \hat{P}_n a monic polynomial, are still linked by recurrent relations similar to (12):

$$\begin{aligned} \hat{P}_{n+1}(z) &= \hat{a}_{n+1}(z)\hat{P}_n(z) + \beta_{n+1} \cdot \hat{P}_{n-1}(z); \\ \hat{Q}_{n+1}(z) &= \hat{a}_{n+1}(z)\hat{Q}_n(z) + \beta_{n+1} \cdot \hat{Q}_{n-1}(z) \end{aligned} \quad (14)$$

where we define, with ρ_n denoting the leading coefficient of P_n ,

$$\hat{a}_{n+1}(z) = \frac{a_{n+1}(z) \cdot \rho_n}{\rho_{n+1}} \quad \text{and} \quad \beta_{n+1} = \frac{\rho_{n-1}}{\rho_{n+1}}.$$

One can easily check from (14) that $\hat{a}_n(z)$ are always monic, while the denominators \hat{Q}_n may be not monic in a general case. However we will see in the next section that for the function \tilde{f}_{TM} both numerators and denominators of the canonical representatives $\hat{P}_n(z)/\hat{Q}_n(z)$ of convergents are monic polynomials.

3 Continuous fraction of the function \tilde{f}_{TM}

Lemma 3.1. Let $P(z)/Q(z)$ be a convergent to $\tilde{f}_{TM}(z)$. Then $P^*(z)/Q^*(z)$ is also a convergent, where

$$P^*(z) = (z-1)P(z^2); \quad Q^*(z) = Q(z^2). \quad (15)$$

□

Proof. It is a consequence of the functional relation for $\tilde{f}_{TM}(z)$. Indeed,

$$\deg(Q^*(z)\tilde{f}_{TM}(z) - P^*(z)) = \deg((z-1)(Q(z^2)\tilde{f}_{TM}(z^2) - P(z^2))) \leq -2\deg(Q) - 1.$$

At the same time, $\deg Q^*(z) = 2\deg Q(z)$. Therefore $Q^*(z)$ and $P^*(z)$ satisfy the condition (13). We conclude that $P^*(z)/Q^*(z)$ is a convergent to \tilde{f}_{TM} by Proposition 2.1. ■

Recursive application of Lemma 3.1 enables us to construct an infinite sequence of convergents to $\tilde{f}_{TM}(z)$ starting from only one convergent. However not every convergent can be constructed in this way. For example, by a direct computation one can find the convergents

$$\frac{1}{1+z} \quad \text{and} \quad \frac{z^2-2}{z^3+z^2}. \quad (16)$$

Then (15) immediately gives us the convergents

$$\frac{z-1}{z^2+1}, \quad \frac{(z-1)(z^2-1)}{z^4+1}, \quad \text{and} \quad \frac{(z-1)(z^4-2)}{z^6+z^4}. \quad (17)$$

However these calculations, using Lemma 3.1 and the initial convergents (16) only, neither allow one to construct the convergent $P(z)/Q(z)$ to $\tilde{f}_{TM}(z)$ with $\deg(Q) = 5$, nor give an information whether such a convergent exists. The next proposition shows that, in fact, for every $n \in \mathbb{N}$ there is a convergent $P_n(z)/Q_n(z)$ to $\tilde{f}_{TM}(z)$ such that $\deg(Q_n) = n$.

Proposition 3.2. Let $[0; a_1(z), a_2(z), \dots]$ be the continued fraction expansion of $\tilde{f}_{TM}(z)$. Then $\forall n \geq 2$, $a_{2n-1}(z) = \alpha_{2n-1} \cdot (z+1)$ and $a_{2n}(z) = \alpha_{2n} \cdot (z-1)$ where $\alpha_i \in \mathbb{Q}$, $i \in \mathbb{N}$, are constants. Moreover for every $n \in \mathbb{N}$, the convergent $Q_{2n}(z)$ is of the form $Q_{2n}(z) = Q_n(z^2)$, in particular it is an even function; $Q_{2n-1}(z)$

is of the form $Q_{2n-1}(z) = (z+1) \cdot Q_{n-1}^+(z^2)$, where $Q_{n-1}^+(X)$ is a polynomial of degree $n-1$ with rational coefficients. \square

Proof. We reason by induction. We already checked this statement for $n=2$. Now we need to check it for $2n+1$ and $2n+2$ given that the statement for $3, 4, \dots, 2n$ is true.

Note that induction hypothesis and (12) imply that $\deg Q_k = k$ for $k=1, 2, 3, \dots, 2n$. In particular, the convergent Q_{n+1} has degree $n+1$.

