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On the linear stability of solitons and hairy black holes with a negative cosmological constant: the even-parity sector

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Abstract

Using a recently developed perturbation formalism based on curvature quantities, we complete our investigation of the linear stability of black holes and solitons with Yang-Mills hair and a negative cosmological constant. We show that those solutions which have no linear instabilities under odd- and even-parity spherically symmetric perturbations remain stable under even-parity, linear, non-spherically symmetric perturbations. Together with the result from a previous work, we have therefore established the existence of stable hairy black holes and solitons with anti-de Sitter asymptotic.

1 Introduction

Soliton and hairy black hole solutions of general relativity coupled to non-Abelian gauge fields have been the subject of intensive investigation for over ten years (see [1] for a comprehensive review of the subject). The field was sparked by the discovery of solitonic [2] and coloured black hole [3] solutions to $\mathfrak{su}(2)$ Einstein-Yang-Mills (EYM) theory. Since then, many examples of both solitons and black holes in various theories involving non-Abelian gauge fields have been discovered, both with and without a cosmological constant.

Many of these examples have been shown to be classically unstable, making the search for stable solitons and hairy black holes vital in this area. An important (and surprising) discovery was that there are solitons and black holes in $\mathfrak{su}(2)$ EYM with a negative cosmological constant which are stable with respect to linear, spherically symmetric, perturbations [4, 5]. This result was surprising because all such solutions in EYM with either a positive or zero cosmological constant must be unstable [6, 7]. For those solutions which are stable with respect to spherically symmetric perturbations, it remains to be seen whether the stability is altered if non-spherically symmetric perturbations are considered. The perturbations fall into two sectors, depending on their behaviour under parity transformations. In [8] we analyzed the odd-parity sector of perturbations and showed that those solutions which are stable under spherically symmetric perturbations remain stable under odd-parity, linear perturbations. The proof of stability will therefore be complete if we can show that there are no instabilities in the even-parity sector. This is the subject of the present article.

In common with [8], we shall use a recently developed perturbation formalism based on curvature quantities. The main advantage of this formalism [9, 10] is that on a static and purely magnetic background it yields a wave equation for the linearized extrinsic curvature and electric YM field where the spatial part of the wave operator is (formally) self-adjoint. This offers the possibility of studying the linear stability of non-rotating black holes and solitons by analytical means. For example, we can use the nodal theorem [11] which in fact will turn out to be an essential tool in our stability proof. Even with these powerful tools at hand, the even-parity sector is considerably less amenable to analysis than the odd-parity sector. As we have shown in our previous work [8], the stability in the odd-parity sector is *topological* in the sense that we do not need to know the exact details of the background solutions in order to show that the system is stable. In the even-parity sector, we will see that the detailed structure of the metric is needed even in some simple cases. Nonetheless, we will present an analytic proof that those solitons and black holes having no instabilities under spherically symmetric perturbations remain stable under even-parity linear perturbations. In short, all soliton and black hole solutions when the cosmological constant is sufficiently large and negative are linearly stable.

This work is organized as follows. In section 2 we remember some important results about the hairy black hole solutions with a negative cosmological constant, recently found in [4], and the corresponding solitonic solutions [5]. In section 3, we briefly review the curvature-based formalism of perturbation theory for a static background. The harmonic decomposition is performed in section 4. Special cases, such as the stability of the Schwarzschild-anti-de Sitter and the Reissner-Nordström-anti-de Sitter metric are discussed in section 5. In section 6, we discuss the general case and prove that the system is stable when $|\Lambda|$ is large enough. Given that the proof is rather involved, we first present an outline of our argument in section 6.1 before working through each step in detail. Technical details such as the discussion of the perturbation potential in the Reissner-Nordström-anti-de Sitter case and the factorization of the spatial perturbation operator in the odd-parity sector are discussed in appendices A and B, respectively.

The metric signature is $(-,+,+,+)$ throughout, and we use the standard notations $2\omega_{(ab)} = \omega_{ab} + \omega_{ba}$ and $2\omega_{[ab]} = \omega_{ab} - \omega_{ba}$ for symmetrizing and anti-symmetrizing, respectively. Throughout the paper, greek letters denote space-time indices taking values in $(0, 1, 2, 3)$, while roman letters will denote spatial indices taking values $(1, 2, 3)$.

2 Solitons and hairy black holes with a negative cosmological constant

In this section we discuss very briefly those details of the spherically symmetric black hole and soliton solutions of $\mathfrak{su}(2)$ Einstein-Yang-Mills theory with a negative cosmological constant that we require later in this paper. Further details can be found in [8] and the original papers [4, 5].

The equilibrium metric is spherically symmetric

$$ds^2 = -N(r)S^2(r)\,dt^2 + N^{-1}(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right),$$

and the gauge field potential has the spherically symmetric form

$$A = (1 - w(r))\left[-\tau_\phi d\theta + \tau_\theta \sin\theta\,d\phi\right].$$

Here the $\mathfrak{su}(2)$ generators $\tau_{r,\theta,\phi}$ are given in terms of the usual Pauli matrices σ_i by $\tau_r = \underline{e}_r \cdot \underline{\sigma}/2i$, etc. Writing $N(r) = 1 - 2m(r)/r - \Lambda r^2/3$, where Λ is the (negative) cosmological constant, the field equations take the form:

$$\begin{aligned}
m_{,r} &= G\left[Nw_{,r}^2 + \frac{1}{2r^2}(w^2-1)^2\right], \\
\frac{S_{,r}}{S} &= \frac{2Gw_{,r}^2}{r}, \\
0 &= Nr^2 w_{,rr} + \left(2m - \frac{2\Lambda r^3}{3} - G\frac{(w^2-1)^2}{r}\right)w_{,r} + (1-w^2)w, \qquad (1)
\end{aligned}$$

where G is Newton's constant, and we have set the gauge coupling constant equal to $\sqrt{4\pi}$ for convenience. Here, and in the rest of the paper, we use a comma in a subscript to denote partial differentiation, i.e. $w_{,r} = \partial_r w$.

For solutions which approach anti-de Sitter (adS) space at infinity, the asymptotic behaviour of the field functions is:

$$\begin{aligned}
m(r) &= M + \frac{M_1}{r} + O(r^{-2}), \\
w(r) &= w_\infty + \frac{w_1}{r} + O(r^{-2}), \\
S(r) &= 1 + O(r^{-4}).
\end{aligned}$$

Due to these boundary conditions, in general the solutions will be globally magnetically charged.

For black hole solutions having a regular event horizon at $r = r_h$, all the field variables have regular Taylor expansions near the event horizon, for example

$$w(r) = w(r_h) + w_{,r}(r_h)(r - r_h) + O(r - r_h)^2.$$

However, there are just two independent parameters, $S(r_h)$ and $w(r_h)$ since $N = 0$ at the event horizon, which gives

$$m(r_h) = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6}.$$

In order for the event horizon to be regular, we shall also require that $N_{,r}(r_h) > 0$, which implies that

$$F_h \equiv 1 - \Lambda r_h^2 - G\frac{(w(r_h) - 1)^2}{r_h^2} > 0.$$

From (1), one has

$$w_{,r}(r_h) = \frac{(w(r_h)^2 - 1)w(r_h)}{r_h F_h}.$$

There are also globally regular (solitonic) solutions, for which the behaviour near the origin is:

$$\begin{aligned} m(r) &= 2Gb^2r^3 + O(r^4), \\ w(r) &= 1 - br^2 + O(r^3), \\ S(r) &= S(0)\left[1 + 4Gb^2r^2 + O(r^3)\right]. \end{aligned}$$

Here the independent parameters are b and $S(0)$.

The simplest solutions to the field equations (1) are the Schwarzschild-adS solution,

$$w = \pm 1, \quad S = 1, \quad m = \text{const.}$$

and the Reissner-Nordström-adS (RN-adS) solution

$$w = 0, \quad S = 1, \quad m = \text{const.} - \frac{G}{2r}.$$

In both cases, the YM field is effectively Abelian. It is however interesting to study the stability of these solutions with respect to non-Abelian perturbations (see section 5).

The solutions of greatest interest in this article are those effectively non-Abelian solutions, for which the gauge function w has no zeros, since these solutions were shown in [4] to be linearly stable to both even- and odd-parity spherical perturbations. In [4] it is proved that for any value of the gauge field at the event horizon, $w(r_h) \neq 0$, for all sufficiently large $|\Lambda|$ there is a black hole solution in which w has no zeros. Similar behaviour is found numerically for the solitonic solutions [5]. For spherical perturbations of these equilibrium configurations, it is proved analytically that all the solutions in which w has no zeros are stable in the odd-parity sector [4]. The even-parity sector is more complicated, but stability can be proven for sufficiently large $|\Lambda|$.

In addition to the boundary conditions on the field variables at the origin (or event horizon, as applicable) and infinity, we shall in section 6 make extensive use of the fact that the equilibrium solutions (both solitonic and black hole) are analytic in both r and the parameter Λ. This is proved in [4] for black hole solutions, a proof which is readily extended to cover the solitonic case (see, for example [12] for the asymptotically flat situation). We shall also apply the result, proven in [4], that

$$w_{,r}(r) \sim o(|\Lambda|^{-\frac{1}{2}}) \qquad \text{as } |\Lambda| \to \infty.$$

Dyonic solutions with a non-vanishing electric field also exist in this model with $\Lambda < 0$ [5]. However, in common with our analysis of the odd-parity perturbations [8], our work here applies only to the purely magnetic equilibrium solutions. In addition, recently hairy black holes with non-spherical event horizon topology have been found in this model [13]. Here we consider only black holes whose event horizon has spherical topology, although we conjecture that all black holes (whatever the topology of their event horizon) will be stable if $|\Lambda|$ is sufficiently large.

3 The pulsation equations

Generalizing previous results [9, 10], we have shown in [8] that linear fluctuations on a static and purely magnetic background are governed by a symmetric wave equation for the linearized extrinsic curvature and electric YM field. Performing an ADM decomposition of the metric and the gauge potential,

$$\begin{aligned} \boldsymbol{g} &= -\alpha^2 dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \\ A &= -\Phi\, \alpha dt + \bar{A}_i(dx^i + \beta^i dt). \end{aligned}$$

(where Φ and $\bar{A}_i$ are both Lie algebra valued), these quantities are given by

$$L_{ij} = \frac{1}{2}\delta\left(\partial_t \bar{g}_{ij} - L_\beta \bar{g}_{ij}\right)$$

and

$$\mathcal{E}_i = -\delta\left(\partial_t \bar{A}_i + \bar{D}_i(\alpha\Phi)\right).$$

(We use the same notation as in [8]: In particular, a bar refers to the 3-metric $\bar{g}_{ij}$ or to the magnetic potential $\bar{A}_i$.) On the background, static coordinates are chosen such that the shift β^i and the time derivative of the 3-metric, $\partial_t \bar{g}_{ij}$ vanish. Similarly, the gauge is chosen such that the electric potential Φ and the time-derivative of the magnetic potential, $\partial_t \bar{A}_i$ vanish. As a consequence, L_{ij} and $\mathcal{E}_i$ are vector-invariant, that is, invariant with respect to both infinitesimal coordinate transformations within the ADM slices and infinitesimal gauge transformations of the gauge potential. More precisely, with respect to a coordinate transformation $\delta x^\mu \mapsto \delta x^\mu + X^\mu$, we have

$$L_{ij} \mapsto L_{ij} + \bar\nabla_{(i}\left(\alpha^2\bar\nabla_{j)} f\right), \tag{2}$$

$$\mathcal{E}_i \mapsto \mathcal{E}_i + \alpha^2 \bar F_{ij}\bar\nabla^j f, \tag{3}$$

where $f = X^t$. The fact that the spatial components, X^i, do not appear in (2) and (3) justifies the name "vector-invariant".

The pulsation equations can be obtained from the following energy functional:

$$E = E_{grav} + E_{YM} + E_{int},$$

where

$$\begin{aligned} E_{grav} &= \frac{1}{2}\int \Bigg(\frac{\bar G^{ijkl}}{\alpha^2}\dot L_{ij}\dot L_{kl} + \bar G^{ijkl}(\bar\nabla^s L_{ij})(\bar\nabla_s L_{kl}) + 2L^{ij}\bar R^k_{\ i} L_{jk} - 2L^{ij}\bar R_{kilj}L^{kl} \\ &\quad + \frac{4}{\alpha}L^{ij}\bar\nabla_i\left(L_{jk}\bar\nabla^k\alpha\right) - \frac{4}{\alpha}L^{ij}(\bar\nabla_i\alpha)\bar\nabla^k L_{kj} - 2L^{ij}\bar\nabla^k\left(\frac{\bar\nabla_i\alpha}{\alpha}\right)L_{jk} \\ &\quad - \frac{4}{\alpha^2}L\bar\nabla^i(\alpha\bar\nabla^j\alpha)L_{ij} + \frac{2}{\alpha^2}(\bar\nabla^k\alpha)(\bar\nabla_k\alpha)L^2 - \frac{2}{\alpha^2}\bar G^{ijkl}L_{ij}\bar\nabla_k\alpha^2\bar\nabla_l\tilde A \\ &\quad - 2\Lambda L^{ij}L_{ij} + 4G\,\mathrm{Tr}\left\{L^{ij}\bar F^k_{\ i}\bar F^l_{\ j}L_{kl} + \frac{1}{4}\bar F_{kl}\bar F^{kl}L_{ij}L^{ij}\right\}\Bigg)\alpha\sqrt{\bar g}\,d\,x^3, \\ E_{YM} &= G\int \mathrm{Tr}\Bigg\{\frac{1}{\alpha^2}\dot{\mathcal{E}}^i\dot{\mathcal{E}}_i + 2\bar D^{[i}\mathcal{E}^{j]}\cdot\bar D_{[i}\mathcal{E}_{j]} + \alpha^2\left[\bar D^j\left(\frac{\mathcal{E}_j}{\alpha}\right)\right]^2 \\ &\quad + \bar F^{ij}[\mathcal{E}_i,\mathcal{E}_j] + 4G\,\mathrm{Tr}(\bar F^l_{\ k}\mathcal{E}_l)\bar F^{ik}\mathcal{E}_i\Bigg\}\alpha\sqrt{\bar g}\,d\,x^3, \\ E_{int} &= -2G\int \mathrm{Tr}\Bigg\{L^{ij}\bar D_k\left(\alpha\bar F_i^{\ k}\mathcal{E}_j\right) - \alpha\bar F^{jk}\mathcal{E}^i\bar\nabla_k L_{ij} + \frac{2}{\alpha}L^{ij}\mathcal{E}_k\bar D_i\left(\alpha^2\bar F_j^{\ k}\right) \\ &\quad - 2\bar F^{kl}(\bar\nabla_k\alpha)L\mathcal{E}_l + \alpha\bar F_{ij}\mathcal{E}^i\bar\nabla^j\tilde A\Bigg\}\sqrt{\bar g}\,d\,x^3. \end{aligned}$$

Here, $\tilde A$ is defined by

$$\tilde A = \frac{\delta\dot\alpha - \delta\beta_j\bar\nabla^j\alpha}{\alpha} - L, \tag{4}$$

which is the difference between a vector-invariant combination of the time-derivative of the lapse and the trace, $L = \bar g^{ij}L_{ij}$, of the extrinsic curvature. By absorbing L into $\tilde A$, one makes sure that the resulting dynamical equations are hyperbolic. In fact, it can be checked that (4) corresponds to a linearized version of a densitized lapse, which is widely used in recent hyperbolic formulations of Einstein's equations (see Ref. [14] for example).

In terms of the variable $\tilde A$, the harmonic gauge reads $\tilde A = 0$. In this case, one has a set of equations with symmetric potential, but the kinetic energy is not positive [9]. This is the reason why the De Witt metric

$$\bar G^{ijkl} = \bar g^{i(k}\bar g^{l)j} - \bar g^{ij}\bar g^{kl}$$

appears in E_{grav} .

As we have discussed in [10], one can also adopt the maximal gauge, $L = 0$, in which case the equations are both hyperbolic and symmetric, but where perturbations of the lapse still appear in $\tilde{A}$. In this case, one recovers the energy functional given in Appendix A of Ref. [8]. At this point we recall that for perturbations with odd parity, $L = 0$ and $\tilde{A} = 0$ anyway since they represent scalar quantities. In the even-parity sector, it will turn out to be useful not to choose a particular gauge *a priori* but to derive the pulsation equations in a general gauge and to choose an appropriate gauge later.

The dynamical equations resulting from the variation of E are subject to the linearized momentum and Gauss constraint equations,

$$
\begin{aligned}
0 = \mathcal{C}_i &\equiv \alpha \bar{G}_i{}^{jkl} \bar{\nabla}_j \left(\frac{L_{kl}}{\alpha} \right) - 2G \mathrm{Tr} \left(\bar{F}^j{}_i \mathcal{E}_j \right), \\
0 = \mathcal{G} &\equiv \alpha \bar{D}^j \left(\frac{\mathcal{E}_j}{\alpha} \right).
\end{aligned}
$$

Additional constraints involving also perturbations of the metric and the gauge potential themselves are the Hamiltonian constraint and all evolution equations, which we had differentiated in time in order to construct the wave operator.

Finally, we will make use of the following fact [10] later in our stability analysis: In any gauge, the terms involving $\tilde{A}$ in the energy functional do not contribute provided that L_{ij} and $\mathcal{E}_i$ satisfy the momentum constraint $\mathcal{C}_i = 0$. Indeed, using partial integration (and homogeneous Dirichlet boundary conditions for the perturbations), the terms involving $\tilde{A}$ are

$$
\begin{aligned}
&\int \left(-\frac{1}{\alpha} \bar{G}^{ijkl} L_{ij} \bar{\nabla}_k \alpha^2 \bar{\nabla}_l \tilde{A} - 2G\alpha \mathrm{Tr} \left\{ \bar{F}_{ij} \mathcal{E}^i \bar{\nabla}^j \tilde{A} \right\} \right) \sqrt{\bar{g}}\, d\, x^3 \\
&= \int \alpha \left(\alpha \bar{\nabla}_k \left\{ \bar{G}^{ijkl} \frac{L_{ij}}{\alpha} \right\} - 2G \mathrm{Tr} \left\{ \bar{F}^{kl} \mathcal{E}_k \right\} \right) \bar{\nabla}_l \tilde{A} \sqrt{\bar{g}}\, d\, x^3,
\end{aligned}
$$

which vanishes if $\mathcal{C}^l = 0$.

4 Even-parity fluctuations

Here, we specialize the pulsation equations given in the last section to a spherically symmetric background with gauge group $\mathfrak{su}(2)$. In this case, it is convenient to expand the linearized extrinsic curvature and electric YM field in terms of spherical tensor harmonics since then, perturbations belonging to different choices of the angular momentum numbers ℓ and m decouple. Furthermore, the tensor harmonics can be divided into parities. Odd-parity perturbations were discussed in [8]. In this work, we consider the even-parity sector.

4.1 The vacuum case

We start by computing E_{grav} . For technical reasons, it is convenient to parametrize the background 3-metric according to

$$\bar{\boldsymbol{g}} = dx^2 + r(x)^2 d\Omega^2,$$

where x is a radial coordinate, and where r and the lapse α depend on x only. In terms of the functions N and S defined in section 2, we have $\alpha^2 = NS^2$ and $N = (\partial_x r)^2$.

Einstein's background vacuum equations are obtained from

$$\begin{aligned}
\frac{(r^2\alpha')'}{r^2\alpha} &= \quad R_{00} &= \frac{2G}{r^2}\left(w'^2 + \frac{(w^2-1)^2}{2r^2}\right) - \Lambda, \\
-2\frac{r''}{r} - \frac{\alpha''}{\alpha} &= \quad R_{xx} &= \frac{2G}{r^2}\left(w'^2 - \frac{(w^2-1)^2}{2r^2}\right) + \Lambda, \\
2\left(1 - r'^2 - rr'' - rr'\frac{\alpha'}{\alpha}\right) &= \quad R^A_A &= \frac{2G}{r^2}\frac{(w^2-1)^2}{r^2} + 2\Lambda.
\end{aligned} \tag{5}$$

Here and in the following, capital indices refer to coordinates on the 2-sphere, and a prime denotes differentiation with respect to x.

In the even-parity sector, we expand L_{ij} according to

$$\begin{aligned}
L_{xx} &= \left(\tilde{p} + \frac{1}{3}\tilde{t}\right) e^{(1)}, \\
L_{xB} &= \tilde{q}\, e^{(2)}_B \,, \\
L_{AB} &= \left(-\frac{1}{2}\tilde{p} + \frac{1}{3}\tilde{t}\right) e^{(3)}_{AB} + \tilde{g}\, e^{(4)}_{AB},
\end{aligned}$$

where

$$\begin{aligned}
& e^{(1)} = \frac{Y}{r}, \quad e^{(2)}_A = \hat{\nabla}_A Y, \\
& e^{(3)}_{AB} = r\hat{g}_{AB} Y, \quad e^{(4)}_{AB} = r\left(\hat{\nabla}_A\hat{\nabla}_B Y + \frac{1}{2}\ell(\ell+1)\hat{g}_{AB} Y\right),
\end{aligned}$$

form a basis of even-parity tensor harmonics, which are orthogonal with respect to the inner product induced by $\bar{\boldsymbol{g}}$. ($Y \equiv Y^{\ell m}$ denote the standard spherical harmonics. The indices ℓm are suppressed in what follows.) Also, we have chosen the parametrization such that $\tilde{t}$ and $\tilde{p}$ correspond to the trace and the radial trace-less part, respectively, of L_{ij}. After the rescaling

$$\tilde{t} = \sqrt{\frac{3}{2}}\, t, \quad \tilde{p} = \sqrt{\frac{2}{3}}\, p, \quad \tilde{q} = \frac{1}{\sqrt{2\mu^2}}\, q, \quad \tilde{g} = \sqrt{\frac{2}{\mu^2\lambda}}\, g,$$

with $\mu^2 = \ell(\ell+1)$, $\lambda = \mu^2 - 2$, the expansion is normalized such that the De Witt metric becomes

$$\int L_{ij}\bar{G}^{ijkl}L_{kl}\sqrt{\bar{g}}\,d\,x^3 = \int(-t^2+p^2+q^2+g^2)dx.$$

The signs reflect the signature of the De Witt metric.

Inserting the expressions above, using the background quantities

$$\begin{array}{rl} \bar{R}^x{}_{AxB} = -rr''\hat{g}_{AB}, & \bar{R}^D_{CAB} = 2(1-r'^2)\delta^D_{[A}\hat{g}_{B]C}, \\ \bar{R}_{xx} = -2\frac{r''}{r}\,, & \bar{R}_{AB} = \left(1-r'^2-rr''\right)\hat{g}_{AB}, \\ \bar{F}_{xB} = -w'\hat{\varepsilon}^A_{\ B}\tau_A\,, & \bar{F}_{AB} = (w^2-1)\tau_r\,\hat{\varepsilon}_{AB}\,, \end{array}$$

and integrating over the spherical variables, the gravitational energy functional becomes, after some calculations,

$$E_{grav} = \frac{1}{2}\int\Big(\langle\dot{V}\,,\,\boldsymbol{T}\dot{V}\rangle + \langle\partial_\rho V\,,\,\boldsymbol{T}\partial_\rho V\rangle + \langle V\,,\,\boldsymbol{S}_{grav}V\rangle - 2\langle V\,,\,\boldsymbol{T}\boldsymbol{b}(\tilde{a})_{grav}\rangle\Big)\,d\rho,$$

where $V \equiv (t,p,q,g)^T$. The matrix $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \mathrm{diag}(-1,1,1,1)$$

and the symmetric matrix $\boldsymbol{S}_{grav}$ by

$$\boldsymbol{S}_{grav} = \begin{pmatrix} S_{tt} & S_{t1} & 0 & 0 \\ S_{t1} & S_{11} & \sqrt{12}\mu\gamma_{,\rho} & 0 \\ 0 & \sqrt{12}\mu\gamma_{,\rho} & S_{22} & 2\sqrt{\lambda}\gamma_{,\rho} \\ 0 & 0 & 2\sqrt{\lambda}\gamma_{,\rho} & S_{33} \end{pmatrix},$$

with

$$\begin{aligned} S_{t1} &= -\frac{4r}{3}(\gamma\alpha')' + \frac{4G}{3}\gamma^2\left[w'^2 - \frac{(w^2-1)^2}{r^2}\right], \\ S_{tt} &= -\gamma^2\left[\mu^2 + rr'' + \frac{13}{3}rr'\frac{\alpha'}{\alpha} + \frac{5}{3}\frac{r^2}{\alpha}\,\alpha'' - \frac{4}{3}\frac{r^2}{\alpha^2}\,\alpha'^2 + \Lambda r^2\right] \\ &\quad + \frac{14G}{3}\gamma^2\left[w'^2 + \frac{(w^2-1)^2}{2r^2}\right], \\ S_{11} &= \gamma^2\left[\mu^2 + 6r'^2 - 5rr'' - \frac{31}{3}rr'\frac{\alpha'}{\alpha} + \frac{4}{3}\frac{r^2}{\alpha^2}(\alpha\alpha')' - 2\Lambda r^2\right] \\ &\quad - \frac{4G}{3}\gamma^2\left[w'^2 - 5\frac{(w^2-1)^2}{2r^2}\right], \\ S_{22} &= \gamma^2\left[\mu^2 + 4r'^2 - 4rr'' - 8rr'\frac{\alpha'}{\alpha} + \frac{r^2}{\alpha^2}(\alpha\alpha')' - 2\Lambda r^2 + 2G\frac{(w^2-1)^2}{r^2}\right], \\ S_{33} &= \gamma^2\left[\mu^2 - 2r'^2 - rr'' - rr'\frac{\alpha'}{\alpha} - 2\Lambda r^2 + 4Gw'^2 - 2G\frac{(w^2-1)^2}{r^2}\right]. \end{aligned}$$

Here, we have also introduced $\gamma = \alpha/r$ and $\partial_\rho = \alpha\partial_x$. The inhomogeneous term $\boldsymbol{b}(\tilde{a})_{grav}$, where $\tilde{a} = \tilde{a}(x)$ is the scalar amplitude parametrizing $\tilde{A}$,

$$\tilde{A} = \tilde{a}Y,$$

is given by $\boldsymbol{b}(\tilde{a})_{grav} = (b_t, b_1, b_2, b_3)(\tilde{a})$, where

$$\begin{aligned}
b_t(\tilde{a}) &= \sqrt{\frac{2}{3}}\left[r(\alpha^2\tilde{a}')' + 2\alpha^2 r'\tilde{a}' - \mu^2\frac{\alpha^2}{r}\,\tilde{a}\right],\\
b_1(\tilde{a}) &= \sqrt{\frac{2}{3}}\left[r(\alpha^2\tilde{a}')' - \alpha^2 r'\tilde{a}' + \frac{\mu^2}{2}\frac{\alpha^2}{r}\,\tilde{a}\right],\\
b_2(\tilde{a}) &= \sqrt{2\mu^2}\,\alpha r\left(\frac{\alpha}{r}\,\tilde{a}\right)',\\
b_3(\tilde{a}) &= \sqrt{\frac{\mu^2\lambda}{2}}\,\frac{\alpha^2}{r}\,\tilde{a}.
\end{aligned}$$

The resulting perturbation equations are

$$\left(\boldsymbol{T}\partial_t^2 - \boldsymbol{T}\partial_\rho^2 + \boldsymbol{S}_{grav}\right)V - \boldsymbol{T}\boldsymbol{b}(\tilde{a})_{grav} = 0. \tag{6}$$

The linearized momentum constraint yields the following two equations:

$$\begin{aligned}
0 = \alpha\mathcal{C}_x^{vac} &= \sqrt{\frac{2}{3}}\left[\frac{\alpha}{r^2}\partial_\rho\left(\frac{r^2 p}{\alpha}\right) - \alpha r\partial_\rho\left(\frac{t}{\alpha r}\right) - \frac{\sqrt{3}\mu}{2}\gamma\, q\right]\frac{Y}{r},\\
0 = \alpha\mathcal{C}_B^{vac} &= \frac{1}{\sqrt{2\mu^2}}\left[\frac{\alpha}{r^2}\partial_\rho\left(\frac{r^2 q}{\alpha}\right) - \frac{2\sqrt{3}\mu}{3}\gamma\, t - \frac{\sqrt{3}\mu}{3}\gamma\, p - \sqrt{\lambda}\gamma\, g\right]\hat{\nabla}_B Y.
\end{aligned}$$

Note that the spatial operator which acts on V is symmetric (with respect to the standard L^2 scalar product). However, compared to the odd-parity case, the following problems arise here: First, the kinetic energy is not positive. In order to fix this, one could multiply equation (6) from the left with $\boldsymbol{T}$. However, one would then lose the symmetry of the spatial operator. Next, the (densitized) lapse still appears into the equations, and one has no evolution equation for $\tilde{a}$. On the other hand, the amplitudes are not fully coordinate-invariant but are subject to the gauge transformations (2). As a consequence, we have

$$V \mapsto V + \boldsymbol{b}(\xi)_{grav} \tag{7}$$

where $\xi(x)$ parameterizes f according to $f = \xi\, Y$. Since the same operator-valued vector $\boldsymbol{b}(.)_{grav}$ appears in both the coordinate transformation and the terms involving $\tilde{a}$ in the dynamical equations, it is clear that any gauge-invariant combination of the equations for t, p, q and g annihilates the terms which depend on $\tilde{a}$.

Using the background equations (5), we can re-express rr'', $r'\alpha'$ and α'' in terms of $N = r'^2$ and matter terms:

$$
\begin{aligned}
rr'' &= \frac{1}{2}(1-N) - \frac{\Lambda}{2}r^2 - Gw'^2 - \frac{G}{2}\frac{(w^2-1)^2}{r^2}\,, \\
rr'\frac{\alpha'}{\alpha} &= \frac{1}{2}(1-N) - \frac{\Lambda}{2}r^2 + Gw'^2 - \frac{G}{2}\frac{(w^2-1)^2}{r^2}\,, \\
r^2\frac{\alpha''}{\alpha} &= -(1-N) + 2G\frac{(w^2-1)^2}{r^2}\,.
\end{aligned}
$$

As a result, one can rewrite the coefficients of $\boldsymbol{S}_{grav}$ as

$$
\begin{aligned}
S_{t1} &= -\frac{4}{3}\frac{\alpha_{,\rho\rho}}{\alpha} + \frac{2\gamma^2}{3}\left[1 - N - \Lambda r^2 + 4Gw'^2 - 3G\frac{(w^2-1)^2}{r^2}\right], \\
S_{tt} &= -\gamma^2\left[\lambda + 1 + N - \Lambda r^2 - 4Gw'^2 + 3G\frac{(w^2-1)^2}{r^2}\right] - S_{t1}\,, \\
S_{11} &= \gamma^2\left[\lambda - 5 + 13N + 5\Lambda r^2 - 4Gw'^2 + 9G\frac{(w^2-1)^2}{r^2}\right] - S_{t1}\,, \\
S_{22} &= \frac{r^2}{\alpha}\left(\frac{\alpha}{r^2}\right)_{,\rho\rho} + \gamma^2\left[\lambda + 4G\frac{(w^2-1)^2}{r^2}\right], \\
S_{33} &= \frac{r_{,\rho\rho}}{r} + \gamma^2\left[\lambda + 4Gw'^2\right].
\end{aligned} \tag{8}
$$

4.2 The pure YM case

Next, we compute E_{YM}. The background gauge potential has the components

$$\bar{A}_x = 0, \quad \bar{A}_B = (1-w)\hat{\varepsilon}^C{}_B\tau_C\,,$$

where w depends on x only. The background YM equation reads

$$\frac{1}{\alpha}\left(\alpha w'\right)' = w\frac{w^2-1}{r^2}\,.$$

The electric YM field is expanded into $\mathfrak{su}(2)$-valued spherical harmonic one-forms with even parity (these one-forms are explained in Appendix D of Ref. [15]):

$$
\begin{aligned}
\mathcal{E}_x &= \frac{\tilde{b}}{r}X_3\,, \\
\mathcal{E}_B &= \hat{\varepsilon}^A_B\left[\tilde{c}\,\tau_r\hat{\nabla}_A Y + \tilde{d}\,Y\tau_A + \tilde{e}\left(\hat{\nabla}_A X_2 + \frac{1}{2}\mu^2 Y\tau_A\right)\right],
\end{aligned}
$$

where in terms of the Pauli matrices $\underline{\sigma} = (\sigma^i)$, $\tau_r = \underline{e}_r\cdot\underline{\sigma}/(2i)$, $\tau_A = \underline{e}_A\cdot\underline{\sigma}/(2i)$. Here X_2 and X_3 are the $\mathfrak{su}(2)$-valued harmonics

$$X_2 = \hat{g}^{AB}\hat{\nabla}_A Y\tau_B\,, \quad X_3 = \hat{\varepsilon}^{AB}\hat{\nabla}_A Y\tau_B\,.$$

After the rescaling

$$\tilde{b} = \frac{b}{\sqrt{2G\mu^2}}\,, \quad \tilde{c} = \frac{c}{\sqrt{2G\mu^2}}\,, \quad \tilde{d} = \frac{d}{2\sqrt{G}}\,, \quad \tilde{e} = \frac{e}{\sqrt{G\mu^2\lambda}}\,,$$

the amplitudes are normalized such that

$$2G\int \mathrm{Tr}\left(\bar{g}^{ij}\mathcal{E}_i\mathcal{E}_j\right)\sqrt{\bar{g}}\,d\,x^3 = \int (b^2+c^2+d^2+e^2)dx.$$

The YM energy functional becomes

$$E_{YM} = \frac{1}{2}\int \Big(\langle \dot{W}\,,\,\dot{W}\rangle + \langle \partial_\rho W\,,\,\partial_\rho W\rangle + \langle W\,,\,\boldsymbol{S}_{YM}W\rangle\Big)\,d\rho,$$

where $W \equiv (b,c,d,e)^T$ and

$$\frac{\boldsymbol{S}_{YM}}{\gamma^2} = \begin{pmatrix} \frac{\gamma_{,\rho\rho}}{\gamma^3}+\lambda+f+\frac{u^2}{\gamma^2} & sym. & sym. & sym. \\ -\frac{2}{\gamma^2}(\gamma w)_{,\rho}-\frac{uv}{\gamma} & \lambda+2f+v^2 & sym. & sym. \\ \sqrt{2}\mu\frac{\gamma_{,\rho}}{\gamma^2} & \sqrt{2}\mu w & \lambda-2+3f+\frac{2u^2}{\gamma^2} & sym. \\ \sqrt{2\lambda}\frac{\gamma_{,\rho}}{\gamma^2} & -\sqrt{2\lambda}w & 0 & \mu^2-f \end{pmatrix}, \tag{9}$$

with

$$f \equiv w^2+1, \quad u \equiv 2\sqrt{G}\,\frac{w_{,\rho}}{r}, \quad v \equiv 2\sqrt{G}\,\frac{w^2-1}{r}\,. \tag{10}$$

(Note that the u here differs from the u in our previous article [8] by a factor of $2\sqrt{G}$.) The linearized Gauss constraint gives

$$0 = \sqrt{2G}\mu\alpha r\mathcal{G} = \left\{\gamma\partial_\rho\left(\frac{b}{\gamma}\right)+\gamma wc-\frac{\mu}{\sqrt{2}}\gamma\,d-\sqrt{\frac{\lambda}{2}}\,\gamma\,e\right\}X_3\,.$$

4.3 The interaction term

We finally compute the interaction energy. Using partial integration and the background equations $\bar{D}^k(\alpha\bar{F}_{jk})=0$, it is convenient to rewrite E_{int} as

$$E_{int} = -2G\int\left\{2L^{ij}\mathrm{Tr}\left(\bar{F}_i{}^k(\bar{D}\mathcal{E})_{kj}\right)+2v^il_i+2\frac{\bar{\nabla}^k\alpha}{\alpha}\,v_kL+v^k\bar{\nabla}_k\tilde{A}\right\}\alpha\sqrt{\bar{g}}\,d\,x^3,$$

where $l_i = \alpha\bar{\nabla}^j(L_{ij}/\alpha)$ and $v^j = \mathrm{Tr}(\bar{F}^{kj}\mathcal{E}_k)$.