By Lemma 3.1 we have that $Q_{n+1}^*(z) = Q_{n+1}(z^2)$ is also a convergent and $\deg Q_{n+1}^* = 2n+2$. Moreover we have $\deg Q_{2n} = 2n$, thus $Q_{n+1}^*(z)$ has to coincide either with $Q_{2n+1}(z)$ or with $Q_{2n+2}(z)$. The first case is actually impossible, because otherwise we would have had, using induction hypothesis and (12),

$$\begin{aligned} Q_{n+1}^*(z) = Q_{n+1}(z^2) &= Q_{2n+1}(z) = a_{2n+1}(z) \cdot Q_{2n}(z) + Q_{2n-1}(z) \\ &= a_{2n+1}(z) \cdot Q_n(z^2) + (z+1)Q_{n-1}^+(z^2). \end{aligned} \quad (18)$$

To show that the equality (18) is unattainable, substitute $-z$ in place of z to (18) and then apply $(-z)^2 = z^2$. We find

$$Q_{n+1}(z^2) = a_{2n+1}(-z) \cdot Q_n(z^2) + (1-z)Q_{n-1}^+(z^2). \quad (19)$$

Subtracting (19) from (18) and dividing by 2 we obtain

$$\frac{a_{2n+1}(-z) - a_{2n+1}(z)}{2} \cdot Q_n(z^2) = zQ_{n-1}^+(z^2). \quad (20)$$

The equality (20) is impossible, because its right hand side is a non-zero polynomial of degree $2n-1$, and the left hand side is either a zero or a polynomial of degree at least $2n$. This contradiction shows that $Q_{n+1}^*(z)$ cannot coincide with $Q_{2n+1}(z)$, thus we have

$$Q_{n+1}^*(z) = Q_{2n+2}(z).$$

In particular we see that $\deg Q_{2n+2} = 2n+2$, thus $\deg Q_{2n+1} = 2n+1$ and so

$$\deg a_{2n+1} = \deg a_{2n+2} = 1. \quad (21)$$

Applying again the induction hypothesis and the formulae (12) for convergents Q_{2n+1} and Q_{2n+2} we have

$$\begin{aligned} Q_{2n+2}(z) &= Q_{n+1}(z^2) = a_{2n+2}(z) \cdot Q_{2n+1}(z) + Q_{2n}(z) \\ &= a_{2n+2}(z) \cdot a_{2n+1}(z) \cdot Q_{2n}(z) + a_{2n+2}(z) \cdot Q_{2n-1}(z) + Q_{2n}(z) \\ &= (a_{2n+2}(z) \cdot a_{2n+1}(z) + 1) \cdot Q_n(z^2) + a_{2n+2}(z) \cdot (z+1)Q_{n-1}^+(z^2). \end{aligned}$$

Again, by substituting $-z$ in place of z and then using $(-z)^2 = z^2$ we deduce

$$\begin{aligned} \frac{a_{2n+2}(z) \cdot a_{2n+1}(z) - a_{2n+2}(-z) \cdot a_{2n+1}(-z)}{2} Q_n(z^2) \\ = \frac{a_{2n+2}(-z) \cdot (-z+1) - a_{2n+2}(z) \cdot (z+1)}{2} Q_{n-1}(z^2) \end{aligned} \quad (22)$$

Both polynomials $a_{2n+2}(z) \cdot a_{2n+1}(z) - a_{2n+2}(-z) \cdot a_{2n+1}(-z)$ and $a_{2n+2}(-z) \cdot (-z+1) - a_{2n+2}(z) \cdot (z+1)$ are odd functions, so they are either 0 or of degree 1 (taking into account (21)). Therefore if the left hand side of (22) is not zero then it has degree $2n+1$, and the right hand side in this case has degree $2n-1$. This is a contradiction, so we conclude that both sides of (22) are 0. This gives us

$$\begin{aligned} a_{2n+2}(-z) \cdot (-z+1) &= a_{2n+2}(z) \cdot (z+1), \\ a_{2n+2}(z) \cdot a_{2n+1}(z) &= a_{2n+2}(-z) \cdot a_{2n+1}(-z). \end{aligned}$$

This system implies that $a_{2n+2}(z)$ is a multiple of $z-1$ and a_{2n+1} is a multiple of $z+1$. Now the fact that $Q_{2n+1}(z) = (z-1)Q_n^+(z^2)$ readily follows from (12). This concludes the proof. \blacksquare

To compute the convergents of $\tilde{f}_{TM}(z)$ we just need to find the precise values of coefficients α_n such that $a_n(z) = \alpha_n(z - (-1)^n)$. To this end, it appears easier to make calculations with canonical representatives of the functional convergents, described at the end of the previous section, that is with the fractions $\hat{P}_n(z)/\hat{Q}_n(z)$ such that $\hat{P}_n(z)$ is a monic polynomial.