Inserting the above expansions for L_{ij} and $\mathcal{E}_i$ and integrating over the spherical variables, we obtain

$$\begin{aligned} E_{int} \;&=\; \int\big(\langle V\,,\,\boldsymbol{A}_{int}\partial_\rho W\rangle - \langle \boldsymbol{A}_{int}^T\partial_\rho V\,,\,W\rangle \\ &\qquad + \langle V\,,\,\boldsymbol{S}_{int}W\rangle - \langle \boldsymbol{b}_{int}(\tilde{a})\,,\,W\rangle\big)\,d\rho, \end{aligned}$$

where

$$\boldsymbol{A}_{int} = \frac{u}{2} \begin{pmatrix} 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix},$$

$$\boldsymbol{S}_{int} = \begin{pmatrix} \frac{\mu\gamma u}{\sqrt{3}} & -\frac{\mu\gamma^2 v}{\sqrt{3}} & -\frac{u_{,\rho}}{\sqrt{6}} - \sqrt{\frac{2}{3}} u \frac{(\gamma^2 r^3)_{,\rho}}{\gamma^2 r^3} & 0 \\ \frac{2\mu\gamma u}{\sqrt{3}} & \frac{\mu\gamma^2 v}{\sqrt{3}} & \frac{7u_{,\rho}}{\sqrt{6}} + \sqrt{\frac{2}{3}} u \frac{(\gamma^2)_{,\rho}}{\gamma^2} & 0 \\ \frac{3}{2}\, u_{,\rho} + \frac{\gamma_{,\rho}}{\gamma}\, u & -\gamma w u - \frac{r}{\gamma}\left(\frac{\gamma^2}{r} v\right)_{,\rho} & \frac{\mu\gamma u}{\sqrt{2}} & \sqrt{\frac{\lambda}{2}}\, \gamma u \\ 0 & -\sqrt{\lambda}\gamma^2 v & 0 & -\frac{u_{,\rho}}{\sqrt{2}} \end{pmatrix}, \tag{11}$$

and

$$\boldsymbol{b}_{int}(\tilde{a}) = \frac{\alpha}{\sqrt{2}} \left(\mu u\, \tilde{a}, -\mu\gamma v\, \tilde{a}, \sqrt{2} r u\, \tilde{a}', 0 \right).$$

At this point, we remind the reader that although the linearized electric YM field is invariant under infinitesimal $\mathfrak{su}(2)$-*gauge* transformations, it is not invariant under infinitesimal *coordinate* transformations, see (3). As a result, it is easy to show that the YM amplitudes W transform according to

$$W \mapsto W + \boldsymbol{b}_{int}(\xi),$$

where $f = \xi\, Y$.

4.4 Summary

Taking together the above results, the total energy functional becomes

$$\begin{aligned} E \;=\; \frac{1}{2} \int \Big(& \langle \dot{U}\,,\, \boldsymbol{T}\dot{U} \rangle + \langle \partial_\rho U\,,\, \boldsymbol{T}\partial_\rho U \rangle + \langle U\,,\, \boldsymbol{A}\partial_\rho U \rangle + \langle U\,,\, \partial_\rho \boldsymbol{A} U \rangle \\ & + \langle U\,,\, \boldsymbol{S} U \rangle - 2\langle U\,,\, \boldsymbol{T}\boldsymbol{b}(\tilde{a})_{grav} \rangle \Big) d\rho, \end{aligned} \tag{12}$$

and the pulsation equations are

$$\left(\boldsymbol{T}\partial_t^2 - \boldsymbol{T}\partial_\rho^2 + \boldsymbol{A}\partial_\rho + \partial_\rho \boldsymbol{A} + \boldsymbol{S} - \boldsymbol{T}\boldsymbol{b}(\tilde{a}) \right) U = 0, \tag{13}$$

where $U = (V, W)^T$, $\boldsymbol{T} = \mathrm{diag}(-1, 1, 1, 1, 1, 1, 1, 1)$,

$$\boldsymbol{A} = \begin{pmatrix} 0 & \boldsymbol{A}_{int} \\ -\boldsymbol{A}_{int}^T & 0 \end{pmatrix}, \quad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{grav} & \boldsymbol{S}_{int} \\ \boldsymbol{S}_{int}^T & \boldsymbol{S}_{YM} \end{pmatrix},$$

and

$$\boldsymbol{b}(\tilde{a}) = \left(\boldsymbol{b}_{grav}(\tilde{a}), \boldsymbol{b}_{int}(\tilde{a}) \right).$$

Infinitesimal coordinate transformations are given by

$$U \mapsto U + \boldsymbol{b}(\xi), \tag{14}$$

and the constraint equations are

$$\begin{aligned} 0 &= \frac{\alpha}{r^2}\partial_\rho\left(\frac{r^2 p}{\alpha}\right) - \alpha r \partial_\rho\left(\frac{t}{\alpha r}\right) - \frac{\sqrt{3}\mu}{2}\gamma\, q - \sqrt{\frac{3}{2}}\, u\, d, && (15)\\ 0 &= \frac{\alpha}{r^2}\partial_\rho\left(\frac{r^2 q}{\alpha}\right) - \frac{2\sqrt{3}\mu}{3}\gamma\, t - \frac{\sqrt{3}\mu}{3}\gamma\, p - \sqrt{\lambda}\gamma\, g - u\, b + \gamma v\, c, && (16)\\ 0 &= \gamma\partial_\rho\left(\frac{b}{\gamma}\right) + \gamma w c - \frac{\mu}{\sqrt{2}}\gamma\, d - \sqrt{\frac{\lambda}{2}}\gamma\, e. && (17) \end{aligned}$$

Finally, as in our previous work, we assume that all perturbations vanish at the boundary points, so that all boundary terms vanish.

5 Special cases

In this section, we discuss the perturbation equations for some special cases. We first show that for $\ell = 0$, one recovers the radial pulsation equation derived in [4]. Then, we introduce fully gauge-invariant amplitudes and discuss the stability of the Schwarzschild-adS and the RN-adS solutions. These examples, which are much simpler than the cases where the gauge potential is effectively non-Abelian, indicate that a topological stability analysis, as performed in the odd-parity sector, is not possible in the even-parity sector.

5.1 Radial perturbations

For $\ell = 0$, the only gravitational amplitudes are t and p. Since these are subject to the one-parameter family of coordinate transformations (14) and to the constraint equation (15), one expects that there are no dynamical gravitational modes. In fact, one can show that for $\ell = 0$, the only physical gravitational perturbations correspond to the variation of the mass. This is also clear in view of Birkhoff's theorem. For the YM field, we have only the amplitude d and the Gauss constraint is void. Therefore, we expect to have an unconstrained wave equation for the YM field when $\ell = 0$. Looking at (14) we see that we can construct a gauge-invariant linear combination:

$$\hat{d} = d + \frac{1}{\sqrt{6}}\frac{ru}{r_{,\rho}}(-t + p).$$

For computation, it is convenient to choose a gauge in which $t = p$ since in this gauge, $\hat{d} = d$. Furthermore, in this gauge, the only constraint equation yields the simple relation

$$r_{,\rho} t = \frac{ru}{\sqrt{6}}\, d.$$

Using this, and taking the combination of the dynamical equations which corresponds to the definition of $\hat{d}$, we obtain the following wave equation for d:

$$\begin{aligned}&\left(\partial_t^2 - \partial_\rho^2\right) d + 2\frac{ru}{r_{,\rho}}\, u_{,\rho}\, d \\ &\quad + \gamma^2 \left[3w^2 - 1 + \left(\frac{ru}{r_{,\rho}}\right)^2 \left(-1 + 2N + \Lambda r^2 + G\frac{(w^2-1)^2}{r^2}\right)\right] d = 0.\end{aligned}$$

Finally, using $ru/r_{,\rho} = 2\sqrt{G}w_{,r}$, $u^2 = 4G\gamma^2 N w_{,r}^2$ and the background YM equation $ru_{,\rho} = 2\sqrt{G}\gamma^2 w(w^2-1) - ur_{,\rho}$, we have

$$\begin{aligned}&\left(\partial_t^2 - \partial_\rho^2 + \gamma^2(3w^2-1)\right) d \\ &\quad + 4G\gamma^2 \left[2ww_{,r}\frac{w^2-1}{r} - w_{,r}^2\left(1 - \Lambda r^2 - G\frac{(w^2-1)^2}{r^2}\right)\right] d = 0, \qquad (18)\end{aligned}$$

which agrees with the equation found in [4]. By replacing d by $\hat{d}$, this equation acquires a gauge-invariant meaning. Using the estimate of section 2 for $\Lambda \to -\infty$, we see that the dominant terms in the potential are $\gamma^2(3w^2-1)$ which is positive for solutions with $w(r_h) > 1/\sqrt{3}$ and large $|\Lambda|$. This shows the existence of solutions which are stable with respect to even-parity radial perturbations [4].

5.2 The gauge-invariant approach

Motivated by the analysis above, one can try to introduce gauge-invariant amplitudes for $\ell \geq 1$. In order to do so, it turns out to be convenient to introduce the amplitude

$$\tau \equiv \frac{1}{\sqrt{6}}(-t+p) + \frac{r'}{\sqrt{2}\mu}\, q.$$

Using the transformation rules (7), we find

$$\tau \mapsto \tau + \frac{\alpha^2}{r}\, f_\tau\, \xi,$$

where

$$f_\tau \equiv \frac{\mu^2}{2} + rr'\frac{\alpha'}{\alpha} - N = \frac{\lambda}{2} + \frac{3m}{r} + Gw'^2 - \frac{G}{2}\frac{(w^2-1)^2}{r^2}\,.$$

The advantage of the new amplitude τ is that – like the amplitude g – it transforms with an additive term which is algebraic in ξ. Provided that f_τ has no zeros, one can use τ in order to construct gauge-invariants. For example, we can construct

$$\zeta = g - \sqrt{\frac{\mu^2\lambda}{2}}\frac{1}{f_\tau}\,\tau, \qquad (19)$$

which is gauge-invariant for $\ell \geq 2$. In fact, we will show below that for vacuum gravity, ζ satisfies the Zerilli equation.

Using the momentum constraint equations (15) and (16), we find the following constraint equation

$$\gamma\partial_\rho\left(\frac{\tau}{\gamma}\right) = \gamma(f_\tau - 2Gw'^2)\frac{q}{\sqrt{2}\mu} + \sqrt{\frac{\lambda}{2\mu^2}}\frac{r_{,\rho}}{r}\,g + \frac{r'u}{\sqrt{2}\mu}\,b - \frac{1}{\sqrt{2}\mu}\frac{r_{,\rho}}{r}\,v\,c + \frac{u}{2}\,d. \quad (20)$$

In the gauge where $\tau = 0$, this yields an algebraic relation between q, g, b, c and d.

Next, we compute an evolution equation for τ: Taking the combination of the evolution equations which corresponds to the definition of τ, we arrive at the equation

$$\begin{aligned} 0 \;=\; & \Box\,\tau + \gamma^2\left(\mu^2 - \frac{6m}{r} + 5G\frac{(w^2-1)^2}{r^2}\right)\tau + \left(4\gamma_{,\rho}f_\tau - \frac{\alpha'}{\gamma}u^2 - \gamma uvw\right)\frac{q}{\sqrt{2}\mu} \\ & +\gamma^2 r'\left(-\frac{6m}{r} + 10Gw'^2 + 5G\frac{(w^2-1)^2}{r^2}\right)\frac{q}{\sqrt{2}\mu} \\ & +\frac{\sqrt{2\lambda}}{\mu}\gamma^2\left(1 - 2N - r^2\Lambda - G\frac{(w^2-1)^2}{r^2}\right)g \\ & +\frac{1}{\sqrt{2}\mu}\left[r'u\partial_\rho b + 2r'u_{,\rho}b + r'\frac{\gamma_{,\rho}}{\gamma}\,ub + \mu^2\gamma u\,b + 2\alpha r''ub\right] \\ & -\frac{1}{\sqrt{2}\mu}\left[\gamma wr'u - \frac{rr'}{\gamma}\left(\frac{\gamma^2}{r}\,v\right)_{,\rho} - 2\alpha r''\gamma v\right]c + \left(u_{,\rho} - \frac{1}{2}\frac{r_{,\rho}}{r}\,u\right)d \\ & +\frac{\sqrt{\lambda}}{2\mu}\frac{r_{,\rho}}{r}\,u\,e - \frac{\alpha^2}{r}\,f_\tau\,\tilde{a}, \end{aligned} \quad (21)$$

where we have defined

$$\Box \equiv \left(\partial_t^2 - \partial_\rho^2\right).$$

We have also used the momentum constraint equation (16) in order to eliminate first derivatives of q. It seems clear from equation (21) that a formulation of the evolution equations in terms of gauge-invariant quantities would be quite messy. Worse than this, we have no reasons to expect that the resulting equations have a symmetric potential. Nevertheless, in some special cases where w is constant, the expressions simplify enough and one can find a fully gauge-invariant description. This will be the subject of the next two subsections.

5.3 Stability of the Schwarzschild-adS solution

If $|w| = 1$ (i.e. the YM field is a pure gauge), we see that the interaction terms $\boldsymbol{A}_{int}$ and $\boldsymbol{S}_{int}$ vanish, and the gravitational and YM perturbations decouple from each other. In order to investigate the gravitational sector, we adopt the gauge-invariant formalism described in the previous subsection. First, we note that for $\ell = 1$, the constraint equation (20) yields, in the gauge $\tau = 0$,

$$q = 0.$$

Then, the evolution equation for τ, (21), gives $\tilde{a} = 0$. This shows that for $\ell = 1$, the gravitational sector is empty.

For $\ell \geq 2$, we introduce the gauge-invariant amplitude

$$\zeta = g - \sqrt{\frac{\mu^2\lambda}{2}}\frac{1}{f_\tau}\tau,$$

where $f_\tau = \lambda/2 + 3m/r$. We derive the equation for ζ in the gauge $\tau = 0$. Taking the combination of the evolution equation for τ, (21), and the equation for g corresponding to the definition of ζ, the terms involving $\tilde{a}$ cancel, and we obtain

$$\Box\, g + \frac{\sqrt{\lambda}}{2}\gamma^2 r' \frac{6m}{rf}\, q + \left[S_{33} - \gamma^2\frac{\lambda}{f_\tau}\left(1 - 2N - r^2\Lambda\right)\right] g = 0.$$

Next, the constraint equation (20) yields

$$q = -\sqrt{\lambda}\,\frac{r'}{f_\tau}\, g.$$

Using this, and $r_{,\rho} = N = 1 - 2m/r - \Lambda r^2/3$, we eventually get the Zerilli equation [16] (with the presence of a cosmological constant)

$$(\Box + V_Z)\, g = 0, \tag{22}$$

where

$$V_Z = N\frac{\lambda^2 r^2[(\lambda+2)r + 6m] + 12m^2(3\lambda r + 6m - 2\Lambda r^3)}{(\lambda r + 6m)^2 r^3}.$$

If we replace g by ζ, this equation becomes manifestly gauge-invariant. Since V_Z is positive for all $r \in (r_h, \infty)$, there are no gravitational instabilities. At this point, let us also mention that equation (22) and its odd-parity counterpart have already been derived and used in order to compute the quasi-normal modes of Schwarzschild-adS black holes (see Ref. [17] and references therein).

The YM perturbations describe a linearized YM field on a Schwarzschild black hole. We will show later (see section 6.2) that those fields cannot possess exponentially growing modes either.

5.4 On the stability of the RN-adS solution

When $w = 0$, we see that the equations decouple into two sets, one set comprising the amplitudes (t, p, q, g, c) (referred to as Abelian amplitudes in the following) and the other set comprising (b, d, e) (referred to as the non-Abelian amplitudes).

We start with the discussion of the Abelian amplitudes: The relevant gauge-invariants are

$$\begin{aligned}\zeta &= g - \sqrt{\frac{\mu^2\lambda}{2}}\frac{1}{f_\tau}\tau,\\ \psi &= c + \frac{\mu}{\sqrt{2}}\frac{v}{f_\tau}\tau.\end{aligned}$$

(For $\ell = 1$, ζ is void.) Taking the combination of the evolution equations corresponding to these amplitudes and eliminating q using the constraint equation (20), i.e. using

$$q = -\frac{r'}{f_\tau}\left[\sqrt{\lambda}g - vc\right],$$

we get the following system of equations:

$$(\Box + V_M)\begin{pmatrix} \zeta \\ \psi \end{pmatrix} = 0, \tag{23}$$

where the potential V_M is symmetric and has the particular form

$$V_M = N\left[U + W\begin{pmatrix} -3M & \sqrt{4G\lambda} \\ \sqrt{4G\lambda} & 3M \end{pmatrix}\right].$$

Using $N = 1 - 2M/r + G/r^2 - \Lambda r^2/3$ and $2f_\tau = \lambda + 6M/r - 4G/r^2$, we can write the functions U and W as (the fact that $f_\tau > 0$ for $r \geq r_h$ is shown in Appendix A)

$$\begin{aligned} U &= \frac{1}{r^2(2f_\tau)^2}\left[\lambda^3 + \left(2 + \frac{9M}{r} - \frac{4G}{r^2}\right)\lambda^2\right. \\ &\quad + 4\left(-\frac{2}{3}G\Lambda + \frac{3M}{r} + \frac{9M^2+2G}{r^2} - \frac{16GM}{r^3} + \frac{6G^2}{r^4}\right)\lambda \\ &\quad + 4\left(-3\Lambda M^2 + \frac{4GM\Lambda}{r} + \frac{9M^2 - \frac{8}{3}G^2\Lambda}{r^2}\right. \\ &\qquad \left.\left. + \frac{9M^3}{r^3} - \frac{39GM^2}{r^4} + \frac{32MG^2}{r^5} - \frac{8G^3}{r^6}\right)\right], \\ W &= \frac{1}{r^3(2f_\tau)^2}\left[\lambda^2 + 4\lambda + \frac{4M}{r}\left(3 - \frac{3M}{r} + \frac{G}{r^2}\right) + \frac{2\Lambda r^2}{3}\left(\frac{6M}{r} - \frac{8G}{r^2}\right)\right]. \end{aligned}$$

For $\Lambda = 0$, these equations reduce to the ones obtained by Moncrief [18] describing even-parity perturbations of a RN black hole. The fact that V_M can be written as the sum of a function and a function times a constant matrix allows the system to be diagonalized. The absence of exponentially growing modes in this system is discussed in Appendix A.

For $\ell = 0$, the non-Abelian modes are described by equation (18) where one sets $w = 0$:

$$\left(\Box - \gamma^2\right)d = 0.$$

In [8], we have shown that this equation possesses exponentially growing modes. Therefore, the RN-adS solution is *unstable* with respect to non-Abelian odd- and even-parity radial perturbations.

For $\ell \geq 1$, there are no instabilities. In order to see this, we consider non-Abelian perturbations, which are described by the system

$$
\begin{aligned}
0 &= \Box\, b + \left[\frac{\gamma_{,\rho\rho}}{\gamma} + \gamma^2(\lambda+1)\right] b + \sqrt{2}\mu\gamma_{,\rho}\, d + \sqrt{2\lambda}\,\gamma_{,\rho}\, e, \\
0 &= \Box\, d + \sqrt{2}\mu\gamma_{,\rho}\, b + \gamma^2(\lambda+1)d, \\
0 &= \Box\, e + \sqrt{2\lambda}\,\gamma_{,\rho}\, b + \gamma^2(\lambda+1)e.
\end{aligned}
$$

(For $\ell = 1$, the equation for e is not present.) We also have the Gauss constraint,

$$
0 = \gamma\partial_\rho \left(\frac{b}{\gamma}\right) - \frac{\mu}{\sqrt{2}}\gamma\, d - \sqrt{\frac{\lambda}{2}}\gamma\, e.
$$

Using this constraint, we can eliminate both d and e in the evolution equation for b. Defining $B = b/\gamma$, one obtains the equation

$$
\Box\, B + \gamma^2(\lambda+1)B = 0,
$$

which has no unstable modes. From the Gauss constraint, it follows that $\mu\, d + \sqrt{\lambda}\, e$ cannot grow exponentially in time either. Finally, we can find an equation for $F \equiv d/\mu - e/\sqrt{\lambda}$:

$$
\Box\, F + \gamma^2(\lambda+1)F = 0,
$$

which is identical to the equation for B. Stability for $\ell \geq 1$ will also follow from the considerations made in section 6.2.

6 The general case

In this section we shall prove that both solitons and black holes in which the gauge function w has no zeros are stable for $|\Lambda|$ sufficiently large. Since the analysis required is rather involved, we shall first outline the steps in the proof, before describing the details. The sector with $\ell = 0$ has already been discussed in [4] and section 5, so we assume that $\ell \geq 1$ in the analysis below.

6.1 Outline of stability proof

In the last section, we have re-formulated the perturbation equations on a Schwarzschild-adS and RN-adS background in terms of fully gauge-invariant amplitudes. Unfortunately, in the general case, this procedure is quite messy and we have no guarantee that the resulting equations are going to be suitable for analytic discussions.

In the following, we adopt a gauge in which $t = p$. Somehow, this gauge has similar properties as $\tau = 0$, but has the advantage of being simpler computationally. In particular, one does not lose the symmetry of the potential. For $t = p$, the momentum constraint, Eq. (15), reduces to an algebraic equation,

$$
\sqrt{6}\,\frac{r_{,\rho}}{r}\, t = \frac{\mu}{\sqrt{2}}\gamma q + ud, \tag{24}
$$

which permits one to re-express t in terms of q and d. (Note that $r_{,\rho} = NS$ is positive everywhere outside the horizon for black holes and positive everywhere for solitons.) Introducing $t = p$ into the energy expression (12), we see that there are no kinetic terms involving t (or p). In particular, the kinetic energy is positive, and $t = p$ is non-dynamical. The energy now has the form

$$\begin{aligned} E &= \frac{1}{2}\int\Big(\langle\dot{\bar{U}}\,,\,\dot{\bar{U}}\rangle + \langle\partial_\rho\bar{U}\,,\,\partial_\rho\bar{U}\rangle + \langle\bar{U}\,,\,\bar{\boldsymbol{A}}\partial_\rho\bar{U}\rangle + \langle\bar{U}\,,\,\partial_\rho\bar{\boldsymbol{A}}\bar{U}\rangle \\ &\qquad + \langle\bar{U}\,,\,\bar{\boldsymbol{S}}\bar{U}\rangle - 2\langle\bar{U}\,,\,\bar{\boldsymbol{b}}(\tilde{a})\rangle + \mathcal{R}\Big)d\rho, \end{aligned} \tag{25}$$

where $\bar{U} = (q, g, b, c, d, e)^T$ and where $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{S}}$ and $\bar{\boldsymbol{b}}$ are obtained from $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{b}$ by removing rows and columns corresponding to t and p. The remainder terms $\mathcal{R}$ then are

$$\begin{aligned} \mathcal{R} &= (S_{tt} + S_{11} + 2S_{t1})\,t^2 + 2\sqrt{12}\mu\gamma_{,\rho}\,t\,q \\ &\quad + 2\sqrt{3}\mu\gamma u\,t\,b + 2\sqrt{6}\left(u_{,\rho} - u\frac{r_{,\rho}}{r}\right)t\,d + 2\,(b_t(\tilde{a}) - b_1(\tilde{a}))\,t. \end{aligned} \tag{26}$$

Variation of these equations with respect to t, q, g, b, c, d, and e yields the same equations as in (13) but with $p = t$ and where the equations for t and p are summed, resulting in an elliptic equation for $\tilde{a}$.

In order to discuss the stability of the system, we proceed as follows: We first give initial data for t, p, q, g, b, c, d, e and their time derivatives such that the constraint equations $t = p$, (16), (17), (24) and their time derivatives are satisfied. Then, we evolve the data using the hyperbolic evolution equations (13). In order to make sure that the gauge condition $t = p$ is preserved during the evolution, we have to solve, at each time-step, the elliptic equation for $\tilde{a}$ described above. In [10] we have shown that the constraint equations propagate. In particular, this is the case for the constraint (24). Next, as we have shown in section 3, the terms which depend on $\tilde{a}$ do not contribute to the energy E. As a consequence, one can check that E is conserved, i.e. $\partial_t E = 0$, provided that homogeneous Dirichlet boundary conditions are given at the horizon and at infinity. Then, stability follows if we can show that E is positive. Indeed, if E is positive, L_{ij} and $\mathcal{E}_i$ cannot grow exponentially in time. In a gauge where the shift and the electric potential are zero, the same follows for $\delta\bar{g}_{ij}$ and $\delta\bar{A}_i$ since

$$\delta\bar{g}(t)_{ij} = \delta\bar{g}(0)_{ij} + \int_0^t 2L(\tau)_{ij}d\tau,$$

$$\delta\bar{A}(t)_i = \delta\bar{A}(0)_i - \int_0^t \mathcal{E}_i(\tau)d\tau.$$

Since the terms involving $\tilde{a}$ do not contribute, stability will hold if the operator

$$\bar{\mathcal{A}} \equiv -\partial_\rho^2 + \bar{\boldsymbol{A}}\partial_\rho + \partial_\rho\bar{\boldsymbol{A}} + \bar{\boldsymbol{S}} \tag{27}$$

together with the remainder term $\bar{\mathcal{R}}$ is positive, where

$$\bar{\mathcal{R}} = \mathcal{R} - 2\,(b_t(\tilde{a}) - b_1(\tilde{a}))\,t.$$

From now on, we therefore ignore all terms dependent on $\tilde{a}$, and shall consider only the energy functional

$$\tilde{E} = E - \frac{1}{2}\int \left[-2\langle \bar{U}, \bar{\boldsymbol{b}}(\tilde{a})\rangle + 2\,(b_t(\tilde{a}) - b_1(\tilde{a}))\,t\right] d\rho, \tag{28}$$

since E is positive (and the system stable) if $\tilde{E}$ is positive.

In the first stage of the proof (subsection 6.2) we show that the operator $\hat{\mathcal{A}}$ (which has the same form as $\bar{\mathcal{A}}$ (27) but with slightly modified matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{S}}$) can be written as

$$\hat{\mathcal{A}} = \bar{\boldsymbol{B}}^{\dagger}\bar{\boldsymbol{B}} + \bar{b}^T\bar{b},$$

where $\bar{\boldsymbol{B}}$ is a first order matrix differential operator and $\bar{b}$ is a vector. This means that $\hat{\mathcal{A}}$ is a positive operator.

As a result of the factorization of $\hat{\mathcal{A}}$, we now have a slightly modified remainder term $\hat{\mathcal{R}}$ given by

$$\begin{aligned}\hat{\mathcal{R}} \;&=\; (S_{tt} + S_{11} + 2S_{t1})\,t^2 + 2\sqrt{12}\mu\gamma_{,\rho}\,t\,q \\ &\quad +2\sqrt{3}\mu\gamma u\,t\,b + 2\sqrt{6}u_{,\rho}t\,d,\end{aligned} \tag{29}$$

(ignoring the terms containing the lapse $\tilde{a}$). Unfortunately, $\hat{\mathcal{R}}$ is not positive (even in the Schwarzschild-adS case), so we do not immediately have that the energy $\tilde{E}$ is positive. However, in the next part of the stability proof (subsection 6.3), we argue that this remainder term is of subleading order compared to $\hat{\mathcal{A}}$ when $|\Lambda| \to \infty$. This is not sufficient to give positivity of $\tilde{E}$ even for all sufficiently large $|\Lambda|$, since the operator $\hat{\mathcal{A}}$ is positive and not necessarily bounded away from zero (this means that it may be that $\hat{\mathcal{A}}$ could be equal to zero for some perturbations and then the subleading terms in $\hat{\mathcal{R}}$ would be dominant, and may be negative). From the fact that $\hat{\mathcal{R}}$ is subleading order as $|\Lambda| \to \infty$, we can however deduce that $\tilde{E}$ is positive for the precise value $|\Lambda| = \infty$ (in a sense to be made exact in section 6.3).

It therefore remains to extend this stability for $|\Lambda| = \infty$ to sufficiently large values of $|\Lambda|$. We will do this via an analyticity argument, using the multi-dimensional equivalent of the nodal theorem [11]. The details of this are quite involved, and cover subsections 6.4–6.6. First of all, as before we ignore the terms depending on $\tilde{a}$ in the energy functional E (25), and consider instead the modified energy functional (28)

$$\begin{aligned}\tilde{E} \;&=\; \frac{1}{2}\int \Big(\langle \dot{\bar{U}}\,,\,\dot{\bar{U}}\rangle + \langle \partial_\rho \bar{U}\,,\,\partial_\rho \bar{U}\rangle + \langle \bar{U}\,,\,\bar{\boldsymbol{A}}\partial_\rho \bar{U}\rangle + \langle \bar{U}\,,\,\partial_\rho \bar{\boldsymbol{A}}\bar{U}\rangle \\ &\qquad + \langle \bar{U}\,,\,\bar{\boldsymbol{S}}\bar{U}\rangle + \bar{\mathcal{R}}\Big) d\rho.\end{aligned}$$

We also express t (which is non-dynamical) in terms of q and d using the constraint equation (24). The remainder term $\bar{\mathcal{R}}$ then takes the form:

$$\bar{\mathcal{R}} = \mathcal{S}_{qq}q^2 + 2\mathcal{S}_{qd}qd + \mathcal{S}_{dd}d^2 + 2\mathcal{S}_{qb}qb + 2\mathcal{S}_{bd}bd,$$

where

$$\begin{aligned}
\mathcal{S}_{qq} &= \frac{\mu^2 r^2 \gamma^4}{2r_{,\rho}^2}\left[-1 + 2N + \Lambda r^2 + \frac{G(w^2-1)^2}{r^2}\right] + \frac{2\mu^2 r\gamma_{,\rho}\gamma}{r_{,\rho}};\\
\mathcal{S}_{qd} &= \frac{\mu u r^2 \gamma^3}{\sqrt{2}r_{,\rho}^2}\left[-1 + 2N + \Lambda r^2 + \frac{G(w^2-1)^2}{r^2}\right] + \frac{\sqrt{2}\mu u r\gamma_{,\rho}}{r_{,\rho}} + \frac{u u_{,\rho} r\gamma}{\sqrt{2}r_{,\rho}} - \frac{u^2}{\sqrt{2}};\\
\mathcal{S}_{dd} &= \frac{u^2 r^2 \gamma^2}{r_{,\rho}^2}\left[-1 + 2N + \Lambda r^2 + \frac{G(w^2-1)^2}{r^2}\right] + \frac{2u u_{,\rho} r}{r_{,\rho}} - 2u^2;\\
\mathcal{S}_{qb} &= \frac{\mu^2 u\alpha\gamma}{2r_{,\rho}};\\
\mathcal{S}_{bd} &= \frac{\mu u^2 \alpha}{\sqrt{2}r_{,\rho}}.
\end{aligned} \tag{30}$$

On varying $\tilde{E}$ with respect to the perturbations q, g, b, c, d and e, the remainder term $\bar{\mathcal{R}}$ will contribute additional terms to the perturbation equations. These additional terms are equivalent to adding the following quantities to the pertinent parts of the potential $\bar{S}$: add $\mathcal{S}_{qq}$ to S_{22} in $\boldsymbol{S}_{grav}$ (8); add $\mathcal{S}_{dd}$ to the (dd)-entry in $\boldsymbol{S}_{YM}$ (9); finally, add $\mathcal{S}_{qd}$, $\mathcal{S}_{qb}$ and $\mathcal{S}_{bd}$ to the corresponding entries in $\boldsymbol{S}_{int}$ (11).

There is one additional complication. For both black hole and soliton solutions, it is found to be necessary to study the perturbation equations arising not from the variation of the whole of $\tilde{E}$, but, instead we shall vary only

$$E_{\mathcal{P}} = \tilde{E} - \mathcal{P},$$

where $\mathcal{P}$ is manifestly positive. Positivity of $\tilde{E}$ is automatic if $E_{\mathcal{P}}$ is positive since we shall always have $\mathcal{P} > 0$. Further, we can deduce that $E_{\mathcal{P}} \geq 0$ if the perturbation equations derived from varying $E_{\mathcal{P}}$ have no unstable modes. The splitting of $\tilde{E}$ in this form will be seen to be necessary in section 6.5 in order to satisfy the technical conditions for the application of the multi-dimensional nodal theorem.

In subsection 6.4 we write the equations derived from the variation of $E_{\mathcal{P}}$ in the form

$$\mathcal{O}\bar{U} = -\frac{d}{dR}\left(\boldsymbol{K}_1(R)\frac{d}{dR}\bar{U}\right) + \left(\frac{\boldsymbol{L}(R)}{R^2} + \boldsymbol{M}(R)\right)\bar{U} = -\boldsymbol{K}_2(R)\partial_t^2\bar{U}. \tag{31}$$

In equation (31), the variable R depends on whether we are considering solitonic or black hole solutions and is given in subsection 6.5, and $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{L}$, and $\boldsymbol{M}$

are real symmetric matrices depending on R. We show, in subsection 6.5, that $\boldsymbol{K}_1$ is uniformly positive definite, $\boldsymbol{K}_2$ is positive definite, $\boldsymbol{L}(0)$ is non-negative and that all four matrices are smooth and uniformly bounded for all R in the interval $[0,\infty)$. The crux of the stability proof is to show that the operator $\mathcal{O}$ has no negative eigenvalues for all $|\Lambda|$ sufficiently large, given that we have already proved that there are no negative eigenvalues if $|\Lambda| = \infty$. Since the matrix $\boldsymbol{K}_2$ is positive definite, stability follows if the operator $\mathcal{O}$ has no negative eigenvalues.

Before applying the multi-dimensional nodal theorem [11], we note that there are additional criteria required of the matrices $\boldsymbol{L}$ and $\boldsymbol{M}$, which ensure that the essential spectrum of the operator $\mathcal{O}$ equals $[0,\infty)$, and that the number of bound states is finite. Sufficient criteria are [11]: $\boldsymbol{M} \to 0$ as $R \to \infty$; and $R^2\boldsymbol{M} + \boldsymbol{L} \geq -\beta_c$ for some $\beta_c < 1/4$ and all sufficiently large R. We will show, in section 6.5, that these additional criteria are also satisfied by our matrices $\boldsymbol{L}$ and $\boldsymbol{M}$.