As we mentioned in the previous section, the formulae (12) do not always (in fact, almost never) produce monic polynomials $P_n(z)$ and $Q_n(z)$. For example, one can check that if we start with $Q_1(z) = 1+z$ and $Q_2(z) = z^2+1$ then $a_3(z) = -z-1$ and therefore

$$Q_3(z) = -(z+1)Q_2(z) + Q_1(z) = -z^3 - z^2.$$

Moreover further calculations show that $P_n(z)$ and $Q_n(z)$ do not always have integer coefficients.

So, as we are interested in monic numerator, in our case formulae (14) together with Proposition 3.2 give

$$\hat{Q}_{n+1}(z) = (z + (-1)^n)\hat{Q}_n(z) + \beta_{n+1}\hat{Q}_{n-1}(z), \quad (23)$$

$$\hat{P}_{n+1}(z) = (z + (-1)^n)\hat{P}_n(z) + \beta_{n+1}\hat{P}_{n-1}(z), \quad (24)$$

where $\beta_{n+1} = \frac{\alpha_n}{\alpha_{n+1}}$, $n \in \mathbb{N}$, $n \geq 2$. Polynomials $\hat{P}_n(z)$ and $\hat{Q}_n(z)$ are linked to the original polynomials $P_n(z)$ and $Q_n(z)$ by $\hat{Q}_n(z) := \frac{Q_n(z)}{\prod_{k=1}^n \alpha_k}$ and $\hat{P}_n(z) := \frac{P_n(z)}{\prod_{k=1}^n \alpha_k}$ and one readily verifies that they both are monic for all $n \in \mathbb{N}$.

Proposition 3.3. The coefficients β_n in (23) and (24) can be computed recursively by the following formulae

$$\beta_3 = -1, \beta_4 = 1, \quad (25)$$

$$\beta_{2n+1} = -\frac{\beta_{n+1}}{\beta_{2n}}, \quad (26)$$

$$\beta_{2n+2} = 1 + (-1)^n - \beta_{2n+1} \quad (27)$$

for every positive integer $n \geq 2$. □

Proof. The values β_3 and β_4 can be computed directly from already known $Q_1(z), Q_2(z), Q_3(z)$ and $Q_4(z)$ (see (16) and (17)). Next, we substitute $z \mapsto z^2$ into the formula (23) for $\hat{Q}_{n+1}(z)$ and use Proposition 3.2 to get that

$$\hat{Q}_{2n+2}(z) = (z^2 + (-1)^n)\hat{Q}_{2n}(z) + \beta_{n+1}\hat{Q}_{2n-2}(z).$$

On the other hand, by directly applying (23) we have

$$\begin{aligned} \hat{Q}_{2n+2}(z) &= (z-1)\hat{Q}_{2n+1}(z) + \beta_{2n+2}\hat{Q}_{2n}(z) \\ &= (z-1)\left((z+1)\hat{Q}_{2n}(z) + \beta_{2n+1}\hat{Q}_{2n-1}(z)\right) + \beta_{2n+2}\hat{Q}_{2n}(z). \end{aligned}$$

By comparing these two formulae for $\hat{Q}_{2n+2}(z)$ we get the equation

$$(\beta_{2n+2} - 1 - (-1)^n)\hat{Q}_{2n}(z) + \beta_{2n+1} \cdot (z-1)\hat{Q}_{2n-1}(z) - \beta_{n+1}\hat{Q}_{2n-2}(z) = 0. \quad (28)$$

By looking at the coefficient of z^{2n} we get $\beta_{2n+2} - 1 - (-1)^n + \beta_{2n+1} = 0$, which proves (27). The formula (26) is achieved by substituting Formula (23) for $\hat{Q}_{2n}(z)$ into Equation (28) and looking at the coefficient of z^{2n-2} :

$$-\beta_{2n+1} \cdot ((z-1)\hat{Q}_{2n-1}(z) + \beta_{2n}\hat{Q}_{2n-2}(z)) + \beta_{2n+1} \cdot (z-1)\hat{Q}_{2n-1}(z) - \beta_n\hat{Q}_{2n-2}(z) = 0.$$

It readily follows $\beta_{2n+1}\beta_{2n} + \beta_{n+1} = 0$. ■

Now we have precise recursive formulae to quickly compute convergents to \tilde{f}_{TM} as far as we want. For example, the first few convergents following $P_4(z)/Q_4(z)$ are

$$\frac{z^4 - z^2 - 1}{(z+1)(z^4 + z^2 + 1)}, \frac{(z-1)(z^4 - 2)}{z^6 + z^4}, \frac{z^6 - 2z^4 - z^2 + 3}{(z+1)(z^6 - z^2 - 1)}, \dots$$

The 9th convergent is of particular value for us so we write it down as well:

$$\hat{P}_9(z) := z^8 - 3z^6 + 2z^4 + 3z^2 - 4, \quad \hat{Q}_9(z) := (z+1)(z^8 - z^6 + z^2 + 2). \quad (29)$$

This convergent plays the central role in our proof that τ_{TM} is not badly approximable.