The multi-dimensional nodal theorem [11] gives a method for determining the number of negative eigenvalues of the differential operator $\mathcal{O}$, as follows. Fix 6 (since we have a 6-dimensional system) linearly independent real 6-vectors $e_1,\ldots,e_6$, and $a_1 > 0$. Denote by $\mathcal{U}_{a_1} = [u_1,\ldots,u_6]$ the (6×6)-matrix whose columns are the solutions of the 6 initial value problems

$$\mathcal{O}u_j = 0, \qquad a_1 < R < \infty, \qquad u_j(a_1) = 0, \qquad \frac{du_j}{dR}(a_1) = e_j, \qquad j = 1,\ldots,6.$$

Then the nodal theorem reads [11]:

Theorem: *If $a_1 > 0$ is sufficiently small and $a_2 > a_1$ is sufficiently large, the number of zeros (counted with multiplicities) in the interval (a_1,a_2) of the function $\mathfrak{F} := R \mapsto \det\mathcal{U}_{a_1}(R)$ equals the number of negative eigenvalues of $\mathcal{O}$ (counted with multiplicities).*

At this point it should be stressed that the function $\mathfrak{F}$ must, by definition, vanish at a_1. However, we already know that, when $|\Lambda| = \infty$, the function $\mathfrak{F}$ has no zeros in the (open) interval (a_1,a_2), for sufficiently small a_1. In subsection 6.6 we show that each of the u_j (and hence $\mathfrak{F}$) are analytic in Λ and a_1 in a neighbourhood of $\Lambda = -\infty$, and for all $a_1 > 0$. Therefore $\mathfrak{F}$ has no zeros in the interval (a_1,a_2) for all $|\Lambda|$ sufficiently large (and all sufficiently small a_1). Therefore $\mathcal{O}$ has no negative eigenvalues for all $|\Lambda|$ sufficiently large, and we have proved stability, since the matrix $\boldsymbol{K}_2$ on the right-hand side of (31) is positive definite.

6.2 Factorization of a subsystem

Having outlined the stability proof, in this section we perform the first stage of this proof, by showing that the operator $\hat{\mathcal{A}}$ is positive, where $\hat{\mathcal{A}}$ has the same form as the operator $\bar{\mathcal{A}}$ (27) but with slightly modified matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{S}}$.

Firstly, we note that the operator $\bar{\mathcal{A}}$ (27) is very similar to the corresponding operator in the odd-parity sector. This resemblance becomes even more striking if we use the constraint equation (24) and replace the term $-2\sqrt{6}u\frac{r_{,\rho}}{r}\,t\,d$ on the right-hand side of equation (26) by

$$-\sqrt{2}\mu\gamma u\,q\,d - 2u^2d^2.$$

Redefining $\bar{U} = (q, -b, g, -c, d, -e)^T$, the energy functional $\tilde{E}$ (28) then has the same form as before, but where now

$$\bar{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A}_1 & 0 \\ 0 & \boldsymbol{A}_2 \end{pmatrix}, \quad \bar{\boldsymbol{S}} = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_t^T \\ \boldsymbol{S}_t & \boldsymbol{S}_2 \end{pmatrix},$$

with

$$\boldsymbol{A}_1 = -\frac{u}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{A}_2 = -\frac{u}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{S}_1 = \begin{pmatrix} \frac{r}{\gamma}\left(\frac{\gamma}{r}\right)_{,\rho\rho} + \gamma^2(\lambda + v^2) & sym. \\ -\frac{3}{2}u_{,\rho} - u\frac{\gamma_{,\rho}}{\gamma} & \frac{\gamma_{,\rho\rho}}{\gamma} + \gamma^2(\lambda + f) + u^2 \end{pmatrix},$$

$$\boldsymbol{S}_t = \begin{pmatrix} 2\sqrt{\lambda}\,\gamma_{,\rho} & 0 \\ \frac{r}{\gamma}\left[\frac{\gamma^2}{r}v\right]_{,\rho} + \gamma w u & -2(\gamma w)_{,\rho} - \gamma\,uv \\ -\frac{\mu}{\sqrt{2}}\gamma\,u & -\sqrt{2}\mu\gamma_{,\rho} \\ -\sqrt{\frac{\lambda}{2}}\,\gamma\,u & \sqrt{2\lambda}\gamma_{,\rho} \end{pmatrix},$$

$$\boldsymbol{S}_2 = \begin{pmatrix} \frac{r_{,\rho\rho}}{r} + \lambda\gamma^2 + u^2 & sym. & sym. & sym. \\ \sqrt{\lambda}\gamma^2 v & \gamma^2(\lambda + 2f + v^2) & sym. & sym. \\ 0 & -\sqrt{2}\mu\gamma^2 w & \gamma^2(\lambda - 2 + 3f) & sym. \\ \frac{u_{,\rho}}{\sqrt{2}} & -\sqrt{2\lambda}\gamma^2 w & 0 & \gamma^2(\mu^2 - f) \end{pmatrix},$$

and

$$\begin{aligned} \hat{\mathcal{R}} &= (S_{tt} + S_{11} + 2S_{t1})\,t^2 + 2\sqrt{12}\mu\gamma_{,\rho}\,t\,q \\ &\quad + 2\sqrt{3}\mu\gamma u\,t\,b + 2\sqrt{6}u_{,\rho}t\,d. \end{aligned}$$

Comparing this with the corresponding expressions in the odd-parity sector (see Eq. (8) of Ref. [8]) with $a = 0$, we see that the only difference lies in the sign of $(\boldsymbol{S}_2)_{23} = (\boldsymbol{S}_2)_{32}$. It is also interesting to observe that in terms of the new variables, the constraint equations (16) and (17) are

$$\begin{aligned} 0 &= \frac{\gamma}{r}\partial_\rho\left(\frac{r}{\gamma}\bar{q}\right) + u\,\bar{b} - \sqrt{\lambda}\gamma\,\bar{g} - \gamma v\,\bar{c} - \sqrt{3}\,\mu\gamma\,t \\ 0 &= \gamma\partial_\rho\left(\frac{1}{\gamma}\,\bar{b}\right) + \gamma w\bar{c} + \frac{\mu}{\sqrt{2}}\,\gamma\,\bar{d} - \sqrt{\frac{\lambda}{2}}\,\gamma\,\bar{e}, \end{aligned}$$

which, apart from the term involving t, are exactly the same as the constraint equations for h and b in the odd-parity sector (see Eq. (9) of Ref. [8]).

Therefore, if $\boldsymbol{P}$ denotes the projector from the even-parity amplitudes $\bar{U} = (\bar{q}, \bar{b}, \bar{g}, \bar{c}, \bar{d}, \bar{e})$ to the odd-parity amplitudes $U_{odd} = (h, a, b, k, c, d, e)$ defined in Ref. [8],

$$\boldsymbol{P} : \bar{U} \mapsto (\bar{q}, 0, \bar{b}, \bar{g}, \bar{c}, \bar{d}, \bar{e}),$$

and if $\boldsymbol{E}_{\bar{c}\bar{d}}$ is the matrix whose entry in the column and row corresponding to $\bar{c}$ and $\bar{d}$, respectively, is 1 while all other entries are 0, we have

$$\hat{\mathcal{A}} = \boldsymbol{P}^T \boldsymbol{B}^\dagger \boldsymbol{B} \boldsymbol{P} - 2\sqrt{2}\mu\gamma^2 w \left(\boldsymbol{E}_{\bar{c}\bar{d}} + \boldsymbol{E}_{\bar{d}\bar{c}}\right),$$

where $\hat{\mathcal{A}}$ has the same form as the operator $\bar{\mathcal{A}}$ (27) but with the matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{S}}$ replaced by their modified forms above. The operator $\boldsymbol{B}$ and its adjoint $\boldsymbol{B}^\dagger$ were found in [8] ($\boldsymbol{B}$ is also given in Appendix B for completeness). Now, introducing also

$$\boldsymbol{Q} : U_{odd} = (h, a, b, k, c, d, e) \mapsto (0, a, 0, 0, 0, 0, 0)$$

such that $\mathbb{1}_{odd} = \boldsymbol{P}\boldsymbol{P}^T + \boldsymbol{Q}\boldsymbol{Q}^T$, and noticing that

$$\boldsymbol{Q}^T \boldsymbol{B} \boldsymbol{P} = \gamma(0, 1, 0, 0, 0, 0, 0)^T (0, 0, 0, -\mu, -\sqrt{2}w, 0),$$

we find

$$\begin{aligned} \hat{\mathcal{A}} &= (\boldsymbol{P}^T \boldsymbol{B}^\dagger \boldsymbol{P})(\boldsymbol{P}^T \boldsymbol{B} \boldsymbol{P}) + \gamma^2 \left(\mu^2 \boldsymbol{E}_{\bar{c}\bar{c}} - \sqrt{2}\mu w \boldsymbol{E}_{\bar{c}\bar{d}} - \sqrt{2}\mu w \boldsymbol{E}_{\bar{d}\bar{c}} + 2w^2 \boldsymbol{E}_{\bar{d}\bar{d}}\right) \\ &= \bar{\boldsymbol{B}}^\dagger \bar{\boldsymbol{B}} + \bar{b}^T \bar{b}, \end{aligned} \tag{32}$$

where $\bar{\boldsymbol{B}} \equiv \boldsymbol{P}^T \boldsymbol{B} \boldsymbol{P}$ and $\bar{b} = \gamma(0, 0, 0, -\mu, +\sqrt{2}w, 0)$. This shows that the operator $\hat{\mathcal{A}}$ is positive. Note that this factorization also works when $\ell = 1$, since in that case the amplitudes $(\bar{q}, \bar{b}, \bar{c}, \bar{d})$ decouple from the amplitudes $(\bar{g}, \bar{e})$ so that the above result is still correct when $\bar{g}$ and $\bar{e}$ are removed.

As an application of (32), consider the Schwarzschild-adS case, $|w| = 1$: It can be checked from the explicit expressions in Appendix B that the YM perturbations $(\bar{b}, \bar{c}, \bar{d}, \bar{e})$ decouple from the gravitational perturbations. Since the remainder term $\hat{\mathcal{R}}$ does not depend on the YM perturbations in that case, (32) shows that the YM perturbations are stable. Similarly, in the RN-adS case, the non-Abelian amplitudes $(\bar{b}, \bar{d}, \bar{e})$ decouple, and (32) shows that there are no instabilities. In contrast, the stability of the gravitational perturbations does not follow from (32) since in that case, the remainder term $\hat{\mathcal{R}}$ has to be taken into account. As a matter of fact, it turns out that $\hat{\mathcal{R}}$ is not positive even in the Schwarzschild-adS case, and one needs to derive the master equations (22) and (23) in order to prove stability. In this sense, the gravitational perturbations are responsible for the stability not being topological in the even-parity sector!

The main idea in the next section will be to show that $\hat{\mathcal{R}}$ is negligible when $|\Lambda|$ is large enough and to argue that in that case, the energy is still positive.

6.3 Stability for infinite cosmological constant

We will now argue that the remainder term $\hat{\mathcal{R}}$ (29) given in the previous subsection is sub-leading order as $|\Lambda| \to \infty$. In order to estimate carefully the magnitudes of all terms in our system of equations, we firstly need to non-dimensionalize all quantities. All the perturbations, q, g, b, c, d and e have dimensions of length. Bearing in mind that Newton's constant G has dimension *length*^{2}, we may write, for example,

$$q = \hat{q}\,\ell_G$$

and similarly for the other quantities, where $\ell_G = \sqrt{G}$. Since our equations are linear, the factors of ℓ_G all cancel and the equations for the hatted variables are the same as those for the unhatted variables. Therefore we shall not distinguish in the following between the dimensionless or dimensionful perturbations.

The time and radial variables are rescaled as:

$$t = \hat{t}\,\ell_G\,, \qquad \rho = \hat{\rho}\,\ell_G\,, \qquad r = \hat{r}\,\ell_G\,.$$

The gauge function w and all metric quantities are dimensionless. The remaining dimensionful quantities are:

$$r_h = \hat{r}_h\,\ell_G\,, \qquad m(r) = \hat{m}\,\ell_G\,, \qquad u = \hat{u}\,\ell_G^{-1} = 2\frac{w_{,\hat{\rho}}}{\hat{r}\,\ell_G}, \qquad \gamma = \hat{\gamma}\ell_G^{-1}.$$

We also define the positive dimensionless parameter

$$L = -\Lambda \ell_G^2\,.$$

All other quantities in the pulsation equations are dimensionless.

We are interested in the limit in which L is very large (and positive), and so we can introduce another positive parameter $\xi = L^{-1}$, so that, equivalently, we may consider small ξ in a neighbourhood of $\xi = 0$. It is worth recalling at this stage that it has been proved that, in this limit, $\hat{u} \sim o\left(\xi^{-1/2}\right)$ (see [4] and section 2). Furthermore, we know that in this limit the metric function $\hat{m}$ behaves like L, and, in turn, this means that $N \sim L$. We therefore define

$$\tilde{N} = \xi N = L^{-1} N,$$

and accordingly,

$$\tilde{\gamma}^2 = \xi \hat{\gamma}^2,$$

where $\tilde{N}$ and $\tilde{\gamma}$ will be finite as $\xi \to 0$. With these scalings, the potential $\bar{\boldsymbol{S}}$ is of order L^2 as $L \to \infty$ (since each derivative with respect to $\hat{\rho}$ introduces a factor of N), so in order to obtain non-trivial equations we must rescale the space-time variables again, namely,

$$\hat{t} = L^{-1}\tilde{t} = \xi \tilde{t}, \qquad \hat{r} = \xi \tilde{r}, \qquad \hat{\rho} = \xi \tilde{\rho}.$$

The pulsation equations now have the form

$$0 = \left[\partial_{\tilde{t}}^2 - \partial_{\tilde{\rho}}^2 + \tilde{\boldsymbol{A}}\partial_{\tilde{\rho}} + \partial_{\tilde{\rho}}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{S}} - \tilde{\boldsymbol{b}}(\tilde{a})\right]\bar{U}. \tag{33}$$

where

$$\bar{\boldsymbol{A}} = \tilde{\boldsymbol{A}}L, \qquad \bar{\boldsymbol{S}} = \tilde{\boldsymbol{S}}L^2, \qquad \bar{\boldsymbol{b}} = \tilde{\boldsymbol{b}}L^2.$$

We also define

$$\bar{\mathcal{A}} = \tilde{\mathcal{A}}L^2 \qquad \mathcal{R} = \tilde{\mathcal{R}}L^2.$$

The pulsation equations now have a well-defined limit as $\xi \to 0$ (or, equivalently, $L \to \infty$).

The remainder term $\tilde{\mathcal{R}}$ (where $\mathcal{R}$ is given by (26)) contains t terms. The variable t is non-dynamical and given in terms of the perturbations q and d by the constraint (24), which reads, in dimensionless variables:

$$t = \frac{\xi\tilde{r}}{\sqrt{6}\tilde{r}_{,\tilde{\rho}}}\left(\frac{\mu}{\sqrt{2\xi}}\tilde{\gamma}q + \hat{u}d\right). \tag{34}$$

We therefore substitute for t to give an expression for $\tilde{\mathcal{R}}$ in terms of q, d and b. From (34), each term in $\tilde{\mathcal{R}}$ contains a factor of $\xi^{1/2}$, whereas the operator $\tilde{\mathcal{A}}$ is $O(1)$ as $\xi \to 0$. Therefore, as $\xi \to 0$, the remainder term $\tilde{\mathcal{R}}$ is subleading order compared to the operator $\tilde{\mathcal{A}}$.

So far in this section we have included the terms in the perturbation equations dependent on the lapse $\tilde{a}$. As discussed in section 6.1, these terms do not contribute to the energy functional E, and so may be ignored in our study of stability. Ignoring the $\tilde{\boldsymbol{b}}$ terms therefore, our conclusions apply equally well to the (unscaled) quantities $\bar{\mathcal{A}}$ and $\bar{\mathcal{R}}$ (see subsection 6.1 for the definitions of these), namely that, as $\xi \to 0$, the remainder term $\bar{\mathcal{R}}$ is subleading order compared to the operator $\bar{\mathcal{A}}$. This can also be extended to the slightly modified operator $\hat{\mathcal{A}}$ constructed in subsection 6.2, and the correspondingly modified remainder term $\hat{\mathcal{R}}$ given in (29). Therefore the (modified) remainder term $\hat{\mathcal{R}}$ is subleading order compared to the (modified) operator $\hat{\mathcal{A}}$, as $L \to \infty$.

By the factorization in subsection 6.2, we know that the operator $\hat{\mathcal{A}}$ is positive, and therefore we have proved that the total energy functional giving rise to the pulsation equations (33) is positive when $\xi = 0$ (or, equivalently, $L = \infty$), so that the system is stable for this value of ξ. Since it is possible for $\hat{\mathcal{A}}$ to be zero for some perturbations, we are not able immediately to say that the system is stable for sufficiently small ξ since it may be the case that for non-zero ξ the remainder term $\hat{\mathcal{R}}$ (which is not positive) is dominant, so that the energy is no longer positive. The remainder of this section will be devoted to extending this stability to sufficiently small ξ, using an analyticity argument.

6.4 Transformation of the equations

We now address the problem of extending our proof that the system of pulsation equations is stable when $\Lambda = -\infty$ to sufficiently large (and negative) Λ. To do this, we shall use an analyticity argument based on the multi-dimensional nodal theorem [11].