4 Rational approximations to the Thue-Morse constants.

We start by extracting a specific subsequence of convergents to \tilde{f}_{TM} .

Definition 4.1. Let $n \in \mathbb{N}$. Define

$$\tilde{P}_n(z) := \prod_{k=0}^n (z^{2^k} - 1) \hat{P}_9(z^{2^{n+1}})$$

and

$$\tilde{Q}_n(z) := \hat{Q}_9(z^{2^{n+1}}),$$

where $\hat{P}_9(z)/\hat{Q}_9(z)$ is the 9th functional convergent to \tilde{f}_{TM} given at the end of the previous section (see (29)). \square

Remark 4.2. Iteratively applying Lemma 3.1 we find that $\tilde{P}_n(z) = \hat{P}_{9 \cdot 2^{n+1}}(z)$ and $\tilde{Q}_n(z) = \hat{Q}_{9 \cdot 2^{n+1}}(z)$. In particular, $\tilde{P}_n(z)/\tilde{Q}_n(z)$ is indeed a convergent to $\tilde{f}_{TM}(z)$. \square

Lemma 4.3. For every $n \in \mathbb{N}$ and $z_0 \in \mathbb{N}$, $z_0 \geq 2$ the integers $\tilde{P}_n(z_0)$ and $\tilde{Q}_n(z_0)$ satisfy the following Diophantine approximation properties:

$$\left| \tilde{f}_{TM}(z_0) - \frac{\tilde{P}_n(z_0)}{\tilde{Q}_n(z_0)} \right| \leq C \cdot z_0^{-36 \cdot 2^n}, \quad (30)$$

$$\tilde{Q}_n(z_0) \leq 2 \cdot z_0^{18 \cdot 2^n}, \quad (31)$$

where the constant $C = C(z_0)$ is independent of n . \square

Proof. Consider the following function:

$$F(z) := \tilde{f}_{TM}(z) - \frac{\hat{P}_9(z)}{\hat{Q}_9(z)}. \quad (32)$$

As $\hat{P}_9(z)/\hat{Q}_9(z)$ is the 9th convergent to $\tilde{f}_{TM}(z)$, it follows from Proposition 2.1 that $\deg(F) \leq -19$.

In fact, one can check by an explicit calculation that $F(z)$, being an infinite series in $\frac{1}{z}$, starts from the term $\frac{6}{z^{19}}$, that is

$$F(z) = \frac{6}{z^{19}} + O\left(\frac{1}{z^{20}}\right). \quad (33)$$

Further, consider $F(z^{2^{n+1}}) \prod_{k=1}^n (z^{2^k} - 1)$ where $n \in \mathbb{N}$. By using the definition (32) of F , functional equation (10) and estimate (33) we find

$$\begin{aligned} F(z^{2^{n+1}}) \prod_{k=1}^n (z^{2^k} - 1) &= \tilde{f}_{TM}(z) - \frac{\hat{P}_9(z^{2^{n+1}}) \prod_{k=1}^n (z^{2^k} - 1)}{\hat{Q}_9(z^{2^{n+1}})} \\ &= \frac{6 \prod_{k=1}^n (z^{2^k} - 1)}{z^{19 \cdot 2^{n+1}}} + O\left(\frac{\prod_{k=1}^n (z^{2^k} - 1)}{z^{20 \cdot 2^{n+1}}}\right), \end{aligned} \quad (34)$$

where the constant implied by the symbol $O(\cdot)$ is independent of n .

By Definition 4.1,

$$\tilde{f}_{TM}(z) - \frac{\hat{P}_9(z^{2^{n+1}}) \prod_{k=1}^n (z^{2^k} - 1)}{\hat{Q}_9(z^{2^{n+1}})} = \tilde{f}_{TM}(z) - \frac{\tilde{P}_n(z)}{\tilde{Q}_n(z)}.$$

At the same time, we have $\prod_{k=1}^n (z^{2^k} - 1) < z^{2^{n+1}}$ for any $z > 1$, hence we infer from (34)

$$\tilde{f}_{TM}(z) - \frac{\tilde{P}_n(z)}{\tilde{Q}_n(z)} = \frac{6}{z^{18 \cdot 2^{n+1}}} + O\left(\frac{1}{z^{19 \cdot 2^{n+1}}}\right) = \frac{6}{z^{36 \cdot 2^n}} + O\left(\frac{1}{z^{38 \cdot 2^n}}\right)$$

and (30) follows.

Further, a straightforward calculation shows that $\hat{Q}_9(z_0) \leq 2z_0^9$ for every $z_0 \geq 2$. Then

$$\tilde{Q}_n(z_0) := \hat{Q}_9(z_0^{2^{n+1}}) \leq 2 \cdot z_0^{9 \cdot 2^{n+1}} = 2 \cdot z_0^{18 \cdot 2^n},$$

which proves (31). ■

This lemma shows that for $z_0 \in \mathbb{N}$, $z_0 \geq 2$, $\tilde{P}_n(z_0)/\tilde{Q}_n(z_0)$ already provides sufficiently good rational approximation to $\tilde{f}_{TM}(z_0)$ (following the scheme presented in the introduction, they provide approximations satisfying (6)). In order to show that $\tilde{f}_{TM}(z_0)$ is not badly approximable it is sufficient for every $r > 1$ to find an $n \in \mathbb{N}$ such that both $\tilde{P}_n(z_0)$ and $\tilde{Q}_n(z_0)$ have a common factor bigger than r . We prove this fact in Lemma 4.8.

In further discussion we will stick to the case $z_0 = 2$, however as we will see in the next section similar ideas should work for other positive integers z_0 .

The following chain of simple lemmas prepares the proof of our essential ingredient, Lemma 4.8. We start with the following classical result which can be found for instance in [7][p. 102].

Lemma 4.4. Let p be an odd prime and g be a primitive root modulo p^2 . Then g is a primitive root modulo p^m for all $m \in \mathbb{N}$. □

One can check that 2 is a primitive root modulo 3^2 therefore a direct corollary of this Lemma is that 2 is a primitive root modulo 3^m for every $m \in \mathbb{N}$.

Lemma 4.5. Let $m \in \mathbb{N}$ and let t be an even integer, $t \not\equiv 0 \pmod{3}$. Then there exists $n \in \mathbb{N}$ such that

$$2^n \equiv t \pmod{2 \cdot 3^m}.$$

Moreover, one can choose such n to verify additionally $n \leq 2 \cdot 3^{m-1}$. \square

Proof. As t is even, there exists $k \in \mathbb{N}$ such that $t = 2k$. Since 2 is a primitive root modulo 3^m and k is coprime to 3, there exists $n \in \mathbb{N}$ such that

$$2^{n-1} \equiv k \pmod{3^m}, \quad (35)$$

where k is as above. Multiplying the congruence (35) by 2 we find

$$2^n \equiv t \pmod{2 \cdot 3^m},$$

hence the claim.

To prove the concluding part of the lemma, note that the size of multiplicative group of residues modulo 3^m is $2 \cdot 3^{m-1}$. So in (35) we can always choose n verifying $n - 1 \leq 2 \cdot 3^{m-1} - 1$, and the second claim of the lemma follows. \blacksquare

In the next lemma we use the following notation.

Notation 4.6. Let $a, b \in \mathbb{Z}$. We write $a || b$ if a divides b , but a^2 does not. \square

Lemma 4.7. Let $m \in \mathbb{N}$, $m \geq 2$ and let t be an integer such that $3 || t - 1$. Then there exists $n \in \mathbb{N}$ such that

$$2^{2^n} \equiv t \pmod{3^m}.$$

Moreover, one can choose such n to verify additionally $n \leq 2 \cdot 3^{m-2}$. \square

Proof. Since 2 is a primitive root modulo 3^m there is a $k \in \mathbb{N}$ such that

$$2^k \equiv t \pmod{3^m}. \quad (36)$$

Reducing congruence (36) modulo 3 we find (using our assumption on t)

$$2^k \equiv 1 \pmod{3},$$

hence k is an even positive integer. Furthermore, by reducing (36) modulo 9, we get $2^k \not\equiv 1 \pmod{9}$ therefore k is not a multiple of 3.

By Lemma 4.5 we have that there exists an $n \in \mathbb{N}$ such that

$$2^n \equiv k \pmod{2 \cdot 3^{m-1}}$$

and $n \leq 2 \cdot 3^{m-2}$.

At the same time, by Euler's theorem we have

$$2^{2 \cdot 3^{m-1}} \equiv 1 \pmod{3^m},$$

as $\phi(3^m) = 3^m - 3^{m-1} = 2 \cdot 3^{m-1}$. Therefore

$$2^{2^n} \equiv 2^k \pmod{3^m},$$

and we conclude by comparing this last congruence with the congruence (36). ■

Lemma 4.8. For any $m \in \mathbb{N}$, $m \geq 3$ there exists an index n_m such that both integers $\tilde{P}_{n_m}(2)$ and $\tilde{Q}_{n_m}(2)$ are divisible by 3^m , and moreover $n_m \leq 3^{m-1}$. □

Proof. We verify by a direct calculation that $Q_9(1) = 6$ is divisible by 3 but not by 9, and

$$Q'_9(1) = 11 \not\equiv 0 \pmod{3}.$$

So by Hensel's lemma, for every $m \in \mathbb{N}$, $m \geq 1$, there exists a solution $x_m \in \mathbb{N}$ to the congruence

$$Q_9(x_m) \equiv 0 \pmod{3^m}$$

such that this solution x_m is congruent to 1 modulo 3 and $x_m \not\equiv 1 \pmod{9}$. By Lemma 4.7, there exists $t_m \in \mathbb{N}$ such that $2^{2^{t_m}} \equiv x_m \pmod{3^m}$, thus

$$\tilde{Q}_{n_m}(2) = Q_9(2^{2^{t_m}}) \equiv 0 \pmod{3^m}. \tag{37}$$

Moreover, by the same Lemma we can choose t_m to satisfy $t_m \leq 2 \cdot 3^{m-2}$.