In order to apply the multi-dimensional nodal theorem [11], we first need to cast our pulsation equations into the appropriate form. We begin with the pulsation equations in the form

$$0 = \left(\partial_t^2 - \partial_\rho^2 + \bar{\boldsymbol{A}}\partial_\rho + \partial_\rho \bar{\boldsymbol{A}} + \bar{\boldsymbol{S}}\right) \bar{U},$$

where we remind the reader that $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{S}}$ are anti-symmetric and symmetric matrices, respectively, and $\bar{U} = (q, g, b, c, d, e)$. Note that we are ignoring the terms in the perturbation equations which contain the lapse, since these terms do not contribute to the energy E. Furthermore, we derive these equations by firstly substituting for the variable $t = p$ using the momentum constraint (24), which gives the remainder term $\bar{\mathcal{R}}$ whose terms are given by (30) . Finally, we vary not the whole modified energy $\tilde{E}$ (28), but rather $E_{\mathcal{P}}$, which is the modified energy $\tilde{E}$ minus a term $\mathcal{P}$ which is manifestly positive. This procedure is outlined in section 6.1, and the details for the particular cases of black hole and soliton equilibrium solutions are given in section 6.5 below.

Define a new perturbation vector $\bar{V}$ by

$$\bar{U} = \boldsymbol{R}\bar{V}$$

where $\boldsymbol{R}$ is a transformation matrix, which may depend on ρ (but not t) and contains no derivative operators. Then the equation for $\bar{V}$ is

$$0 = \left[\boldsymbol{R}^T\boldsymbol{R}\partial_t^2 - \partial_\rho \boldsymbol{R}^T\boldsymbol{R}\partial_\rho + {}^1\bar{\boldsymbol{A}}\partial_\rho + \partial_\rho {}^1\bar{\boldsymbol{A}} + {}^1\bar{\boldsymbol{S}}\right] \bar{V}, \tag{35}$$

where the transformed matrices ${}^1\bar{\boldsymbol{A}}$ and ${}^1\bar{\boldsymbol{S}}$ are

$$\begin{aligned}
{}^1\bar{\boldsymbol{A}} &= \boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R} + \frac{1}{2}\left(\boldsymbol{R}^T_{,\rho}\boldsymbol{R} - \boldsymbol{R}^T\boldsymbol{R}_{,\rho}\right), \\
{}^1\bar{\boldsymbol{S}} &= \boldsymbol{R}^T\boldsymbol{S}\boldsymbol{R} + \boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}_{,\rho} - \boldsymbol{R}^T_{,\rho}\boldsymbol{A}\boldsymbol{R} - \frac{1}{2}\left(\boldsymbol{R}^T\boldsymbol{R}_{,\rho\rho} + \boldsymbol{R}^T_{,\rho\rho}\boldsymbol{R}\right).
\end{aligned}$$

Since

$$\bar{\boldsymbol{A}} = \begin{pmatrix} 0 & \bar{\boldsymbol{A}}_{int} \\ -\bar{\boldsymbol{A}}^T_{int} & 0 \end{pmatrix},$$

we set

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix},$$

where $\boldsymbol{F}$ is a 2×4 matrix. If we choose $\boldsymbol{F}$ such that

$$\bar{\boldsymbol{A}}_{int} = -\frac{1}{2}\boldsymbol{F}_{,\rho},$$

then ${}^1\bar{\boldsymbol{A}} = 0$, and the pulsation equations simplify. With this choice, the matrix $\boldsymbol{F}$ takes the form

$$\boldsymbol{F} = \mathcal{F}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

where the scalar function $\mathcal{F}(\rho)$ is given by

$$\mathcal{F}(\rho) = -2\sqrt{G}\int^{\rho} \frac{w_{,\rho}}{r}\, d\rho = -2\sqrt{G}\int^{r} \frac{w_{,r}}{r}\, dr. \tag{36}$$

We write the transformed potential as:

$$^{1}\bar{\boldsymbol{S}} = \boldsymbol{R}^{T} \begin{pmatrix} ^{1}\bar{\boldsymbol{S}}_{grav} & ^{1}\bar{\boldsymbol{S}}_{int} \\ ^{1}\bar{\boldsymbol{S}}_{int} & ^{1}\bar{\boldsymbol{S}}_{YM} \end{pmatrix} \boldsymbol{R}, \tag{37}$$

where

$$\begin{aligned} ^{1}\bar{\boldsymbol{S}}_{grav} &= \bar{\boldsymbol{S}}_{grav} - 4\bar{\boldsymbol{A}}_{int}\bar{\boldsymbol{A}}_{int}^{T}, \\ ^{1}\bar{\boldsymbol{S}}_{int} &= \bar{\boldsymbol{S}}_{int} + \bar{\boldsymbol{A}}_{int\,,\rho}, \\ ^{1}\bar{\boldsymbol{S}}_{YM} &= \bar{\boldsymbol{S}}_{YM}. \end{aligned} \tag{38}$$

The transformation matrix $\boldsymbol{R}$ has all its eigenvalues equal to unity, but it is not necessarily positive definite because it may contain a Jordan block. However, if we introduce the vector $\boldsymbol{X} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$, then

$$\begin{aligned} \boldsymbol{X}^{T}\boldsymbol{R}\boldsymbol{X} &= (x_1 + \mathcal{F}(\rho)x_3)^2 + \left(x_6 + \sqrt{2}\mathcal{F}(\rho)x_2\right)^2 \\ &\quad + x_4^2 + x_5^2 + \left(1 - 2\mathcal{F}^2(\rho)\right)x_2^2 + \left(1 - \mathcal{F}^2(\rho)\right)x_3^2. \end{aligned}$$

Therefore $\boldsymbol{R}$ is positive definite if $\mathcal{F}^2(\rho) < 1/2$. Since (see section 2 and [4]), $w_{,r}(r) \sim o(|\Lambda|^{-\frac{1}{2}})$ as $|\Lambda| \to \infty$, for sufficiently large $|\Lambda|$ (and an appropriate choice of integration constant) we can make $\mathcal{F}$ as small as we like. Therefore $\boldsymbol{R}$ is positive definite for sufficiently large $|\Lambda|$.

We now make a further transformation by introducing a diagonal matrix $\boldsymbol{Y}$ (depending only on ρ and not on t), which commutes with $\boldsymbol{R}$. In order to commute with $\boldsymbol{R}$, the elements of $\boldsymbol{Y}$ must be such that

$$\boldsymbol{Y} = \mathrm{diag}(y_1, y_2, y_1, y_3, y_4, y_2). \tag{39}$$

Defining $\bar{W} = \boldsymbol{Y}^{-1}\bar{V}$, the pulsation equations (35) now take the form

$$0 = \boldsymbol{Y}\boldsymbol{R}^{T}\boldsymbol{R}\boldsymbol{Y}\partial_t^2\bar{W} - \partial_\rho\left(\boldsymbol{Y}\boldsymbol{R}^{T}\boldsymbol{R}\boldsymbol{Y}\partial_\rho\bar{W}\right) + {}^{2}\bar{\boldsymbol{S}}\bar{W}, \tag{40}$$

where the new potential is

$$\begin{aligned} ^{2}\bar{\boldsymbol{S}} &= \boldsymbol{Y}\left[^{1}\bar{\boldsymbol{S}} - \partial_\rho\left(\boldsymbol{R}^{T}\boldsymbol{R}\right)\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho} - \boldsymbol{R}^{T}\boldsymbol{R}\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho\rho}\right]\boldsymbol{Y} \\ &= \boldsymbol{Y}\boldsymbol{R}^{T} \begin{pmatrix} ^{2}\bar{\boldsymbol{S}}_{grav} & ^{2}\bar{\boldsymbol{S}}_{int} \\ ^{2}\bar{\boldsymbol{S}}_{int}^{T} & ^{2}\bar{\boldsymbol{S}}_{YM} \end{pmatrix} \boldsymbol{R}\boldsymbol{Y}, \end{aligned} \tag{41}$$

and

$$\begin{aligned} ^{2}\bar{\boldsymbol{S}}_{grav} &= {}^{1}\bar{\boldsymbol{S}}_{grav} - \boldsymbol{Y}_1^{-1}\boldsymbol{Y}_{1\,,\rho\rho}, \\ ^{2}\bar{\boldsymbol{S}}_{int} &= {}^{1}\bar{\boldsymbol{S}}_{int} - \boldsymbol{F}_{,\rho}\boldsymbol{Y}_2^{-1}\boldsymbol{Y}_{2\,,\rho}, \\ ^{2}\bar{\boldsymbol{S}}_{YM} &= {}^{1}\bar{\boldsymbol{S}}_{YM} - \boldsymbol{Y}_2^{-1}\boldsymbol{Y}_{2\,,\rho\rho}, \end{aligned} \tag{42}$$

and we have defined diagonal matrices $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$, which form the blocks of the matrix $\boldsymbol{Y}$, as follows:

$$\begin{aligned}\boldsymbol{Y}_1 &= \mathrm{diag}(y_1, y_2), \\ \boldsymbol{Y}_2 &= \mathrm{diag}(y_1, y_3, y_4, y_2).\end{aligned}$$

At the moment we leave the functions in the matrix $\boldsymbol{Y}$ to be quite general. They will be fixed in the next subsection 6.5, when we discuss the form of the equations in detail for black holes and solitons in turn.

For the moment, we write the pulsation equations (40) as

$$-\boldsymbol{K}(\rho)\partial_t^2 \bar{W} = -\partial_\rho \left(\boldsymbol{K}(\rho)\partial_\rho \bar{W} \right) + {}^2\bar{\boldsymbol{S}}\bar{W}, \tag{43}$$

where

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{Y}.$$

Equation (43) should be compared with the form (31) required for the application of the nodal theorem [11]. The first issue is whether our coordinate ρ has the required domain $[0, \infty)$. This will be addressed in the following subsection for the black hole and soliton solutions. For the time being, we assume that we can, if necessary, change to a coordinate R (which will be a function of ρ) which does have the required range of values. If

$$\frac{dR}{d\rho} = Q, \tag{44}$$

then the equations (43) become

$$-\boldsymbol{K}Q^{-1}\partial_t^2 \bar{W} = -\frac{d}{dR}\left(\boldsymbol{K}Q\frac{d\bar{W}}{dR} \right) + {}^2\bar{\boldsymbol{S}}Q^{-1}\bar{W},$$

which is of the form (31), with $\boldsymbol{K}_1 = \boldsymbol{K}Q$ and $\boldsymbol{K}_2 = \boldsymbol{K}Q^{-1}$. In our work in the following section, the function Q will always be of one sign, and so we consider instead (by multiplying throughout by -1 if necessary)

$$-\boldsymbol{K}|Q|^{-1}\partial_t^2 \bar{W} = -\frac{d}{dR}\left(\boldsymbol{K}|Q|\frac{d\bar{W}}{dR} \right) + {}^2\bar{\boldsymbol{S}}|Q|^{-1}\bar{W}, \tag{45}$$

so that the matrices $\boldsymbol{K}_1 = \boldsymbol{K}|Q|$ and $\boldsymbol{K}_2 = \boldsymbol{K}|Q|^{-1}$ are positive definite. As discussed in section 6.1, in order to prove stability we require that $\boldsymbol{K}_1$ is uniformly bounded and uniformly positive definite, a condition which will be considered in the next subsection. In order to apply the generalized multi-dimensional nodal theorem, it remains to show that the matrix ${}^2\bar{\boldsymbol{S}}|Q|^{-1}$ can be written in the form

$${}^2\bar{\boldsymbol{S}}|Q|^{-1} = \frac{1}{R^2}\boldsymbol{L}(R) + \boldsymbol{M}(R), \tag{46}$$

where $\boldsymbol{L}$ and $\boldsymbol{M}$ are uniformly bounded matrices on $R \in [0, \infty)$, and $\boldsymbol{L}(0)$ is non-negative. Furthermore, we also require $\boldsymbol{M} \to 0$ as $R \to \infty$, and $\boldsymbol{M}R^2 + \boldsymbol{L} \geq -\beta_c$ for some $\beta_c < 1/4$ and all sufficiently large R. This is the subject of the next subsection.

6.5 The nodal theorem matrices

Having cast our pulsation equations into a form (45) which will allow us to apply the (extended) multi-dimensional nodal theorem, we now have to verify that the matrices in (45) do in fact satisfy the conditions of the nodal theorem. Namely, $\boldsymbol{K}|Q|$ and $\boldsymbol{K}|Q|^{-1}$ must be uniformly bounded and positive definite, and the potential ${}^2\bar{\boldsymbol{S}}|Q|^{-1}$ must have the form (46). In addition, $\boldsymbol{M} \to 0$ as $R \to \infty$, and $\boldsymbol{M}R^2 + \boldsymbol{L} \geq -\beta_c$ for some $\beta_c < 1/4$ and all sufficiently large R, which will be the case if ${}^2\bar{\boldsymbol{S}}|Q|^{-1} \to 0$ as $R \to \infty$, and ${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2 \geq -\beta_c$ for all sufficiently large R. So far we have also left open the precise identity of the coordinate R for use in the application of the nodal theorem.

In this subsection we shall address these issues. The two cases, black hole and soliton solutions, need to be addressed separately because of the different boundary conditions on the equilibrium solutions. The black hole perturbation equations are more readily cast into an appropriate form, so we deal with them first.

6.5.1 Black holes

For black hole equilibrium solutions, the usual "tortoise" coordinate ρ has the domain $(-\infty, \rho_{max}]$, tending to $-\infty$ close to the event horizon, and approaching the finite value ρ_{max} at infinity. By a suitable choice of integration constant in the definition of ρ, we may set ρ_{max} to be equal to zero. Therefore $\rho \in (-\infty, 0]$. Our application of the multi-dimensional nodal theorem requires a variable R having domain $[0, \infty)$, so we define

$$R = -\rho.$$

It should be noted that $R = 0$ corresponds now to infinity, and $R \to \infty$ corresponds to the event horizon.

In this situation, the quantity Q (see (44)) is equal to -1, and so $|Q| = 1$. We set the transformation matrix $\boldsymbol{Y}$ (39) to be equal to the identity, so that the perturbation equations are now of the form (45)

$$-\boldsymbol{K}\partial_t^2 \bar{V} = -\frac{d}{dR}\left(\boldsymbol{K}\frac{dV}{dR}\right) + {}^1\bar{\boldsymbol{S}}\bar{V},$$

where

$$\boldsymbol{K} = \boldsymbol{R}^T \boldsymbol{R},$$

and ${}^1\bar{\boldsymbol{S}}$ is given by (37).

The first necessary condition is that the matrix $\boldsymbol{K}$ is uniformly bounded and uniformly positive definite on $R \in [0, \infty)$. Recall that $\boldsymbol{R}$ depends on the function $\mathcal{F}(R)$, defined by (36). We are interested in those black holes for which the gauge function w has no zeros, in which case, from the field equations, the function w is a monotonic function and so $w_{,r}$ is of one sign. To see this, suppose that $w > 0$. First note that the field equations (1) imply that w cannot have a minimum in the interval $0 < w < 1$ nor a maximum if $w > 1$ [12]. Furthermore, a point of inflection is possible only if $w = 1$, and it was proven in [8] that for solutions in which w has no nodes, the function w cannot cross ± 1. Taken together, these statements mean that w can have no stationary points, and this argument applies equally well to soliton solutions or if $w < 0$ everywhere.

Given the asymptotic behaviour of w at infinity (see section 2), in this case the integral defining $\mathcal{F}$ (36) is convergent at infinity (which corresponds to $R = 0$). This means that the matrix $\boldsymbol{R}$ is uniformly bounded on $R \in [0, \infty)$, so that $\boldsymbol{K}$ is also uniformly bounded. It is also straightforward to show that $\boldsymbol{K}$ is uniformly positive definite on $R \in [0, \infty)$ in this case.

Since the matrix $\boldsymbol{R}$ is uniformly bounded and positive definite, to show that the potential ${}^1\bar{\boldsymbol{S}}$ can be written in the form (46), it is sufficient to show that this is the case for the potential matrix

$$\begin{pmatrix} {}^1\bar{\boldsymbol{S}}_{grav} & {}^1\bar{\boldsymbol{S}}_{int} \\ {}^1\bar{\boldsymbol{S}}_{int}^T & {}^1\bar{\boldsymbol{S}}_{YM} \end{pmatrix},$$

whose components are given in (38). Remember that $R = 0$ (which is where the potential can be singular) corresponds to infinity. It is simplest to consider each part of the potential separately.

Firstly, from (38), there are terms in the potential proportional to $\bar{\boldsymbol{A}}_{int}$ and/or its derivative with respect to ρ. These depend only on the gauge function u (10) and its derivative, respectively. Both these functions are regular everywhere outside the event horizon and uniformly bounded for $R \in [0, \infty)$. Therefore those parts of the potential containing $\bar{\boldsymbol{A}}_{int}$ or its derivative may be included in the matrix $\boldsymbol{M}$ (46).

It now remains to consider only the terms $\bar{\boldsymbol{S}}_{grav}$, $\bar{\boldsymbol{S}}_{int}$ and $\bar{\boldsymbol{S}}_{YM}$, remembering to include the additional contributions from the remainder term (30). Careful analysis of each component in these matrices reveals that all components are regular and uniformly bounded everywhere outside the event horizon (so that they may be included in the matrix $\boldsymbol{M}$) with the sole exception of the quantity

$$\frac{r_{,\rho\rho}}{r}$$

which arises in the S_{33} component of the $\bar{\boldsymbol{S}}_{grav}$ matrix (see (8)). At infinity ($R \to 0$), we have

$$\frac{r_{,\rho\rho}}{r} = \frac{2}{R^2} + O(1).$$

We therefore define a matrix $\boldsymbol{L}$ which has as its only non-zero element

$$\boldsymbol{L}_{33} = \frac{R^2 r_{,\rho\rho}}{r}.$$

Near the event horizon $R \sim -\log(r - r_h) \to \infty$, and so $\boldsymbol{L}_{33} \to 0$. Thus the matrix $\boldsymbol{L}$ is symmetric, regular and uniformly bounded on $R \in [0, \infty)$. In addition,

$\boldsymbol{L}_{33}(0) = 2$, so that the matrix $\boldsymbol{L}(0)$ is non-negative, as required. We comment here that a crude estimate of the behaviour of

$$\frac{r^2}{\alpha}\left(\frac{\alpha}{r^2}\right)_{,\rho\rho}$$

which arises in the S_{22} component of $\bar{\boldsymbol{S}}_{grav}$ suggests that this is divergent at infinity. However, a precise calculation, using the asymptotic forms of the metric functions (section 2) shows that in fact this function is regular everywhere outside the event horizon.