One can easily check that if t_m is a solution to the equation (37) then every integer $t > 1$ such that $t \equiv t_m \pmod{\phi(2 \cdot 3^{m-1})}$ is also a solution. So for any $l \in \mathbb{N}$ a number $t_m + l \cdot 2 \cdot 3^{m-2}$ also provides a solution to (37).

If $t_m > m$ then we choose $n_m := t_m$, otherwise $n_m := t_m + 2 \cdot 3^{m-2}$. Note that for $m \geq 3$ we have $3^{m-2} \geq m$, thus our definition of n_m assures that for $m \geq 3$ we have $n_m > m$ and $n_m \leq 3^{m-1}$.

Further, note that for any $k \in \mathbb{N}$ we have $2^{2^k} \equiv 1 \pmod{3}$. Therefore

$$\prod_{k=0}^{n_m-1} (2^{2^k} - 1) \equiv 0 \pmod{3^{n_m-1}}$$

and we readily have that

$$\tilde{P}_{n_m}(2) := \prod_{k=0}^{n_m-1} (2^{2^k} - 1)P_9(2^{2^{n_m+1}}) \equiv 0 \pmod{3^{n_m-1}}.$$

Finally, we infer from $n_m > m$ that

$$\tilde{P}_{n_m}(2) \equiv 0 \pmod{3^m}. \quad (38)$$

Congruences (37) and (38) show that n_m indeed verifies the properties claimed in the statement of the lemma, and this completes the proof. \blacksquare

Theorem 4.9. Thue-Morse constant τ_{TM} is not badly approximable. Moreover, there exists a constant $c > 0$ such that the inequality

$$|\tau_{TM} - p/q| \leq \frac{c}{q(\log \log q)^2}. \quad (39)$$

has infinitely many solutions $(p, q) \in \mathbb{N}^2$. \square

Proof. We are going to prove that an analogue of (39) is satisfied for $\tilde{f}_{TM}(2)$. Then the relation (11) would straightforwardly imply the same condition on τ_{TM} too.

By Lemma 4.3 we have that the sequence $p_n := \tilde{P}_n(2)$ and $q_n := \tilde{Q}_n(2)$ provide sufficiently good rational approximations to $\tilde{f}_{TM}(2)$, that is for any n we have

$$\left| \tilde{f}_{TM}(2) - \frac{p_n}{q_n} \right| \leq \frac{C}{q_n^2},$$

where the constant C is independent of n . Moreover, by Lemma 4.8 we have a subsequence of indices $(n_m)_{m \in \mathbb{N}}$ such that both integers $\tilde{P}_{n_m}(2)$ and $\tilde{Q}_{n_m}(2)$ are divisible by 3^m . Therefore the integers $\tilde{p}_{n_m} := \frac{\tilde{P}_{n_m}(2)}{3^m}$ and $\tilde{q}_{n_m} := \frac{\tilde{Q}_{n_m}(2)}{3^m}$ satisfy

$$\left| \tilde{f}_{TM}(2) - \frac{\tilde{p}_{n_m}}{\tilde{q}_{n_m}} \right| \leq \frac{C}{3^{2m}\tilde{q}_{n_m}^2}, \quad (40)$$

which readily implies that the number $\tilde{f}_{TM}(2)$ is not badly approximable.

To justify (39), note that because of the bound $n_m \leq 3^{m-1}$ and the explicit formula

$$\tilde{Q}_{n_m}(2) := Q_9(2^{2^{n_m+1}}),$$

there exists a constant c_1 independent of m such that $\tilde{q}_{n_m} \leq c_1 2^{9 \cdot 2^{3m}}$, so for m sufficiently large we have $\log \log \tilde{q}_{n_m} \leq 2 \cdot 3^m$ and (39) follows from (40). \blacksquare

5 Constants $\tilde{f}_{TM}(a)$ for arbitrary $a \in \mathbb{N}$

The proposed chain of lemmata suggests the method for checking whether the value $\tilde{f}_{TM}(a)$ is badly approximable for an arbitrary $a \in \mathbb{N}$, $a > 1$. We formulate it as the following theorem.

Theorem 5.1. Assume that there exist positive integers n, t, p such that

1. p is a prime such that $p \parallel a^{2^n} - 1$ (recall Notation 4.6);
2. 2 is a primitive root modulo p^2 ;
3. $p \parallel \hat{Q}_t(1)$;
4. $\hat{Q}'_t(1) \not\equiv 0 \pmod{p}$.

Then $\tilde{f}_{TM}(a)$ is not badly approximable. Moreover, there exists a constant $c > 0$ such that the inequality

$$\left| \tilde{f}_{TM}(a) - p/q \right| \leq \frac{c}{q(\log \log q)^2}. \quad (41)$$

has infinitely many solutions $(p, q) \in \mathbb{N}^2$. \square

Proof. For any $n, t \in \mathbb{N}$ we define

$$\tilde{P}_{n,t}(z) := \prod_{k=0}^n (z^{2^k} - 1) \hat{P}_t(z^{2^{n+1}})$$

and

$$\tilde{Q}_{n,t}(z) := \hat{Q}_t(z^{2^{n+1}}).$$

Then the same arguments as in Lemma 4.3 imply that for every $n \in \mathbb{N}$ and $z_0 \in \mathbb{N}$ with $z_0 \geq 2$ one has

$$\left| \tilde{f}_{TM}(z_0) - \frac{\tilde{P}_{n,t}(z_0)}{\tilde{Q}_{n,t}(z_0)} \right| \leq C \cdot z_0^{-4t \cdot 2^n}, \quad (42)$$

$$\tilde{Q}_{n,t}(z_0) \leq C \cdot z_0^{2t \cdot 2^n}, \quad (43)$$

where the constant $C = C(z_0, t)$ is independent of n .

Values $\tilde{P}_{n,t}(a)$ and $\tilde{Q}_{n,t}(a)$ are not necessarily integer. However $\tilde{P}_{n,t}(a) \in 1/d_P \cdot \mathbb{Z}$ and $\tilde{Q}_{n,t}(a) \in 1/d_Q \cdot \mathbb{Z}$ where d_P (respectively d_Q) is the least common multiple of all denominators of the rational coefficients of the polynomial $\hat{P}_t(z)$ (respectively of $\hat{Q}_t(z)$). So for every pair of integers p_n, q_n where $p_n = d_P d_Q \cdot \tilde{P}_{n,t}(a)$,

$q_n = d_p d_Q \cdot \tilde{Q}_{n,t}(a)$ the following inequality takes place:

$$\left| \tilde{f}_{TM}(a) - \frac{p_n}{q_n} \right| \leq \frac{d_Q^2 \cdot C^3}{q_n^2}.$$

Note that the value $d_Q^2 C^3$ depends only on a and t and does not depend on n . Therefore it is enough to find arbitrarily large $r \in \mathbb{Z}$ and some $n \in \mathbb{N}$ such that values p_n and q_n have a common factor r . We will show that positive integer power of p can play the role of such common factor r .

By Condition 1 of the theorem, $p \mid a^{2^n} - 1$. Therefore for every $m > n$ we have $p^{m-n} \mid \tilde{P}_{m,t}(a)$.

Conditions 3 and 4 and Hensel's lemma imply that the equation $\hat{Q}_t(x) = 0$ has a solution $x \in \mathbb{Z}_p$ such that $x \equiv 1 \pmod{p}$ and $x \not\equiv 1 \pmod{p^2}$. Next, since 2 is a primitive root modulo p^2 (in view of condition 2), then by Lemma 4.4 it is also a primitive root modulo every power p^m , $m \in \mathbb{N}$.

For every $m \in \mathbb{N}$, the multiplicative group $\mathcal{R}_{p^m}^*$ of residues modulo p^m has the order $\phi(p^m) = (p-1)p^{m-1}$. As the element a^{2^n} is congruent to 1 modulo p , it lies in the kernel of the canonical projection $\mathcal{R}_{p^m}^* \rightarrow \mathcal{R}_p^*$. The multiplicative group \mathcal{R}_p^* of residues modulo p has the order $p-1$, so the residue a^{2^n} has the order p^l in $\mathcal{R}_{p^m}^*$, for some $l \leq m-1$. If the value l is strictly smaller than $m-1$, then we necessarily have $a^{2^n} \equiv 1 \pmod{p^2}$, which contradicts the Condition 1, hence the multiplicative order of a^{2^n} modulo p^m is exactly p^{m-1} and thus the set of residues $\{a^{2^{n \cdot s}} \pmod{p^m} : s \in \mathbb{N}, \gcd(s, p) = 1\}$ coincides with the set of residues modulo p^m congruent to 1 modulo p but not congruent to 1 modulo p^2 . So, there is an $s \in \mathbb{N}$ such that $a^{2^{n \cdot s}} \equiv x \pmod{p^m}$ and $s \not\equiv 0 \pmod{p}$.