Having written the potential ${}^1\bar{\boldsymbol{S}}$ in the required form (46), it remains to show that this potential vanishes sufficiently quickly as $R \to \infty$, that is,

$${}^1\bar{\boldsymbol{S}}R^2 \geq -\beta_c$$

for all sufficiently large R, where β_c is a constant which is less than $1/4$. Recall that $R \to \infty$ corresponds to the event horizon of the black hole. As the event horizon is approached, all terms in the potential ${}^1\bar{\boldsymbol{S}}$ tend to zero at least as rapidly as $(r - r_h)^{1/2}$, with the exception of the following terms:

$$\begin{aligned}
\mathcal{S}_{qq} &= \frac{\mu^2 S^2(r_h) N_{,r}(r_h)}{2r_h} + O(r - r_h); \\
S_{22} &= \frac{1}{4} S^2(r_h) N_{,r}^2(r_h) + O(r - r_h);
\end{aligned}$$

where $\mathcal{S}_{qq}$ is the extra contribution to S_{22} in $\boldsymbol{S}_{grav}$ (8) given in (30). In addition, the (bb)-component of $\boldsymbol{S}_{YM}$ (9) has the same behaviour as S_{22}. Therefore there are two entries in the potential matrix which approach a positive constant as $R \to \infty$ (since $N_{,r}(r_h) > 0$). We deal with these by defining a manifestly positive quantity $\mathcal{P}$:

$$\mathcal{P} = \left[\frac{\mu^2 S^2(r_h) N_{,r}(r_h)}{2r_h} + \frac{1}{4} S^2(r_h) N_{,r}^2(r_h)\right] q^2 + \frac{1}{4} S^2(r_h) N_{,r}^2(r_h) b^2 > 0$$

which we subtract from the modified energy functional $\tilde{E}$ before varying $E_{\mathcal{P}} = \tilde{E} - \mathcal{P}$ to yield the perturbation equations. These equations are identical to the previous ones apart from two terms in the untransformed potential $\bar{\boldsymbol{S}}$: the (qq)-term in $\bar{\boldsymbol{S}}_{grav}$, which has the positive constant

$$\left[\frac{\mu^2 S^2(r_h) N_{,r}(r_h)}{2r_h} + \frac{1}{4} S^2(r_h) N_{,r}^2(r_h)\right]$$

subtracted, and the (bb)-term in $\bar{\boldsymbol{S}}_{YM}$, which has the positive constant

$$\frac{1}{4} S^2(r_h) N_{,r}^2(r_h)$$

subtracted. This does not affect any of the analysis so far in this subsection, in particular the definition of the matrix $\boldsymbol{L}$ is unchanged, and the matrix $\boldsymbol{M}$ simply has appropriate positive constants subtracted from its (qq) and (bb) entries. Therefore both $\boldsymbol{L}$ and $\boldsymbol{M}$ remain uniformly bounded for all $R \in [0, \infty)$.

However, now all entries in the potential ${}^1\bar{\boldsymbol{S}}$ vanish at least as quickly as $(r - r_h)^{1/2}$ as $r \to r_h$ (or, equivalently, $R \to \infty$). Since, near the event horizon, $R \sim -\log(r - r_h)$, this means that the potential is vanishing like $e^{-R/2}$ as $R \to \infty$. Therefore ${}^1\bar{\boldsymbol{S}}R^2 \to 0$ as $R \to \infty$, so it is certainly the case that

$$ {}^1\bar{\boldsymbol{S}}R^2 \geq -\beta_c $$

where $\beta_c < 1/4$ for all sufficiently large R.

Therefore, for black hole equilibrium solutions, we have constructed matrices satisfying the conditions necessary for the application of the multi-dimensional nodal theorem.

6.5.2 Solitons

The corresponding analysis for soliton solutions is rather more complex. We shall begin by defining $\mathcal{P} = (\gamma^4\mu^2 r^2/2r_{,\rho}^2)q^2$ as it is manifestly positive, and consider the perturbation equations derived from the variation of $E_{\mathcal{P}} = \tilde{E} - \mathcal{P}$ (as outlined in section 6.1). This makes some of the algebra later in this subsection more tractable. We can prove stability for soliton solutions if these pulsation equations have no unstable modes. Therefore, for the pulsation equations for solitons, we remove a term

$$ \frac{\mu^2\gamma^4 r^2}{2r_{,\rho}^2} $$

from S_{22} in $\bar{\boldsymbol{S}}_{grav}$. All other terms in the potential are unchanged.

For soliton solutions, the "tortoise" co-ordinate ρ has a finite range of values, so we shall instead begin with the co-ordinate r, which has the domain $[0, \infty)$. We define a new co-ordinate R by

$$ R = r^{-\frac{1}{2}}, \tag{47} $$

which also has the domain $[0, \infty)$. However, $R = 0$ now corresponds to infinity $(r \to \infty)$, while $R \to \infty$ corresponds to the origin $(r = 0)$. This mapping of infinity to $R = 0$ can be motivated from our experience in dealing with the black hole solutions in the previous subsection, as we already know in detail the behaviour of the potential at $R = 0$, which is crucial for applying the multi-dimensional nodal theorem. The specific choice of co-ordinate (47) will be seen below to enable us to put the potential in such a form that the precise conditions necessary for the application of the nodal theorem are satisfied.

With R defined by (47), the quantity Q (44) is given by

$$ Q = -\frac{1}{2}r^{-\frac{3}{2}}NS. $$

Since this is negative, we shall use $|Q|$ in the perturbation equations (as in (45)).

As with the black hole solutions in the previous subsection, $\mathcal{F}$ (36) is regular and uniformly bounded on the whole of $r \in [0, \infty)$ and therefore so too is the transformation matrix $\boldsymbol{R}$.

For our stability proof, we require the matrix $\boldsymbol{K}|Q|$ to be uniformly bounded and uniformly positive definite. However, $|Q| \sim r^{1/2}$ as $r \to \infty$, while $\boldsymbol{R}$ is uniformly bounded, so we need to introduce a second transformation matrix $\boldsymbol{Y}$ (see section 6.4) such that $\boldsymbol{K}|Q| = \boldsymbol{Y}\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{Y}|Q|$ is uniformly positive definite and bounded. Let

$$\boldsymbol{Y} = \frac{1}{2\sqrt{2}\mu} r^{\frac{3}{4}} (NS)^{-1/2} \boldsymbol{I},$$

which satisfies the requirement that $\boldsymbol{Y}$ should commute with $\boldsymbol{R}$. The factor $1/2\sqrt{2}\mu$ will be seen later in this subsection to ensure that the behaviour of the potential as $R \to \infty$ satisfies the conditions of the nodal theorem. Then

$$\boldsymbol{K}|Q| = \frac{1}{16\mu^2} \boldsymbol{R}^T \boldsymbol{R}$$

which is uniformly bounded and positive definite on $R \in [0, \infty)$, as required.

With the additional transformation given by the matrix $\boldsymbol{Y}$, the potential is now ${}^2\bar{\boldsymbol{S}}|Q|^{-1}$, where ${}^2\bar{\boldsymbol{S}}$ is given by (41). Since the matrix $\boldsymbol{R}$ is uniformly bounded and positive definite, as in the previous subsection, we only need to show that the matrix

$$|Q|^{-1}\boldsymbol{Y} \begin{pmatrix} {}^2\bar{\boldsymbol{S}}_{grav} & {}^2\bar{\boldsymbol{S}}_{int} \\ {}^2\bar{\boldsymbol{S}}_{int}^T & {}^2\bar{\boldsymbol{S}}_{YM} \end{pmatrix} \boldsymbol{Y}$$

can be written in the form (46). The elements of this matrix are given by (42). As with the black hole solutions, we consider the various contributions to the potential in turn.

Firstly,

$$\boldsymbol{Y} = \frac{1}{2\sqrt{2S(0)}\mu} r^{\frac{3}{4}} \boldsymbol{I} + O(r^{\frac{11}{4}})$$

as $r \to 0$ ($R \to \infty$) and

$$\boldsymbol{Y} = \frac{\sqrt{3}}{2\sqrt{-2\Lambda}\mu} r^{-\frac{1}{4}} \boldsymbol{I} + O(r^{-\frac{9}{4}})$$

as $r \to \infty$ ($R \to 0$), where we have used the behaviour of the metric functions in the asymptotic regions (see section 2). Therefore, as $r \to 0$,

$$\begin{aligned} \boldsymbol{Y}\left(\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho}\right)\boldsymbol{Y}|Q|^{-1} &= \frac{3}{16\mu^2} r^2 S(0)^{-1} \boldsymbol{I} + O(r^4); \\ \boldsymbol{Y}\left(\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho\rho}\right)\boldsymbol{Y}|Q|^{-1} &= -\frac{3}{64\mu^2} r \boldsymbol{I} + O(r^3); \end{aligned} \tag{48}$$

and as $r \to \infty$,

$$\begin{aligned}
\boldsymbol{Y}\left(\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho}\right)\boldsymbol{Y}|Q|^{-1} &= \frac{3}{16\Lambda\mu^2}\boldsymbol{I} + O(r^{-2}); \\
\boldsymbol{Y}\left(\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho\rho}\right)\boldsymbol{Y}|Q|^{-1} &= -\frac{3}{64\mu^2}r\boldsymbol{I} + O(r^{-1}).
\end{aligned}$$

Using the fact that the matrix $\boldsymbol{F}$ and its derivative with respect to ρ are both regular and uniformly bounded, for $R, r \in [0, \infty)$, the contribution to the potential $\boldsymbol{F}_{,\rho}\boldsymbol{Y}_2^{-1}\boldsymbol{Y}_{2,\rho}$ is also regular and uniformly bounded, and so can be included in the matrix $\boldsymbol{M}$. However, as $r \to \infty$ (i.e. $R \to 0$), the matrix $\boldsymbol{Y}\left(\boldsymbol{Y}^{-1}\boldsymbol{Y}_{,\rho\rho}\right)\boldsymbol{Y}|Q|^{-1}$ diverges like $r \sim R^{-2}$. Therefore this part of the potential will need to be included in the $\boldsymbol{L}$ matrix, which we will consider in detail below.

Next we consider the contribution due to the $\boldsymbol{R}$ transformation, given in (38). The matrix $\bar{\boldsymbol{A}}_{int}$ and its derivative $\bar{\boldsymbol{A}}_{int,\rho}$ are both regular and uniformly bounded on $r \in [0, \infty)$, and so those terms in the potential (38) depending on $\bar{\boldsymbol{A}}_{int}$ can be included in the matrix $\boldsymbol{M}$.

It remains to consider the following part of the potential

$$|Q|^{-1}\boldsymbol{Y}\bar{\boldsymbol{S}}\boldsymbol{Y} = |Q|^{-1}\boldsymbol{Y}\begin{pmatrix} \bar{\boldsymbol{S}}_{grav} & \bar{\boldsymbol{S}}_{int} \\ \bar{\boldsymbol{S}}_{int}^T & \bar{\boldsymbol{S}}_{YM} \end{pmatrix}\boldsymbol{Y},$$

whose components are given in section 4. From our analysis of the potential for black hole solutions (subsection 6.5.1), we know that the potential component S_{33} in $\bar{\boldsymbol{S}}_{grav}$ diverges like r^2 as $r \to \infty$, all other terms in $\bar{\boldsymbol{S}}$ being regular. When multiplied by $\boldsymbol{Y}^2|Q|^{-1}$, this gives a divergent term of the form

$$\frac{1}{2\mu^2}r + O(r^{-1}) = \frac{1}{2\mu^2}R^{-2} + O(R^2).$$

Therefore we define a matrix $\boldsymbol{L}$ (46) to be equal to ${}^1\boldsymbol{L} + {}^2\boldsymbol{L}$, where the only non-zero entry in ${}^1\boldsymbol{L}$ is

$${}^1\boldsymbol{L}_{33} = \frac{1}{2\mu^2};$$

and

$${}^2\boldsymbol{L} = \frac{3}{64\mu^2}\boldsymbol{I}.$$

Therefore the matrix $\boldsymbol{L}$ is uniformly bounded on $R \in [0, \infty)$, and, furthermore, is constant. Since $\mu^2 > 0$, we also have that $\boldsymbol{L}$ is positive definite, so in particular $\boldsymbol{L}(R = 0)$ is positive definite, as required by the nodal theorem.

We subtract $\boldsymbol{L}R^{-2}$ from the full, transformed potential ${}^2\bar{\boldsymbol{S}}|Q|^{-1}$ to give the matrix $\boldsymbol{M}$, which is then uniformly bounded for $R \in [0, \infty)$. The final criterion we need to check before the multi-dimensional nodal theorem can be applied in the next subsection is that the potential ${}^2\bar{\boldsymbol{S}}|Q|^{-1}$ vanishes sufficiently quickly as $R \to \infty$ (i.e. $r \to 0$) that

$${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2 \geq -\beta_c,$$

for all sufficiently large R, where β_c is a constant less than 1/4. This requires a detailed study of the behaviour of the potential near the origin $r = 0$.

We have already considered the form at the origin of the contributions to the potential due to the $\boldsymbol{Y}$ transformation (48). Those parts of the potential (38) arising from the $\boldsymbol{R}$ transformation, as already observed, are bounded at the origin, and so when multiplied by $\boldsymbol{Y}^2|Q|^{-1} \sim r^3$ as $r \to 0$, they vanish at least as quickly as R^{-6}, and so need not be considered further here.

The behaviour of the untransformed potential $\bar{\boldsymbol{S}}$ at the origin $r = 0$ is complicated due to the presence of γ and its derivatives. Near the origin,

$$\gamma = \frac{S(0)}{r} + O(1).$$

By considering each term in the potential in turn, we arrive at the following explicit expression for the untransformed potential near the origin:

$$\begin{aligned} \bar{\boldsymbol{S}} &= \frac{S^2(0)}{r^2} \begin{pmatrix} 2-\lambda & -2\sqrt{\lambda} & 0 & 0 & 0 & 0 \\ -2\sqrt{\lambda} & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda+4 & 2 & -\sqrt{2}\mu & -\sqrt{2\lambda} \\ 0 & 0 & 2 & \lambda+4 & \sqrt{2}\mu & -\sqrt{2\lambda} \\ 0 & 0 & -\sqrt{2}\mu & \sqrt{2}\mu & \lambda+4 & 0 \\ 0 & 0 & -\sqrt{2\lambda} & -\sqrt{2\lambda} & 0 & \lambda \end{pmatrix} + O(r^{-1}) \\ &= \frac{S^2(0)}{r^2}\mathcal{M} + O(r^{-1}); \end{aligned}$$

where, in the second line above, we have defined the matrix $\mathcal{M}$. Notice that the entry in the first row and column in the matrix $\mathcal{M}$ which is negative for $\ell \geq 2$ originates from the (qq)-term in the potential ${}^2\bar{\boldsymbol{S}}$, including contributions from both the untransformed potential term S_{22} (8) and the remainder term $\mathcal{S}_{qq}$ (30). Multiplying this by $\boldsymbol{Y}^2|Q|^{-1}$ and adding the contribution to the potential from the $\boldsymbol{Y}$ transformation (whose behaviour near the origin is given by (48)), we have, for the complete potential ${}^2\bar{\boldsymbol{S}}$,

$${}^2\bar{\boldsymbol{S}} = \frac{1}{4\mu^2}\mathcal{M}r + \frac{3}{64\mu^2}\boldsymbol{I}r + O(r^2) = \left(\frac{1}{4\mu^2}\mathcal{M} + \frac{3}{64\mu^2}\boldsymbol{I}\right)R^{-2} + O(R^{-4}),$$

as $R \to \infty$. Therefore the potential vanishes like R^{-2} as $R \to \infty$.

In order to check whether ${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2 \geq -\beta_c$ for all sufficiently large R, where $\beta_c < 1/4$, we need to calculate the eigenvalues of the matrix $\mathcal{M}$. Firstly, arising from the bottom right hand part of the matrix,

$$\ell(\ell-1), \qquad (\ell+1)(\ell+2),$$

both these eigenvalues repeated twice. For $\ell \geq 1$, these eigenvalues are non-negative. From the top left-hand part of the matrix we have the single eigenvalues

$$-\lambda, \qquad \lambda + 2 = \mu^2.$$

One of these eigenvalues is positive, the other negative, leading to a negative eigenvalue of ${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2$. However, this negative eigenvalue of ${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2$ takes the form

$$-\frac{1}{4}+\frac{35}{64\mu^2}=-\beta_c>-\frac{1}{4}.$$

Therefore ${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2 > -\beta_c$ for all sufficiently large R, where we have defined the constant $\beta_c < 1/4$ in the equation above.