As 2 is a primitive root modulo p^{m-1} and a^{2^n} has order p^{m-1} modulo p^m , we have that the set of residues $\{a^{2^{n \cdot 2^s}} \pmod{p^m} : s \in \mathbb{N}\}$ coincides with the set of residues $\{a^{2^{n \cdot s}} \pmod{p^m} : s \in \mathbb{N}, \gcd(s, p) = 1\}$. In particular, there exists s_1 such that $a^{2^{n+s_1}} \equiv x \pmod{p^m}$.

Moreover, as s_1 is defined modulo $\phi(p^{m-1}) = (p-1)p^{m-2}$, one can choose such s_1 that $n_m := n + s_1$ verifies

$$m < n_m \leq m + (p-1)p^{m-2}. \quad (44)$$

So we get that $\tilde{Q}_{n_m,t}(a)$ is divisible by p^m and therefore both p_{n_m} and q_{n_m} have common divisor p^{m-n} . be taken arbitrary large this finishes the proof of the theorem. We deduce that there exists a constant C_1 such that for any $m > n$

$$\left| \tilde{f}_{TM}(a) - \frac{\tilde{p}_m}{\tilde{q}_m} \right| \leq \frac{C_1}{p^{2n_m} \tilde{q}_m^2},$$

where $\tilde{p}_m = p_{n_m}/p^m$ and $\tilde{q}_m = q_{n_m}/p^m$ are integers. The upper inequality in (44) implies $\log \log q_{n_m} \ll n_m \ll p^m$, so moreover we have $\log \log \tilde{q}_m \ll p^m$ and the inequality (41) follows. \blacksquare

Remark. Conditions 2 – 4 of Theorem 5.1 do not depend on a at all. One can look at them as conditions on a prime number p . We call p *acceptable* if there exists $t \in \mathbb{N}$ such that the Conditions 2 – 4 are satisfied.

Then Theorem 5.1 can be reformulated as follows: if there exist $n \in \mathbb{N}_0$ and an acceptable prime p such that $p \mid a^{2^n} - 1$ then $\tilde{f}_{TM}(a)$ is not badly approximable. By testing various polynomials $\hat{Q}_t(z)$ it is easy to find many acceptable primes. In the previous section we have already checked that 3 is acceptable. By considering

$$\hat{Q}_{11}(z) = x^{11} + x^{10} + \frac{2x^9}{3} + \frac{2x^8}{3} + \frac{4x^7}{3} + \frac{4x^6}{3} + x^5 + x^4 + \frac{2x^3}{3} + \frac{2x^2}{3} + \frac{x}{3} + \frac{1}{3},$$

one can check that 5 is acceptable too. This remark already leads us to the following corollary, which generalizes Theorem 4.9.

Corollary 5.2. Let $a \in \mathbb{N}$ be a positive integer which is not divisible by 15. Then $\tilde{f}_{TM}(a)$ is not badly approximable. \square

Proof. As we have noted just before this corollary, primes 3 and 5 are acceptable, that is they verify conditions 2–4 of Theorem 5.1.

It is an easy exercise, which we leave to the reader, to check that for $a \not\equiv 0 \pmod{15}$ either 3 or 5 satisfies the condition 1 of Theorem 5.1 and therefore by this theorem we obtain that $\tilde{f}_{TM}(a)$ is not badly approximable. \blacksquare

So the remaining uncovered case is $\tilde{f}_{TM}(a)$ where a is divisible by 15.

Unfortunately, Theorem 5.1 can not be applied to show that $\tilde{f}_{TM}(15)$ is not badly approximable. Indeed, the numbers $15 - 1, 15^2 - 1$ and $15^4 - 1$ have prime divisors 2, 7 and 113, and 2 is not a primitive root for neither of them. Other prime divisors of $15^{2^n} - 1$ for some $n \in \mathbb{N}$ must also divide $15^{2^m} + 1$ for some $m \geq 2$. It is a classical result that such primes p satisfy the condition $p \equiv 1 \pmod{8}$. Since 2 is a quadratic residue modulo such primes p then the condition 2 is never satisfied.

However Theorem 5.1 can still work for a equal to some of the multiples of 15. For example, for $a = 30$ we have $29 \mid 30 - 1$ and 29 is an acceptable prime (one can take $t = 35$). Also,

$$11 \mid 45 - 1; \quad 61 \mid 60^2 - 1; \quad 19 \mid 75^2 - 1; \quad 13 \mid 90^2 - 1$$

and primes 11, 61, 19, 13 are acceptable. One may check this by taking $t = 43, 49, 19$ and 33 respectively. So, all the values $a \in \mathbb{N}$ such that $2 \leq a \leq 104$ and $a \neq 15$ assure that $\tilde{f}_{TM}(a)$ is not badly approximable.

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