As an aside, we note that the eigenvector corresponding to the negative eigenvalue of $\mathcal{M}$ has the form

$$(q=\sqrt{\lambda}g, g, 0, 0, 0, 0). \tag{49}$$

We are working in a gauge in which $t = p$. This condition does not completely specify the gauge, and some freedom remains in our quantities q, g, b, c, d, e. Physically we are interested only in gauge-invariant quantities. One such gauge-invariant is ζ, given in (19). Since we are working in a gauge in which $t = p$, near the origin ζ has the form

$$\zeta = g - \frac{1}{\sqrt{\lambda}}q.$$

Therefore the eigenvector (49) corresponds to a perturbation in which the gauge-invariant ζ vanishes near the origin, and so represents a pure gauge perturbation. The reader may wonder, at this point, whether the existence of the negative eigenvalue in $\mathcal{M}$ is due to our leaving part of the potential term S_{22} in the remainder $\mathcal{P}$. In fact, even if we had not performed that step, we would still obtain a negative eigenvalue in the corresponding $\mathcal{M}$ matrix. However, in that case the eigenvalue is considerably more algebraically complicated than the value $-\lambda$ we have in this case, and, furthermore, the associated eigenvector is not so easily interpreted as being pure gauge.

We note furthermore that, in order to construct the potential in such a way that the conditions for the application of the nodal theorem are satisfied, it was essential that we used a co-ordinate R which was zero at infinity ($r \to \infty$), and tended to infinity at the origin ($r = 0$). Otherwise, we would have been forced to construct an $\boldsymbol{L}$ matrix such that, at $R = 0$, this matrix had a negative eigenvalue, whereas the nodal theorem requires that $\boldsymbol{L}$ is positive definite at $R = 0$. By setting $R \to \infty$ at the origin $r = 0$, the matrix having the negative eigenvalue is instead the limit of ${}^2\bar{\boldsymbol{S}}|Q|^{-1}R^2$ as $R \to \infty$, which need not be positive definite, but instead must simply be bounded below by a constant strictly greater than $-1/4$. Including the factor of $(2\sqrt{2}\mu)^{-1}$ in $\boldsymbol{Y}$, we have been able to ensure that the only negative eigenvalue of this matrix at $R \to \infty$ is strictly greater than $-1/4$, thus satisfying the criteria.

To summarize, in this subsection we have shown that, for soliton solutions, the potential has the form which enables the application of the multi-dimensional nodal theorem. We are now in a position to complete our stability proof.

6.6 Analyticity argument

In the previous subsection we cast the pulsation equations in a form which enables us to use the multi-dimensional nodal theorem. From section 6.3, we know that there are no unstable modes of the system when $|\Lambda| = \infty$. We now wish to extend this result to sufficiently large $|\Lambda|$. To do this, we consider the function $\mathfrak{F} := R \mapsto \det \mathcal{U}_{a_1}(R)$ (see section 6.1 for definitions), which we know has no nodes in the interval (a_1, a_2) for sufficiently small a_1 and sufficiently large a_2 when $|\Lambda| = \infty$. In this subsection we shall show that $\mathfrak{F}$ is analytic in $|\Lambda|$ (or, equivalently, $\xi = -\Lambda^{-1}$), and therefore, in a neighbourhood of $\xi = 0$, the function $\mathfrak{F}$ will also have no zeros (in an appropriate interval (a_1, a_2)), which means that our system of pulsation equations has no instabilities.

Since the function $\mathfrak{F}$ is defined in terms of the matrix $\mathcal{U}_{a_1}$, having columns $u_j(R)$, where the u_j are the solutions of the initial value problems:

$$\mathcal{O} u_j = 0, \qquad a_1 < R < \infty, \qquad u_j(a_1) = 0, \qquad \frac{du_j}{dR}(a_1) = e_j, \qquad j = 1, \ldots, 6 \tag{50}$$

(where the operator $\mathcal{O}$ is defined in (31)), $\mathfrak{F}$ will be analytic if we can prove that each $u_j(R)$ is analytic in ξ. From [4], the background solutions are analytic in ξ in a neighbourhood of $\xi = 0$, so all the elements in the matrices in $\mathcal{O}$ are analytic in ξ for sufficiently small ξ. The differential equation for u_j has regular singular points at $R = 0$ and ∞, but all $R \in [a_1, a_2]$ with $a_1 > 0$ and $a_2 > a_1$ are regular points of the differential equation (50).

Define a new independent variable $\hat{R}$ by $\hat{R} = R - a_1$, and make R and ξ into dependent variables. Then we consider the differential equations in (50) (but now with $\hat{R}$ as the variable) together with

$$\frac{dR}{d\hat{R}} = 1, \qquad \frac{d\xi}{d\hat{R}} = 0.$$

The initial conditions are now:

$$u_j(0) = 0, \qquad \frac{du_j}{d\hat{R}}(0) = e_j, \qquad R(0) = a_1, \qquad \xi(0) = \xi.$$

With this set-up, $\hat{R} = 0$ is a regular point of the system of differential equations, and so standard theorems (see, for example, proposition 1 of [12]) tell us that we have a solution of the initial value problem, at least in a neighbourhood of $\hat{R} = 0$, which is analytic in $\hat{R}$ and the initial parameters a_1 and ξ, for all $a_1 > 0$ and ξ sufficiently small. Furthermore, since the differential equations are linear, and the only regular singular points are at $\hat{R} = -a_1$ and $\hat{R} = \infty$, any solution which exists in a neighbourhood of $\hat{R} = 0$ can be extended to $\hat{R}$ sufficiently large and positive, and will remain analytic in $\hat{R}$, a_1 and ξ.

To summarize, for all $a_1 > 0$ and sufficiently small ξ, we have solutions u_j to the initial value problems (50) which are defined on $R \in [a_1, a_2]$ for all $a_2 > a_1$, and are analytic in ξ and R. Therefore, for all $a_1 > 0$ and sufficiently small ξ we have a function $\mathfrak{F}$ which is defined on $R \in [a_1, a_2]$ for all $a_2 > a_1$ and analytic in ξ and R. We have already shown that, when $\xi = 0$, the function $\mathfrak{F}$ has no zeros in the interval (a_1, a_2) for all sufficiently small a_1 and sufficiently large a_2. We stress once again that $\mathfrak{F}$ must vanish at $R = a_1$ (i.e. $\hat{R} = 0$) by the initial conditions on the u_j (50). In other words, when considered as a function of $\hat{R}$, the function $\mathfrak{F}$ has no zeros on $\hat{R} \in (0, \infty)$, for sufficiently small a_1, when $\xi = 0$. We now invoke analyticity to state that for sufficiently small ξ and a_1, the function $\mathfrak{F}$ will still have no zeros on $\hat{R} \in (0, \infty)$, i.e. in the interval $R \in (a_1, a_2)$ for sufficiently large a_2. Therefore the system of pulsation equations remains stable for sufficiently small (non-zero) ξ.

7 Conclusions

This paper completes the proof that solitons and black hole solutions to $\mathfrak{su}(2)$ EYM theory with a negative cosmological constant are stable if the magnitude of the cosmological constant is sufficiently large. Here we considered the even-parity sector of perturbations, which proved to be considerably more complex than the odd-parity sector studied in [8]. It is testament to the power of the curvature-based formalism employed here that we have been able to complete this proof. Unlike a metric-based formulation, this formalism yields a symmetric wave operator for the perturbations. Using the nodal theorem, the idea is then to count the number of nodes of the 'determinant of the zero modes' which gives the number of bound states of the spatial part of the wave operator. Here, we have shown analytically that this determinant has no nodes by taking the limit $\Lambda \to -\infty$. Using the fact that the background solutions depend analytically on Λ we have been able to extend this result to large $|\Lambda|$. However, by counting the nodes using a numerical code, it would be straightforward to extend the stability analysis to any value of Λ. Furthermore, using a numerical code, we also expect that the non-spherical stability of the asymptotically flat black holes and solitons [2, 3] and of the topological black holes [13] could be discussed on the same lines as in the present article. On the other hand, our analysis also revealed the limitations of the curvature-based formalism: Since one does not fix the residual gauge completely, there can still be gauge modes. For solitons, we have seen that these modes make it difficult to write the perturbation equations in the form required by the nodal theorem.

Physically, our result shows that the presence of a large, negative cosmological constant stabilizes black hole hair, and also non-Abelian solitons. This is important in the light of the recent revival in the study of asymptotically anti-de Sitter spacetimes, due in part to the (rather less recent) idea of "holography". The concept of holography (and its more recent relative, the adS/CFT (conformal field theory) correspondence, see, for example, [19] for a review) centres around the idea that behaviour (for example, of a supergravity theory) in the bulk of an asymptotically adS spacetime is governed by the behaviour of a related theory on the boundary. In terms of these ideas, the black holes considered in this paper are manifestly "non-holographic", since their structure is not determined by quantities (such as magnetic charge) measurable at infinity. The consequences of the existence of these stable black holes for the adS/CFT correspondence remain to be investigated.

On a classical and semi-classical level, there are various open avenues of study. Firstly, we would like to extend our current work to seek rotating black holes and solitons in this model. If the rotation is slow, one could use the linearized theory to find such solutions (they are linked to the zero modes of the system considered here). In the asymptotically flat case, slowly rotating solutions were found in [20] and some of their rapidly rotating counterparts in [21]. Secondly, the behaviour of quantum fields on hairy black hole backgrounds has received little attention to date (no doubt partly due to the classical instability of most hairy black holes). These questions shall be the subject of future work.

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A On the stability of the RN-adS solution

In this appendix, we discuss the positiveness of the operator

$$\mathcal{A}_+ \equiv -\partial_\rho^2 + N\left[U + W\begin{pmatrix} -3M & \sqrt{4G\lambda} \\ \sqrt{4G\lambda} & 3M \end{pmatrix}\right]$$

where the functions N, U and W are given in section 5.4.

We start by showing that the function $f_\tau(r) = \lambda/2 + 3M/r - 2G/r^2$ which appears in the denominators of U and W is positive for all $\lambda \geq 0$ and all $r \geq r_h$. Introducing the dimensionless quantities $x = r/M$, $q^2 = G/M^2$ and $l = -3/(\Lambda M^2)$, we can write N as

$$N(x) = 1 - \frac{2}{x} + \frac{q^2}{x^2} + \frac{x^2}{l}\,.$$

Using the fact that $N_{,x}(x_h) > 0$ and $N(x_h) = 0$, one can show that $3x_h - 2q^2 > x_h^2 > 0$. Therefore, the function $3/x - 2q^2/x^2$ is positive outside the horizon, and $f_\tau(r) > 0$ for all $r \geq r_h$.

When $\ell \geq 2$, one can generalize Chandrasekhar's arguments [22] in order to relate the operator $\mathcal{A}_+$ to the corresponding operator $\mathcal{A}_-$ in the odd-parity sector by a supersymmetric-like transformation. This works as follows: First, we diagonalize the operator $\mathcal{A}_+$, obtaining the two decoupled operators

$$\mathcal{A}_{i+} = -\partial_\rho^2 + N\left[U + \sigma_i W\right]$$

where $\sigma_1 = -\sqrt{9M^2 + 4G\lambda}$, $\sigma_2 = +\sqrt{9M^2 + 4G\lambda}$. One can check that $\mathcal{A}_{i+}$ can be factorized according to

$$\mathcal{A}_{i+} = \mathcal{B}_i \mathcal{B}_i^\dagger - \omega_i^2,$$

where

$$\mathcal{B}_i \equiv \partial_\rho + \frac{q_i N}{r(\lambda r + q_i)} + \omega_i\,, \quad \mathcal{B}_i^\dagger \equiv -\partial_\rho + \frac{q_i N}{r(\lambda r + q_i)} + \omega_i\,,$$

with $q_i = 3M - \sigma_i$ and where $\omega_i = (\ell-1)\ell(\ell+1)(\ell+2)/2q_i$ are the algebraic special frequencies. (Note that $(\lambda r + q_1)(\lambda r + q_2) = 2\lambda r^2 f_\tau > 0$, so $\lambda r + q_2$ is positive.) The supersymmetric partners of $\mathcal{A}_{i+}$ are

$$\mathcal{A}_{i-} = \mathcal{B}_i^\dagger \mathcal{B}_i - \omega_i^2.$$

Explicitly, one finds

$$\mathcal{A}_{i-} = -\partial_\rho^2 + N\left[U_- + \sigma_i W_-\right]$$

where $U_- = (\ell(\ell+1) - 3M/r + 4G/r^2)/r^2$ and $W_- = 1/r^3$, which is equivalent to the operator in the odd-parity sector! Since we have shown in [8] that there are no unstable modes in the odd-parity sector, and since for perturbations with compact support, $\mathcal{A}_{i+}$ and $\mathcal{A}_{i-}$ have the same spectrum, it follows that there are no unstable modes in the even-parity sector either. At this point, it is interesting to note that the functions

$$u_i = \frac{e^{\omega_i \rho}}{\lambda + \frac{q_i}{r}}$$

fulfill $\mathcal{B}_i^\dagger u_i = 0$ and therefore are eigenfunctions of the operators $\mathcal{A}_{i+}$ with negative eigenvalue $-\omega_i^2$. However, these functions do not satisfy homogeneous Dirichlet boundary conditions at infinity and should, therefore, not be considered in our stability analysis. Nevertheless, the fact that u_i are finite as $r \to \infty$ could affect the equivalence of quasi-normal modes in the odd- and even-parity sector. It has been shown recently [17] that – unlike for the Schwarzschild or RN case – the quasi-normal frequencies of the Schwarzschild-adS solutions are different in the odd- and even-parity sector.

Finally, we discuss the case $\ell = 1$, where $\mathcal{A}_+$ reduces to

$$-\partial_\rho^2 + N\left[U + 3MW\right].$$

Explicitly, one finds

$$r^2(2f_\tau)^2\left[U + 3MW\right] = \frac{8}{lx^6}\left[9lx^4 + 4q^4x^4 - 18lq^2x^2 + 16lq^4x - 4lq^6\right].$$

Using the fact that $q^2 \leq 1$ for a black hole, it is not difficult to show that this is positive when $x \geq 1$. This shows the stability of all black holes with $r_h \geq M$.

B Factorization of the spatial operator in the odd-parity case

In the odd-parity sector [8], we were able to factorize the spatial operator such that the perturbation equations assume the form

$$\ddot{U}_{odd} + \boldsymbol{B}^\dagger \boldsymbol{B} U_{odd} = 0,$$

where U_{odd} is the vector containing the odd-parity perturbations. For $\ell \geq 2$, the operator $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \partial_\rho + \begin{pmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_t^T \\ \boldsymbol{C}_t & \boldsymbol{D}\left(\boldsymbol{X}_0 + T(\rho)\boldsymbol{X}_1\right)\boldsymbol{D} - \boldsymbol{A}_2 \end{pmatrix},$$

where

$$\boldsymbol{C}_1 = \begin{pmatrix} \frac{\gamma}{r}\left(\frac{r}{\gamma}\right)_{,\rho} & 0 & u \\ 0 & -\frac{\gamma_{,\rho}}{\gamma} & 0 \\ 0 & 0 & -\frac{\gamma_{,\rho}}{\gamma} \end{pmatrix},$$

$$\boldsymbol{C}_t^T = \gamma \begin{pmatrix} -\sqrt{\lambda} & -v & 0 & 0 \\ 0 & -\mu & -\sqrt{2}w & 0 \\ 0 & w & \frac{\mu}{\sqrt{2}} & -\sqrt{\frac{\lambda}{2}} \end{pmatrix}.$$

In addition, $\boldsymbol{D} = \mathrm{diag}(\mu\sqrt{\lambda}, \mu, \sqrt{2}, \mu\sqrt{\lambda/2})^{-1}$, and the matrices $\boldsymbol{X}_0$, $\boldsymbol{X}_1$ and $\boldsymbol{A}_2$ are

$$\boldsymbol{X}_0 = \begin{pmatrix} -\lambda\mu^2\frac{r_{,\rho}}{r} + f_1 w\,uv & sym. & sym. & sym. \\ -f_1 w\,u & 2ww_{,\rho} & sym. & sym. \\ f_1 u & -2w_{,\rho} & 0 & sym. \\ 2w^2(1-w^2)u & 2w^2 w_{,\rho} & -2ww_{,\rho} & -f_2 ww_{,\rho} \end{pmatrix},$$

$$\boldsymbol{X}_1 = \begin{pmatrix} 2w^2v^2 & sym. & sym. & sym. \\ -2w^2 v & 2w^2 & sym. & sym. \\ 2wv & -2w & 2 & sym. \\ f_2 wv & -f_2 w & f_2 & \frac{1}{2}f_2^2 \end{pmatrix},$$

$$\boldsymbol{A}_2 = -\frac{u}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

where $f_1 = \lambda + 2w^2$ and $f_2 = \mu^2 - 2w^2$.

Finally, the function $T(\rho)$ has to satisfy the differential equation

$$-\partial_\rho T + \mathcal{A}T^2 + \mathcal{B}T + \mathcal{C} = 0, \tag{51}$$

with

$$\mu^2\lambda\mathcal{A} = \frac{8G}{r^2}w^2(1-w^2)^2 + 4\left(w^2 - 1 - \frac{\lambda}{4}\right)^2 + 4\lambda + \frac{7}{4}\lambda^2,$$

$$\mu^2\lambda\mathcal{B} = 8\left[\frac{G}{r^2}(\lambda + 2w^2) + 1\right](w^2-1)ww_{,\rho}\,,$$

$$\mu^2\lambda\mathcal{C} = \left[\frac{2G}{r^2}(\lambda + 2w^2)^2 + 2\lambda + 4w^2\right]w_{,\rho}^2 - \mu^2\lambda\left(\frac{\lambda}{2} + w^2\right)\gamma^2.$$

In [8], we have shown that equation (51) admits global solutions, at least when $|\Lambda|$ is sufficiently large. (There are two small errors in the expressions for $\mathcal{B}$ and $\mathcal{C}$ which are published in Ref. [8]. However, these errors do not affect our proof that there exist global solutions to equation (51).) Following the same lines, it is also easy to show that equation (51) admits global solutions when $|w| = 1$ or $w = 0$.

In the sector with $\ell = 1$, the matrix-valued operator $\boldsymbol{B}$ can even be given explicitly, see [8].

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